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Note on NLSM tree amplitudes and soft theorems

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Abstract This note provides a new point of view for bootstrapping the tree amplitudes of the nonlinear sigma model (NLSM). We use the universality of single soft behavior, together with the double copy structure, to completely determine the tree amplitudes of the NLSM. We first observe Adler's zero for four-point NLSM amplitudes, by considering kinematics. Then we assume the universality of Adler's zero and use this requirement to construct general tree amplitudes of the NLSM in the expanded formula, i.e., the formula of expanding NLSM amplitudes to bi-adjoint scalar amplitudes, which allows us to give explicit expressions of amplitudes with arbitrary numbers of external legs. The construction does not require the assumption of quartic diagrams. We also derive double soft factors for NLSM tree amplitudes based on the resulting expanded formula, and the results are consistent with those in the literature.

1 Introduction

Soft theorems describe the universal infrared (IR) behavior of scattering amplitudes when one or more external massless momenta are taken to near zero. This limit can be achieved by rescaling the massless momenta via a soft parameter as $k^\mu \to \tau k^\mu$, and taking the limit $\tau \to 0$. Soft theorems then state the factorization of amplitudes. For instance, when one of the external gravitons is taken to be soft, the $(n+1)$-point general relativity (GR) amplitude factorizes as [1–3]

$$\mathcal{A}_{n+1} \to \left(\tau^{-1}\, S_h^{(0)} + \tau^0\, S_h^{(1)} + \tau\, S_h^{(2)} + \cdots\right) \mathcal{A}_n, \qquad (1)$$

where $\mathcal{A}_n$ is the sub-amplitude of $\mathcal{A}_{n+1}$, which is generated from $\mathcal{A}_{n+1}$ by removing the soft external graviton. The universal operators $S_h^{(0)}$, $S_h^{(1)}$, and $S_h^{(2)}$ are called soft factors, or soft operators, at leading, sub-leading, and sub-sub-leading orders.

Soft theorems have been exploited in the construction of tree amplitudes, such as the related on-shell recursion relations, the inverse soft theorem program, and so on [4,6–12]. Impressively, in [10], it was shown that soft theorems uniquely fix tree amplitudes. However, the previous constructions mentioned above require the explicit forms of soft factors. To derive soft factors, one needs a certain expression for amplitudes, for example, summations of contributions from Feynman diagrams [13,14], Britto–Cachzo–Feng–Witten (BCFW) on-shell recursion relations [1,15–17], or Cachazo–He–Yuan (CHY) contour integrals [2,3,18–22]. Then the question arises: of the tree amplitude and the soft factor, which one determines the other one?

On the other hand, the factorization in (1) has an intuitive physical picture. Roughly speaking, in the soft limit, the soft particle can be thought of as vanishing, leaving a lower-point amplitude with the soft external leg removed, and the universal soft factors carried by the soft particle. Thus, to avoid the logical confusion mentioned previously, it is natural to ask whether we can take the soft behavior in (1) as the principle and use it to construct tree amplitudes without knowing the explicit forms of soft factors. In the recent work of one of the authors for the present note, it was shown that such construction can be realized at least for tree amplitudes of the Yang–Mills scalar, pure Yang–Mills, Einstein–Yang–Mills, and pure gravitational theories [23]. The factorization behaviors and the universality of soft factors, together with the double copy structure [24–28], completely determine tree amplitudes of these theories.

This note is a generalization of the previous work in [23] and applies a similar idea to consider the tree amplitudes of the NLSM. The NLSM tree amplitudes vanish in the single soft limit (one of the external legs being soft), known as Adler's zero [29]. For such a case, the factorization behavior in (1) does not occur. However, one can still

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talk about the universality of the single soft behavior. As will be seen in Sect. 3.1, it is easy to determine Adler's zero for the four-point NLSM tree amplitude by considering kinematics. Then, the universality of soft behavior indicates that all NLSM tree amplitudes vanish in the single soft limit. This condition, together with the double copy structure and the requirement for manifest permutation invariance among external legs, uniquely determines the general NLSM tree amplitudes. We obtain the general expression of NLSM tree amplitudes with an arbitrary even number of external legs in the expanded formula, i.e., the formula of expanding NLSM amplitudes to bi-adjoint scalar (BAS) amplitudes. Using the resulting expanded formula, we also re-derive double soft factors for the NLSM tree amplitudes [30,31], which describe the behavior when two external scalars are taken to be soft simultaneously.

It is well known that Adler's zero together with other constraints completely determines the tree-level NLSM amplitudes [32–34]—for instance, using the Adler's zero condition with the singularity structure [4,5,34] or using Adler's zero with the double soft theorem for the NLSM [10]. Meanwhile, imposing Bern–Carrasco–Johansson (BCJ) relations and the assumption of quartic diagrams also uniquely fixes NLSM tree amplitudes [34,35]. Compared with constructions in the literature, our method in this note is based on different assumptions, which are listed as follows:

- The amplitude describes the scattering of massless scalars with a single coupling constant.
- The kinematic part of the amplitude has a mass dimension of 2.
- The amplitude is color-ordered.
- The universality of single soft behavior: the single soft behavior of low-point amplitudes holds for general higher-point ones.
- The double copy structure: when expanding to a double-ordered BAS basis, coefficients depend on only one ordering.
- The manifest permutation symmetry among external legs.

First, our method does not assume the locality, i.e, does not assume quartic diagrams. Secondly, we do not employ the double soft theorem. An advantage of our construction is that we obtain the explicit expression for general NLSM amplitudes with an arbitrary number of external legs, while in other studies, the explicit formula is absent. This is because we have chosen the expanded formula to represent general amplitudes, and the coefficients have an elegant universal form.

Let us offer some remarks for the universality of soft behaviors. As will be seen, Adler's zero plays a central role in our bottom-up construction. However, from the bottom-up perspective, without the aid of a Lagrangian, one needs to explain why the amplitudes should exhibit Adler's zero. In this note, Adler's zero is observed from the four-point amplitude determined by bootstrapping, then imposed to higher-point amplitudes due to the universality of soft behaviors. In other words, logically, we did not know Adler's zero at the beginning. For example, suppose that the four-point NLSM amplitudes behave similarly to the gravitational ones in (1) when taking one of the external momenta to be soft; then we will impose the universal soft factors to higher-point amplitudes. We emphasize that Adler's zero for four-point amplitudes is well known and obvious, and technically there is no difference between assuming the universality of soft behaviors and assuming Adler's zero. However, it seems that the universality is a more general feature since it also holds for the Yang–Mills scalar, pure Yang–Mills, Einstein–Yang–Mills, and pure gravitational amplitudes [23]. Furthermore, we want to determine whether the method developed in our previous work [23] based on the universality of soft behaviors and double copy structure can be applied to other theories. Thus, we assume the universality instead of Adler's zero.

The remainder of this note is organized as follows. In Sect. 2, we provide a quick introduction to the necessary background, including BAS tree amplitudes and expansions of other amplitudes to them. In Sect. 3, we use the the universality of single soft behavior to construct the NLSM tree amplitudes in the expanded formula. In Sect. 4, we derive the double soft factor for NLSM tree amplitudes based on the expanded formula obtained in Sect. 3. Finally, we close with a brief summary in Sect. 5.

2 Background

In this section we briefly review the necessary background. In Sect. 2.1, we introduce the tree-level amplitudes of bi-adjoint scalar (BAS) theory. Some notations and techniques which will be used in subsequent sections are also included. In Sect. 2.2, we introduce the expansions of tree amplitudes to BAS amplitudes, including the choice of basis, as well as the double copy structure for coefficients.

2.1 Tree-level BAS amplitudes

The BAS theory describes the bi-adjoint scalar field $\phi_{a\bar{a}}$ with the Lagrangian

$$\mathcal{L}_{\rm BAS} = \frac{1}{2}\,\partial_\mu\phi^{a\bar{a}}\,\partial^\mu\phi^{a\bar{a}} - \frac{\lambda}{3!}\,f^{abc}\,f^{\bar{a}\bar{b}\bar{c}}\,\phi^{a\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}}, \tag{2}$$

where the structure constant f^{abc} and generator T^a satisfy

$$[T^a, T^b] = i f^{abc} T^c, \tag{3}$$

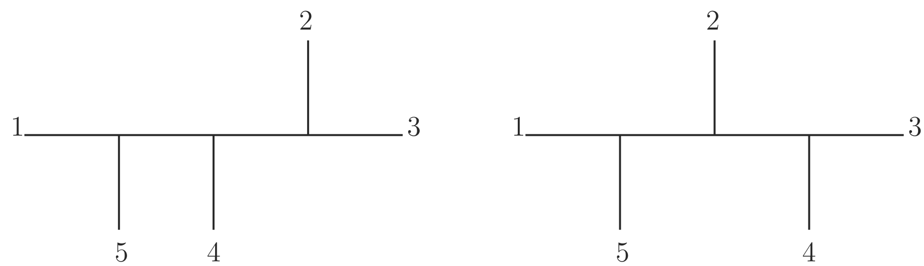

Fig. 1 Two five-point diagrams

and the dual-color algebra encoded by $f^{\bar{a}\bar{b}\bar{c}}$ and $T^{\bar{a}}$ is analogous. The tree-level amplitudes of this theory contain only propagators and can be decomposed into double-color-ordered partial amplitudes via the standard technique. Each double-color-ordered partial amplitude is simultaneously planar with respect to two color orderings, arising from expanding the full n-point amplitude to $\mathrm{Tr}(T^{a_{\sigma_1}} \cdots T^{a_{\sigma_n}})$ and $\mathrm{Tr}(T^{\bar{a}_{\bar{\sigma}_1}} \cdots T^{\bar{a}_{\bar{\sigma}_n}})$, respectively, where σ_i and $\bar{\sigma}_i$ denote the permutations among all external scalars. Here we give the five-point example $\mathcal{A}_S(1,2,3,4,5|1,4,2,3,5)$. In Fig. 1, the first diagram satisfies both of the two color orderings $(1,2,3,4,5)$ and $(1,4,2,3,5)$, while the second one satisfies the ordering $(1,2,3,4,5)$ but not $(1,4,2,3,5)$. Thus, the first diagram is allowed by the dual-color orderings $(1,2,3,4,5|1,4,2,3,5)$, while the second one is not. It is easy to see that other diagrams are also forbidden by the ordering $(1,4,2,3,5)$; thus, the first diagram in Fig. 1 is the only diagram that contributes to the amplitude $\mathcal{A}_S(1,2,3,4,5|1,4,2,3,5)$.

It is convenient to calculate double-color-ordered partial amplitudes via the diagrammatical rules proposed by Cachazo, He, and Yuan [20]. For the above example, one can draw a disk diagram as follows:

- Draw points on the boundary of the disk according to the first ordering $(1,2,3,4,5)$.
- Draw a loop of line segments connecting the points according to the second ordering $(1,4,2,3,5)$.

The obtained disk diagram is shown in the first diagram in Fig. 2. One can see that two orderings share the boundaries $\{1,5\}$ and $\{2,3\}$. These co-boundaries indicate channels $1/s_{15}$ and $1/s_{23}$, therefore the first Feynman diagram in Fig. 1. Then the BAS amplitude $\mathcal{A}_S(1,2,3,4,5|1,4,2,3,5)$ can be computed as

$$\mathcal{A}_S(1,2,3,4,5|1,4,2,3,5) = \frac{1}{s_{23}}\frac{1}{s_{51}}, \tag{4}$$

up to an overall sign. The Mandelstam variable $s_{i\cdots j}$ is defined as

$$s_{i\cdots j} \equiv K_{i\cdots j}^2, \quad K_{i\cdots j} \equiv \sum_{a=i}^{j} k_a, \tag{5}$$

where k_a is the momentum carried by the external leg a.

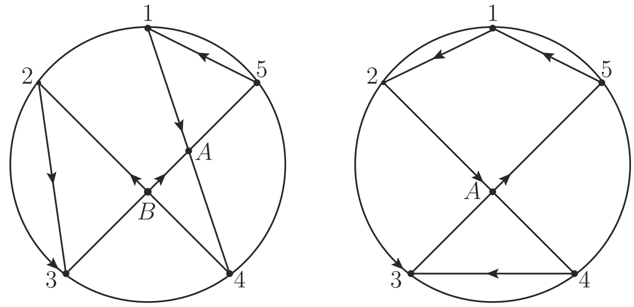

Fig. 2 Diagram for $\mathcal{A}_S(1,2,3,4,5|1,4,2,3,5)$ and $\mathcal{A}_S(1,2,3,4,5|1,2,4,3,5)$

As another example, let us consider the BAS amplitude $\mathcal{A}_S(1,2,3,4,5|1,2,4,3,5)$. The corresponding disk diagram is shown in the second configuration in Fig. 2, and one can see that the two orderings have co-boundaries $\{3,4\}$ and $\{5,1,2\}$. The co-boundary $\{3,4\}$ indicates the channel $1/s_{34}$. The co-boundary $\{5,1,2\}$ indicates the channel $1/s_{512}$, which is equivalent to $1/s_{34}$, as well as sub-channels $1/s_{12}$ and $1/s_{51}$. Using the above decomposition, one can calculate $\mathcal{A}_S(1,2,3,4,5|1,2,4,3,5)$ as

$$\mathcal{A}_S(1,2,3,4,5|1,2,4,3,5) = \frac{1}{s_{34}}\left(\frac{1}{s_{12}} + \frac{1}{s_{51}}\right), \tag{6}$$

up to an overall sign.

The overall sign, determined by color algebra, can be fixed by the following rules:

- Each polygon with an odd number of vertices contributes a plus sign if its orientation is the same as that of the disk, and a minus sign if opposite.
- Each polygon with an even number of vertices always contributes a minus sign.
- Each intersection point contributes a minus sign.

We now apply these rules to previous examples. In the first diagram in Fig. 2, the polygons are three triangles, namely, $51A$, $A4B$, and $B23$, which contribute a plus sign, minus sign, and plus sign, respectively, while two intersection points A and B contribute two minus signs. In the second one in Fig. 2, the polygons are $512A$ and $A43$, which contribute two minus signs, while the intersection point A contributes

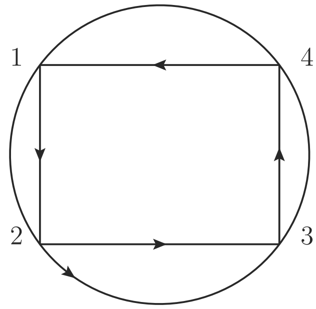

Fig. 3 The overall sign + under the new convention

a minus sign. Then we arrive at the full results

$$\mathcal{A}_{\rm S}(1,2,3,4,5|1,4,2,3,5) = -\frac{1}{s_{23}}\frac{1}{s_{51}},$$
$$\mathcal{A}_{\rm S}(1,2,3,4,5|1,2,4,3,5) = -\frac{1}{s_{34}}\Big(\frac{1}{s_{12}}+\frac{1}{s_{51}}\Big). \tag{7}$$

In the remainder of this note, we adopt another convention for the overall sign. If the line segments form a convex polygon, and the orientation of the convex polygon is the same as that of the disk, then the overall sign is a plus sign. For instance, the disk diagram in Fig. 3 indicates that the overall sign is a plus sign under the new convention, while the old convention gives a minus sign according to the square formed by the four line segments. Note that the diagrammatical rules described previously still give the related sign between different disk diagrams. For example, the two disk diagrams in Fig. 2 show that the relative sign between $\mathcal{A}_{\rm S}(1,2,3,4,5|1,4,2,3,5)$ and $\mathcal{A}_{\rm S}(1,2,3,4,5|1,2,4,3,5)$ is a plus sign. The advantage of the new convention is that when removing a soft external scalar, the resulting sub-amplitude carries the same sign as the original one.

When considering the soft limit, the two-point channels play the central role. Since the partial BAS amplitude carries two color orderings, if the two-point channel contributes $1/s_{ab}$ to the amplitude, the external legs a and b must be adjacent to each other in both orderings. Suppose the first color ordering is $(\cdots,a,b,\cdots)$; then $1/s_{ab}$ is allowed by this ordering. To denote whether it is allowed by the other one, we introduce the symbol δ_{ab}, whose ordering of the two subscripts a and b is determined by the first color ordering.[1] The value of δ_{ab} is $\delta_{ab}=1$ if the other color ordering is $(\cdots,a,b,\cdots)$, and $\delta_{ab}=-1$ if the other color ordering is $(\cdots,b,a,\cdots)$, due to the anti-symmetry of the structure constant, i.e., $f^{abc}=-f^{bac}$, and $\delta_{ab}=0$ otherwise. From the definition, it is straightforward to see that $\delta_{ab}=-\delta_{ba}$, and a simple but useful identity

$$\sum_{b\neq a}\delta_{ab}=0. \tag{8}$$

Before ending this subsection, we discuss the single soft behavior of BAS amplitudes at the leading order. Consider the double-color-ordered BAS amplitude $\mathcal{A}_{\rm S}(1,\ldots,n|\sigma_n)$. We rescale k_i as $k_i\to\tau k_i$ and expand the amplitude in τ. The leading-order contribution manifestly arises from two-point channels $1/s_{1(i+1)}$ and $1/s_{(i-1)i}$, which provide the $1/\tau$-order contributions, namely,

$$\begin{aligned}&\mathcal{A}^{(0)}_{\rm S}(1,\ldots,n|\sigma_n)\\&\quad=\frac{1}{\tau}\Big(\frac{\delta_{i(i+1)}}{s_{i(i+1)}}+\frac{\delta_{(i-1)i}}{s_{(i-1)i}}\Big)\\&\qquad\mathcal{A}_{\rm S}(1,\ldots,i-1,\ni,i+1,\ldots,n|\sigma_n\backslash i)\\&\quad=S^{(0)}_s(i)\,\mathcal{A}_{\rm S}(1,\ldots,i-1,\ni,i+1,\ldots,n|\sigma_n\backslash i),\end{aligned} \tag{9}$$

where $\ni$ stands for removal of the leg i, and $\sigma_n\backslash 1$ signifies the color ordering generated from σ_n by eliminating i. The leading soft operator $S^{(0)}_s(i)$ for the scalar i is extracted as

$$S^{(0)}_s(i)=\frac{1}{\tau}\Big(\frac{\delta_{i(i+1)}}{s_{i(i+1)}}+\frac{\delta_{(i-1)i}}{s_{(i-1)i}}\Big), \tag{10}$$

which acts on external scalars that are adjacent to i in two-color orderings.

2.2 Expanding tree-level amplitudes to the BAS basis

Tree-level amplitudes for massless particles and cubic interactions can be expanded to double-color-ordered BAS amplitudes, given the observation that each Feynman diagram for pure propagators can be mapped to at least one disk diagram whose polygons are all triangles. An illustrative example is given in Fig. 4. For higher-point vertices, one can decompose them into cubic ones via the well-known technique, i.e., inserting $1=D/D$ with the propagator $1/D$ and the numerator D. An example is shown in Fig. 5. Based on such insertions, one can decompose each tree amplitude to tree Feynman diagrams with only cubic interactions. Since each Feynman diagram contributes propagators which can be provided by BAS amplitudes, along with a numerator that is dependent on kinematic variables, one can conclude that each tree amplitude for massless particles can be expanded to double-color-ordered partial BAS amplitudes, with coefficients which are polynomials that are dependent on Lorentz-invariant combinations of external kinematic variables.

To realize the expansion, one needs to find the basis that consists of BAS amplitudes. Such basis can be determined by the well-known Kleiss–Kuijf (KK) relation [36]

$$\mathcal{A}_{\rm S}(1,\vec{\boldsymbol\alpha},n,\vec{\boldsymbol\beta}|\sigma_n)=(-)^{|\boldsymbol\alpha|}\,\mathcal{A}_{\rm S}(1,\vec{\boldsymbol\alpha}\shuffle\vec{\boldsymbol\beta}^{\,T},n|\sigma_n). \tag{11}$$

[1] The Kronecker symbol will not appear in this paper; thus we hope that the notation δ_{ab} will not confuse the readers.

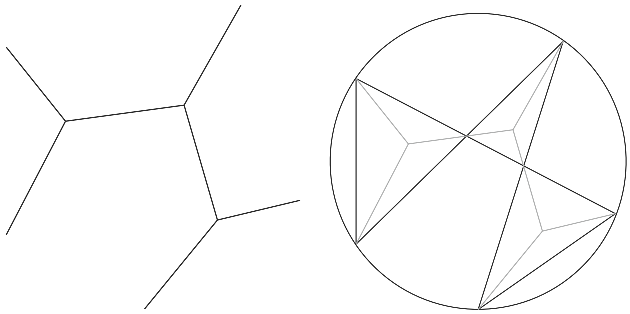

Fig. 4 Map between Feynman diagram and disk diagram

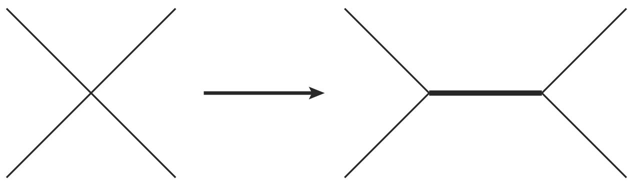

Fig. 5 Turning the four-point vertex to three-point vertices. The bold line corresponds to the inserted propagator $1/D$. This manipulation turns the original numerator N to DN and splits the original coupling constant g to two $\sqrt{g}$ for two cubic vertices

Here, $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$ are two ordered subsets of external scalars, and $\vec{\boldsymbol{\beta}}^T$ represents the ordered set generated from $\vec{\boldsymbol{\beta}}$ by reversing the original ordering. The n-point BAS amplitude $\mathcal{A}_S(1, \vec{\boldsymbol{\alpha}}, n, \vec{\boldsymbol{\beta}}|\sigma_n)$ on the l.h.s of (11) carries two color orderings: one is $(1, \vec{\boldsymbol{\alpha}}, n, \vec{\boldsymbol{\beta}})$, and the other is denoted by σ_n. The symbol $\sqcup\!\sqcup$ means summing over all possible shuffles of two ordered sets $\vec{\boldsymbol{\beta}}_1$ and $\vec{\boldsymbol{\beta}}_2$, i.e., all permutations in the set $\vec{\boldsymbol{\beta}}_1 \cup \vec{\boldsymbol{\beta}}_2$, while preserving the orderings of $\vec{\boldsymbol{\beta}}_1$ and $\vec{\boldsymbol{\beta}}_2$. For instance, suppose that $\vec{\boldsymbol{\beta}}_1 = \{1, 2\}$ and $\vec{\boldsymbol{\beta}}_2 = \{3, 4\}$; then

$$\begin{aligned} &\mathcal{A}(\vec{\boldsymbol{\beta}}_1 \sqcup\!\sqcup \vec{\boldsymbol{\beta}}_2) \\ &= \mathcal{A}(1,2,3,4) + \mathcal{A}(1,3,2,4) + \mathcal{A}(1,3,4,2) \\ &\quad + \mathcal{A}(3,1,2,4) + \mathcal{A}(3,1,4,2) + \mathcal{A}(3,4,1,2). \end{aligned} \tag{12}$$

The analogous KK relation holds for the other color ordering σ_n. The KK relation implies that different double-color-ordered BAS amplitudes are not independent; thus, the basis can be chosen as BAS amplitudes $\mathcal{A}_S(1, \sigma_1, n|1, \sigma_2, n)$, with 1 and n fixed at two ends in each color ordering. We call such a basis the KK BAS basis. Based on the discussion above, the KK BAS basis can provide any structure of massless propagators; thus, any amplitude which includes only massless particles can be expanded to this basis.[2] In other words, the basis provides propagators, and the coefficients in expansions provide numerators. From this point of view, one can regard the BAS KK basis as the complete set of different structures of propagators, and disregard the corresponding Lagrangian in (2).

In this note, we will consider the expansion of NLSM amplitudes. The color-ordered NLSM amplitude $\mathcal{A}_N(1, \sigma_{n-2}, n)$ can be expanded to the KK BAS basis as

$$\begin{aligned} &\mathcal{A}_N(1, \sigma_{n-2}, n) \\ &= \sum_{\sigma'_{n-2}} \mathcal{C}(\sigma'_{n-2}, k_i)\, \mathcal{A}_S(1, \sigma'_{n-2}, n|1, \sigma_{n-2}, n), \end{aligned} \tag{13}$$

where σ_{n-2} and σ'_{n-2} are permutations among $(n-2)$ external legs in $\{2, \cdots, n-1\}$. The double copy structure [24–28] indicates that the coefficient $\mathcal{C}(\sigma'_{n-2}, k_i)$ is dependent on momenta k_i carried by external scalars, and permutations σ'_{n-2}, but is independent of the permutation σ_{n-2}.[3] Thus, suppose we replace $(1, \sigma_{n-2}, n)$ by the more general ordering σ_n among all external legs, without fixing 1 and n at any position; the expansion in (13) still holds. The coefficients $\mathcal{C}(\sigma'_{n-2}, k_i)$ will be constructed in the next section.

Note that the expanded formula (13) indicates the BCJ relations among NLSM amplitudes with different σ_n, since amplitudes $\mathcal{A}_S(1, \sigma'_{n-2}, n|\sigma_n)$ with fixed σ'_{n-2} satisfy BCJ relations individually. This means that we do not need to solve constraints from BCJ relations as in [35]. Indeed, using BCJ relations, together with the appropriate structure of poles, one can also completely determine the NLSM amplitudes, as shown in [34,35]. In this note, we take a different path: we regard the universality of soft behaviors as the principle, and do not use unitarity and locality as tools for construction.

3 Expanded NLSM amplitudes

Tree amplitudes can be expressed in various formulas, and in this note we choose the expansions of tree amplitudes to the KK BAS basis as discussed in Sect. 2.2. To determine the NLSM tree amplitudes, it is sufficient to fix coefficients $\mathcal{C}(\sigma'_{n-2}, k_i)$ in (13). This is the goal in the present section.

The standard NLSM Lagrangian in the Cayley parameterization is given as

$$\mathcal{L}_N = \frac{1}{8\lambda^2}\,\mathrm{Tr}(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U}), \tag{14}$$

[2] The well-known Bern–Carrasco–Johansson (BCJ) relation [25–28] links BAS amplitudes in the KK basis together, and the independent BAS amplitudes can be obtained by fixing three legs at three particular positions in the color orderings. However, in the BCJ relation, the coefficients of BAS amplitudes depend on Mandelstam variables, which leads to poles in coefficients when expanding to the BCJ basis. On the other hand, when expanding to the KK basis, one can find the expanded formula in which the coefficients contain no poles. In this paper, we choose the KK basis since we hope that all poles of tree amplitudes are included in the basis, and the coefficients serve only as numerators.

Footnote 2 continued

[3] Originally, the double copy meant that the GR amplitude could be factorized as $\mathcal{A}_G = \mathcal{A}_Y \times \mathcal{S} \times \mathcal{A}_Y$, where the kernel $\mathcal{S}$ was obtained by inverting propagators. Our assumption that the coefficients depend on only one color ordering is equivalent to the original version; see [23].

with

$$\mathrm{U} = (\mathbb{I} + \lambda\Phi)\,(\mathbb{I} - \lambda\Phi)^{-1}, \tag{15}$$

where $\mathbb{I}$ is the identity matrix, and $\Phi = \phi_I T^I$, with T^I the generators of $U(N)$. Fields ϕ_I describe massless scalars, and the accompanying generators T^I indicate the color ordering for the corresponding partial tree amplitudes. From the Lagrangian in (14), we see that the mass dimension of coupling constant λ is $(2-d)/2$, in d-dimensional spacetime. The mass dimension of the n-point amplitude is $d - \frac{d-2}{2}n$, and the coupling constants contribute $(2-d)(n-2)/2$; thus the kinematic part must have mass dimension 2.

Our purpose, therefore, is to find the n-point amplitude (kinematic part) $\mathcal{A}_{\mathrm{N}}(\sigma_n)$ for pure massless scalars that carry the color ordering σ_n among n external scalars, which has mass dimension 2. As will be seen, the requirements mentioned above, together with the universality of soft behavior and the permutation symmetry among external scalars, are sufficient to determine the NLSM tree amplitudes completely, without using any other information. In this sense, in the remainder of this section, one can disregard the traditional Lagrangian and Feynman rules and concentrate only on the color ordering and mass dimension. Since the propagators contribute the mass dimension $-2(n-3)$, the mass dimension of numerators can be fixed as $2(n-2)$. This is the mass dimension of coefficients $\mathcal{C}(\sigma'_{n-2}, k_i)$ in the expansion (13), which plays an important role in the subsequent subsections.

We will argue that the three-point NLSM tree amplitude does not exist, while the four-point one has vanishing single soft behavior at the τ^0 order, via the general consideration of Mandelstam variables. Such consideration also yields the statement that the non-vanishing NLSM tree amplitudes only have an even number of external legs. Then, by imposing the universality of single soft behavior observed from the four-point case, we will construct the general $\mathcal{C}(\sigma'_{n-2}, k_i)$ for $n \geqq 6$. The whole process uses only the mass dimension, the universality of soft behavior, and the permutation invariance among external scalars.

3.1 Four-point NLSM amplitude

The three-point NLSM tree amplitude has mass dimension 2 and contains no pole. However, one can never use three on-shell massless momenta satisfying momentum conservation to construct any non-vanishing Lorentz invariant with mass dimension 2. Thus, the three-point NLSM amplitude does not exist.

The simplest NLSM amplitudes are the four-point ones $\mathcal{A}_{\mathrm{N}}(\sigma_4)$. The absence of the three-point amplitude implies that the four-point ones have no pole. Then, the mass dimension requires the four-point amplitudes to be linear combinations of Mandelstam variables s, t, and u, where $s = s_{12}$, $t = s_{14}$, and $u = s_{13}$, satisfying $s + u + t = 0$. Such combinations can be fixed via the symmetry. For $\mathcal{A}_{\mathrm{N}}(1,2,3,4)$, the color ordering indicates symmetry between s and t; thus, $\mathcal{A}_{\mathrm{N}}(1,2,3,4)$ is proportional to $s+t$ or u. We can choose

$$\mathcal{A}_{\mathrm{N}}(1,2,3,4) = u = -(s+t), \tag{16}$$

via an overall rescaling of amplitude. Similarly, we have

$$\begin{aligned} &\mathcal{A}_{\mathrm{N}}(1,3,2,4) = s = -(t+u),\\ &\mathcal{A}_{\mathrm{N}}(1,2,4,3) = t = -(u+s). \end{aligned} \tag{17}$$

It is straightforward to observe that all of the above amplitudes vanish when taking any external momentum to be soft, due to the definition of s, t, and u. Thus, we conclude that the single soft behavior of the four-point NLSM amplitude does not exist at τ^{-1} and τ^0 orders. Such behavior is known as Adler's zero.

The vanishing of the four-point amplitude in the single soft limit is further evidence for the vanishing of the three-point one. Suppose that the n-point and $(n+1)$-point NLSM amplitudes exist; by taking an external leg to be soft, one can always factorize the latter one as the product of the former one and a non-vanishing leading soft factor. Thus, the vanishing of the $(n+1)$-point amplitude in the single soft limit indicates the vanishing of the n-point amplitude.

In this note, one of the basic assumptions is the universality of soft behaviors. As a consequence of universality, the vanishing of amplitude at τ^{-1} and τ^0 orders holds for any NLSM amplitude with an arbitrary number of external legs. If we adopt the existence of the four-point amplitudes, then the five-point ones must vanish; otherwise the vanishing of five-point amplitudes in the single soft limit will imply the vanishing of four-point ones. One can further generalize the above argument and conclude that the number of external legs for each non-vanishing NLSM amplitude should be even.

The four-point NLSM tree amplitudes can be expanded to the KK BAS basis; the double copy assumption requires the following expanded formula:

$$\mathcal{A}_{\mathrm{N}}(\sigma_4) = \mathcal{C}_1\, \mathcal{A}_{\mathrm{S}}(1,2,3,4|\sigma_4) + \mathcal{C}_2\, \mathcal{A}_{\mathrm{S}}(1,3,2,4|\sigma_4), \tag{18}$$

where the coefficients $\mathcal{C}_1$ and $\mathcal{C}_2$ have mass dimension 4. Using (16) and (17), we obtain the following equations:

$$\begin{aligned} &\mathcal{C}_1\, \mathcal{A}_{\mathrm{S}}(1,2,3,4|1,2,3,4) + \mathcal{C}_2\, \mathcal{A}_{\mathrm{S}}(1,3,2,4|1,2,3,4) = u,\\ &\mathcal{C}_1\, \mathcal{A}_{\mathrm{S}}(1,2,3,4|1,3,2,4) + \mathcal{C}_2\, \mathcal{A}_{\mathrm{S}}(1,3,2,4|1,3,2,4) = s,\\ &\mathcal{C}_1\, \mathcal{A}_{\mathrm{S}}(1,2,3,4|1,2,4,3) + \mathcal{C}_2\, \mathcal{A}_{\mathrm{S}}(1,3,2,4|1,2,4,3) = t. \end{aligned} \tag{19}$$

After evaluating the BAS amplitudes via the diagrammatical rules introduced in Sect. 2.1=, the above equations can be reduced to

$$\frac{\mathcal{C}_1}{s} + \frac{\mathcal{C}_2}{u} = t. \tag{20}$$

The solution of this equation is not unique, and it is hard to choose a particular one since all these solutions lead to the correct four-point amplitudes. Thus, we do not give the explicit formula of $\mathcal{C}_1$ and $\mathcal{C}_2$ in the current subsection. In Sect. 3.2, we will use the universality of soft behaviors to determine the n-point NLSM amplitudes in the expanded formula, with $n \geq 6$. In the next section, we will use this expanded formula to derive the double soft factors for NLSM amplitudes at τ^0 and τ^1 orders. Imposing the universality of the double soft factor, the four-point amplitude can be generated by removing two soft legs from the six-point one; then the expansion for the four-point case can be fixed through such manipulation, as will be shown in Sect. 4.1.

3.2 The n-point case

As noted at the end of Sect. 3.1, the universality of soft behaviors indicates that each NLSM amplitude vanishes at τ^{-1} and τ^0 orders when one of the external legs is soft. This subsection aims to find $\mathcal{C}(\sigma'_{n-2}, k_i)$ for the n-point NLSM amplitude with $n \geq 6$, by imposing the above requirement, as well as the permutation symmetry among external legs in $\{2, \ldots, n-1\}$, which is evident in (13).

Coefficients $\mathcal{C}(\sigma'_{n-2}, k_i)$ have mass dimension $2(n-2)$ and depend on color orderings $(1, \sigma'_{n-2}, n)$ carried by $\mathcal{A}_{\rm S}(1, \sigma'_{n-2}, n|\sigma_n)$. Consider $k_2 \to \tau k_2$, and expand $\mathcal{A}_{\rm N}(\sigma_n)$ in τ. In order to construct $\mathcal{A}_{\rm N}(\sigma_n)$ which vanishes at τ^{-1} and τ^0 orders, the simplest idea is to require each term in $\mathcal{C}(\sigma'_{n-2}, k_i)$ to contain the factor $(k_2 \cdot \mathcal{K}_{a_2})\,(k_2 \cdot \mathcal{K}_{b_2})$, where $\mathcal{K}_{a_2}$ and $\mathcal{K}_{b_2}$ are combinations of external momenta. Then, considerations for $k_i \to \tau k_i$ for other $i \in \{2, \ldots, n-1\}$ yield the conclusion that each term in $\mathcal{C}(\sigma'_{n-2}, k_i)$ also contains $(k_i \cdot \mathcal{K}_{a_i})\,(k_i \cdot \mathcal{K}_{b_i})$. The total number of k_i required by the above construction is $2(n-2)$, which satisfies the correct mass dimension of $\mathcal{C}(\sigma'_{n-2}, k_i)$. This means that each $\mathcal{C}(\sigma'_{n-2}, k_i)$ can be decomposed as the linear combination of building blocks $\prod_{j=1}^{n-2} k_{a_j} \cdot k_{b_j}$, where the set $\{a_j, b_j\}$ contains two i for each $i \in \{2, \ldots, n-1\}$, since when expanding to the KK basis, the coefficients do not contain any pole. When taking $k_1 \to \tau k_1$ or $k_n \to \tau k_n$ and expanding in τ, the coefficients $\mathcal{C}(\sigma'_{n-2}, k_i)$ constructed in this way are at the τ^0 order, which causes the single soft behavior of $\mathcal{A}_{\rm N}(\sigma_n)$ to be at the τ^{-1} order, therefore violating the universality of the soft behavior. Consequently, the above naive construction is not correct.

To find the correct answer, we consider $k_2 \to \tau k_2$ and express the leading-order contribution of $\mathcal{A}_{\rm N}(\sigma_n)$ as

$$\mathcal{A}^{(0)}_{\rm N}(\sigma_n) = \sum_{\sigma'_{n-2}} \mathcal{C}^{(0)}(\sigma'_{n-2}, k_i)\, S^{(0)}_{\rm S}(2)\, \mathcal{A}_{\rm S}(1, \sigma'_{n-2} \setminus 2, n|\sigma_n \setminus 2), \tag{21}$$

where the soft theorem (9) for the external BAS scalar has been used. The leading-order contribution $\mathcal{C}^{(0)}(\sigma'_{n-2}, k_i)$ is obtained from $\mathcal{C}(\sigma'_{n-2}, k_i)$ as follows. For each $\mathcal{K}_a \cdot \mathcal{K}_b$ contained in $\mathcal{C}(\sigma'_{n-2}, k_i)$, where $\mathcal{K}_a$ and $\mathcal{K}_b$ are again two combinations of external momenta, we turn it as $\mathcal{K}_a \cdot \mathcal{K}_b \to \mathcal{K}^{(0)}_a \cdot \mathcal{K}^{(0)}_b$. The combinatorial momentum $\mathcal{K}^{(0)}_a$ is defined as

$$\mathcal{K}^{(0)}_a = \begin{cases} \tau k_2 & \text{if } \mathcal{K}_a = k_2, \\ \mathcal{K}_a - k_2 & \text{otherwise.} \end{cases} \tag{22}$$

The definition of $\mathcal{K}^{(0)}_b$ is analogous. The universality of soft behavior requires $\mathcal{A}^{(0)}_{\rm N}(1, \ldots, n)$ to be of the τ^1 order. The simplest and most natural solution to the above condition can be obtained by assuming the independence of BAS amplitudes in the KK BAS basis. This assumption indicates that at τ^{-1} and τ^0 orders, coefficients for different $\mathcal{A}_{\rm S}(1, \sigma'_{n-2} \backslash 2, n|\sigma_n \backslash 2)$ vanish individually, that is,

$$\begin{aligned} 0 &= \mathcal{C}^{(0)}(2 \sqcup \{\sigma_3, \ldots, \sigma_{n-1}\}, k_i)\, S^{(0)}_{\rm S}(2) \\ &\quad \mathcal{A}_{\rm S}(1, \not{2} \sqcup \{\sigma_3, \ldots, \sigma_{n-1}\}, n|\sigma_n \backslash 2) \\ &= C(1, \not{2})\, \frac{1}{\tau} \left(\frac{\delta_{12}}{s_{12}} + \frac{\delta_{2\sigma_3}}{s_{2\sigma_3}} \right) \mathcal{A}_{\rm S}(1, \not{2}, \sigma_3, \cdots, \sigma_{n-1}, n|\sigma_n \backslash 2) \\ &\quad + \sum_{j=3}^{n-2} C(\sigma_j, \not{2})\, \frac{1}{\tau} \left(\frac{\delta_{\sigma_j 2}}{s_{\sigma_j 2}} + \frac{\delta_{2\sigma_{j+1}}}{s_{2\sigma_{j+1}}} \right) \\ &\quad \mathcal{A}_{\rm S}(1, \sigma_3, \ldots, \sigma_j, \not{2}, \sigma_{j+1}, \ldots, \sigma_{n-1}, n|\sigma_n \backslash 2) \\ &\quad + C(\sigma_{n-1}, \not{2})\, \frac{1}{\tau} \left(\frac{\delta_{\sigma_{n-1} 2}}{s_{\sigma_{n-1} 2}} + \frac{\delta_{2n}}{s_{2n}} \right) \\ &\quad \mathcal{A}_{\rm S}(1, \sigma_3, \ldots, \sigma_{n-1}, \not{2}, n|\sigma_n \backslash 2), \end{aligned} \tag{23}$$

therefore,

$$\begin{aligned} 0 &= C(1, \not{2}) \left(\frac{\delta_{12}}{s_{12}} + \frac{\delta_{2\sigma_3}}{s_{2\sigma_3}} \right) + C(\sigma_{n-1}, \not{2}) \left(\frac{\delta_{\sigma_{n-1} 2}}{s_{\sigma_{n-1} 2}} + \frac{\delta_{2n}}{s_{2n}} \right) \\ &\quad + \sum_{j=3}^{n-2} C(\sigma_j, \not{2}) \left(\frac{\delta_{\sigma_j 2}}{s_{\sigma_j 2}} + \frac{\delta_{2\sigma_{j+1}}}{s_{2\sigma_{j+1}}} \right). \end{aligned} \tag{24}$$

Here, we abbreviated $\mathcal{C}^{(0)}(2 \sqcup \{\sigma_3, \ldots, \sigma_{n-1}\}, k_i)$ as $C(a, \not{2})$, where $(a, \not{2})$ emphasizes the position of external leg 2 in the color ordering $(\cdots, a, 2, \cdots)$. The symbol $\sqcup$ in the first line of (23) means the summation over particular permutations, which preserves the ordering $(\sigma_3, \cdots, \sigma_{n-1})$, as explained below (11). The notation $\not{2}$ means delaying the leg 2, and thus all $\mathcal{A}_{\rm S}(1, \not{2} \sqcup \{\sigma_3, \ldots, \sigma_{n-1}\}, n|\sigma_n \backslash 2)$ are the same BAS amplitude. Note that since our first attempt has failed, we must assume that $C(a, \not{2})$ are at the τ^0 or τ^1 order.

Since the ordering of σ_n is arbitrary, which means that the values of δ_{ab} have not been fixed, the only way to obtain the nonzero solution of $C(a, \not{2})$ to the Eq. (24) is to employ the identity

$$\sum_{i \neq 2} \delta_{i2} = 0, \tag{25}$$

which is the special case of identity (8). In other words, Eq. (24) should be reduced to

$$0=\widetilde{C}\sum_{i\neq 2}\delta_{i2}, \tag{26}$$

where $\widetilde{C}$ is Lorentz-invariant, constructed from external momenta. The formula (26) requires that all δ_{i2} have a common coefficient $\widetilde{C}$. Applying this condition to δ_{12}, we immediately find that $C(1,\not{2})=2(k_2\cdot k_1)\,\widetilde{C}$. Then, applying the same condition to $\delta_{\sigma_3 2}$, we have

$$\widetilde{C}=\frac{C(\sigma_3,\not{2})-C(1,\not{2})}{s_{2\sigma_3}}=\frac{C(\sigma_3,\not{2})-2\,(k_2\cdot k_1)\,\widetilde{C}}{s_{2\sigma_3}}, \tag{27}$$

and the only solution is $C(\sigma_3,\not{2})=2(k_2\cdot K_{1\sigma_3})\,\widetilde{C}$, where $K_{a_1\cdots a_m}\equiv\sum_{i=1}^{m}k_{a_i}$. Repeating the above process recursively, we obtain

$$C(a,\not{2})=2(k_2\cdot K_{1\sigma_3\cdots\sigma_a})\,\widetilde{C}, \tag{28}$$

for arbitrary $a\in\{3,\ldots,n-1\}$.

Some remarks are in order. First, when expanding the NLSM amplitude to the KK basis, the coefficients do not contain any poles. Thus, $\widetilde{C}$ is Lorentz-invariant without any pole; otherwise, one cannot ensure that no $C(a,\not{2})$ contains any pole. Secondly, we previously assumed that $C(a,\not{2})$ are at the τ^0 or τ^1 order. Since $C(a,\not{2})$ includes the factor $k_2\cdot K_{1\sigma_3\cdots\sigma_a}$ accompanied by τ under the rescaling of $k_2\to\tau k_2$, we now exclude the τ^0 case and conclude that $C(a,\not{2})$ and $\widetilde{C}$ are at the τ^1 order. Finally, for a given color ordering $(1,\sigma_3,\ldots,\sigma_{n-1},n)$ carried by $\mathcal{A}_{\rm S}(1,\sigma_3,\ldots,\sigma_{n-1},n|\sigma_n\backslash 2)$, $\widetilde{C}$ is independent of the position of leg 2 in the original orderings $(1,2\sqcup\{\sigma_3,\ldots,\sigma_{n-1}\},n)$.

The solution (28) indicates that the coefficients $\mathcal{C}(\sigma'_{n-2},k_i)$ contain the component $k_2\cdot X_2$, where the combinatorial momentum X_i is defined as the summation of momenta carried by external legs on the l.h.s of i in the color ordering. This result can be generalized as $\mathcal{C}(\sigma'_{n-2},k_i)\propto k_i\cdot X_i$ for each $i\in\{2,\ldots,n-1\}$, because of the permutation symmetry among external legs $i\in\{2,\ldots,n-1\}$ in the expansion (13). Since the coefficients $\mathcal{C}(\sigma'_{n-2},k_i)$ have mass dimension $2(n-2)$, the above information fixes them completely, namely,

$$\mathcal{C}(\sigma'_{n-2},k_i)=\alpha\prod_{i=2}^{n-1}k_i\cdot X_i, \tag{29}$$

where α is an constant with mass dimension 0. Since $\mathcal{C}(\sigma'_{n-2},k_i)$ contains no pole, α is independent of any kinematic variable. Furthermore, α is also independent of the color ordering $(1,\sigma'_{n-2},n)$, due to the permutation invariance. Substituting the solution (29) into (13), the expanded formula of the n-point NLSM amplitude is found to be

$$\mathcal{A}_{\rm N}(\sigma_n)=\sum_{\sigma'_{n-2}}\left(\prod_{i=2}^{n-1}k_i\cdot X_i\right)\mathcal{A}_{\rm S}(1,\sigma'_{n-2},n|\sigma_n). \tag{30}$$

Here we have fixed the constant α as $\alpha=1$ via an overall rescaling of the amplitude. The expression in (30) is the desired expanded formula of the NLSM amplitude with $n\geq 6$. In Sect. 4.1, we will show that (30) is also correct for the four-point case.

It is not hard to verify that $\mathcal{A}_{\rm N}(\sigma_n)$ given in (30) vanishes at τ^{-1} and τ^0 orders under the rescaling $k_1\to\tau k_1$ or $k_n\to\tau k_n$; thus the expanded formula (30) satisfies the correct single soft behavior for any external leg being soft. Taking $k_2\to\tau k_2$, $\widetilde{C}$ in the solution (28) can be calculated from(30) as

$$\widetilde{C}=\frac{\tau}{2}\prod_{i=3}^{n-1}k_i\cdot K_{1\sigma_3\cdots\sigma_{i-1}}, \tag{31}$$

which satisfies our expectations: $\widetilde{C}$ is Lorentz-invariant, with no pole, and is independent of the position where leg 2 is inserted. The situations for other $k_i\to\tau k_i$ are analogous.

The solution (29) is obtained by assuming the independence of BAS amplitudes in the KK BAS basis. Indeed, such independence is violated by the well-known Bern–Carrasco–Johansson (BCJ) relations among BAS amplitudes. BCJ relations allow for more solutions to Eq. (24). For instance, one can modify the expanded formula in (30) by adding terms which vanish automatically, such as

$$\begin{aligned}\mathcal{A}_{\rm N}(\sigma_n)=&\sum_{\sigma'_{n-2}}\left(\prod_{i=2}^{n-1}k_i\cdot X_i\right)\mathcal{A}_{\rm S}(1,\sigma'_{n-2},n|\sigma_n)\\&+(k_2\cdot X_2)\,\mathcal{T}\,\mathcal{A}_{\rm S}(1,\sigma'_{n-2},n|\sigma_n),\end{aligned} \tag{32}$$

where $\mathcal{T}$ is Lorentz-invariant with mass dimension $2(n-3)$. The above modification is guaranteed by the fundamental BCJ relation

$$0=(k_2\cdot X_2)\,\mathcal{A}_{\rm S}(1,2\sqcup\{\sigma_3,\ldots,\sigma_{n-1}\},n|\sigma_n). \tag{33}$$

Thus, when considering BCJ relations, the expanded formula (30) should be understood as an equivalent class rather than a unique expression.

The disadvantage of the modified formula (32) is the breaking of manifest permutation symmetry among legs in $\{2,\ldots,n-1\}$. Let us focus only on the expanded formulas that manifest such permutation invariance, and ask whether the BCJ relations lead to different solutions to Eq. (24) which satisfy this symmetry, and the coefficients $\mathcal{C}(\sigma'_{n-2},k_i)$ do not contain any poles. Quite surprisingly, for the six-point and eight-point cases, such a new solution which is nonequivalent to (30) cannot be found. Thus, we conjecture that this situation is a general one and claim that the expansion in (30) is the only solution which manifests the desired permutation invariance, although the general proof is absent.

4 Double soft factors

As discussed in Sect. 3.1, using the single soft theorem, one can generate the n-point tree amplitude from the $(n+1)$-point one by removing the soft external leg. The NLSM tree amplitude does not have such luxuries, since the number of external legs must be even. However, it is natural to think of the n-point NLSM amplitude as the resulting object of removing two soft legs from the $(n+2)$-point one. This picture leads to the consideration of the double soft behavior of NLSM tree amplitudes.

The double soft theorems for NLSM tree amplitudes have been obtained by different methods [30,31]. The leading soft factor is at the τ^0 order, while the sub-leading one is at the τ^1 order. Such leading-order behavior coincides with the picture of generating the n-point amplitude from the $(n+2)$-point one, since such manipulation forbids the vanishing of the $(n+2)$-point amplitude in the limit $\tau \to 0$. In this section, we propose an efficient new approach to derive the leading and sub-leading double soft factors, based on the expansion in (30).

Without loss of generality, we consider $k_a \to \tau k_a$, $k_b \to \tau k_b$ and expand $\mathcal{A}_{\rm N}(1, \cdots, n, a, b)$ by τ. We chose as the basis BAS amplitudes whose external legs 1 and n are fixed at two ends in the first color ordering; thus the $n+2$-point amplitude $\mathcal{A}_{\rm N}(1, \ldots, n, a, b)$ can be expanded as

$$\begin{aligned}&\mathcal{A}_{\rm N}(1, \ldots, n, a, b)\\&\quad= (k_a \cdot X_a)\,(k_b \cdot X_b)\left(\prod_{i=2}^{n-1} k_i \cdot X_i\right)\\&\qquad \mathcal{A}_{\rm S}(1, a \shuffle b \shuffle 2 \shuffle \cdots \shuffle n-1, n | 1, \ldots, n, a, b).\end{aligned} \tag{34}$$

According to the meaning of $\shuffle$ mentioned below (11), one can rewrite the expansion in (30) as

$$\mathcal{A}_{\rm N}(\sigma_n) = \left(\prod_{i=2}^{n-1} k_i \cdot X_i\right) \mathcal{A}_{\rm S}(1, 2 \shuffle \cdots \shuffle n-1, n | \sigma_n). \tag{35}$$

We used this notation in (34) to emphasize the positions of special legs a and b in the first color ordering carried by $\mathcal{A}_{\rm S}(1, \sigma_1, \cdots, \sigma_n, n | 1, \ldots, n, a, b)$. For later convenience, in the remainder of this section, we denote the BAS amplitude $\mathcal{A}_{\rm S}(1, \sigma_1, \ldots, \sigma_n, n | 1, \ldots, n, a, b)$ as $\mathrm{A}(\sigma_1, \ldots, \sigma_n)$, where $\{\sigma_1, \ldots, \sigma_n\} = \{a, b, 2, \ldots, n-1\}$.

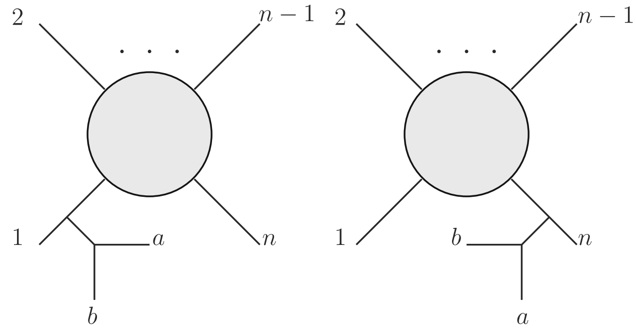

Fig. 6 First type of Feynman diagram

4.1 Leading order

To derive the double soft factor at leading order, let us consider Feynman diagrams for BAS amplitudes $\mathrm{A}(a \shuffle b \shuffle 2 \shuffle \cdots \shuffle n-1)$ whose external legs a and b are coupled to a common vertex. Such diagrams allowed by the second color ordering are shown in Fig. 6. Among terms on the r.h.s of the expansion (34), contributions from the first configuration in Fig. 6 are contained in

$$\begin{aligned}&(k_a \cdot k_1)\,(k_b \cdot K_{1a})\left(\prod_{i=2}^{n-1} k_i \cdot X_i\right)\\&\quad \mathrm{A}(a, b, 2 \shuffle \cdots \shuffle n-1)\\&\quad +(k_b \cdot k_1)\,(k_a \cdot K_{1b})\left(\prod_{i=2}^{n-1} k_i \cdot X_i\right)\\&\quad \mathrm{A}(b, a, 2 \shuffle \cdots \shuffle n-1),\end{aligned} \tag{36}$$

while contributions from the second configuration are included in

$$\begin{aligned}&(k_b \cdot k_n)\,(k_a \cdot K_{bn})\left(\prod_{i=2}^{n-1} k_i \cdot X_i\right)\\&\quad \mathrm{A}(2 \shuffle \cdots \shuffle n-1, a, b)\\&\quad +(k_a \cdot k_n)\,(k_b \cdot K_{an})\left(\prod_{i=2}^{n-1} k_i \cdot X_i\right)\\&\quad \mathrm{A}(2 \shuffle \cdots \shuffle n-1, b, a).\end{aligned} \tag{37}$$

After taking $k_a \to \tau k_a$, $k_b \to \tau k_b$ and expanding in τ, the leading-order contribution from (36) can be calculated as

$$\begin{aligned}L_1 &= \tau^2\,(k_a \cdot k_1)\,\big(k_b \cdot (k_1 + \tau k_a)\big)\\&\quad \times\Big(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\Big)\,\frac{1}{s_{ab}^{(0)}}\,\frac{1}{s_{ab1}^{(0)}}\,\mathrm{A}(2 \shuffle \cdots \shuffle n-1)\\&\quad -\tau^2\,(k_b \cdot k_1)\,\big(k_a \cdot (k_1 + \tau k_b)\big)\\&\quad \times\Big(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\Big)\,\frac{1}{s_{ab}^{(0)}}\,\frac{1}{s_{ab1}^{(0)}}\,\mathrm{A}(2 \shuffle \cdots \shuffle n-1)\end{aligned}$$

$$= \frac{(k_a - k_b)\cdot k_1}{4(k_a + k_b)\cdot k_1}\left[\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right) \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)\right]$$
$$= \frac{(k_a - k_b)\cdot k_1}{4(k_a + k_b)\cdot k_1}\, \mathcal{A}_{\mathrm{N}}(1,\ldots,n), \tag{38}$$

where $X_i^{(0)}$ arises from X_i by deleting k_a and k_b, and $s_{ab}^{(0)}$ and $s_{ab1}^{(0)}$ stand for leading-order contributions of s_{ab} and s_{ab1}, respectively. Therefore, $s_{ab}^{(0)} = 2\tau^2 k_a \cdot k_b$, and $s_{ab1}^{(0)} = 2\tau\,(k_a + k_b)\cdot k_1$. Here, the relative minus sign in the first equality can be determined via the diagrammatic rules, and a more direct way to see it is by employing the anti-symmetry of the structure constant f^{abc} of the Lie group, which indicates a minus sign when swapping legs a and b. Because of this minus sign, terms at the τ^{-1} order cancel each other, leaving the non-vanishing L_1 at the τ^0 order. The last equality uses the observation

$$\mathcal{A}_{\mathrm{N}}(1,\cdots,n) = \left(\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}\right) \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1), \tag{39}$$

based on the expansion (30) and the definition of $X_i^{(0)}$.

The consideration for (37) is analogous and gives

$$L_2 = \frac{(k_b - k_a)\cdot k_n}{4(k_a + k_b)\cdot k_n}\, \mathcal{A}_{\mathrm{N}}(1,\ldots,n), \tag{40}$$

which is also at the τ^0 order. We claim that (38) and (40) provide the full double soft behavior of $\mathcal{A}_{\mathrm{N}}(1,\ldots,n,a,b)$ at leading order. The reasoning can be explained as follows. For diagrams whose legs a and b are coupled to different vertices, the double soft limit can be achieved by taking single soft limits for a and b successively. Since the single soft behavior for a or b vanishes at the τ^0 order, one can conclude that taking two single soft limits consecutively contributes nothing at the τ^0 order. Thus, the leading double soft factor can be found by combining (38) and (40), which gives

$$\mathcal{A}_{\mathrm{N}}^{(0)}(1,\ldots,n,a,b) = S_{\mathrm{N}}^{(0)}(a,b)\,\mathcal{A}_{\mathrm{N}}(1,\ldots,n), \tag{41}$$

where

$$S_{\mathrm{N}}^{(0)}(a,b) = \frac{(k_a - k_b)\cdot k_1}{4(k_a + k_b)\cdot k_1} + \frac{(k_b - k_a)\cdot k_n}{4(k_a + k_b)\cdot k_n}. \tag{42}$$

This result is the same as that obtained in [30,31] by different approaches.

The universality of the leading soft factor (42) indicates that the expanded formula (30) is valid for the four-point NLSM amplitude. The reasoning is as follows. The universality imposes

$$\mathcal{A}_{\mathrm{N}}^{(0)}(1,2,3,4,a,b) = S_{\mathrm{N}}^{(0)}(a,b)\,\mathcal{A}_{\mathrm{N}}(1,2,3,4), \tag{43}$$

which is a special case of (41). Since the relation (41) is based on the expansion (39), $\mathcal{A}_{\mathrm{N}}(1,2,3,4)$ in (43) must satisfy the expansion (39), which is equivalent to (30).

4.2 Sub-leading order

The sub-leading double soft factor is more complicated, since Feynman diagrams whose a and b are coupled to different vertices also have non-vanishing contributions at the τ^1 order. We will consider the corresponding diagrams one by one.

Let us start with the first configuration in Fig. 6 and express corresponding terms contained in (36) as

$$\Bigg[(k_a\cdot k_1)\,(k_b\cdot K_{1a})\left(\prod_{i=2}^{n-1} k_i\cdot X_i\right)$$
$$-(k_b\cdot k_1)\,(k_a\cdot K_{1b})\left(\prod_{i=2}^{n-1} k_i\cdot X_i\right)\Bigg]\frac{1}{s_{ab}}\,\frac{1}{s_{ab1}}\,\mathcal{M}$$
$$= \frac{(k_a - k_b)\cdot k_1}{2 s_{ab1}}\left(\prod_{i=2}^{n-1} k_i\cdot X_i\right)\mathcal{M}. \tag{44}$$

The relative minus sign was interpreted in Sect. 4.1 around (38). When $k_a \to \tau k_a$, $k_b \to \tau k_b$, (44) behaves as

$$\frac{(k_a - k_b)\cdot k_1}{4(k_a + k_b)\cdot k_1 + 4\tau\, k_a\cdot k_b}\left(\prod_{i=2}^{n-1} k_i\cdot X_i(\tau)\right)\mathcal{M}(\tau). \tag{45}$$

Our purpose is to extract the τ^1-order terms from (45). First, one can expand $\mathcal{M}(\tau)$ as

$$\mathcal{M}(\tau) = \mathcal{M}(0) + \tau\,\frac{\partial}{\partial\tau}\,\mathcal{M}(\tau)\big|_{\tau=0} + \cdots \tag{46}$$

and pick up the second term. One can observe that the parameter τ enters $\mathcal{M}(\tau)$ only through the combinatorial momentum $k_1 + \tau(k_a + k_b)$; thus

$$\frac{\partial}{\partial\tau} = \frac{1}{\tau}\,(k_a + k_b)\cdot\frac{\partial}{\partial(k_a + k_b)} = (k_a + k_b)\cdot\frac{\partial}{\partial k_1}, \tag{47}$$

which leads to

$$\frac{\partial}{\partial\tau}\,\mathcal{M}(\tau)\big|_{\tau=0} = (k_a + k_b)\cdot\frac{\partial}{\partial k_1}\,\mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1), \tag{48}$$

where the observation $\mathcal{M}(0) = \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)$ was used. Consequently, the first piece is found to be

$$\mathcal{P}_{1;1} = \tau\,\frac{(k_a - k_b)\cdot k_1}{4(k_a + k_b)\cdot k_1}\left(\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}\right)$$
$$\times\Big[(k_a + k_b)\cdot\partial_{k_1}\,\mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)\Big], \tag{49}$$

where

$$\partial_{k_1} \equiv \frac{\partial}{\partial k_1}. \tag{50}$$

Secondly, one can expand the denominator in (45) and pick up the second terms on the r.h.s of

$$\frac{1}{4(k_a+k_b)\cdot k_1+4\tau\,k_a\cdot k_b} = \frac{1}{4(k_a+k_b)\cdot k_1} - \frac{\tau\,k_a\cdot k_b}{4\big((k_a+k_b)\cdot k_1\big)^2} + \cdots\,, \qquad (51)$$

which gives the second piece

$$\mathcal{P}_{2;1} = -\tau\,\frac{\big((k_a-k_b)\cdot k_1\big)\,(k_a\cdot k_b)}{4\big((k_a+k_b)\cdot k_1\big)^2} \times\Big(\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}\Big)\,\mathrm{A}(2\shuffle\cdots\shuffle n-1). \qquad (52)$$

Finally, each $X_i(\tau)$ contains $\tau\,(k_a+k_b)$; thus the third piece is found to be

$$\begin{aligned}\mathcal{P}_{3;1} &= \tau\,\frac{(k_a-k_b)\cdot k_1}{4(k_a+k_b)\cdot k_1} \times\Bigg[\sum_{j=2}^{n-1}\Bigg(k_j\cdot(k_a+k_b)\,\frac{\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}}{k_j\cdot X_j^{(0)}}\Bigg)\Bigg] \\ &\quad \mathrm{A}(2\shuffle\cdots\shuffle n-1) \\ &= \tau\,\frac{(k_a-k_b)\cdot k_1}{4(k_a+k_b)\cdot k_1}\Bigg[(k_a+k_b)\cdot\partial_{k_1}\Bigg(\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}\Bigg)\Bigg] \\ &\quad \mathrm{A}(2\shuffle\cdots\shuffle n-1). \end{aligned} \qquad (53)$$

Similarly, considering the second configuration in Fig. 6 yields

$$\mathcal{P}_{1;2} = \tau\,\frac{(k_b-k_a)\cdot k_n}{4(k_a+k_b)\cdot k_n}\Big(\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}\Big) \times\Big[(k_a+k_b)\cdot\partial_{k_n}\,\mathrm{A}(2\shuffle\cdots\shuffle n-1)\Big], \qquad (54)$$

$$\mathcal{P}_{2;2} = -\tau\,\frac{\big((k_b-k_a)\cdot k_n\big)\,(k_a\cdot k_b)}{4\big((k_a+k_b)\cdot k_n\big)^2}\Big(\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}\Big)\,\mathrm{A}(2\shuffle\cdots\shuffle n-1)\,, \qquad (55)$$

as well as

$$\begin{aligned}\mathcal{P}_{3;2} &= \tau\,\frac{(k_b-k_a)\cdot k_n}{4(k_a+k_b)\cdot k_n} \times\Bigg[\sum_{j=2}^{n-1}\Bigg(k_j\cdot(k_a+k_b)\,\frac{\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}}{k_j\cdot X_j^{(0)}}\Bigg)\Bigg] \\ &\quad \mathrm{A}(2\shuffle\cdots\shuffle n-1) \\ &= \tau\,\frac{(k_b-k_a)\cdot k_n}{4(k_a+k_b)\cdot k_n} \times\Bigg[(k_a+k_b)\cdot\partial_{k_n}\Bigg(\prod_{i=2}^{n-1} k_i\cdot X_i^{(0)}\Bigg)\Bigg] \\ &\quad \mathrm{A}(2\shuffle\cdots\shuffle n-1). \end{aligned} \qquad (56)$$

The process is parallel to those for obtaining (49), (52), and (53). To derive (56), one should use momentum conservation to replace the explicit form of X_i by the equivalent one $\mathcal{X}_i = X_i - \big(k_a + k_b + \sum_{i=1}^{n} k_i\big)$. The special expressions for X_i imply that the individual results in (53) and (56) are inconsistent with momentum conservation. In (56), suppose one uses momentum conservation to remove k_n in $X_i^{(0)}$; then $k_i\cdot X_i^{(0)}$ will be annihilated by the operator ∂_{k_n}. A similar phenomenon occurs for (53). However, after combining (53) and (56), the resulting object is consistent with momentum conservation. One can use momentum conservation to freely modify the explicit expression of any X_i; the combination $\mathcal{P}_{4;1}+\mathcal{P}_{4;2}$ always gives the correct result. Similarly, the individual pieces (49) and (54) are inconsistent with momentum conservation, while the combination $\mathcal{P}_{1;1}+\mathcal{P}_{1;2}$ does not have this problem.

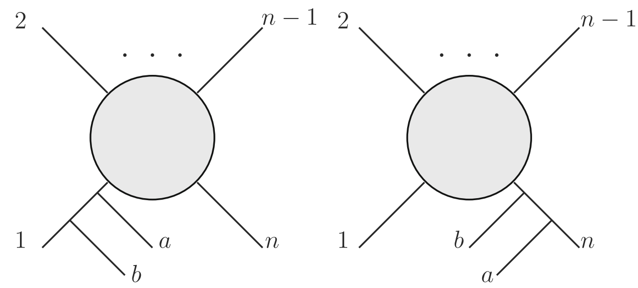

Fig. 7 Second type of Feynman diagram

Now we turn to a new type of Feynman diagram, which can be seen in Fig. 7. We focus on the first configuration in Fig. 7, and the second one can be treated similarly. On the r.h.s of (34), contributions from such diagrams are included in

$$(k_b\cdot k_1)\,(k_a\cdot K_{1b})\,\Big(\prod_{i=2}^{n-1} k_i\cdot X_i\Big)\,\mathrm{A}(b,a,2\shuffle\cdots\shuffle n-1), \qquad (57)$$

and can thus be expressed as

$$\begin{aligned}&(k_b\cdot k_1)\,(k_a\cdot K_{1b})\Big(\prod_{i=2}^{n-1} k_i\cdot X_i\Big)\frac{1}{s_{b1}}\,\frac{1}{s_{ab1}}\,\mathcal{N} \\ &\quad = \frac{k_a\cdot K_{1b}}{2s_{ab1}}\Big(\prod_{i=2}^{n-1} k_i\cdot X_i\Big)\mathcal{N}. \end{aligned} \qquad (58)$$

Taking $k_a\to\tau k_a$, $k_b\to\tau k_b$ turns (58) into

$$\frac{k_a\cdot k_1+\tau\,k_a\cdot k_b}{4(k_a+k_b)\cdot k_1+4\tau\,k_a\cdot k_b}\Big(\prod_{i=2}^{n-1} k_i\cdot X_i(\tau)\Big)\mathcal{N}(\tau), \qquad (59)$$

and one can extract four pieces at the τ^1 order. In the first one, expanding $\mathcal{N}(\tau)$ by τ gives

$$\mathcal{P}_{4;1} = -\tau \frac{k_a \cdot k_1}{4(k_a+k_b)\cdot k_1}\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right) \times\left[(k_a+k_b)\cdot\partial_{k_1} \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)\right], \quad (60)$$

where the observation $\mathcal{N}(0) = -\mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)$ is used, and the minus sign can be verified via diagrammatical rules. The second piece arises from expanding the denominator of (59), which is given by

$$\mathcal{P}_{5;1} = -\tau \frac{k_a \cdot k_b}{4(k_a+k_b)\cdot k_1} \times\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right) \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1). \quad (61)$$

The third one is obtained by expanding the denominator of (59), which is found to be

$$\mathcal{P}_{6;1} = \tau \frac{(k_a \cdot k_1)\,(k_a \cdot k_b)}{4\big((k_a+k_b)\cdot k_1\big)^2}\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right) \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1). \quad (62)$$

The final one comes from $\tau\,(k_a+k_b)$ contained in $X_i(\tau)$, and thus is given as

$$\mathcal{P}_{7;1} = -\tau \frac{k_a \cdot k_1}{4(k_a+k_b)\cdot k_1} \times\left[(k_a+k_b)\cdot\partial_{k_1}\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right)\right] \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1). \quad (63)$$

Considering the second configuration in Fig. 7 gives analogous results

$$\mathcal{P}_{4;2} = -\tau \frac{k_b \cdot k_n}{4(k_a+k_b)\cdot k_n}\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right) \times\left[(k_a+k_b)\cdot\partial_{k_n} \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)\right], \quad (64)$$

$$\mathcal{P}_{5;2} = -\tau \frac{k_a \cdot k_b}{4(k_a+k_b)\cdot k_n}\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right) \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)\,, \quad (65)$$

$$\mathcal{P}_{6;2} = \tau \frac{(k_b \cdot k_n)\,(k_a \cdot k_b)}{4\big((k_a+k_b)\cdot k_n\big)^2}\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right) \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1)\,, \quad (66)$$

and

$$\mathcal{P}_{7;2} = -\tau \frac{k_b \cdot k_n}{4(k_a+k_b)\cdot k_n}\left[(k_a+k_b)\cdot\partial_{k_n}\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right)\right] \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1). \quad (67)$$

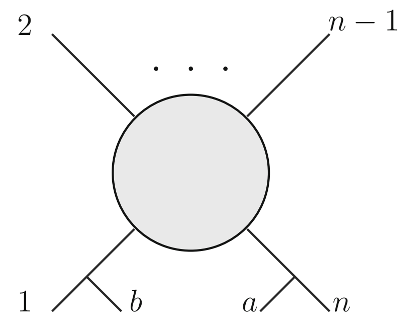

Fig. 8 Third type of Feynman diagram

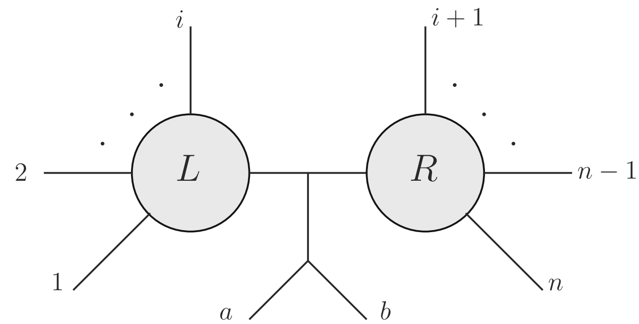

Fig. 9 Fourth type of Feynman diagram

Combinations $\mathcal{P}_{4;1}+\mathcal{P}_{4;2}$ and $\mathcal{P}_{7;1}+\mathcal{P}_{7;2}$ allow the expressions of $X_i^{(0)}$ to be rewritten via momentum conservation, while individual pieces do not.

We then consider the configuration in Fig. 8, which corresponds to

$$-\,(k_b \cdot k_1)\,(k_a \cdot k_n)\left(\prod_{i=2}^{n-1} k_i \cdot X_i\right) \mathrm{A}(b, 2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1, a) \quad (68)$$

on the r.h.s of (34). Here, we have used on-shell and momentum conservation conditions to rewrite X_a as $-k_n$. Under the rescaling $k_a \to \tau k_a$, $k_b \to \tau k_b$, the leading-order contributions from these terms are the τ^1 order, which provides

$$\mathcal{P}_8 = \frac{\tau}{4}\left[\Big(k_b \cdot \partial_{k_1} + k_a \cdot \partial_{k_n}\Big)\left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)}\right)\right] \mathrm{A}(2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup n-1). \quad (69)$$

Only one of the operators ∂_{k_1} and ∂_{k_n} is effective, since the definition of $X_i^{(0)}$ implies that k_1 and k_n cannot be included in $X_i^{(0)}$ simultaneously. The formula (69) gives a correct result for any expression of $X_i^{(0)}$, and thus is consistent with momentum conservation.

Next we deal with the configuration in Fig. 9, in which legs a and b are coupled to a common vertex and then coupled to an internal line. On the r.h.s of (34), this type of Feynman diagram corresponds to

$$
\begin{aligned}
&(k_a \cdot K_{1\cdots i})\,(k_b \cdot K_{1\cdots ia})\left(\prod_{j=2}^{n-1} k_j \cdot X_j\right)\\
&\quad \mathrm{A}(2 \shuffle \cdots \shuffle i, a, b, i+1 \shuffle \cdots \shuffle n-1)\\
&\quad +(k_b \cdot K_{1\cdots i})\,(k_a \cdot K_{1\cdots ib})\left(\prod_{j=2}^{n-1} k_j \cdot X_j\right)\\
&\quad \mathrm{A}(2 \shuffle \cdots \shuffle i, b, a, i+1 \shuffle \cdots \shuffle n-1).
\end{aligned} \tag{70}
$$

When $k_a \to \tau k_a$, $k_b \to \tau k_b$, the leading-order contributions corresponding to Fig. 9 are at the τ^1 order and can be calculated as

$$
\begin{aligned}
&\tau\,\frac{(k_a-k_b)\cdot K_{1\cdots i}}{2s_{1\cdots i}^2}\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\mathcal{M}_L\,\mathcal{M}_R\\
&= -\frac{\tau}{4}\left((k_a-k_b)\cdot\partial_{k_1}\frac{1}{s_{1\cdots i}}\right)\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\mathcal{M}_L\,\mathcal{M}_R\\
&= -\frac{\tau}{4}\left((k_b-k_a)\cdot\partial_{k_n}\frac{1}{s_{i+1\cdots n}}\right)\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\mathcal{M}_L\,\mathcal{M}_R.
\end{aligned} \tag{71}
$$

The last equality is obtained by replacing $K_{1\cdots i}$ with $-K_{i+1\cdots n}$ via momentum conservation. Note that since $s_{1\cdots i}$ is defined as $s_{1\cdots i} = K_{1\cdots i}^2$ which contains k_1^2, we have $\partial_{k_{1,\mu}} s_{1\cdots i} = 2K_{1\cdots i}^{\mu}$, although $k_1^2 = 0$, and we usually omit it when expressing $s_{1\cdots i}$ explicitly. The analogous situation holds for $\partial_{k_{n,\mu}} s_{i+1\cdots n}$. Summing over all possible i leads to the piece

$$
\begin{aligned}
\mathcal{P}_9 = &-\frac{\tau}{4}\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\\
&\times\Bigg[\Big((k_a-k_b)\cdot\partial_{k_1} + (k_b-k_a)\cdot\partial_{k_n}\Big)\\
&\quad \mathrm{A}(2 \shuffle \cdots \shuffle n-1)\Bigg].
\end{aligned} \tag{72}
$$

The operator $(k_a-k_b)\cdot\partial_{k_1} + (k_b-k_a)\cdot\partial_{k_n}$ is introduced for two reasons. First, this operator is consistent with momentum conservation. Secondly, this operator annihilates terms from Feynman diagrams in Fig. 10, which also contributes to $\mathrm{A}(2 \shuffle \cdots \shuffle n-1)$ and leaves only the desired contributions corresponding to the configuration in Fig. 9.

Finally, we should consider diagrams provided in Fig. 11. Let us focus on the first configuration in Fig. 11. On the r.h.s of (34), the corresponding terms are

$$
\begin{aligned}
&(k_b \cdot k_1)\,(k_a \cdot K_{b1\cdots i})\left(\prod_{j=2}^{n-1} k_j \cdot X_j\right)\\
&\quad \mathrm{A}(1, b, 2 \shuffle \cdots \shuffle i, a, i+1 \shuffle \cdots \shuffle n-1).
\end{aligned} \tag{73}
$$

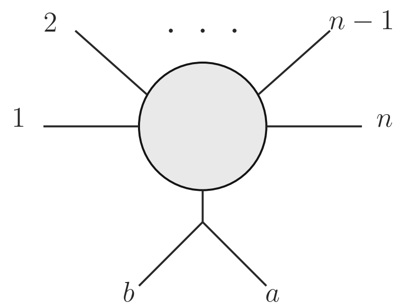

Fig. 10 Feynman diagrams which should be excluded

When $k_a \to \tau k_a$, $k_b \to \tau k_b$, leading contributions are at the τ^1 order and are given by

$$
\begin{aligned}
&-\tau\,\frac{k_a \cdot K_{1\cdots i}}{2s_{1\cdots i}^2}\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\mathcal{N}_L\,\mathcal{N}_R\\
&= \frac{\tau}{4}\left(k_a\cdot\partial_{k_1}\frac{1}{s_{1\cdots i}}\right)\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\mathcal{N}_L\,\mathcal{N}_R.
\end{aligned} \tag{74}
$$

Summing over all possible i, we obtain

$$
\mathcal{P}_{10;1} = \frac{\tau}{4}\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\Big(k_a\cdot\partial_{k_1}\,\mathrm{A}(2 \shuffle \cdots \shuffle n-1)\Big). \tag{75}
$$

Applying the same manipulation to the second configuration in Fig. 11 gives

$$
\mathcal{P}_{10;2} = \frac{\tau}{4}\left(\prod_{j=2}^{n-1} k_j \cdot X_j^{(0)}\right)\Big(k_b\cdot\partial_{k_n}\,\mathrm{A}(2 \shuffle \cdots \shuffle n-1)\Big). \tag{76}
$$

Again, the combination $\mathcal{P}_{10;1} + \mathcal{P}_{10;2}$ is consistent with momentum conservation, while individual pieces are not.

Now we are ready to determine the sub-leading double soft operator. To realize this, we need to regroup the sub-leading contributions at the τ^1 order as

$$
\mathcal{A}_{\mathrm{N}}^{(1)}(1,\ldots,n,a,b) = S_{\mathrm{N}}^{(1)}(a,b)\,\mathcal{A}_{\mathrm{N}}(1,\ldots,n), \tag{77}
$$

where $S_{\mathrm{N}}^{(1)}(a,b)$ is an operator at the τ^1 order. We first add $\mathcal{P}_{2;1}, \mathcal{P}_{5;1}, \mathcal{P}_{6;1}, \mathcal{P}_{2;2}, \mathcal{P}_{5;2}$, and $\mathcal{P}_{6;2}$ in (52), (61), (62), (55), (65), and (66) together to get

$$
\begin{aligned}
\mathcal{R}_1 = &-\tau\,(k_a\cdot k_b)\\
&\times\left(\frac{k_a\cdot k_1}{4\big((k_a+k_b)\cdot k_1\big)^2} + \frac{k_b\cdot k_n}{4\big((k_a+k_b)\cdot k_n\big)^2}\right)\\
&\mathcal{A}_{\mathrm{N}}(1,\cdots,n).
\end{aligned} \tag{78}
$$

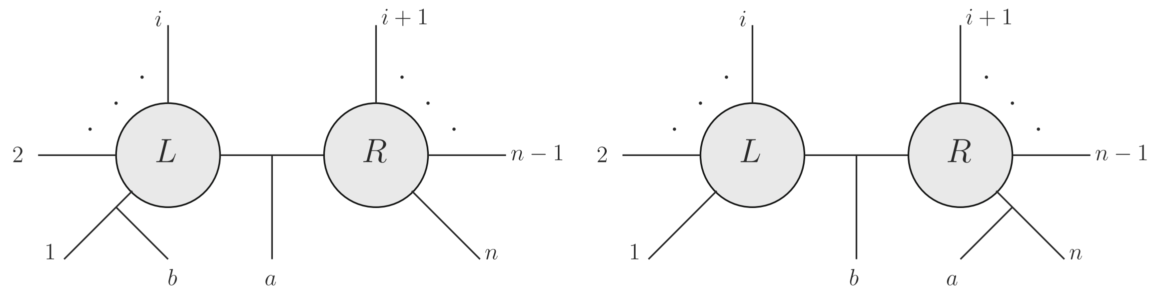

Fig. 11 Fifth type of Feynman diagram

Then by combining $\mathcal{P}_{3;1}$, $\mathcal{P}_{7;1}$, $\mathcal{P}_{3;2}$, $\mathcal{P}_{7;2}$, and $\mathcal{P}_8$ in (53), (63), (56), (67), and (69), we obtain

$$\mathcal{R}_{21} = \tau \left[\mathcal{J} \left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)} \right) \right] \mathrm{A}(2 \sqcup \cdots \sqcup n-1), \tag{79}$$

where

$$\begin{aligned}\mathcal{J} &= \frac{(k_a \cdot k_1)\,(k_b \cdot \partial_{k_1}) - (k_b \cdot k_1)\,(k_a \cdot \partial_{k_1})}{4(k_a + k_b) \cdot k_1} \\ &\quad + \frac{(k_b \cdot k_n)\,(k_a \cdot \partial_{k_n}) - (k_a \cdot k_n)\,(k_b \cdot \partial_{k_n})}{4(k_a + k_b) \cdot k_n} \\ &= \frac{k_a \cdot J_1 \cdot k_b}{4(k_a + k_b) \cdot k_1} + \frac{k_b \cdot J_n \cdot k_a}{4(k_a + k_b) \cdot k_n}. \end{aligned} \tag{80}$$

In the second line on the r.h.s, $J_a^{\mu\nu}$ is the angular momentum operator of the scalar particle a, which is defined by

$$J_a^{\mu\nu} \equiv k_a^{\mu} \frac{\partial}{\partial k_{a,\nu}} - k_a^{\nu} \frac{\partial}{\partial k_{a,\mu}}. \tag{81}$$

Putting pieces $\mathcal{P}_{1;1}$, $\mathcal{P}_{4;1}$, $\mathcal{P}_{10;1}$, $\mathcal{P}_{1;2}$, $\mathcal{P}_{4;2}$, $\mathcal{P}_{10;2}$, and $\mathcal{P}_9$ in (49), (60), (75), (54), (64), (76), and (72) together, we find

$$\mathcal{R}_{22} = \tau \left(\prod_{i=2}^{n-1} k_i \cdot X_i^{(0)} \right) \left(\mathcal{J}\, \mathrm{A}(2 \sqcup \cdots \sqcup n-1) \right). \tag{82}$$

One can combine $\mathcal{R}_{21}$ and $\mathcal{R}_{22}$ to obtain

$$\mathcal{R}_2 = \mathcal{R}_{21} + \mathcal{R}_{22} = \tau\, \mathcal{J}\, \mathcal{A}_{\mathrm{N}}(1, \ldots, n). \tag{83}$$

Consequently, we can express $\mathcal{A}_{\mathrm{N}}^{(1)}(1, \ldots, n, a, b)$ as

$$\mathcal{A}_{\mathrm{N}}^{(1)}(1, \ldots, n, a, b) = \mathcal{R}_1 + \mathcal{R}_2 = S_{\mathrm{N}}^{(1)}(a, b)\, \mathcal{A}_{\mathrm{N}}^{(1)}(1, \ldots, n), \tag{84}$$

where the soft operator $S_{\mathrm{N}}^{(1)}(a, b)$ is given as

$$\begin{aligned} S_{\mathrm{N}}^{(1)}(a, b) &= -\tau \Bigg[(k_a \cdot k_b) \left(\frac{k_a \cdot k_1}{4\big((k_a + k_b) \cdot k_1\big)^2} + \frac{k_b \cdot k_n}{4\big((k_a + k_b) \cdot k_n\big)^2} \right) \\ &\quad + \frac{k_b \cdot J_1 \cdot k_a}{4(k_a + k_b) \cdot k_1} + \frac{k_a \cdot J_n \cdot k_b}{4(k_a + k_b) \cdot k_n} \Bigg], \end{aligned} \tag{85}$$

which is again the same as the result found in [30,31].

5 Summary

In this note, we bootstrapped NLSM tree amplitudes based on the assumptions listed in Sect. 1. Note that the existence of expansions and the characteristic wherein coefficients do not involve any pole are proven rather than assumed, as can be seen in Sect. 2.2. Within the above assumptions, we first observed Adler's zero for four-point NLSM amplitudes by considering kinematics. Then we determined the expanded formula of general NLSM tree amplitudes, which manifests the permutation invariance among external legs, by using the universality of Adler's zero. The whole process does not require the assumption of quartic diagrams. We also re-derived double soft factors for NLSM tree amplitudes at leading and sub-leading orders, via the resulting expanded formula. The obtained double soft factors are coincident with those in the literature.

Our soft bootstrap method, based on the universality of soft behaviors and a double copy structure, is proven to be useful for constructing tree amplitudes of Yang–Mills scalar, Yang–Mills, Einstein–Yang–Mills, gravity [23], and nonlinear sigma models. It will be interesting to apply this method to a wider range of theories. Another potential future direction is to generalize this method from the tree level to the loop level.

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