



Scalarized black holes in new massive gravity dressed by a nonminimally coupled scalar

Yun Soo Myung^a

Institute of Basic Sciences and Department of Computer Simulation, Inje University, Gimhae 50834, Korea

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Abstract We investigate scalarized black holes in new massive gravity dressed by a nonminimally coupled scalar. For this purpose, we find the Gregory–Laflamme (GL) and tachyonic instability bounds of bald BTZ black hole expressed in terms of m^2 a massive spin-2 parameter and α a scalar coupling parameter to Ricci scalar by making use of the linearized theory around black hole. On the other hand, we obtain a solution bound of $0.161 < \alpha < 3/16$ for achieving non-BTZ black holes with scalar hair analytically and a thermodynamic bound of $1/8 < \alpha < 13/80$ for obtaining consistent thermodynamic quantities. Without imposing the GL instability bound ($m^2 < 1/2\ell^2$), we find a very narrow bound of $0.161 < \alpha < 0.1625$ which is located inside tachyonic instability bound of $\alpha > 1/8 + m_\phi^2 \ell^2/6$ for obtaining scalarized black holes where m_ϕ^2 is a scalar mass parameter.

1 Introduction

Recently, there was a significant progress on obtaining black holes with scalar hair called as spontaneous scalarization [1]. This corresponds to an explicit example for evading no-hair theorem. In this direction, the tachyonic instability of bald black holes is regarded as the onset of scalarized black holes when introducing a scalar coupling $f(\phi)$ to the source term: the Gauss–Bonnet (GB) term for Schwarzschild black hole with mass M [2–4] and Kerr black hole [5] or Maxwell term for Reissner–Nordström (RN) black hole with mass M and charge Q [6] and Kerr–Newman black hole [7]. For the GB coupling, the coupling function was usually chosen to be either $f(\phi) = (1 - e^{-6\phi^2})/6$ or ϕ^2 which leads to the same form when linearizing equations. Explicitly, the tachyonic instability for a scalar perturbation can be found when the negative region of potential is developed in the near horizon by including a negative mass term

$\mu_S^2 = -2\alpha \bar{R}_{\text{GB}}^2 = -\frac{96\alpha M^2}{r^6}$ with a positive scalar coupling parameter α in the Einstein–Gauss–Bonnet–scalar (EGBS) theory or $\mu_{\text{RN}}^2 = \alpha \bar{F}^2/2 = -\frac{\alpha(Mq)^2}{r^4}$ with $q = Q/M$ in Einstein–Maxwell–scalar theory. Another source terms might be geometric invariant sources of Ricci scalar (R) and Chern–Simons term ($*RR$). The first branch of scalarized black holes crosses the Schwarzschild black hole at the threshold ($\alpha_{th} = 0.363$ for $M = 1$) of tachyonic instability, while the first branch of scalarized charged black holes crosses the RN black hole at the threshold ($\alpha_{th} = 8.45$ for $q = 0.7$) of tachyonic instability. However, it is worth noting that most of scalarized black hole solutions were constructed numerically because their analytic solutions are hardly found. This implies that a completely thermodynamic study of these scalarized black holes is handicapped. For this purpose, it is desirable to find an analytic black hole solution with scalar hair.

On the other hand, a fourth-order gravity [Einstein–Weyl theory: $R - \frac{1}{2m_2^2}(R_{\mu\nu}^2 - R^2/3)$] with a massive spin-2 parameter m_2^2 has provided the non-Schwarzschild solution which crosses the Schwarzschild solution at the threshold point ($m_2^{th} = 0.876$ for $r_+ = 2M = 1$) [8]. This numerical solution represents a black hole with Ricci tensor hair, comparing to Schwarzschild with zero Ricci tensor. Even though its approximate analytic solutions were found by making use of the continued-fraction expansion [9], one could not find any analytic form of non-Schwarzschild solution. At this stage, we note that the instability of Schwarzschild black hole was found in the dRGT massive gravity [10, 11] and the instability bound of Schwarzschild black hole was found as $m_2 < 0.876/r_+$ when solving the Lichnerowicz equation for the linearized Ricci tensor $\delta R_{\mu\nu}$ [12]. It corresponds exactly to the linearized Einstein equation for $h_{\mu\nu}^{(4)}$ a four-dimensional metric tensor around a five-dimensional black string where the Gregory–Laflamme (GL) instability was firstly observed

^ae-mail: ysmyung@inje.ac.kr (corresponding author)

[13]. More recently, it was shown that the long-wave length instability bound for non-Schwarzschild solution is given by $m_2 < 0.876/r_+$ [14], which is the same bound as the GL instability for Schwarzschild solution. However, this instability bound is not consistent with that predicted by thermodynamic analysis of non-Schwarzschild solution [15].

Now, it is very interesting to note that the new massive gravity with a positive massive spin-2 parameter m^2 [16] is considered as a three-dimensional version of Einstein–Weyl gravity with a negative cosmological constant. In this case, the GL instability bound was found to be $m^2 < \frac{1}{2\ell^2}$ with ℓ the AdS₃ curvature radius [17]. Also, it indicates a clear connection (CSC: correlated stability conjecture) between thermodynamic instability and GL instability for the BTZ black hole regardless of the horizon radius r_+ [18]. On later, analytic black hole solutions were found in new massive gravity dressed by a nonminimally coupled scalar to Ricci scalar R [19]. However, an interpretation of asymptotically AdS black hole solutions is not still clear as well as their thermodynamic analysis is incomplete. Furthermore, these solutions have received less attention than asymptotically Lifshitz black hole solutions in three dimensions [20, 21].

In the present work, we wish to revisit these AdS black hole solutions as non-BTZ black holes with scalar hair and find all thermodynamic quantities. Without imposing the GL instability bound, we find a very narrow bound of $0.161 < \alpha < 0.1625$ which is located inside tachyonic instability bound of $\alpha > \alpha_{th}$ with $\alpha_{th} = 1/8 + m_\phi^2 \ell^2/6$ for obtaining scalarized black holes by analyzing both solution and thermodynamic bounds. On the other hand, imposing the GL instability bound, one could not find any consistent bound.

2 BTZ black holes

We start with the new massive gravity dressed by a nonminimally coupled scalar (NMGdbs) [19]

$$S_{\text{NMGdbs}} = S_{\text{NMG}} + S_\Phi, \tag{1}$$

$$S_{\text{NMG}} = \frac{1}{2} \int d^3x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{m^2} \left(R_{\rho\sigma} R^{\rho\sigma} - \frac{3}{8} R^2 \right) \right], \tag{2}$$

$$S_\Phi = -\frac{1}{2} \int d^3x \sqrt{-g} \left[(\partial\Phi)^2 + \alpha R\Phi^2 + m_\phi^2 \Phi^2 + \frac{\lambda}{12} \Phi^4 \right], \tag{3}$$

where m^2 is a positive massive spin-2 parameter, α is a positive scalar coupling to Ricci scalar R , and m_ϕ^2 is a scalar mass parameter [$m, m_\phi, \alpha \in (0, \infty)$]. Here, it is important to note that we introduce a quadratic scalar coupling ϕ^2 to Ricci

scalar R but not to fourth-order term $K = R_{\rho\sigma} R^{\rho\sigma} - 3R^2/8$. The reason for this choice is that the former is easy to find an analytic black hole solution with scalar hair than the latter.

The Einstein equation is given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} - T_{\mu\nu}^\Phi = 0, \tag{4}$$

where

$$K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2} \nabla_\mu \nabla_\nu R - \frac{1}{2} \Box R g_{\mu\nu} + 4R_{\mu\rho\nu\sigma} R^{\rho\sigma} - \frac{3}{2} R R_{\mu\nu} - R_{\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \frac{3}{8} R^2 g_{\mu\nu} \tag{5}$$

and the stress tensor for a scalar Φ is defined by

$$T_{\mu\nu}^\Phi = \partial_\mu \Phi \partial_\nu \Phi - \frac{g_{\mu\nu}}{2} \left[(\partial\Phi)^2 + m_\phi^2 \Phi^2 + \frac{\lambda}{12} \Phi^4 \right] + \alpha \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + G_{\mu\nu} \right) \Phi^2. \tag{6}$$

On the other hand, the scalar equation takes the form as

$$\Box \Phi - \alpha R \Phi - m_\phi^2 \Phi - \frac{\lambda}{6} \Phi^3 = 0. \tag{7}$$

Our primary concern is the non-rotating BTZ black hole solution to Eqs. (4) and (7) given by

$$ds_{\text{BTZ}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \tag{8}$$

$$f(r) = -M + \frac{r^2}{\ell^2},$$

$$\Phi = 0$$

under the condition of $1/\ell^2 + \Lambda + 1/(4m^2\ell^4) = 0$. Here M is an integration constant related to the ADM mass of BTZ black hole and ℓ denotes the curvature radius of AdS₃ spacetimes. The horizon radius $r_+ = \sqrt{M}\ell$ is determined by the condition of $f(r) = 0$. Its Hawking temperature is found to be

$$T_H(r_+) = \frac{f'(r_+)}{4\pi} = \frac{r_+}{2\pi\ell^2}. \tag{9}$$

Using the Abbott–Deser–Tekin method [22, 23], one can derive all (m^2 -dependent) thermodynamic quantities of its mass, heat capacity ($C = \frac{dM_{\text{ADT}}}{dT_H}$), entropy, and Helmholtz free energy as [18]

$$M_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2} M(r_+), \quad C_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2} C(r_+),$$

$$S_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2} S_{\text{BH}}(r_+), \quad F_{\text{ADT}} = \frac{\mathcal{M}^2}{m^2} F(r_+), \tag{10}$$

where \mathcal{M}^2 is defined by

$$\mathcal{M}^2(m^2) = m^2 - \frac{1}{2\ell^2}. \tag{11}$$

Here, thermodynamic quantities in Einstein gravity are given by

$$M(r_+) = \frac{r_+^2}{\ell^2}, \quad C(r_+) = 4\pi r_+, \quad S_{BH}(r_+) = 4\pi r_+,$$

$$F(r_+) = M - T_H S_{BH} = -\frac{r_+^2}{\ell^2}. \tag{12}$$

We note that these quantities are positive regardless of the horizon size r_+ except that the free energy is always negative. This means that the BTZ black hole is thermodynamically stable in Einstein gravity. Also, it is easy to check that the first-law of thermodynamics is satisfied as

$$dM_{\text{ADT}} = T_H dS_{\text{ADT}} \tag{13}$$

as the first-law is satisfied in Einstein gravity

$$dM = T_H dS_{\text{BH}} \tag{14}$$

where ‘ d ’ denotes the differentiation with respect to the horizon size r_+ only. It is interesting to observe that in the limit of $m^2 \rightarrow \infty$ one recovers thermodynamics of the BTZ black hole in Einstein gravity, while in the limit of $m^2 \rightarrow 0$ we recover the black hole thermodynamics in purely fourth-order gravity.

We are in a position to discuss the linear stability of BTZ black hole in NMGdbs by considering metric perturbation $h_{\mu\nu}$ and scalar perturbation ϕ in $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and $\Phi = 0 + \phi$. Firstly, the linearized Einstein equation to (4) upon choosing the transverse-traceless gauge ($\bar{\nabla}^\mu h_{\mu\nu} = 0$ and $h^\mu{}_\mu = 0$) leads to the fourth-order equation for the metric perturbation $h_{\mu\nu}$

$$\left[\bar{\Delta}_L + \frac{2}{\ell^2} - \mathcal{M}^2(m^2) \right] \left(\bar{\Delta}_L + \frac{2}{\ell^2} \right) h_{\mu\nu} = 0, \tag{15}$$

where $\bar{\Delta}_L$ denotes the Lichnerowicz operator defined as

$$\bar{\Delta}_L h_{\mu\nu} = -\bar{\square} h_{\mu\nu} - \bar{R}_{\rho\mu\sigma\nu} h^{\rho\sigma} - \frac{2}{\ell^2} h_{\mu\nu}. \tag{16}$$

This might imply the two second-order linearized equations

$$\left(\bar{\Delta}_L + \frac{2}{\ell^2} \right) h_{\mu\nu} = 0, \tag{17}$$

$$\left[\bar{\Delta}_L + \frac{2}{\ell^2} - \mathcal{M}^2(m^2) \right] h_{\mu\nu}^{\mathcal{M}} = 0. \tag{18}$$

Here, the mass squared \mathcal{M}^2 of a massive spin-2 is given by Eq. (11). Importantly, Eq. (17) describes a massless spin-2 (gauge degrees of freedom), whereas Eq. (18) describes a massive spin-2 with 2 DOF propagating around the BTZ black hole under the transverse-traceless gauge. Solving a coupled first-order equations for $s(n = 0)$ -mode H_{tr} and $H_- = H_{tt}/f(r) - f(r)H_{rr}$ derived from Eq. (18) with $h_{\mu\nu} = e^{\Omega t} e^{in\varphi} H_{\mu\nu}$, the classical instability (Gregory–Laflamme instability) bound of the BTZ black hole on m^2

was found by

$$\mathcal{M}^2 < 0 \rightarrow m^2 < \frac{1}{2\ell^2} \rightarrow m < m_{th}, \quad m_{th} = \frac{0.7}{\ell} \tag{19}$$

regardless of the horizon radius r_+ [17].

It is well known that the local thermodynamic stability is determined by the positive heat capacity ($C_{\text{ADT}} > 0$) in NMGdbs. For $\mathcal{M}^2 > 0 (m^2 > m_{th}^2)$, all thermodynamic quantities have the same property as those for Einstein gravity, whereas for $\mathcal{M}^2 < 0 (m^2 < m_{th}^2)$, all thermodynamic quantities have the same property as those for fourth-order term. We observe that for $\mathcal{M}^2 > 0$, the BTZ black hole is thermodynamically stable, regardless of the horizon radius r_+ because of $C_{\text{ADT}} > 0$. The case of $\mathcal{M}^2 = 0 (m^2 = m_{th}^2)$ corresponds to the critical gravity where all thermodynamic quantities are zero and logarithmic modes appear. For $\mathcal{M}^2 < 0$, the BTZ black hole is thermodynamically unstable because of $C_{\text{ADT}} < 0$ as well as it is classically unstable against metric perturbation. Hence, it shows a clear connection (CSC: correlated stability conjecture) between thermodynamic and classical instability for the BTZ black hole regardless of the horizon radius r_+ in NMGdbs. However, it is important to note that there is no such connection in Einstein gravity.

Finally, let us study the tachyonic instability for scalar perturbation. By linearizing Eq. (7), one finds a linearized scalar equation with $\bar{R} = -6/\ell^2$

$$\bar{\square}\phi - \left(-\frac{6\alpha}{\ell^2} + m_\Phi^2 \right) \phi = 0. \tag{20}$$

Considering the separation of variables as

$$\phi(t, \varphi, r) \propto e^{-i\omega t + in\varphi} \frac{\psi(r)}{\sqrt{r}}, \tag{21}$$

one finds the Schrödinger equation with a tortoise coordinate defined by $dr_* = dr/f(r)$ as

$$\frac{d^2\psi}{dr_*^2} + \left[\omega^2 - V_{\psi,\alpha}(r) \right] \psi = 0, \tag{22}$$

where the scalar potential is given by

$$V_{\psi,\alpha}(r) = f(r) \left[\frac{M}{4r^2} + \frac{n^2}{r^2} - \frac{6}{\ell^2} \left(\alpha - \frac{1}{8} - \frac{m_\Phi^2 \ell^2}{6} \right) \right]. \tag{23}$$

The appearance of negative region in the far region indicates tachyonic instability because the last term of Eq. (23) plays the role of an effective mass term.

Observing Fig. 1, one finds that the tachyonic instability bound for $s(n = 0)$ -mode scalar is determined by

$$\alpha > \alpha_{th}, \quad \alpha_{th} = \frac{1}{8} + \frac{m_\Phi^2 \ell^2}{6}. \tag{24}$$

Consequently, we find two instability bounds (19) and (24) from stability analysis of BTZ black holes in NMGdbs.

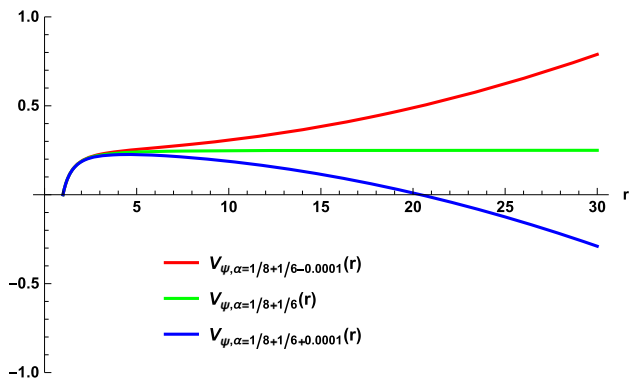


Fig. 1 Scalar potential $V_{\psi, \alpha}(r \in [r_+ = 1, 30])$ for $s(n = 0)$ -mode ϕ with $M = \ell^2 = m_{\Phi}^2 = 1$ and $\alpha = 1/8 + 1/6 - 0.0001$ (stable), $\alpha = 1/8 + 1/6 (= \alpha_{th})$ (threshold of tachyonic instability), and $\alpha = 1/8 + 1/6 + 0.0001$ (unstable). The negative region increases as α increases because the last term of Eq. (23) plays the role of an effective mass term

3 Scalarized black holes

One finds an analytic solution for scalarized black holes by solving (4) and (7) as [19]

$$ds_{\text{sbh}}^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\varphi^2,$$

$$F(r) = -M \left(\frac{r}{\ell} \right)^{\frac{32\alpha-5}{16\alpha-3}} + \frac{r^2}{\ell^2}$$

$$\Phi_{\alpha}(r) = \left(\frac{r}{\ell} \right)^{\frac{0.5}{16\alpha-3}} \tilde{\Phi}_{\alpha}, \quad \tilde{\Phi}_{\alpha} = \sqrt{\frac{8M(32\alpha-5)}{256\alpha^2-32\alpha-1}}, \quad (25)$$

where M denotes the ADM mass of scalarized black hole. We regard Eq. (25) as non-BTZ black holes with scalar hair in three dimensions. One finds that $256\alpha^2 - 32\alpha - 1 = 256(\alpha - \frac{1+\sqrt{2}}{16})(\alpha - \frac{1-\sqrt{2}}{16})$. Importantly, we observe that a case of $\alpha = \frac{5}{32}$ recovers the BTZ black hole Eq. (8) with $\Phi = 0$. Imposing a vanishing $\Phi_{\alpha}(r)$ at infinity and a positive finite scalar hair on the horizon, we find two bounds for α

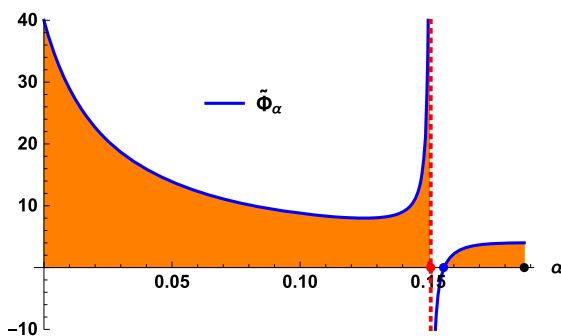


Fig. 2 (Left) $\tilde{\Phi}_{\alpha}$ as a function of α with $M = 1$. The shaded region represents $\tilde{\Phi}_{\alpha} > 0$, it blows up at $\alpha = 1.151$ (dotted line) and it is zero at $\alpha = 0.156, 0.1875$. (Right) Scalar hairs $\Phi_{\alpha}(r)$ as function

(see Fig. 2)

$$0 < \alpha < \frac{1 + \sqrt{2}}{16} (= 0.151),$$

$$\frac{5}{32} (= 0.156) < \alpha < \frac{3}{16} (= 0.1875). \quad (26)$$

For this solution, other parameters take the forms as

$$m^2(\alpha) = \frac{256\alpha^2 - 32\alpha - 1}{2(16\alpha - 3)^2 \ell^2},$$

$$\Lambda(\alpha) = -\frac{(16\alpha - 1)(48\alpha - 7)}{2(256\alpha^2 - 32\alpha - 1)\ell^2},$$

$$m_{\Phi}^2(\alpha) = \frac{(8\alpha - 1)(768\alpha^2 - 192\alpha + 11)}{4(16\alpha - 3)^2 \ell^2},$$

$$\lambda(\alpha) = -\frac{3(8\alpha - 1)(256\alpha^2 - 32\alpha - 1)(768\alpha^2 - 152\alpha + 9)}{16(16\alpha - 3)^2(32\alpha - 5)\ell^2}. \quad (27)$$

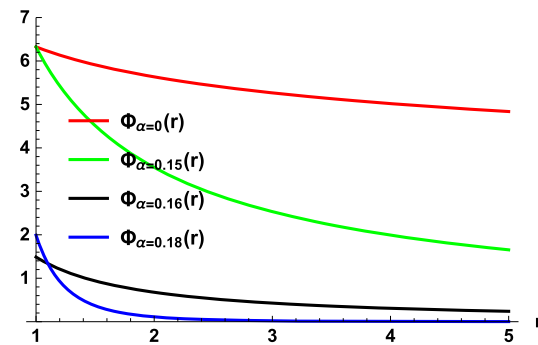
Taking a positive $m^2(\alpha)$, one has a bound of $\alpha > \frac{1+\sqrt{2}}{16}$ [see (Left) Fig. 3]. For a negative $\Lambda(\alpha)$, one has two bounds: $\frac{1}{16} (= 0.063) < \alpha < \frac{7}{48} (= 0.146)$ and $\alpha > \frac{1+\sqrt{2}}{16}$ [see (Right) Fig. 3]. Two roots of $768\alpha^2 - 192\alpha + 11 = 0$ are given by $\alpha = 0.089, 0.161$. For a positive $m_{\Phi}^2(\alpha)$, we find two bounds: $0.089 < \alpha < \frac{1}{8} (= 0.125)$ and $\alpha > 0.161$ (see Fig. ??). However, there is no restriction on $\lambda(\alpha)$. A common bound from $m^2(\alpha) > 0$, $\Lambda(\alpha) < 0$, and $m_{\Phi}^2(\alpha) > 0$ leads to

$$\alpha > 0.161. \quad (28)$$

Finally, combining Eq. (28) with Eq. (26), one finds a solution bound for obtaining scalarized black hole solutions

$$0.161 < \alpha^{\text{sol}} < \frac{3}{16} (= 0.1875). \quad (29)$$

Here, we obtain three isolated points for positive α : $\alpha_{\text{iso}} = \frac{1+\sqrt{2}}{16}, \frac{5}{32}, \frac{3}{16}$ because of $\alpha = \frac{1-\sqrt{2}}{16} < 0$. These all are excluded from the allowed parameter space.



of $r \in [r_+ = 1, 5]$ with $\ell = 1$ and $\alpha = 0, 0.15, 0.16, 0.18$. The forbidden region is between $\alpha = 0.151$ and $\alpha = 0.156$

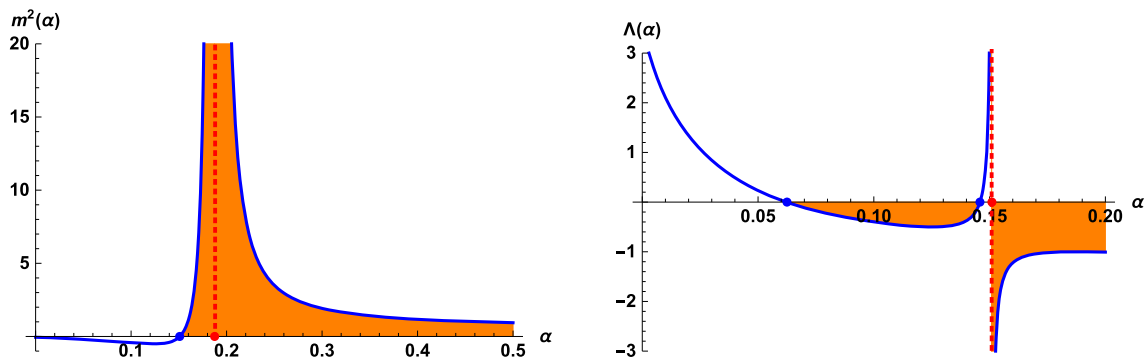


Fig. 3 (Left) Mass $m^2(\alpha)$ as a function of α with $\ell = 1$. The shaded region represents $m^2(\alpha) > 0$ and it blows up at $\alpha = 0.1875$ (dotted line). (Right) Cosmological constant $\Lambda(\alpha)$ as function of α with $\ell = 1$. The shaded region represents $\Lambda(\alpha) < 0$ and it blows up at $\alpha = 0.151$ (dotted line)

4 Complete analysis for thermodynamics of scalarized black holes

First of all, the (α -dependent) Hawking temperature is defined by

$$T_H(r_+, \alpha) = \frac{F'(r_+)}{4\pi} = \frac{r_+}{4\pi\ell^2(3 - 16\alpha)}. \tag{30}$$

Other (α -dependent) thermodynamic quantities are found to be

$$\begin{aligned} M(r_+, \alpha) &= \frac{r_+^2}{\ell^2} \left(\frac{r_+}{\ell}\right)^{\frac{32\alpha-5}{16\alpha-3}}, & C(r_+, \alpha) &= 4\pi r_+ \left(\frac{r_+}{\ell}\right)^{\frac{32\alpha-5}{16\alpha-3}}, \\ S_{BH}(r_+, \alpha) &= 2\pi r_+ \left(\frac{16\alpha-3}{8\alpha-1}\right) \left(\frac{r_+}{\ell}\right)^{\frac{32\alpha-5}{16\alpha-3}}, \\ F_\alpha(r_+) &= M - T_H S_{BH} = \frac{r_+^2}{\ell^2} \left(\frac{80\alpha-13}{16\alpha-2}\right) \left(\frac{r_+}{\ell}\right)^{\frac{32\alpha-5}{16\alpha-3}}. \end{aligned} \tag{31}$$

In deriving the entropy S_{BH} , we use the first-law of thermodynamics as

$$dM = T_H dS_{BH}. \tag{32}$$

For an isolated point ($\alpha = \frac{5}{32}$), we observe that the above reduces to those in Eq. (12) for BTZ black hole exactly. For an isolated point ($\alpha = \frac{3}{16}$), any thermodynamic quantities are not defined properly because they blow up. Requiring that $T_H(r_+, \alpha) > 0 \rightarrow 0 < \alpha < \frac{3}{16}$, $S_{BH}(r_+, \alpha) > 0 \rightarrow \frac{1}{8} < \alpha < \frac{3}{16}$, and $F_\alpha(r_+) < 0 \rightarrow \frac{1}{8} < \alpha < \frac{13}{80}$, one finds a thermodynamic bound for obtaining appropriate thermodynamic quantities for scalarized black hole as

$$\frac{1}{8} (= 0.125) < \alpha^{\text{ther}} < \frac{13}{80} (= 0.1625). \tag{33}$$

We note that in view of having appropriately thermodynamic quantities, two cases of $\alpha = \frac{3}{16}$ and $\frac{1}{8}$ (stealth black hole) are isolated points which are excluded from the allowed parameter space α .

Consequently, combining Eq. (29) with (33), one has a very specific bound for obtaining scalarized black hole with

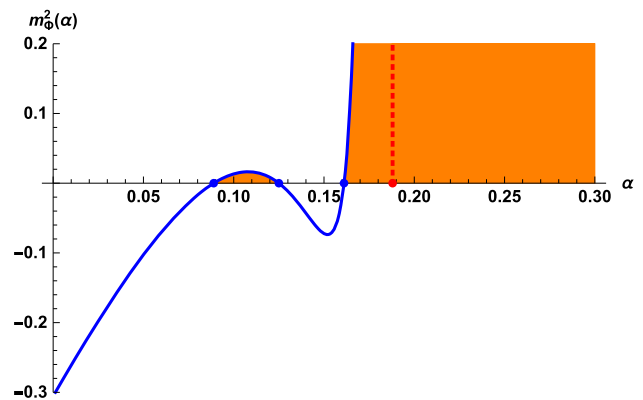


Fig. 4 (Left) Mass of scalar $m_\phi^2(\alpha)$ as a function of α with $\ell = 1$. Two shaded regions denote $m_\phi^2(\alpha) > 0$ and it blows up at $\alpha = 0.1875$ (dotted line)

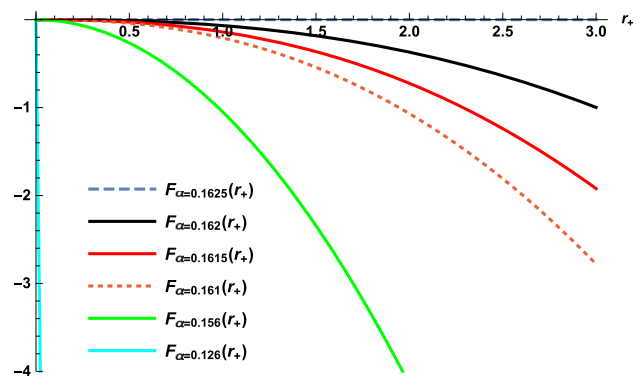


Fig. 5 Helmholtz free energy $F_\alpha(r_+ \in [0, 10])$ with $\ell^2 = 1$ and $\alpha = 0.1625$ (zero: r_+ -axis), $\alpha = 0.162$ (within bound (34)), $\alpha = 0.1615$ (within bound (34)), $\alpha = 0.161$, $\alpha = 0.156$ (BTZ), and $\alpha = 0.126$ (F_α -axis)

appropriate thermodynamic quantities as

$$0.161 < \alpha < 0.1625, \tag{34}$$

which is regarded as our key result. We note that this bound belongs to the tachyonic bound (24). However, if the GL

instability bound (19) is imposed, the mass bound on m^2 is modified as $\frac{1+\sqrt{2}}{16} < \alpha < \frac{5}{32}$. In this case, we have no such bound like as Eq. (34). In this sense, the GL instability bound might have nothing to do with constructing scalarized black holes.

Finally, we observe the Helmholtz free energy depicted in Fig. 5 to study phase transition between BTZ and scalarized black holes since all specific heats $C(r_+, \alpha)$ are positive definite. We find that phase transition from BTZ black hole to scalarized black holes within the bound Eq. (34) is unlikely to occur. The BTZ black hole is always favored than scalarized black hole because of $F_{\alpha=0.156(BTZ)} < F_{0.161 < \alpha < 0.625}$. It needs to perform stability analysis of scalarized black holes within the bound Eq. (34).

5 Discussions

We have investigated the new massive gravity dressed by a nonminimally coupled scalar (NMGdbs) in Eq. (1). In this case, a quadratic scalar coupling to Ricci scalar R was included, instead of a quadratic scalar coupling to fourth-order term $K = R_{\rho\sigma}R^{\rho\sigma} - 3R^2/8$: $S_{\text{NMGs}} = \frac{1}{2} \int d^3x \sqrt{-g} [R - 2\Lambda - (\partial\Phi)^2 - (1 - \Phi^2)K/m^2]$ with m^2 a single massive spin-2 parameter. We note that K plays a role of the Gauss–Bonnet term in four dimensions because the Gauss–Bonnet term is identically zero in three dimensions. Even though the former coupling does not belong to the general setup like the latter coupling, we have considered the former coupling because it may admit an analytic black hole solution with scalar hair.

From the NMGdbs, we have found the non-rotating BTZ black hole as the bald black hole without scalar hair. We have made the linearized theory around the BTZ black hole to find the condition for instability and to trigger the spontaneous scalarization. Linearizing two equations (4) and (7) leads to two linearized equations: one is the linearized Einstein equation and the other is the linearized scalar equation. The linearized scalar equation (20) differs from the four-dimensional linearized scalar equation. The mass term of the NMGdbs is a constant term like as $'-6\alpha/\ell^2 + m_\Phi^2'$, whereas the mass term of the EGBS theory is given by $\mu_S^2 = -\frac{96\alpha M^2}{r^6}$. The former induces negative potential in the far region, while the latter provides negative potential in the near horizon. Also, the linearized Einstein equation (18) in NMGdbs is the same form as in NMG, whereas the linearized Einstein equation in the EGBS theory is the same as that for Einstein theory. We have obtained the Gregory–Laflamme instability bound ($m < m_{th} = 0.7/\ell$) and tachyonic instability bound ($\alpha > \alpha_{th} = 1/8 + m_\Phi^2 \ell^2/6$) of bald BTZ black hole by making use of the linearized theory around BTZ black hole.

Solving two full equations (4) and (7) directly, one has found an analytic (α -dependent) scalarized black hole (25) with four parameters $m^2(\alpha)$, $\Lambda(\alpha)$, $m_\Phi^2(\alpha)$, and $\lambda(\alpha)$. We have obtained a solution bound of $0.161 < \alpha^{\text{sol}} < 3/16$ for achieving black holes with scalar hair analytically and a thermodynamic bound of $1/8 < \alpha^{\text{ther}} < 13/80$ for finding consistent thermodynamic quantities. Without imposing the GL instability bound ($m < \frac{0.7}{\ell}$), we have found a very narrow bound of $0.161 < \alpha < 0.1625$ which is located inside tachyonic instability bound of $\alpha > \alpha_{th}$ with $\alpha_{th} = 1/8 + m_\Phi^2 \ell^2/6$ for obtaining scalarized black holes. However, if the GL instability bound is imposed, the mass bound on $m^2(\alpha)$ is modified as $\frac{1+\sqrt{2}}{16} < \alpha < \frac{5}{32}$. In this case, there is no such bound like Eq. (34). In this sense, it seems that the GL instability bound has nothing to do with constructing scalarized black holes.

On the other hand, the GL instability supports correlated stability conjecture (CSC) between thermodynamic instability and classical instability for bald BTZ black hole regardless of the horizon radius r_+ in NMGdbs.

Consequently, the scalarized (α -dependent) black hole exists within a very narrow band of $0.161 < \alpha < 0.1625$ in NMGdbs.

Data availability statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The author confirms that the data will not be deposited because this work is a theoretical one.]

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