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Moderately non-local $\eta\bar{\eta}$ vertices in the AdS_4 higher-spin gauge theory

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Abstract A new concept of moderate non-locality in higher-spin gauge theory is introduced. Based on the recently proposed differential homotopy approach, a moderately non-local scheme, that is softer than those resulting from the shifted homotopy approach available in the literature so far, is worked out in the mixed $\eta\bar{\eta}$ sector of the Vasiliev higher-spin theory. To calculate moderately non-local vertices $\Upsilon^{\eta\bar{\eta}}(\omega, C, C, C)$ for all ordering of the fields ω and C we apply an interpolating homotopy, that respects the moderate non-locality in the perturbative analysis of the higher-spin equations.

Contents

1 Introduction

Higher-spin (HS) gauge theory describes interacting systems of massless fields of all spins, resulting from a nonlinear deformation of the Fronsdal theory of free HS fields [1]. Such theories play a role in various contexts from holography [2] to cosmology [3]. A useful way of description of HS dynamics in AdS_4 is provided by the generating Vasiliev system of HS equations [4]. The latter contains a free complex parameter

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η. Reconstructing on-shell HS vertices order by order one obtains vertices proportional to various powers of η and $\bar{\eta}$.

Since the HS gauge theory contains infinite tower of gauge fields of all spins and the number of space-time derivatives increases with the spins of fields in the vertex [5–8], the theory exhibits certain degree of non-locality. The level of non-locality of HS gauge theory is debatable in the literature.

In the papers [9–15] vertices in the holomorphic (anti-holomorphic) sector up to η^2 ($\bar{\eta}^2$), were reconstructed from the generating Vasiliev system in the spin-local form. (See also [16] for a higher-order extension of these results.) The shifted homotopy approach used in [9–14] demands careful choice of the homotopy scheme compatible with the spin-locality of the vertices (for more detail on the notion of spin-locality see Sect. 4.1).

Being efficient in the lowest orders, the original shifted homotopy approach turns out to be less powerful at higher orders. This way, it has not been yet possible to find spin-local vertices in the so called mixed $\eta\bar{\eta}$ sector of equations for zero-form fields.

From the perspective of HS theory in the bulk it is hard to identify the minimal level of non-locality of the theory unless a constructive scheme that supports some its specific level is presented. The aim of this paper is to present such a scheme that supports a moderate non-locality of the HS theory in the mixed $\eta\bar{\eta}$ sector, that is less non-local than those resulting from the shifted homotopy approach available in the literature so far. Specifically, we will use the differential homotopy approach proposed recently by Vasiliev in [17] to obtain moderately non-local vertices $\Upsilon^{\eta\bar{\eta}}(\omega, C, C, C)$ for the zero-form equations in the mixed sector.

Since the moderately non-local vertices obtained in this paper minimize the level of non-locality of the known HS vertices, it would be interesting to compare it with the level of non-locality of the vertices obtained in [18] via holographic duality based on the Klebanov–Polyakov conjecture [2] (see also [19–21]). A priori, there are two options:

(i) Moderately non-local vertices may have the same (or even worse) level of nonlocality than that deduced in [18].

(ii) Moderately non-local vertices of this paper may be less nonlocal than those of [18].

The option (i) is in fact inconclusive since it is not guaranteed that there is no better scheme allowing to soften further the level of vertex non-locality. On the other hand, the option (ii) would imply that the HS holographic duality has to be modified one way or another, for instance along the lines of [22]. Though the analysis of this issue is very interesting, it is not straightforward because of the difference between the formalisms underlying the space-time analysis of [18] and the unfolded analysis of this paper in terms of auxiliary spinor variables. Hence it is postponed for the future study.

The paper is organized as follows. In Sect. 2 we recall the form of HS equations. In Sect. 3 the Vasiliev concept of differential homotopy and the Ansatz for the linear in η deformations of [17] are recalled. In Sect. 3.3 the Ansatz for the linear in $\eta\bar{\eta}$ deformations is introduced, as a straightforward generalization of that of [17]. In Sect. 4.1 we briefly discuss a locality issue and introduce a notion of 'moderate spin-non-locality' (MNL), also introducing 'interpolating homotopy' (IH) that respects MNL. In Sect. 5 the derivation of the MNL B_3 is considered in detail. In Sect. 6 the resulting MNL vertices $\Upsilon^{\eta\bar{\eta}}(\omega, C, C, C)$ are introduced. Conclusions are summarized in Sect. 7. Appendices A–C collect previously known results of the lowest-order computations while Appendices D and E contain vertex $\Upsilon_{\omega CCC}$ and $\Upsilon_{C\omega CC}$ calculation details, respectively.

2 Higher-spin equations

2.1 Original form

The nonlinear HS equations of [4]

$$d_x W + W * W = 0\,, \tag{2.1}$$
$$d_x S + W * S + S * W = 0\,, \tag{2.2}$$
$$d_x B + [W, B]_* = 0\,, \tag{2.3}$$
$$S * S = i(\theta^A \theta_A + B * (\eta\gamma + \bar{\eta}\bar{\gamma}))\,, \tag{2.4}$$
$$[S, B]_* = 0 \tag{2.5}$$

reproduce field equations on dynamical HS fields in any gauge and choice of field variables. The field $B(Z; Y; K|x)$ is a zero-form, x are space-time coordinates, $Z_A = (z_\alpha, \bar{z}_{\dot{\alpha}})$, $Y_A = (y_\alpha, \bar{y}_{\dot{\alpha}})$ are auxiliary commuting spinor variables ($\alpha, \beta = 1, 2$; $\dot{\alpha}, \dot{\beta} = 1, 2$), η is a free complex parameter ($\bar{\eta}$ is its complex conjugate) and $K = (k, \bar{k})$ are involutive Klein operators obeying

$$\{k, y_\alpha\} = \{k, z_\alpha\} = 0, \qquad [k, \bar{y}_{\dot{\alpha}}] = [k, \bar{z}_{\dot{\alpha}}] = 0,$$
$$k^2 = 1, \qquad [k, \bar{k}] = 0. \tag{2.6}$$

Analogously for $\bar{k}$.

The field $W(Z; Y; K|x)$ is a space-time one-form, i.e., $W = dx^\nu W_\nu$, while $S(Z; Y; K|x)$ is a one-form in Z spinor directions, i.e., $S = \theta^\alpha S_\alpha + \bar{\theta}^{\dot{\alpha}} S_{\dot{\alpha}}$, $\theta^\alpha := dz^\alpha$, $\bar{\theta}^{\dot{\alpha}} := d\bar{z}^{\dot{\alpha}}$. The wedge symbol is implicit in this paper since all products are exterior.

The star product is

$$(f * g)(Z, Y) = \frac{1}{(2\pi)^4} \int dU dV f(Z + U; Y + U) g \times (Z - V; Y + V) \exp(i U_A V^A). \tag{2.7}$$

Indices are raised and lowered by the symplectic form $C_{BA} = (\epsilon_{\beta\alpha}, \epsilon_{\dot{\beta}\dot{\alpha}})$,

$$X^A = C^{AB} X_B, \qquad X_A = X^B C_{BA}. \tag{2.8}$$

Elements γ and $\bar{\gamma}$,

$$\gamma = \exp(iz_\alpha y^\alpha)k\theta^\alpha\theta_\alpha, \qquad \bar{\gamma} = \exp(i\bar{z}_{\dot\alpha}\bar{y}^{\dot\alpha})\bar{k}\bar{\theta}^{\dot\alpha}\bar{\theta}_{\dot\alpha}, \tag{2.9}$$

are central with respect to the star product since $\theta^3 = \bar{\theta}^3 = 0$.

Following [4], to analyse Eqs. (2.1)–(2.5) perturbatively one starts with the vacuum solution

$$B_0 = 0, \qquad S_0 = \theta^A Z_A = \theta^\alpha z_\alpha + \bar{\theta}^{\dot\alpha}\bar{z}_{\dot\alpha}. \tag{2.10}$$

Plugging this into (2.1)–(2.5) and using that

$$[S_0,]_* = -2i\mathrm{d}_Z, \qquad \mathrm{d}_Z := \theta^A\frac{\partial}{\partial Z^A}, \tag{2.11}$$

one finds that W_0 should be Z-independent, $W_0 = \omega(Y; K|x)$, and satisfy (2.1). Similarly, at the next order one gets $B_1 = C(Y; K|x)$ from $[S_0, B_1] = 0$ and that C satisfies (2.3). This way one reconstructs the first terms on the *r.h.s.'s* of the unfolded equations of the form originally proposed in [23]

$$\begin{aligned}\mathrm{d}_x\omega + \omega * \omega &= \Upsilon^\eta(\omega,\omega,C)\\ &+\Upsilon^{\bar\eta}(\omega,\omega,C) + \Upsilon^{\eta\eta}(\omega,\omega,C,C)\\ &+\Upsilon^{\bar\eta\bar\eta}(\omega,\omega,C,C) + \Upsilon^{\eta\bar\eta}(\omega,\omega,C,C)\ldots,\end{aligned} \tag{2.12}$$

$$\begin{aligned}\mathrm{d}_xC + [\omega, C]_* &= \Upsilon^\eta(\omega,C,C) + \Upsilon^{\bar\eta}(\omega,C,C)\\ &+\Upsilon^{\eta\eta}(\omega,C,C,C) + \Upsilon^{\bar\eta\bar\eta}(\omega,C,C,C)\\ &+\Upsilon^{\eta\bar\eta}(\omega,C,C,C)\ldots.\end{aligned} \tag{2.13}$$

As in [23], the resulting perturbative expansion is in powers of the zero-forms C.

To obtain dynamical Eqs. (2.12), (2.13) one should plug obtained B_i, W_i into Eqs. (2.1), (2.3). For instance, Eq. (2.3) up to the third order in C-field is

$$\begin{aligned}\mathrm{d}_xC + [\omega, C]_* = &-\mathcal{D}B_2 - [W_1, C]_*\\ &-\mathcal{D}B_3 - [W_1, B_2]_* - [W_2, C]_*,\end{aligned} \tag{2.14}$$

where

$$\mathcal{D}A := \mathrm{d}_xA + [\omega, A]_*. \tag{2.15}$$

For more detail we refer the reader to the review [24].

2.2 Free equations in AdS_4

AdS_4 vacuum one-form connection W_0 is

$$W_0 = \frac{1}{2}w^{AB}(x)Y_AY_B, \quad \mathrm{d}w^{AB} + w^{AC}C_{CD}w^{DB} = 0, \tag{2.16}$$

where C_{AB} is the $sp(4)$ invariant form, $w^{AB} = (\omega^{\alpha\beta}, \bar\omega^{\dot\alpha\dot\beta}, e^{\alpha\dot\alpha})$ describes Lorentz connection, $\omega^{\alpha\beta}$, $\bar\omega^{\dot\alpha\dot\beta}$, and vierbein, $e^{\alpha\dot\alpha}$. The unfolded system for free massless fields $\omega(y,\bar{y}; K|x)$ and $C(y,\bar{y}; K|x)$ reads as [23]

$$\begin{aligned}R_1(y,\bar{y}; K \mid x) = \frac{i}{4}\Big(&\eta\overline{H}^{\dot\alpha\dot\beta}\bar\partial_{\dot\alpha}\bar\partial_{\dot\beta}C(0,\bar{y}; K \mid x)k\, n\\ &+\bar\eta H^{\alpha\beta}\partial_\alpha\partial_\beta C(y, 0; K \mid x)\bar{k}\Big),\end{aligned} \tag{2.17}$$

$$\tilde{D}_0C(y,\bar{y}; K \mid x) = 0, \tag{2.18}$$

where

$$\partial_\alpha := \frac{\partial}{\partial y^\alpha}, \qquad \bar\partial_{\dot\alpha} := \frac{\partial}{\partial\bar{y}^{\dot\alpha}}, \tag{2.19}$$

$$H_{\alpha\beta} := e_{\alpha\dot\alpha}e_\beta{}^{\dot\alpha}, \qquad \overline{H}_{\dot\alpha\dot\beta} := e_{\alpha\dot\alpha}e^\alpha{}_{\dot\beta}, \tag{2.20}$$

$$R_1(y,\bar{y}; K \mid x) := D_0^{ad}\omega(y,\bar{y}; K \mid x)$$

$$D_0^{ad} = D^L - e^{\alpha\dot\beta}\Big(y_\alpha\bar\partial_{\dot\beta} + \partial_\alpha\bar{y}_{\dot\beta}\Big), \tag{2.21}$$

$$D^L = \mathrm{d}_x - \Big(\omega^{\alpha\beta}y_\alpha\partial_\beta + \bar\omega^{\dot\alpha\dot\beta}\bar{y}_{\dot\alpha}\bar\partial_{\dot\beta}\Big),$$

$$\tilde{D}_0 = D^L + e^{\alpha\dot\beta}\Big(y_\alpha\bar{y}_{\dot\beta} + \partial_\alpha\bar\partial_{\dot\beta}\Big). \tag{2.22}$$

The massless fields obey

$$\begin{aligned}&\omega(y,\bar{y}; -k, -\bar{k} \mid x) = \omega(y,\bar{y}; k,\bar{k} \mid x),\\ &C(y,\bar{y}; -k, -\bar{k} \mid x) = -C(y,\bar{y}; k,\bar{k} \mid x).\end{aligned} \tag{2.23}$$

System (2.17), (2.18) decomposes into subsystems of different spins, with a spin s described by the one-forms $\omega(y,\bar{y}; K|x)$ and zero-forms $C(y,\bar{y}; K|x)$ obeying

$$\begin{aligned}&\omega(\mu y, \mu\bar{y}; K \mid x) = \mu^{2(s-1)}\omega(y,\bar{y}; K \mid x),\\ &C(\mu y, \mu^{-1}\bar{y}; K \mid x) = \mu^{\pm 2s}C(y,\bar{y}; K \mid x),\end{aligned} \tag{2.24}$$

where $+$ and $-$ correspond to helicity $h = \pm s$ selfdual and anti-selfdual parts of the generalized Weyl tensors $C(y,\bar{y}; K|x)$.

We consider Eq. (2.18) on the gauge invariant zero-forms C

$$C(Y; K|x) = \sum_{A=0}^{1}\sum_{n,m=0}^{\infty}\frac{1}{2n!m!}C^{A\,1-A}_{\alpha_1\ldots\alpha_n,\dot\alpha_1\ldots\dot\alpha_m}(x)y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot\alpha_1}\ldots\bar{y}^{\dot\alpha_m}k^A\bar{k}^{1-A}.$$

Spin-s zero-forms are $C^{A\,1-A}_{\alpha_1\ldots\alpha_n,\dot\alpha_1\ldots\dot\alpha_m}(x)$ with

$$n - m = \pm 2s. \tag{2.25}$$

Eq. (2.18) rewritten in the form

$$D^LC = e^{\alpha\dot\beta}\frac{\partial^2}{\partial y^\alpha\partial\bar{y}^{\dot\beta}}C + \text{lower-derivative and nonlinear terms} \tag{2.26}$$

(discarding indices A) implies that higher-order terms in y and $\bar{y}$ in the zero-forms $C(y,\bar{y}|x)$ describe higher-derivative descendants of the primary components $C(y, 0|x)$ and $C(0,\bar{y}|x)$ relating second derivatives in $y, \bar{y}$ to the x derivatives of $C(Y; K|x)$ of lower degrees in Y. Generally, $C_{\alpha_1\ldots\alpha_n,\dot\alpha_1\ldots\dot\alpha_m}(x)$ contains $\frac{n+m}{2} - \{s\}$ space-time derivatives of the spin-s dynamical field. As a result, the zero-forms C in the HS vertices may induce infinite towers of derivatives and, hence, non-locality.

3 Vasiliev's differential homotopy approach

Here we recall the concept of differential homotopy of [17] and the Ansatz for (anti)holomorphic deformation linear in $(\bar\eta)\eta$ and discuss its $\eta\bar\eta$ generalization used in this paper.

3.1 Differential homotopy

Shifted contracting homotopy $\triangle_q$ and cohomology projector h_q act as follows [10]

$$\triangle_q \phi(Z,Y,\theta) = \int_0^1 \frac{dt}{t}(Z+q)^A \frac{\partial}{\partial\theta^A}\phi(tZ-(1-t)q, t\theta),$$
$$h_q\phi(Z,Y,\theta) = \phi(-q,Y,0) \tag{3.1}$$

obeying the standard resolution of identity

$$\{\mathrm{d}_Z, \triangle_q\} + h_q = Id\,. \tag{3.2}$$

Here a shift q must be independent of Z and t but can depend on some parameters and/or integration variables. Moreover, further contracting homotopies lead to multiple integrals $\int dt^1 \int dt^2 \ldots$. All of these parameters were interpreted in [17] as additional coordinates t^a of some manifold $\mathcal{M}$ with the total differential

$$\mathrm{d} = \mathrm{d}_Z + \mathrm{d}_t, \tag{3.3}$$
$$\mathrm{d}_Z = \theta^A \frac{\partial}{\partial Z^A}, \qquad \mathrm{d}_t = dt^a \frac{\partial}{\partial t^a}, \tag{3.4}$$

where θ^A and dt^a are anticommuting differentials and the homotopy coordinates t^a belong to a unite hypercube confining integration to a compact $\mathcal{M}$,

$$0 \le t^a \le 1. \tag{3.5}$$

In these terms, perturbative equations to be solved acquire the form

$$\mathrm{d}f(Z,t,\theta,dt) = g(Z,t,\theta,dt), \qquad \mathrm{d}g(Z,t,\theta,dt) = 0, \tag{3.6}$$

where the second one expresses the compatibility of the first with $\mathrm{dd}=0$.

Functions like f and g contain theta and delta functions like $\theta(t^a)\theta(1-t^a)$ restricting the t integration to a locus inside a unit hypercube. Physical fields and equations in HS theory are supported by d_t cohomology carried by the integrals over t^a.

Differential homotopy is based on the removal of the integrals. Namely, following [17] let

$$\mathrm{d}_Z f_{int} = g_{int}, \qquad f_{int} = \int_{\mathcal{M}} f(Z,\theta,t,dt),$$
$$g_{int} = \int_{\mathcal{M}} g(Z,\theta,t,dt), \tag{3.7}$$

where $\mathcal{M}$ is a manifold with homotopy variables like t as local coordinates, resulting in

$$\mathrm{d}_Z f = g + \mathrm{d}_t h + g^{weak}, \tag{3.8}$$

where g^{weak} is any differential form of a degree different from $\dim\mathcal{M}$, which therefore does not contribute to the integral. Setting $g^{weak} = \mathrm{d}_Z h - \mathrm{d}_t f$ (taking into account that $\deg h = \dim\mathcal{M} - 1$ and $\deg f = \dim\mathcal{M}$) and replacing $f \to f - h$, we obtain (3.6).

$\mathcal{M}$ can be treated as $\mathbb{R}^n$. Following [17], for $\int_{\mathcal{M}}$ we use notation $\int_{t^1}\ldots\int_{t^k} := \int_{t^1\ldots t^k}$ with the convention that it is totally antisymmetric in t^a. Though the integrals are removed from the equations, to avoid a sign ambiguity due to (anti)commutativity of differential forms, every differential expression is accompanied with integrals $\int_{t^1\ldots t^k}$ that can be written anywhere in the expression for the differential form to be integrated with the convention

$$\mathrm{d}\int_{t^1\ldots t^k} = (-1)^k \int_{t^1\ldots t^k}\mathrm{d}. \tag{3.9}$$

3.2 Differential homotopy Ansatz for the η deformation

As shown in [17] direct computation within the differential homotopy approach gives the following form for the lowest order holomorphic deformation linear in η in the perturbative analysis

$$f_\mu = \eta \int_{u^2v^2\tau\sigma\beta\rho} \mu(\tau,\sigma,\beta,\rho,u,v)$$
$$\mathrm{d}\Omega^2 \mathcal{E}(\Omega) G_l(g(r))\big|_{r=0}, \tag{3.10}$$
$$\mathrm{d}\Omega^2 := \mathrm{d}\Omega^\alpha \mathrm{d}\Omega_\alpha, \tag{3.11}$$
$$\mathcal{E}(\Omega) := \exp i\big(\Omega_\beta(y^\beta + p_+^\beta + u^\beta) + u_\alpha v^\alpha - \sum_{k\ge j>i\ge 1} p_{i\beta}p_j^\beta\big) \tag{3.12}$$
$$G_l(g) := g_1(r_1)\ldots g_l(r_l)k, \tag{3.13}$$
$$p_{+\alpha} = \sum_{i=1}^k p_{i\alpha}, \qquad p_{j\alpha} = -i\frac{\partial}{\partial r^{j\alpha}}, \tag{3.14}$$

$g_i(y)$ are some functions of y_α (e.g., $C(y)$ or $\omega(y)$) (antiholomorphic variables $\bar y_{\dot\alpha}$, Klein operators $K = (k,\bar k)$ and the antiholomorphic star product $\bar *$ are implicit).

$$\mathrm{d} = dZ^A\frac{\partial}{\partial Z^A} + d\tau\frac{\partial}{\partial\tau} + d\rho\frac{\partial}{\partial\rho}$$
$$+ d\sigma_i\frac{\partial}{\partial\sigma_i} + d\beta\frac{\partial}{\partial\beta} + du^\alpha\frac{\partial}{\partial u^\alpha} + dv^\alpha\frac{\partial}{\partial v^\alpha} \tag{3.15}$$

and

$$\mu(\tau,\sigma,\beta,\rho,u,v) = \mu(\tau,\sigma,\beta,\rho)d^2u d^2v, \tag{3.16}$$

where du^α and dv^α are anticommuting differentials,

$$d^2u = du^\alpha du_\alpha, \qquad d^2v = dv^\alpha dv_\alpha, \\ \times \int d^2ud^2v \exp iu_\alpha v^\alpha = 1, \tag{3.17}$$

Ω_α has the form

$$\Omega_\alpha := \tau z_\alpha - (1-\tau)(p_\alpha(\sigma) \\ -\beta v_\alpha + \rho(y_\alpha + p_{+\alpha} + u_\alpha)), \tag{3.18}$$

where

$$p_\alpha(\sigma) = \sum_{i=1}^{k} p_{i\alpha}\sigma_i \tag{3.19}$$

with some parameters σ_i. We use the convention of [17] that it does not matter where the symbol of integral is situated; the integration over d^2u and d^2v in (3.10) also accounts for the u, v–dependent measure factor $\mathrm{d}\Omega^2$.

The measure $\mu \mathrm{d}\Omega^2$ may contain so called *weak* terms that do not contribute under the integration if the number of integrations does not match the number of respective differentials. This issue plays important role in the computations of [17].

Due to the identity $(\mathrm{d}\Omega)^3 = 0$ being a consequence of the anticommutativity of $\mathrm{d}\Omega_\alpha$ and two-componentness of the spinor indices α, formula (3.10) has the following remarkable property [17]

$$\mathrm{d}[\mathrm{d}^2u\mathrm{d}^2v\mathrm{d}\Omega^2\mathcal{E}(\Omega)] = \mathrm{d}\Big(\mathrm{d}^2u\mathrm{d}^2v\mathrm{d}\Omega^2 \exp i\Big(\Omega_\beta(y+p_+ + u)^\beta \\ +u_\alpha v^\alpha - \sum_{k\geq j>i\geq 1} p_{i\beta}p_j^\beta\Big)\Big) = 0. \tag{3.20}$$

As a result,

$$\mathrm{d} f_\mu = (-1)^N f_{\mathrm{d}\mu}, \tag{3.21}$$

where N is the number of the integration parameters $\tau, \sigma_i, \beta, \rho$. By virtue of (3.21), Eq. (3.6) amounts to

$$f_{\mathrm{d}\mu_f} = g_{\mu_g}. \tag{3.22}$$

This demands

$$\mathrm{d}\mu_f \cong \mu_g, \tag{3.23}$$

where $\cong$ denotes the weak equality up to possible weak terms, that do not contribute under the integrals in $f_{\mathrm{d}\mu_f}$ and g_{μ_g}. Since g in (3.6) is d closed

$$\mathrm{d}\mu_g \cong 0. \tag{3.24}$$

In most cases this implies that

$$\mu_g \cong \mathrm{d}h_g \tag{3.25}$$

allowing to set

$$\mu_f = h_g. \tag{3.26}$$

3.3 Ansatz for the $\eta\bar\eta$ deformations

In this paper we use a particular case of Vasiliev's Ansatz (3.10) with $\rho = \beta = 0$ allowing to discard the dependence on u and v, that trivializes at $\beta = 0$.

Firstly, recall that HS equations remain consistent with the fields W and B valued in any associative algebra [23]. As a result, the components of W and B do not commute and different orderings of the fields can be considered independently. Hence, functions $G_l(g, K)$ under consideration with $l = 3$ and $l = 4$, being at least linear in ω, can be represented as a sum of expressions with different positions of ω. For the future convenience we denote arguments of ω as $r_0, \bar r_0$ for any ordering. Namely, for $l = 3, 4$

$$G_l(g) = \begin{cases} C(r^1,\bar r^1)C(r^2,\bar r^2)C(r^3,\bar r^3)k\bar k, \\ \omega(r^0,\bar r^0)C(r^1,\bar r^1)C(r^2,\bar r^2)C(r^3,\bar r^3)k\bar k, \\ C(r^1,\bar r^1)\omega(r^0,\bar r^0)C(r^2,\bar r^2)C(r^3,\bar r^3)k\bar k, \\ C(r^1,\bar r^1)C(r^2,\bar r^2)\omega(r^0,\bar r^0)C(r^3,\bar r^3)k\bar k, \\ C(r^1,\bar r^1)C(r^2,\bar r^2)C(r^3,\bar r^3)\omega(r^0,\bar r^0)k\bar k. \end{cases} \tag{3.27}$$

To simplify formulae we will use shorthand notations ωCCC instead of $\omega(r^0,\bar r^0)C(r^1,\bar r^1)C(r^2,\bar r^2)C(r^3,\bar r^3)|_{r^i=\bar r^i=0}$ *etc.*

In this paper, we introduce Ansatz in the bilinear $\eta\bar\eta$ deformation with

$$F = \sum_i F^i \quad \text{where}$$

$$F^i = \eta\bar\eta \int_{\tau\bar\tau\sigma(n)} \mu^i(\tau,\bar\tau,\sigma)\mathbf{E}(\Omega^i|\bar\Omega^i)G_l(g) \tag{3.28}$$

with the some compact measure factors $\mu^i(\tau,\bar\tau,\sigma)$, $G_l(g)$ (3.27),

$$\mathbf{E}(\Omega^i,\bar\Omega^i) = (\mathrm{d}\Omega^i)^2(\mathrm{d}\bar\Omega^i)^2\mathcal{E}(\Omega^i)\bar{\mathcal{E}}(\bar\Omega^i) \tag{3.29}$$

with

$$\mathcal{E}(\Omega) := \exp i\Big(\Omega_\beta(y^\beta + p_+^\beta) \\ - \sum_{3\geq j>i\geq 1} p_{i\beta}p_j^\beta - \sum_{3\geq j\geq 1} s_j p_{0\beta}p_j^\beta\Big), \tag{3.30}$$

$$\bar{\mathcal{E}}(\bar\Omega) := \exp i\Big(\bar\Omega_{\dot\beta}(\bar y + \bar p_+)^{\dot\beta} - \sum_{3\geq j>i\geq 1} \bar p_{i\dot\beta}\bar p_j^{\dot\beta} \\ - \sum_{3\geq j\geq 1} \bar s_j \bar p_{0\dot\beta}\bar p_j^{\dot\beta}\Big), \tag{3.31}$$

$$\Omega^i_\alpha := \tau z_\alpha - (1-\tau)a^{ij}(\sigma)\, p_{j\alpha},$$

$$\bar{\Omega}^i_{\dot{\alpha}} := \bar{\tau}\bar{z}_{\dot{\alpha}} - (1-\bar{\tau})\bar{a}^{\,ij}(\sigma)\,\bar{p}_{j\dot{\alpha}}, \tag{3.32}$$

where s_j, $\bar{s}_j$ are sign factors that depend on the ordering of fields C and ω (3.27), σ are integration parameters and $a^{ij}(\sigma)$, $\bar{a}^{ij}(\sigma)$ are some rational functions that satisfy inequalities $|a^{ij}(\sigma)| \le 1$, $|\bar{a}^{ij}(\sigma)| \le 1$. The notation $\sigma(n)$ at the integral symbol is used for the ordered string of variables $\sigma_1, \sigma_2, \ldots \sigma_n$.

Introducing additional integration parameters σ'^{ij} and new measure factors

$$\mu'^i(\tau,\sigma,\sigma') = \mu^i(\tau,\sigma)\prod_{j=0}^{l} d\sigma'^{ij}\delta(\sigma'^{ij} - a^{ij}(\sigma)), \tag{3.33}$$

one brings Ω^i_α to the form (3.18). Note that in [17] it was proposed to consider polyhedra as integration domains, while Eq. (3.33) provides some variety embedded into a polyhedron. In this paper it is more convenient to use (3.28) with Ω, $\bar{\Omega}$ (3.32) with some polyhedra as integration domains.

Another difference compared to the approach of [17] is that in this paper we discard the weak terms, reconstructing the final results from the compatibility conditions. Though we agree with the idea of [17] that it is useful to keep the weak terms inducing non-zero contribution at the further stages of the computations preserving the form of the Ansatz we find it simpler to discard the weak terms in this paper since our aim is just to illustrate how moderately non-local vertex can be obtained in the mixed sector without going too much into the computation details.

4 Moderate spin-non-locality

4.1 Spin-locality and moderate spin-non-locality

To check whether F^i (3.28) is spin-local or not we consider the coefficients in front of $p_{k\alpha}p_j{}^\alpha$ and $\bar{p}_{k\dot{\alpha}}\bar{p}_j{}^{\dot{\alpha}}$ in the exponents of $\mathbf{E}(\Omega^i|\bar{\Omega}^i)$, which yield, schematically,

$$\exp i\Big(\tau z_\alpha y^\alpha + \cdots + \frac{1}{2}P^{kj}p_k{}^\alpha p_{j\alpha} + \bar{\tau}\bar{z}_{\dot{\alpha}}\bar{y}^{\dot{\alpha}} + \cdots + \frac{1}{2}\bar{P}^{kj}\bar{p}_k{}^{\dot{\alpha}}\bar{p}_{j\dot{\alpha}}\Big). \tag{4.1}$$

By the Z-dominance Lemma of [10] (see also [25]), only the coefficients at $\tau = \bar{\tau} = 0$ matter.

• Spin-locality

Space-time spin-locality demands [10] that truncation of all vertices to any finite subset of fields be local at any given order of the perturbation expansion, containing at most a finite number of space-time derivatives. By virtue of (2.26) and taking into account that, by virtue of (2.25), for any given spin s the degree in y^α is limited once that in $\bar{y}^{\dot{\alpha}}$ is, this can be reformulated in terms of spinor variables y^α, $\bar{y}^{\dot{\alpha}}$ as a condition that any vertex represented as a power series in y_j, $\bar{y}_j$-derivatives p_j, $\bar{p}_j$ contains at most a finite power of $(p_j\bar{p}_j)^n$ for any j. To check whether F^i (3.28) with $G_l(g)$ (3.27) is spin-local or not one has to analyse coefficients in front of the terms bilinear in spinor derivatives $p_{i\alpha}p_j{}^\alpha$ and $\bar{p}_{i\dot{\alpha}}\bar{p}_j{}^{\dot{\alpha}}$ with respect to arguments of the zero-forms C_i (i.e., with with $i, j > 0$) in the exponents of $\mathcal{E}(\Omega)$ and $\bar{\mathcal{E}}(\bar{\Omega})$ in (3.29). To achieve spin-locality it is enough to demand that

$$P^{ij}\bar{P}^{ij}|_{\tau=\bar{\tau}=0} = 0 \qquad \forall\, i, j > 0. \tag{4.2}$$

Being formulated in terms of spinor derivatives p_j and $\bar{p}_j$ this condition is referred to as spinor spin-locality. Note that, being equivalent at a given order of the perturbative expansion, space-time and spinor definitions of spin-locality may differ when the lower-order contributions are taken into account. For more detail on this issue we refer to [26] where the concept of projectively-compact spin-local vertices has been introduced for which spinor spin-locality implies space-time spin-locality at all orders of the perturbative expansion.

• Spin-non-locality

Violation of this condition for at least one pair of $i, j > 0$ implies spin-non-locality,

$$\exists\, i, j > 0 \qquad P^{ij}\bar{P}^{ij}|_{\tau=\bar{\tau}=0} \neq 0. \tag{4.3}$$

• Moderate spin-non-locality

Here we introduce the concept of *moderate spin-non-locality (MNL)* with the coefficients P^{ij} and $\bar{P}^{ij}$ obeying the conditions

$$(|P^{ij}| + |\bar{P}^{ij}|)|_{\tau=\bar{\tau}=0} \le 1 \qquad \forall i, j > 0\,. \tag{4.4}$$

Note, that the concept of spin-locality simply demands that power series in y, $\bar{y}$ does not contain an infinite number of $(p_j\bar{p}_j)^n$ for any j. Hence, its formal definition does not demand (4.4). Indeed, e.g., the case of $\bar{P} = 0$ and $P = 2$ is also spin-local. Nevertheless, all known examples of spin-local perturbative contributions to Vasiliev nonlinear equations obey the moderately spin non-locality condition (4.4). It is this property that induces the inequality (4.4) hence playing the key role in the construction of this paper of the moderately non-local vertex $\Upsilon^{\eta\bar{\eta}}(\omega, C, C, C)$.

For instance, the lower-order computations for vertices bilinear in C in the (anti)holomorphic sectors [11,12,27,28] imply that they satisfy both condition (4.2) and (4.4),

$$P^{12}\bar{P}^{12}|_{\tau=\bar{\tau}=0} = 0, \qquad (|P^{12}| + |\bar{P}^{12}|)|_{\tau=\bar{\tau}=0} = 1. \tag{4.5}$$

It is not hard to find P^{ij} and $\bar{P}^{ij}$ (4.1) for the Ansatz (3.28). For instance, for

$$\Omega^\alpha|_{\tau=0} = -\big(a^0 p_0 + a^1 p_1 + \cdots + a^n p_n\big)^\alpha,$$
$$\bar{\Omega}^{\dot{\alpha}}|_{\bar{\tau}=0} = -(\bar{a}^0\bar{p}_0 + \bar{a}^1\bar{p}_1 + \cdots + \bar{a}^n\bar{p}_n)^{\dot{\alpha}} \tag{4.6}$$

one obtains

$$P^{ij}|_{\tau=\bar\tau=0} = a^i - a^j + 1, \quad \bar P^{ij}|_{\tau=\bar\tau=0} = \bar a^i - \bar a^j + 1 \quad \forall \le i < j \le n. \tag{4.7}$$

Note that the star product $C(y,\bar y) * C(y,\bar y)$ (2.7) yields $|P^{12}| + |\bar P^{12}| = 2$.

4.2 Moderate non-locality compatible interpolating homotopy

Consider equation of the form

$$\mathrm{d}A = F, \qquad \mathrm{d}F = 0. \tag{4.8}$$

Let F be (i) of the form (3.28) and (ii) MNL. To proceed we need a scheme allowing to solve (4.8) within the same class. This is achieved by a MNL compatible interpolating homotopy (IH) introduced in this section.

Let two expressions F^a and F^b be of the form (3.28) and

$$F^a + F^b = \eta\bar\eta \int_{\tau\bar\tau\sigma(n)} \Big\{\mu^a(\tau,\bar\tau,\sigma)\mathbf{E}(\Omega^a|\bar\Omega^a) - \mu^b(\tau,\bar\tau,\sigma)\mathbf{E}(\Omega^b|\bar\Omega^b)\Big\} G_l(g,K). \tag{4.9}$$

Suppose that there exists such a measure $\mu(\nu,\tau,\bar\tau,\sigma)$ depending on an additional parameter ν, that

$$\mu(\nu,\tau,\bar\tau,\sigma)|_{\nu=1} = \mu^b(\tau,\bar\tau,\sigma),$$
$$\mu(\nu,\tau,\bar\tau,\sigma)|_{\nu=0} = \mu^a(\tau,\bar\tau,\sigma).$$

Since

$$\mathrm{d}[\theta(\nu)\theta(1-\nu)] = d\nu(\delta(\nu) - \delta(1-\nu)), \tag{4.10}$$

$$F^a + F^b = \eta\bar\eta \times \int_{\tau\bar\tau\sigma(n)\nu} \mu'(\nu,\tau,\bar\tau,\sigma)\mathbf{E}(\Omega^\nu|\bar\Omega^\nu)G_l(g,K), \tag{4.11}$$

where

$$\mu'(\nu,\tau,\bar\tau,\sigma) = \mathrm{d}[\theta(\nu)\theta(1-\nu)]\mu(\nu,\tau,\bar\tau,\sigma), \tag{4.12}$$

$$\Omega^\nu = \nu\Omega^b + (1-\nu)\Omega^a,$$
$$\bar\Omega^\nu = \nu\bar\Omega^b + (1-\nu)\bar\Omega^a. \tag{4.13}$$

In these terms, the total differential d (3.3) acquires the form

$$\mathrm{d} = \theta^\alpha \frac{\partial}{\partial z^\alpha} + \bar\theta^{\dot\alpha}\frac{\partial}{\partial \bar z^{\dot\alpha}} + d\tau\frac{\partial}{\partial\tau} + d\bar\tau\frac{\partial}{\partial\bar\tau} + d\sigma_i\frac{\partial}{\partial\sigma_i} + d\nu\frac{\partial}{\partial\nu}. \tag{4.14}$$

Since the property (3.20) is still true,

$$\mathrm{d}\Big[\mathbf{E}(\Omega^\nu|\bar\Omega^\nu)\Big] = \mathrm{d}\Big[(\mathrm{d}\Omega^\nu)^2(\mathrm{d}\bar\Omega^\nu)^2\mathcal{E}(\Omega^\nu)\bar{\mathcal{E}}(\bar\Omega^\nu)\Big] = 0, \tag{4.15}$$

(4.12) allows us to represent $F^{a,b}$ (4.11) in the form

$$F^a + F^b = \mathrm{d}G^{a,b} + F^{a,b}, \tag{4.16}$$

$$G^{a,b} = \eta\bar\eta \int_{\tau\bar\tau\sigma(n)\nu} \mu' \; (\nu,\tau,\bar\tau,\sigma)\mathbf{E}(\Omega^\nu|\bar\Omega^\nu)G_l(g,K), \tag{4.17}$$

$$F^{a,b} = -\eta\bar\eta \int_{\tau\bar\tau\sigma(n)\nu} \theta(\nu)\theta(1-\nu)\mathrm{d}\Big[\mu(\nu,\tau,\bar\tau,\sigma)\Big] \times\mathbf{E}(\Omega^\nu|\bar\Omega^\nu)G_l(g,K). \tag{4.18}$$

If F^a and F^b (4.9) are MNL, i.e., P^{aij} and $\bar P^{aij}$ of $\mathbf{E}(\Omega^a|\bar\Omega^a)$ as well as P^{bij} and $\bar P^{bij}$ of $\mathbf{E}(\Omega^b|\bar\Omega^b)$ obey (4.4),

$$(|P^{aij}| + |\bar P^{aij}|)|_{\tau=\bar\tau=0} \le 1,$$
$$(|P^{bij}| + |\bar P^{bij}|)|_{\tau=\bar\tau=0} \le 1, \qquad i,j>0, \tag{4.19}$$

this is also true for $P^{\nu ij}$ and $\bar P^{\nu ij}$ of $\mathbf{E}(\Omega^\nu|\bar\Omega^\nu)$ with Ω^ν, $\bar\Omega^\nu$(4.13) for any $\nu\in[0,1]$. Indeed, according to (4.13), (3.30), (3.31) and (3.29)

$$\begin{aligned}\mathbf{E}(\Omega^\nu,\bar\Omega^\nu) = {}& \mathrm{d}(\Omega^\nu)^2\mathrm{d}(\bar\Omega^\nu)^2 \exp i\Bigg[\nu\Bigg\{\Omega^a_\beta(y^\beta + p^\beta_+) - \sum_{l\ge j>i\ge1} p_{i\beta}p^\beta_j - \sum_{l\ge j\ge1} s_j p_{0\beta}p^\beta_j\Bigg\}\\ &+(1-\nu)\Bigg\{\Omega^b_\beta(y^\beta + p^\beta_+) - \sum_{l\ge j>i\ge1} p_{i\beta}p^\beta_j - \sum_{l\ge j\ge1} s_j p_{0\beta}p^\beta_j\Bigg\}\Bigg]\\ &\times\exp i\Bigg[\nu\Bigg\{\bar\Omega^a_{\dot\beta}(\bar y + \bar p_+)^{\dot\beta} - \sum_{l\ge j>i\ge1}\bar p_{i\dot\beta}\bar p^{\dot\beta}_j - \sum_{l\ge j\ge1}\bar s_j\bar p_{0\dot\beta}\bar p^{\dot\beta}_j\Bigg\}\\ &+(1-\nu)\Bigg\{\bar\Omega^b_{\dot\beta}(\bar y + \bar p_+)^{\dot\beta} - \sum_{l\ge j>i\ge1}\bar p_{i\dot\beta}\bar p^{\dot\beta}_j - \sum_{l\ge j\ge1}\bar s_j\bar p_{0\dot\beta}\bar p^{\dot\beta}_j\Bigg\}\Bigg].\end{aligned} \tag{4.20}$$

Rewriting exponents in the form (4.1), one obtains

$$P^{\nu ij} = \nu P^{aij} + (1-\nu)P^{bij},$$
$$\bar P^{\nu ij} = \nu\bar P^{aij} + (1-\nu)\bar P^{bij}. \tag{4.21}$$

Since $\nu \in [0,1]$, (4.19) and (4.21) imply $(|P^{\nu ij}| + |\bar P^{\nu ij}|)|_{\tau=\bar\tau=0} \le 1$. The essence of the idea is that if the coefficients $a^{ij}(\sigma)$ for any i,j on the *r.h.s.* of (3.32) satisfy

$$|a^{ij}(\sigma)| \le 1 \tag{4.22}$$

then

$$|\nu a^{aj}(\sigma)+(1-\nu)a^{bj}(\sigma)|\le 1 \tag{4.23}$$

as well. In the sequel it will be shown, that inequality (4.22) holds true for a set of functions Ω, $\bar{\Omega}$ under consideration, thus forming a convex set.

Picking up an appropriate pair F^a and F^b on the *r.h.s.* of (4.8) we apply IH to single out the corresponding d-exact part setting $A=G^{a,b}+A'$ we are left with the equation

$$dA'=\sum_{i\neq a,b}F^i+F^{a,b}, \tag{4.24}$$

with $F^{a,b}$ (4.18). The *r.h.s.* of (4.24) is evidently (i) d-closed, (ii) of the form (3.28) and (iii) MNL as the *r.h.s.* of (4.8).

To arrive at the final result we repeat this procedure as many times as needed for the leftover MNL terms until all of them cancel. Note that, at every step, the choice of a proper pair is to large extent ambiguous and it is not a priory guaranteed that the process ends at some stage. For instance, the choice $\mu^b=0$ can unlikely yield a reasonable result.

Nevertheless, for some reason to be better understood, it works. Let us stress that in this paper we manage to choose all appropriate pairs of the *r.h.s.'s* under consideration with the same measure factors $\mu^a=\mu^b$, that simplifies the calculations making $\mu(\nu,\tau,\bar\tau,\sigma)$ ν-independent.

This interpolating homotopy approach underlies the construction of MNL solutions. Specifically, it is used below to solve for S_2 the following consequences of (2.2)

$$2i\mathrm{d}S_2^{\eta\bar\eta}=-\left\{i\bar\eta B_2^{\eta}*\bar\gamma+i\eta B_2^{\bar\eta}*\gamma-\{S_1^{\bar\eta}\,S_1^{\eta}\}_*\right\} \tag{4.25}$$

in such a way that the *r.h.s.'s* of the following consequences of (2.1), (2.3)

$$\begin{aligned}2i\mathrm{d}W_2^{\eta\bar\eta}=&\Big\{\mathrm{d}_xS_1^{\eta}+\mathrm{d}_xS_1^{\bar\eta}+\{W_1^{\eta},S_1^{\bar\eta}\}_*+\{W_1^{\bar\eta},S_1^{\eta}\}_*\\&+\mathrm{d}_xS_2^{\eta\eta}+\{\omega,S_2^{\eta\bar\eta}\}_*\Big\},\end{aligned} \tag{4.26}$$

$$2i\mathrm{d}B_3^{\eta\bar\eta}=\left\{[S_1^{\eta},B_2^{\bar\eta}]_*+[S_1^{\bar\eta},B_2^{\eta}]_*+[S_2^{\eta\bar\eta},C]_*\right\} \tag{4.27}$$

as well as $[\mathrm{d}W_2^{\eta\bar\eta},C]_*$ be MNL.

This allows us to find by IH such $B_3^{\eta\bar\eta}$ that the *r.h.s.* of

$$\begin{aligned}\Upsilon^{\eta\bar\eta}(\omega,C,C,C)=&[W_2^{\eta\bar\eta},C]_*-[W_1^{\bar\eta},B_2^{\eta}]_*\\&-[W_1^{\eta},B_2^{\bar\eta}]_*-\mathrm{d}_xB_3^{\eta\bar\eta}\\&-[\omega,B_3^{\eta\bar\eta}]_*-\mathrm{d}_xB_2^{\eta}(\Upsilon^{\bar\eta}(\omega,C,C))\\&-\mathrm{d}_xB_2^{\bar\eta}(\Upsilon^{\eta}(\omega,C,C))\end{aligned} \tag{4.28}$$

in its turn becomes MNL, allowing to eliminate step by step manifest Z-dependence using IH. Namely, choosing an appropriate pair from the *r.h.s.* of (4.28) we apply IH to drop the d-exact part since it does not contribute in the Z, dZ-independent sector. Then this procedure is repeated as many times as needed until all leftover MNL terms cancel except for the cohomological terms producing MNL physical vertices (see Sect. 6).

Note that the interpolating homotopy can be treated as certain generalization of the general homotopy of [17].

5 Moderately non-local $B_3^{\eta\bar\eta}$

To compute the MNL form of $\Upsilon^{\eta\bar\eta}(\omega,C,C,C)$ vertex we have to find a MNL B_3. This is the aim of this section. In the sequel we use notations of [17]

$$\begin{aligned}\Box(\tau,\bar\tau)&=l(\tau)l(\bar\tau),\qquad l(\nu)=\theta(\nu)\theta(1-\nu),\\\mathbb{D}(\nu)&=d\nu\delta(\nu),\\\nabla(\alpha(n)):=&\prod_{i=1}^n\theta(\alpha_i)\mathbb{D}\left(1-\sum_{i=1}^n\alpha_i\right).\end{aligned} \tag{5.1}$$

Equation for $B_3^{\eta\bar\eta}$ in the mixed sector resulting from (2.5) has the form (4.27). To obtain MNL B_3 we need the *r.h.s.* of (4.27) to be of that class. Straightforwardly, using $S_{1,2}$ and B_2 of [12], one can make sure that this is true for $[S_1^{\eta},B_2^{\bar\eta}]_*+[S_1^{\bar\eta},B_2^{\eta}]_*$, while $[S_2^{\eta\bar\eta},C]_*$ is not MNL. The key observation of this paper is that, as we show now, there exists an alternative $S_2^{\eta\bar\eta}$ such that $[S_2^{\eta\bar\eta},C]_*$ is MNL.

5.1 $S_2^{\eta\bar\eta}$

$\mathrm{d}S_2^{\eta\bar\eta}$ is determined by (4.25). One can make sure straightforwardly that $[\mathrm{d}S_2^{\eta\bar\eta},C]_*$ is both spin-local and MNL. The problem is that all spin-local terms of $[\mathrm{d}S_2^{\eta\bar\eta},C]_*$ have different structure and it is not clear how to find such a solution for $S_2^{\eta\bar\eta}$ that $[S_2^{\eta\bar\eta},C]_*$ be spin-local. However, since $[\mathrm{d}S_2^{\eta\bar\eta},C]_*$ is MNL, the interpolating homotopy of Sect. 4.2 allows us to find such $S_2^{\eta\bar\eta}$ that $[S_2^{\eta\bar\eta},C]_*$ is MNL as well.

Indeed, one can see that

$$-\{S_1^{\bar\eta},S_1^{\eta}\}_*=\frac{\eta\bar\eta}{4}\int_{\tau\bar\tau}\Box(\tau,\bar\tau)[\mathbf{E}(\Omega^1|\bar\Omega^1)-\mathbf{E}(\Omega^2|\bar\Omega^2)]\,CCk\bar k, \tag{5.2}$$

$$\begin{aligned}\Omega^1_\alpha&=\tau z_\alpha-(1-\tau)[-p_1]_\alpha,\\\bar\Omega^1_{\dot\alpha}&=\bar\tau\bar z_{\dot\alpha}-(1-\bar\tau)[\bar p_2]_{\dot\alpha},\end{aligned} \tag{5.3}$$

$$\begin{aligned}\Omega^2_\alpha&=\tau z_\alpha-(1-\tau)[p_2]_\alpha,\\\bar\Omega^2_{\dot\alpha}&=\bar\tau\bar z_{\dot\alpha}-(1-\bar\tau)[-\bar p_1]_{\dot\alpha}.\end{aligned} \tag{5.4}$$

Applying IH to the *r.h.s.* of (5.2) one finds S_2 in the form (3.28). Namely, one can see that

$$-\{S_1^{\bar\eta},S_1^{\eta}\}_*=\mathrm{d}\left[\frac{\eta\bar\eta}{4}\int_{\tau\bar\tau\sigma(2)}\nabla(\sigma(2))\Box(\tau,\bar\tau)\mathbf{E}(\Omega|\bar\Omega)\right.\,CCk\bar k \tag{5.5}$$

$$-\frac{\eta\bar{\eta}}{4}\int_{\tau\bar{\tau}\sigma(2)} d[\square(\tau,\bar{\tau})]\nabla(\sigma(2))\mathbf{E}(\Omega|\bar{\Omega})CCk\bar{k}, \tag{5.6}$$

$$\Omega_\alpha = \tau z_\alpha - (1-\tau)[-\sigma_1 p_1 + \sigma_2 p_2]_\alpha,$$
$$\bar{\Omega}_{\dot\alpha} = \bar\tau\bar z_{\dot\alpha} - (1-\bar\tau)[-\sigma_2\bar p_1 + \sigma_1\bar p_2]_{\dot\alpha}. \tag{5.7}$$

Differentiation of $\square(\tau,\bar\tau)$ in (5.6) yields

$$\frac{\eta\bar\eta}{4}\int_{\tau\bar\tau\sigma(2)}\nabla(\sigma(2))\Big[\mathbb{D}(1-\tau)l(\bar\tau)\mathbf{E}(\Omega'|\bar\Omega)$$
$$+\mathbb{D}(1-\bar\tau)l(\tau)\mathbf{E}(\Omega|\bar\Omega')\Big]CCk\bar k, \tag{5.8}$$

where $\Omega'_\alpha = z_\alpha$, $\bar\Omega'_{\dot\alpha} = \bar z_{\dot\alpha}$, while the (weak) terms with $\mathbb{D}(\tau)$ or $\mathbb{D}(\bar\tau)$ vanish because, e.g.,

$$\mathbb{D}(\tau)\mathbb{D}(1-\sigma_1-\sigma_2)d\Omega_\alpha d\Omega^\alpha$$
$$\sim d\tau d(\sigma_1+\sigma_2)d\sigma_2 d\sigma_1\delta(\tau)\delta(1-\sigma_1-\sigma_2)p_{1\alpha}p_2{}^\alpha = 0.$$

As a result, (5.8) just equals to $-i\bar\eta B_2^\eta * \bar\gamma - i\eta B_2^{\bar\eta} * \gamma$ while Eq. (4.25) acquires the form

$$2i\,dS_2^{\eta\bar\eta} = -d\Big[\frac{\eta\bar\eta}{4}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))\mathbf{E}(\Omega|\bar\Omega)\Big]CCk\bar k \tag{5.9}$$

allowing to set

$$S_2^{\eta\bar\eta} = \frac{i\eta\bar\eta}{8}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))\mathbf{E}(\Omega|\bar\Omega)CCk\bar k. \tag{5.10}$$

By construction, $S_2^{\eta\bar\eta}$ (5.10) is spin-local, while $[S_2^{\eta\bar\eta}, C]_*$ is MNL. Indeed, consider for instance the exponent of $S_2^{\eta\bar\eta} * C$ in the form (4.1), i.e.,

$$\exp i\big(\ldots + \frac{1}{2}P^{ij}p_i{}^\alpha p_{j\alpha} + \frac{1}{2}\bar P^{ij}\bar p_i{}^{\dot\alpha}\bar p_{j\dot\alpha}\big). \tag{5.11}$$

Equation (5.10) straightforwardly yields by virtue of Eq. (5.7)

$$P^{12}|_{\tau=\bar\tau=0} = 0, \qquad P^{13}|_{\tau=\bar\tau=0} = \sigma_1,$$
$$P^{23}|_{\tau=\bar\tau=0} = -\sigma_2,$$
$$\bar P^{12}|_{\tau=\bar\tau=0} = 0, \qquad \bar P^{13}|_{\tau=\bar\tau=0} = \sigma_2,$$
$$\bar P^{23}|_{\tau=\bar\tau=0} = -\sigma_1. \tag{5.12}$$

Thanks to $\Delta(1-\sigma_1-\sigma_2)$ on the *r.h.s.* of Eq. (5.10) inequalities (4.4) hold true.

5.2 $dB_3^{\eta\bar\eta}$

Substituting S_1,W_1, B_2 (A.1)–(A.9), S_2 (5.10) we obtain using (5.1)

$$\frac{1}{2i}S_2{}^{\eta\bar\eta} * C = -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))$$
$$\mathbf{E}(\Omega\,|\bar\Omega)CCCk\bar k, \tag{5.13}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)[-\sigma_1(p_1+p_2)+p_3+p_2]^\alpha,$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-\sigma_2(\bar p_1+\bar p_2)+\bar p_3+\bar p_2]^{\dot\alpha}, \tag{5.14}$$

$$-\frac{1}{2i}C * S_2{}^{\eta\bar\eta} = \frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))$$
$$\mathbf{E}(\Omega\,|\bar\Omega)CCCk\bar k, \tag{5.15}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)[-p_1-p_2+\sigma_2(p_3+p_2)]^\alpha,$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-\bar p_1-\bar p_2+\sigma_1(\bar p_3+\bar p_2)]^{\dot\alpha}, \tag{5.16}$$

$$-\frac{1}{2i}B_2^\eta * \bar S_1^{\bar\eta} = \frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))$$
$$\mathbf{E}(\Omega\,|\bar\Omega)CCCk\bar k, \tag{5.17}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)[-\sigma_1(p_1+p_2)+p_2+p_3]^\alpha,$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-\bar p_1-\bar p_2]^{\dot\alpha}, \tag{5.18}$$

$$\frac{1}{2i}\bar S_1^{\bar\eta} * B_2^\eta = -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))$$
$$\mathbf{E}(\Omega\,|\bar\Omega)CCCk\bar k, \tag{5.19}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)[-p_1-p_2+\sigma_2(p_3+p_2)]^\alpha,$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[\bar p_2+\bar p_3]^{\dot\alpha}, \tag{5.20}$$

$$-\frac{1}{2i}\bar B_2^{\bar\eta} * S_1^\eta = -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))$$
$$\mathbf{E}(\Omega\,|\bar\Omega)CCCk\bar k, \tag{5.21}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)[-p_1-p_2]^\alpha,$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-\sigma_1(\bar p_1+\bar p_2)+\bar p_2+\bar p_3]^{\dot\alpha}, \tag{5.22}$$

$$\frac{1}{2i}S_1^\eta * \bar B_2^{\bar\eta} = \frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\sigma(2))$$
$$\mathbf{E}(\Omega\,|\bar\Omega)CCCk\bar k, \tag{5.23}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)[p_2+p_3]^\alpha,$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-\bar p_1-\bar p_2+\sigma_2(\bar p_3+\bar p_2)]^{\dot\alpha}. \tag{5.24}$$

As mentioned in Sect. 5.1, the *r.h.s.'s* of Eqs. (5.13) and (5.15) are MNL. Straightforwardly one can check that the *r.h.s.'s* of Eqs. (5.17), (5.19), (5.21) and (5.23) are also MNL. Indeed, consider for instance the *r.h.s.* of (5.23). According to Eqs. (3.29)–(3.32) the exponent is

$$\exp i\Bigg((\tau z - (1-\tau)[p_2+p_3])_\beta(y+p_+)^\beta$$
$$-\sum_{3\ge j>i\ge 1}p_{i\beta}p_j^\beta\Bigg)$$
$$\times\exp i\Bigg((\bar\tau\bar z - (1-\bar\tau)[-\bar p_1-\bar p_2$$
$$+\sigma_2(\bar p_3+\bar p_2)])_{\dot\beta}(\bar y+\bar p_+)^{\dot\beta} - \sum_{3\ge j>i\ge 1}\bar p_{i\dot\beta}\bar p_j^{\dot\beta}\Bigg). \tag{5.25}$$

Discarding the τ, $\bar{\tau}$, y and $\bar{y}$-dependent terms one is left with

$$i\big(\ldots - p_{2\beta}p_3{}^{\beta} - \sigma_2\bar{p}_{3\dot{\beta}}\bar{p}_1{}^{\dot{\beta}} + (1-\sigma_2)\bar{p}_{2\dot{\beta}}\bar{p}_1^{\dot{\beta}}\big).$$

Since the coefficients in front of $p_{i\beta}p_j^{\beta}$ and $\bar{p}_{i\dot{\beta}}\bar{p}_j^{\dot{\beta}}$ satisfy inequalities (4.4) $S_1^{\eta} * \bar{B}_2^{\bar{\eta}}$ is MNL. Note that it is also spin-local.

That the *r.h.s.'s* of Eqs. (5.17), (5.19) and (5.21) are MNL can be checked analogously. Once $\mathrm{d}B_3^{\eta\bar{\eta}}$ is shown to be MNL one can look for MNL B_3 applying IH.

5.3 Solving for moderately non-local $B_3^{\eta\bar{\eta}}$

Applying IH to the sum of (5.17) and (5.21) and then of (5.19) and (5.23), using (4.16) one can see that the terms (5.13) and (5.15) cancel out and (4.27) yields using notation (5.1)

$$\mathrm{d}B_3^{\eta\bar{\eta}} = \frac{\eta\bar{\eta}}{16}\int_{\nu(2)\tau\bar{\tau}\sigma(2)}\Big\{\mathrm{d}\big[\Box(\tau,\bar{\tau})\nabla(\nu(2))\nabla(\sigma(2))\big] \tag{5.26}$$

$$-\mathrm{d}[\Box(\tau,\bar{\tau})]\nabla(\nu(2))\nabla(\sigma(2))\Big\}$$

$$\Big[\mathbf{E}(\Omega_1\,|\bar{\Omega}_1) - \mathbf{E}(\Omega_2\,|\bar{\Omega}_2)\Big]CCCk\bar{k}, \tag{5.27}$$

where

$$\begin{aligned}\Omega_1^{\alpha} &:= \tau z^{\alpha} - (1-\tau)[-\nu_2(p_1+p_2)^{\alpha}\\ &\quad +(1-\nu_2\sigma_1)(p_3+p_2)^{\alpha}],\\ \bar{\Omega}_1^{\dot{\alpha}} &:= \bar{\tau}\bar{z}^{\dot{\alpha}} - (1-\bar{\tau})[-\nu_1(\bar{p}_1+\bar{p}_2)^{\dot{\alpha}}\\ &\quad +(1-\nu_1\sigma_1)(\bar{p}_3+\bar{p}_2)^{\dot{\alpha}}],\end{aligned} \tag{5.28}$$

$$\begin{aligned}\Omega_2^{\alpha} &:= \tau z^{\alpha} - (1-\tau)[-(1-\nu_2\sigma_2)(p_1+p_2)^{\alpha}\\ &\quad +\nu_2(p_3+p_2)^{\alpha}],\\ \bar{\Omega}_2^{\dot{\alpha}} &:= \bar{\tau}\bar{z}^{\dot{\alpha}} - (1-\bar{\tau})[-(1-\nu_1\sigma_2)(\bar{p}_1+\bar{p}_2)^{\dot{\alpha}}\\ &\quad +\nu_1(\bar{p}_3+\bar{p}_2)^{\dot{\alpha}}].\end{aligned} \tag{5.29}$$

Since it is shown that Eqs. (5.17), (5.19), (5.21) and (5.23) are MNL, by the reasoning of Sect. 4.1 all terms on the *r.h.s.'s* of (5.26) and (5.27) are MNL as well.

Equation (5.26) determines a part of $B_3^{\eta\bar{\eta}}$ with the integrand containing $\Box(\tau,\bar{\tau})$ without derivatives. Following [17], such terms will be referred to as 'bulk' in contrast to those with $\mathrm{d}\Box(\tau,\bar{\tau})$ referred to as 'boundary',

$$\mathrm{d}\Box(\tau,\bar{\tau}) = [\mathbb{D}(\tau) + \mathbb{D}(1-\tau)]l(\bar{\tau}) + c.c. \tag{5.30}$$

The terms proportional to $\mathbb{D}(1-\tau)$ or $\mathbb{D}(1-\bar{\tau})$ do not contribute to (5.27) (are weakly zero in terminology of [17]) because of the lack of differentials. Indeed, consider for instance the Ω_1-dependent term with $\sim\mathbb{D}(1-\bar{\tau})$. Due to (3.29) along with (5.28), (5.1) it yields

$$\begin{aligned}&\ldots d\bar{\tau}\delta(1-\bar{\tau})l(\tau)\nabla(\nu(2))\nabla(\sigma(2))(\mathrm{d}\Omega^1)^2\\ &(\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}})^2\mathcal{E}(\Omega^1)\bar{\mathcal{E}}(\bar{\Omega}^1)\ldots\ldots\end{aligned} \tag{5.31}$$

Since non-weak terms of $(\mathrm{d}\Omega^1)^2$ must contain $\mathrm{d}\tau$, modulo weak terms it equals to

$$\begin{aligned}2d\quad &\tau\Big\{z + [-(1-\nu_2\sigma_2)(p_1+p_2) + \nu_2(p_3+p_2)]\Big\}_{\alpha}\\ &\Big\{\tau dz - (1-\tau)[d(\nu_2\sigma_2)(p_1+p_2)\\ &+d\nu_2(p_3+p_2)]\Big\}^{\alpha}.\end{aligned} \tag{5.32}$$

To be non-weak it must contain a factor of $d\sigma_2 d\nu_2$ which is absent in (5.32).

Hence non-zero 'boundary' terms are those proportional to either $\mathbb{D}(\tau)$ or $\mathbb{D}(\bar{\tau})$. Firstly, consider the terms with $\mathbb{D}(\bar{\tau})$. To see, that the sum of such terms is d−closed, it is useful to make the following change of variables:

$$\nu_1\sigma_1 := \xi_1,\qquad \nu_1\sigma_2 := \xi_2,\qquad \nu_2 = \xi_3,\qquad \sum\xi_i = 1 \tag{5.33}$$

with Ω_1, $\bar{\Omega}_1$ (5.28). To change variables in the Ω_2, $\bar{\Omega}_2$ part (5.29) we use the following cyclic permutation of (5.33)

$$\nu_1\sigma_1 := \xi_2,\qquad \nu_1\sigma_2 := \xi_3,\qquad \nu_2 = \xi_1,\qquad \sum\xi_i = 1. \tag{5.34}$$

As a result, using notations (5.1), the $\mathbb{D}(\bar{\tau})$−proportional part of (5.27) acquires the form

$$\frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\xi(3)}\nabla(\xi(3))\mathbb{D}(\bar{\tau})l(\tau)\Big[\mathbf{E}(\Omega_1\,|\bar{\Omega}_1) - \mathbf{E}(\Omega_2\,|\bar{\Omega}_2)\Big]CCCk\bar{k}\,, \tag{5.35}$$

where

$$\begin{aligned}\Omega_1^{\alpha} &:= \tau z^{\alpha} - (1-\tau)[-\xi_3(p_1+p_2)^{\alpha}\\ &\quad +\Big(1-\xi_1\xi_3(1-\xi_3)^{-1}\Big)(p_3+p_2)^{\alpha}],\\ \bar{\Omega}_1^{\dot{\alpha}} &:= -[-(1-\xi_3)(\bar{p}_1+\bar{p}_2)\\ &\quad +(1-\xi_1)(\bar{p}_3+\bar{p}_2)]^{\dot{\alpha}},\end{aligned} \tag{5.36}$$

$$\begin{aligned}\Omega_2^{\alpha} &:= \tau z^{\alpha} - (1-\tau)[-\Big(1-\xi_1\xi_3(1-\xi_1)^{-1}\Big)\\ &\quad (p_1+p_2)^{\alpha} + \xi_1(p_3+p_2)^{\alpha}],\\ \bar{\Omega}_2^{\dot{\alpha}} &:= -[-(1-\xi_3)(\bar{p}_1+\bar{p}_2)\\ &\quad +(1-\xi_1)(\bar{p}_3+\bar{p}_2)]^{\dot{\alpha}}.\end{aligned} \tag{5.37}$$

Analogously, changing the variables in the $\mathbb{D}(\tau)$ part of (5.27) with Ω_1, $\bar{\Omega}_1$ (5.28)

$$\nu_2\sigma_1 := \xi_1,\qquad \nu_2\sigma_2 := \xi_2,\qquad \nu_1 = \xi_3,\qquad \sum\xi_i = 1 \tag{5.38}$$

and the cyclically transformed change of variables $\xi_3 \to \xi_1 \to \xi_2 \to \xi_3$ in Ω_2 and $\bar{\Omega}_2$ (5.29), we obtain

$$\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\xi(3)}\nabla(\xi(3))\mathbb{D}(\tau)l(\bar\tau)\Big[\mathbf{E}(\Omega_1\,|\bar\Omega_1)-\mathbf{E}(\Omega_2\,|\bar\Omega_2)\Big] CCCk\bar k, \tag{5.39}$$

where

$$\begin{aligned}\Omega_1^\alpha &:= \tau z^\alpha-(1-\tau)[-(1-\xi_3)(p_1+p_2)^\alpha\\ &\quad+(1-\xi_1)(p_3+p_2)^\alpha],\\ \bar\Omega_1^{\dot\alpha} &:= \bar\tau\bar z^{\dot\alpha}-(1-\bar\tau)[-\xi_3(\bar p_1+\bar p_2)^{\dot\alpha}\\ &\quad+(1-\xi_3\xi_1(1-\xi_3)^{-1})(\bar p_3+\bar p_2)^{\dot\alpha}],\end{aligned} \tag{5.40}$$

$$\begin{aligned}\Omega_2^\alpha &:= \tau z^\alpha-(1-\tau)[-(1-\xi_3)(p_1+p_2)^\alpha\\ &\quad+(1-\xi_1)(p_3+p_2)^\alpha],\\ \bar\Omega_2^{\dot\alpha} &:= \bar\tau\bar z^{\dot\alpha}-(1-\bar\tau)[-(1-\xi_1\xi_3(1-\xi_1)^{-1})\\ &\quad(\bar p_1+\bar p_2)^{\dot\alpha}+\xi_1(\bar p_3+\bar p_2)^{\dot\alpha}].\end{aligned} \tag{5.41}$$

One can easily make sure that the expressions (5.35) and (5.39) are d-closed. For instance, applying d to (5.39) one can see that the only potentially non-zero term is that with $\mathbb{D}(1-\bar\tau)$. However, Eqs. (5.40), (5.41) yield $\Big[\mathbf{E}(\Omega_1\,|\bar\Omega_1)-\mathbf{E}(\Omega_2\,|\bar\Omega_2)\Big]\Big|_{\bar\tau=1}=0$. The case of (5.35) is analogous.

Application of IH of Sect. 4.2 to the MNL pairs of (5.35) and (5.39) brings the 'boundary' part of Eq. (5.46) to the form

$$\frac{\eta\bar\eta}{16}\mathrm{d}\Big\{\int_{\alpha(2)\tau\bar\tau\xi(3)}\nabla(\alpha(2))\nabla(\xi(3))\Big[-\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3\,|\bar\Omega_3) +\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_4\,|\bar\Omega_4)\Big]CCCk\bar k\Big\}, \tag{5.42}$$

where

$$\begin{aligned}\Omega_3^\alpha &:= \tau z^\alpha-(1-\tau)\\ &\Big[-\Big\{\alpha_1\xi_3+\alpha_2(1-\xi_1\xi_3(1-\xi_1)^{-1})\Big\}\\ &(p_1+p_2)^\alpha\\ &+\Big\{\alpha_1(1-\xi_3\xi_1(1-\xi_3)^{-1})+\alpha_2\xi_1\Big\}(p_3+p_2)^\alpha\Big],\\ \bar\Omega_3^{\dot\alpha} &:= -[-(1-\xi_3)(\bar p_1+\bar p_2)\\ &+(1-\xi_1)(\bar p_3+\bar p_2)]^{\dot\alpha},\end{aligned} \tag{5.43}$$

$$\begin{aligned}\Omega_4^\alpha &:= -[-(1-\xi_3)(p_1+p_2)^\alpha+(1-\xi_1)(p_3+p_2)^\alpha],\\ \bar\Omega_4^{\dot\alpha} &:= \bar\tau\bar z^{\dot\alpha}-(1-\bar\tau)[-\Big\{\alpha_1\xi_3+\alpha_2(1-\xi_1\xi_3(1-\xi_1)^{-1})\Big\}\\ &(\bar p_1+\bar p_2)^{\dot\alpha}\\ &+\Big\{\alpha_1(1-\xi_3\xi_1(1-\xi_3)^{-1})+\alpha_2\xi_1\Big\}(\bar p_3+\bar p_2)^{\dot\alpha}].\end{aligned} \tag{5.44}$$

Equations (5.26) and (5.42) yield the following final result for MNL B_3:

$$\begin{aligned}B_3^{\eta\bar\eta} &= B_3^{\eta\bar\eta}|_{blk}+B_3^{\eta\bar\eta}|_{bnd},\\ B_3^{\eta\bar\eta}|_{blk} &= \frac{\eta\bar\eta}{16}\int_{\nu(2)\tau\bar\tau\sigma(2)}\square(\tau,\bar\tau)\nabla(\nu(2))\nabla(\sigma(2))\\ &\quad\times\Big[\mathbf{E}(\Omega_1\,|\bar\Omega_1)-\mathbf{E}(\Omega_2\,|\bar\Omega_2)\Big]CCCk\bar k,\end{aligned} \tag{5.45}$$

$$\begin{aligned}B_3^{\eta\bar\eta}|_{bnd} &= \frac{\eta\bar\eta}{16}\int_{\alpha(2)\tau\bar\tau\xi(3)}\nabla(\alpha(2))\nabla(\xi(3))\\ &\quad\times\Big[\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3\,|\bar\Omega_3)-\mathbb{D}(\tau)l(\bar\tau)\\ &\quad\mathbf{E}(\Omega_4\,|\bar\Omega_4)\Big]CCCk\bar k\end{aligned} \tag{5.46}$$

with Ω_1, $\bar\Omega_1$ (5.28), Ω_2, $\bar\Omega_2$ (5.29), Ω_3, $\bar\Omega_3$ (5.43) and Ω_4, $\bar\Omega_4$ (5.44). $B_3^{\eta\bar\eta}|_{blk}$ (5.45) and $B_3^{\eta\bar\eta}|_{bnd}$ (5.46) are MNL by construction. This allows us to construct the MNL vertex $\Upsilon^{\eta\bar\eta}(\omega,C,C,C)$.

6 Moderately non-local vertex $\Upsilon^{\eta\bar\eta}(\omega, C, C, C)$

According to Eq.(3.27) the vertex $\Upsilon^{\eta\bar\eta}(\omega,C,C,C)$ in the zero-form sector can be represented in the form

$$\Upsilon^{\eta\bar\eta}(\omega,C,C,C)=\Upsilon^{\eta\bar\eta}_{\omega CCC}+\Upsilon^{\eta\bar\eta}_{C\omega CC}+\Upsilon^{\eta\bar\eta}_{CC\omega C}+\Upsilon^{\eta\bar\eta}_{CCC\omega} \tag{6.1}$$

with the subscripts referring to the orderings of the product factors.

As a consequence of consistency of the HS equations, though having the form of the sum of Z-dependent terms, the *r.h.s.* of (4.28) must be Z, dZ-independent. Hence in the vertex analysis we discard the dZ-dependent terms which are weakly zero anyway.

In this section we present the final form of the MNL vertices $\Upsilon^{\eta\bar\eta}_{\omega CCC}$ and $\Upsilon^{\eta\bar\eta}_{C\omega CC}$. Technical details are elaborated in Appendices D and E, respectively. The vertices $\Upsilon^{\eta\bar\eta}_{CC\omega C}$ and $\Upsilon^{\eta\bar\eta}_{CCC\omega}$ can be worked out analogously. (Note that these can be obtained from the vertices $\Upsilon^{\eta\bar\eta}_{\omega CCC}$ and $\Upsilon^{\eta\bar\eta}_{C\omega CC}$ by the HS algebra antiautomorphism [23,29,30].)

The sketch of the calculation scheme is as follows.

Firstly, we write down the *r.h.s.* of Eq. (4.28) for $\Upsilon^{\eta\bar\eta}(\omega,C,C,C)$. To this end we use the previously known W_1^η, $W_1^{\bar\eta}$, B_2^η and $B_2^{\bar\eta}$ rewritten in the form (3.28) in Appendix A, MNL $B_3^{\eta\bar\eta}$ of Sect. 5, $W_2^{\eta\bar\eta}$ obtained in Appendix B in such a way that $[W_2^{\eta\bar\eta}, C]_*$ is MNL, and the spin-local vertices $\Upsilon^\eta(\omega,C,C)$ written in the form (3.10) with $\rho=\beta=0$ in Appendix C, and their conjugated.

Plugging these terms into the *r.h.s.* of (4.28) one can make sure that the resulting expressions have the form of Ansatz (3.28) and are MNL for every ordering of ω and C's.

Let us emphasize that the full expression for $\Upsilon^{\eta\bar\eta}(\omega,C,C,C)$ (4.28) must be Z-independent for each ordering. In principle, one could find manifestly Z-independent expression by setting for instance $Z=0$. The result would not be manifestly MNL, since τ and $\bar\tau$ would not be zero. According to Z-dominance Lemma, the Z dependence can be eliminated by adding to the integrand d-exact expressions giving zero upon integration in the sector in question so that $\tau = \bar{\tau} = 0$ in the end. For this we will again use IH of Sect. 4.2.

Namely, for each ordering, picking up an appropriate pair of terms from the *r.h.s.* of (4.28) we apply IH dropping the corresponding d-exact part. For the leftover terms, that are MNL, this procedure is repeated as many times as needed until all of them cancel except for some cohomological ones producing the physical vertices.

The resulting MNL vertices are presented in the next subsections. Note that it may not be manifest that they are indeed MNL. The easiest way to see this is to prove inequalities (4.4) at the first step of calculations then using repeatedly the simple inequality

$$|\alpha A + (1-\alpha)B| + |\alpha A' + (1-\alpha)B'| \le \alpha(|A| + |A'|) + (1-\alpha)(|B| + |B'|), \qquad \alpha \in [0,1].$$

6.1 $\Upsilon^{\eta\bar{\eta}}_{\omega CCC}$

According to (4.28)

$$\Upsilon^{\eta\bar{\eta}}|_{\omega CCC} = -(W_2^{\eta\bar{\eta}}|_{\omega CC}) * C - (W_1^{\bar{\eta}}|_{\omega C}) * B_2^{\eta} - (W_1^{\eta}|_{\omega C}) * B_2^{\bar{\eta}} - \omega * B_3^{\eta\bar{\eta}} - \mathrm{d}_x B_3^{\eta\bar{\eta}}|_{\omega CCC} - \mathrm{d}_x B_2^{\eta}|_{\omega CCC} - \mathrm{d}_x B_2^{\bar{\eta}}|_{\omega CCC}. \tag{6.2}$$

Using IH and formulae (5.45), (5.46), (A.1), (A.3), (A.7), (A.9), (B.1), (C.3) one obtains from Eq. (6.2) moderately non-local $\Upsilon^{\eta\bar{\eta}}|_{\omega CCC}$,

$$\begin{aligned}\Upsilon^{\eta\bar{\eta}}|_{\omega CCC} =& \frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\nu(2)\alpha(2)\xi(3)} \mathbb{D}(\tau)\mathbb{D}(\bar{\tau})\mu_1 \\ &\times\Big[\mathbf{E}(\Omega_1\,|\bar{\Omega}_1) - \mathbf{E}(\Omega_2\,|\bar{\Omega}_2)\Big]\omega CCCk\bar{k} \\ &+\frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\beta(2)\alpha(2)\nu(2)\sigma(2)} \mathbb{D}(\tau)\mathbb{D}(\bar{\tau})\mu_2 \\ &\times\Big[\mathbf{E}(\Omega_3\,|\bar{\Omega}_3) + \mathbf{E}(\Omega_4\,|\bar{\Omega}_4)\Big]\omega CCCk\bar{k}\end{aligned} \tag{6.3}$$

where

$$\mu_1 = \nabla(\alpha(2))\nabla(\nu(2))\nabla(\xi(3)),$$
$$\mu_2 = \nabla(\beta(2))\nabla(\nu(2))\nabla(\sigma(2))\nabla(\alpha(2))$$

with ∇ (5.1), and

$$\begin{aligned}\Omega_1{}^{\alpha} :=& -\Big[-(\nu_2 + \nu_1\{\alpha_2\xi_2(1-\xi_1)^{-1} + \xi_3\})p_0 \\ &-\{\alpha_2\xi_2(1-\xi_1)^{-1} + \xi_3\}(p_1+p_2) \\ &+\{\alpha_1\xi_2(1-\xi_3)^{-1} + \xi_1\}(p_3+p_2)]^{\alpha}, \\ \bar{\Omega}_1{}^{\dot{\alpha}} :=& -[-(1-\nu_1\xi_3)\bar{p}_0 - (1-\xi_3) \\ &(\bar{p}_1+\bar{p}_2) + (1-\xi_1)(\bar{p}_3+\bar{p}_2)]^{\dot{\alpha}},\end{aligned} \tag{6.4}$$

$$\begin{aligned}\Omega_2{}^{\alpha} :=& -[-(1-\nu_1\xi_3)p_0 - (1-\xi_3) \\ &(p_1+p_2) + (1-\xi_1)(p_3+p_2)]^{\alpha}, \\ \bar{\Omega}_2{}^{\dot{\alpha}} :=& -[-(\nu_2 + \nu_1\{\xi_3 + \alpha_1\xi_2(1-\xi_1)^{-1}\}) \\ &\bar{p}_0 - \{\xi_3 + \alpha_1\xi_2(1-\xi_1)^{-1}\}(\bar{p}_1+\bar{p}_2) \\ &+\Big\{\alpha_2\xi_2(1-\xi_3)^{-1} + \xi_1\Big\}(\bar{p}_3+\bar{p}_2)]^{\dot{\alpha}},\end{aligned} \tag{6.5}$$

$$\begin{aligned}\Omega_3{}^{\alpha} :=& -\beta_1[-(1-\nu_2\sigma_2\alpha_2)p_0 \\ &-(1-\nu_2\alpha_2)p_1 + (-\nu_1+\nu_2\alpha_2)p_2 \\ &-\nu_1 p_3]^{\alpha} - p_3{}^{\alpha}, \\ \bar{\Omega}_3{}^{\dot{\alpha}} :=& -[-(1-\sigma_2\alpha_1)\bar{p}_0 - \alpha_2(\bar{p}_1 \\ &+\bar{p}_2) + (\bar{p}_3+\bar{p}_2)]^{\dot{\alpha}},\end{aligned} \tag{6.6}$$

$$\begin{aligned}\Omega_4{}^{\alpha} :=& -[-(1-\sigma_2\alpha_1)p_0 \\ &-\alpha_2(p_1+p_2) + (p_3+p_2)]^{\alpha}, \\ \bar{\Omega}_4{}^{\dot{\alpha}} :=& -\beta_1[-(1-\nu_2\sigma_2\alpha_2)\bar{p}_0 \\ &-(1-\nu_2\alpha_2)\bar{p}_1 + (\nu_2(\alpha_2+1)-1) \\ &\bar{p}_2 - \nu_1\bar{p}_3]^{\dot{\alpha}} - \bar{p}_3{}^{\dot{\alpha}}.\end{aligned} \tag{6.7}$$

6.2 $\Upsilon^{\eta\bar{\eta}}_{C\omega CC}$

According to (4.28),

$$\Upsilon^{\eta\bar{\eta}}|_{C\omega CC} = C * (W_2^{\eta\bar{\eta}}|_{\omega CC}) - (W_2^{\eta\bar{\eta}}|_{C\omega C}) * C - (W_1^{\bar{\eta}} * B_2^{\eta} + W_1^{\eta} * B_2^{\bar{\eta}})|_{C\omega CC} - \mathrm{d}_x B_3^{\eta\bar{\eta}}|_{C\omega CC} - \mathrm{d}_x B_2^{\eta}|_{C\omega CC} - \mathrm{d}_x B_2^{\bar{\eta}}|_{C\omega CC}. \tag{6.8}$$

Using IH and formulae (5.45), (5.46), (A.1), (A.3), (A.5), (A.7), (A.9), (B.1), (B.5) and (C.3), (C.4) one obtains from Eq. (6.8) moderately non-local $\Upsilon_{C\omega CC}$,

$$\begin{aligned}\Upsilon^{\eta\bar{\eta}}_{C\omega CC} =& \frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\nu(2)\alpha(2)\xi(3)} \mathbb{D}(\tau)\mathbb{D}(\bar{\tau})\mu_1 \\ &\times\Big[\mathbf{E}(\Omega_1\,|\bar{\Omega}_1) - \mathbf{E}(\Omega_2\,|\bar{\Omega}_2)\Big]C\omega CCk\bar{k} \\ &+\frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\beta(2)\alpha(2)\nu(2)\sigma(2)} \mathbb{D}(\tau)\mathbb{D}(\bar{\tau})\mu_2\Big[\mathbf{E}(\Omega_3\,|\bar{\Omega}_3) \\ &+\mathbf{E}(\Omega_4\,|\bar{\Omega}_4) - \mathbf{E}(\Omega_5\,|\bar{\Omega}_5) - \mathbf{E}(\Omega_6\,|\bar{\Omega}_6)\Big]C\omega CCk\bar{k}\end{aligned} \tag{6.9}$$

with

$$\mu_1 = \nabla(\alpha(2))\nabla(\nu(2))\nabla(\xi(3)),$$
$$\mu_2 = \nabla(\beta(2))\nabla(\nu(2))\nabla(\sigma(2))\nabla(\alpha(2))$$

and

$$\begin{aligned}\Omega_1{}^{\alpha} :=& -\Big[(\nu_1\{\alpha_2\xi_2(1-\xi_3)^{-1} + \xi_1\} \\ &-\{\alpha_1\xi_2(1-\xi_1)^{-1} + \xi_3\})p_0 \\ &-\{\alpha_1\xi_2(1-\xi_1)^{-1} + \xi_3\}(p_1+p_2) \\ &+\{\alpha_2\xi_2(1-\xi_3)^{-1} + \xi_1\}(p_3+p_2)\Big]^{\alpha},\end{aligned}$$

$$\bar{\Omega}_1{}^{\dot\alpha} := -[(\nu_1(1-\xi_1)-(1-\xi_3))\bar{p}_0 -(1-\xi_3)(\bar{p}_1+\bar{p}_2)+(1-\xi_1)(\bar{p}_3+\bar{p}_2)]^{\dot\alpha}, \tag{6.10}$$

$$\Omega_2{}^{\alpha} := -[(\nu_1(1-\xi_1)-(1-\xi_3))p_0-(1-\xi_3)(p_1+p_2)+(1-\xi_1)(p_3+p_2)]^{\alpha},$$

$$\bar{\Omega}_2{}^{\dot\alpha} := -[(\nu_1\{\alpha_1\xi_2(1-\xi_3)^{-1}+\xi_1\} -\{\xi_3+\alpha_2\xi_2(1-\xi_1)^{-1}\})\bar{p}_0 -\{\xi_3+\alpha_2\xi_2(1-\xi_1)^{-1}\}(\bar{p}_1+\bar{p}_2) +\{\alpha_1\xi_2(1-\xi_3)^{-1}+\xi_1\}(\bar{p}_3+\bar{p}_2)]^{\dot\alpha}, \tag{6.11}$$

$$\Omega_3{}^{\alpha} := -([\nu_2-\sigma_1]p_0-\sigma_1 p_1+\sigma_2 p_2+p_3)^{\alpha},$$

$$\bar{\Omega}_3{}^{\dot\alpha} := -\{\beta_1(\alpha_1[\sigma_1-\nu_1]\bar{p}_0-\alpha_1\sigma_2\bar{p}_1+\alpha_1\sigma_1\bar{p}_2+\bar{p}_3) +\beta_2(-\alpha_1\bar{p}_0-\alpha_1\bar{p}_1-\alpha_1\bar{p}_2+\alpha_2\bar{p}_3)\}^{\dot\alpha}, \tag{6.12}$$

$$\Omega_4{}^{\alpha} := -\{\beta_1(\alpha_1[\sigma_1-\nu_1]p_0-\alpha_1\sigma_2 p_1+\alpha_1\sigma_1 p_2+p_3) +\beta_2(-\alpha_1 p_0-\alpha_1 p_1-\alpha_1 p_2+\alpha_2 p_3)\}^{\alpha},$$

$$\bar{\Omega}_4{}^{\dot\alpha} := -([\nu_2-\sigma_1]\bar{p}_0-\sigma_1\bar{p}_1+\sigma_2\bar{p}_2+\bar{p}_3)^{\dot\alpha}, \tag{6.13}$$

$$\Omega_5{}^{\alpha} := -[(\nu_2\sigma_2-1)p_0-p_1-\nu_1 p_2+\nu_2 p_3]^{\alpha},$$

$$\bar{\Omega}_5{}^{\dot\alpha} := -[(\alpha_1+\beta_2(-\sigma_1-\sigma_2\alpha_2\nu_2))\bar{p}_0 -\alpha_2\bar{p}_1+\{-\beta_2\alpha_2\nu_2 +\alpha_1\}\bar{p}_2+\{\beta_2\alpha_2\nu_1+\alpha_1\}\bar{p}_3]^{\dot\alpha}, \tag{6.14}$$

$$\Omega_6{}^{\alpha} := -[(\alpha_1+\beta_2(-\sigma_1-\sigma_2\alpha_2\nu_2))p_0 -\alpha_2 p_1+\{-\beta_2\alpha_2\nu_2+\alpha_1\}p_2+\{\beta_2\alpha_2\nu_1+\alpha_1\}p_3]^{\alpha},$$

$$\bar{\Omega}_6{}^{\dot\alpha} := -[(\nu_2\sigma_2-1)\bar{p}_0-\bar{p}_1-\nu_1\bar{p}_2+\nu_2\bar{p}_3]^{\dot\alpha}. \tag{6.15}$$

6.3 $\Upsilon^{\eta\bar\eta}_{CC\omega C}$

According to (4.28),

$$\Upsilon^{\eta\bar\eta}|_{CC\omega C} = C*(W_2^{\eta\bar\eta}|_{C\omega C})-(W_2^{\eta\bar\eta}|_{CC\omega}) *C+(B_2^{\eta}*W_1^{\bar\eta}+B_2^{\bar\eta}*W_1^{\eta})|_{CC\omega C} -\mathrm{d}_x B_3^{\eta\bar\eta}|_{CC\omega C}-\mathrm{d}_x B_2^{\eta}|_{CC\omega C}-\mathrm{d}_x B_2^{\bar\eta}|_{CC\omega C}. \tag{6.16}$$

Using IH and formulae (5.45), (5.46), (A.1), (A.3), (A.5), (A.7), (A.9), (B.5), (B.11), (C.4), (C.5) one obtains from Eq. (6.16) moderately non-local $\Upsilon_{CC\omega C}$,

$$\Upsilon^{\eta\bar\eta}_{CC\omega C} = -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\alpha(2)\xi(3)} \mathbb{D}(\tau)\mathbb{D}(\bar\tau)\mu_1 \times\Big[\mathbf{E}(\Omega_1\,|\bar\Omega_1)-\mathbf{E}(\Omega_2\,|\bar\Omega_2)\Big]CC\omega Ck\bar k -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\beta(2)\alpha(2)\nu(2)\sigma(2)} \mathbb{D}(\tau)\mathbb{D}(\bar\tau)\mu_2 \times\Big[\mathbf{E}(\Omega_3\,|\bar\Omega_3)+\mathbf{E}(\Omega_4\,|\bar\Omega_4)-\mathbf{E}(\Omega_5\,|\bar\Omega_5) -\mathbf{E}(\Omega_6\,|\bar\Omega_6)\Big]CC\omega Ck\bar k \tag{6.17}$$

with

$$\mu_1 = \nabla(\alpha(2))\nabla(\nu(2))\nabla(\xi(3)),$$
$$\mu_2 = \nabla(\beta(2))\nabla(\nu(2))\nabla(\sigma(2))\nabla(\alpha(2)),$$

$$\Omega_1{}^{\alpha} := -\Big[(\{\alpha_2\xi_2(1-\xi_3)^{-1}+\xi_1\} -\nu_1\{\alpha_1\xi_2(1-\xi_1)^{-1}+\xi_3\})p_0 -\{\alpha_1\xi_2(1-\xi_1)^{-1}+\xi_3\}(p_1+p_2) +\{\alpha_2\xi_2(1-\xi_3)^{-1}+\xi_1\}(p_3+p_2)\Big]^{\alpha},$$

$$\bar{\Omega}_1{}^{\dot\alpha} := -[((1-\xi_1)-\nu_1(1-\xi_3))\bar{p}_0 -(1-\xi_3)(\bar{p}_1+\bar{p}_2)+(1-\xi_1)(\bar{p}_3+\bar{p}_2)]^{\dot\alpha}, \tag{6.18}$$

$$\Omega_2{}^{\alpha} := -[((1-\xi_1)-\nu_1(1-\xi_3))p_0-(1-\xi_3)(p_1+p_2)+(1-\xi_1)(p_3+p_2)]^{\alpha},$$

$$\bar{\Omega}_2{}^{\dot\alpha} := -[(\{\alpha_1\xi_2(1-\xi_3)^{-1}+\xi_1\} -\nu_1\{\xi_3+\alpha_2\xi_2(1-\xi_1)^{-1}\})\bar{p}_0 -\{\xi_3+\alpha_2\xi_2(1-\xi_1)^{-1}\}(\bar{p}_1+\bar{p}_2) +\{\alpha_1\xi_2(1-\xi_3)^{-1}+\xi_1\}(\bar{p}_3+\bar{p}_2)]^{\dot\alpha}, \tag{6.19}$$

$$\Omega_3{}^{\alpha} := -[(-\sigma_1+\nu_2) p_0-p_1-\sigma_1 p_2+\sigma_2 p_3]^{\alpha},$$

$$\bar{\Omega}_3{}^{\dot\alpha} := -\Big[\alpha_2(1-\beta_2(\sigma_2+\nu_1)\bar{p}_0 -(1-\alpha_2\beta_1)\bar{p}_1+\alpha_2(\beta_1-\sigma_2\beta_2)\bar{p}_2+\alpha_2(1-\sigma_2\beta_2)\bar{p}_3\Big]^{\dot\alpha}, \tag{6.20}$$

$$\Omega_4{}^{\alpha} := -\Big[\alpha_2(1-\beta_2(\sigma_2+\nu_1)p_0-(1-\alpha_2\beta_1)p_1 +\alpha_2(\beta_1-\sigma_2\beta_2)p_2+\alpha_2(1-\sigma_2\beta_2)p_3)\Big]^{\alpha},$$

$$\bar{\Omega}_{4\dot\alpha} := -[(-\sigma_1+\nu_2)\bar{p}_0-\bar{p}_1-\sigma_1\bar{p}_2+\sigma_2\bar{p}_3]^{\dot\alpha}, \tag{6.21}$$

$$\Omega_5{}^{\alpha} := -[(1-\sigma_1\nu_1)p_0 -\nu_1(p_1+p_2)+(p_3+p_2)]^{\alpha},$$

$$\bar{\Omega}_5{}^{\dot\alpha} := -[(-\alpha_1+\beta_2\sigma_2+\beta_2\sigma_1\alpha_2\nu_1)\bar{p}_0+(-\alpha_1 -\alpha_2\beta_2\nu_2)\bar{p}_1+\{-\alpha_1+\beta_2\alpha_2\nu_1\}\bar{p}_2+\alpha_2\bar{p}_3]^{\dot\alpha}, \tag{6.22}$$

$$\Omega_6{}^{\alpha} := -[(-\alpha_1+\beta_2\sigma_2+\beta_2\sigma_1\alpha_2\nu_1)p_0 +(-\alpha_1-\alpha_2\beta_2\nu_2)p_1+\{-\alpha_1+\beta_2\alpha_2\nu_1\}p_2+\alpha_2 p_3]^{\alpha},$$

$$\bar{\Omega}_6{}^{\dot\alpha} := -[(1-\sigma_1\nu_1)\bar{p}_0-\nu_1(\bar{p}_1+\bar{p}_2)+(\bar{p}_3+\bar{p}_2)]^{\dot\alpha}. \tag{6.23}$$

6.4 $\Upsilon^{\eta\bar\eta}_{CCC\omega}$

According to (4.28),

$$\Upsilon^{\eta\bar\eta}|_{CCC\omega} = C*(W_2^{\eta\bar\eta}|_{CC\omega}) +B_2^{\eta}*(W_1^{\bar\eta}|_{C\omega})+B_2^{\bar\eta}*(W_1^{\eta}|_{C\omega}) +B_3^{\eta\bar\eta}*\omega-\mathrm{d}_x B_3^{\eta\bar\eta}|_{CCC\omega} -\mathrm{d}_x B_2^{\eta}|_{CCC\omega}-\mathrm{d}_x B_2^{\bar\eta}|_{CCC\omega}. \tag{6.24}$$

Using IH and formulae (5.45), (5.46), (A.1), (A.5), (A.7), (A.9), (B.11), (C.5) one obtains from Eq. (6.24) moderately non-local $\Upsilon_{CCC\omega}$

$$\Upsilon_{CCC\omega} = \frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\alpha(2)\xi(3)} \mathbb{D}(\tau)\mathbb{D}(\bar\tau)\mu_1 \times\Big[\mathbf{E}(\Omega_1\,|\bar\Omega_1)+\mathbf{E}(\Omega_2\,|\bar\Omega_2)\Big]CCC\omega k\bar k -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\beta(2)\alpha(2)\nu(2)\sigma(2)} \mathbb{D}(\tau)\mathbb{D}(\bar\tau)\mu_2 \times\Big[\mathbf{E}(\Omega_3\,|\bar\Omega_3)+\mathbf{E}(\Omega_4\,|\bar\Omega_4)\Big]CCC\omega k\bar k, \tag{6.25}$$

with

$$\mu_1 = \nabla(\alpha(2))\nabla(\nu(2))\nabla(\xi(3)),$$
$$\mu_2 = \nabla(\beta(2))\nabla(\nu(2))\nabla(\sigma(2))\nabla(\alpha(2))$$

and

$$\Omega_1^\alpha := -\Big[(\nu_1 + \nu_2\{\alpha_2\xi_2(1-\xi_3)^{-1} + \xi_1\})p_0 - \{\alpha_1\xi_2(1-\xi_1)^{-1} + \xi_3\}(p_1+p_2)^\alpha + \{\alpha_2\xi_2(1-\xi_3)^{-1} + \xi_1\}(p_3+p_2)^\alpha\Big],$$
$$\bar\Omega_1^{\dot\alpha} := -[(\nu_1 + \nu_2(1-\xi_1))\bar p_0 - (1-\xi_3)(\bar p_1 + \bar p_2) + (1-\xi_1)(\bar p_3 + \bar p_2)]^{\dot\alpha}, \tag{6.26}$$

$$\Omega_2^\alpha := -[(\nu_1 + \nu_2(1-\xi_1))p_0 - (1-\xi_3)(p_1+p_2) + (1-\xi_1)(p_3+p_2)]^\alpha,$$
$$\bar\Omega_2^{\dot\alpha} := -[(\nu_1 + \nu_2\{\alpha_2\xi_2(1-\xi_3)^{-1} + \xi_1\})\bar p_0 \times\{\alpha_1\xi_2(1-\xi_1)^{-1} + \xi_3\}(\bar p_1 + \bar p_2) + \Big\{\alpha_2\xi_2(1-\xi_3)^{-1} + \xi_1\Big\}(\bar p_3 + \bar p_2)]^{\dot\alpha}, \tag{6.27}$$

$${\Omega_3}^\alpha := -[(1-\sigma_1\nu_1)p_0 - p_1 - \nu_1 p_2 + \nu_2 p_3]^\alpha,$$
$${\bar\Omega_3}^{\dot\alpha} := -\big(\beta_2(1-\alpha_1\sigma_1\nu_2)\bar p_0 - (1-\alpha_2\beta_2)\bar p_1 + \beta_2(\alpha_2 - \alpha_1\nu_2)\bar p_2 + \beta_2(1-\alpha_1\nu_2)\bar p_3\big)^{\dot\alpha}, \tag{6.28}$$
$${\Omega_4}^\alpha := -\big(\beta_2(1-\sigma_1\nu_2\alpha_1)p_0 - (1-\alpha_2\beta_2)p_1 + \beta_2(\alpha_2 - \alpha_1\nu_2)p_2 + \beta_2(1-\alpha_1\nu_2)p_3\big)^\alpha,$$
$${\bar\Omega_4}^{\dot\alpha} := -[(1-\sigma_1\nu_1)\bar p_0 - \bar p_1 - \nu_2\bar p_2 + \nu_1\bar p_3]^{\dot\alpha}. \tag{6.29}$$

7 Conclusion

In this paper we introduce the concept of moderate non-locality and calculate moderately non-local vertices $\Upsilon^{\eta\bar\eta}(\omega, C, C, C)$ in the mixed $\eta\bar\eta$ sector of HS gauge theory in AdS_4 for all orderings of the fields ω and C. Our calculation is based on the differential homotopy Ansatz of [17] for the lowest order holomorphic deformation linear in η of the perturbative analysis of the holomorphic sector. To solve the problem we use the interpolating homotopy that preserves moderate non-locality in the process of perturbative analysis of the HS equations.

The degree of non-locality of vertices is expressed by the coefficients P^{ij} and $\bar P^{ij}$ in front of, respectively, convolutions $p_{i\alpha}{p_j}^\alpha$ and $\bar p_{i\dot\alpha}{\bar p_j}^{\dot\alpha}$ in the exponents $\mathbf{E}$ (3.29). Moderately non-local vertices obey the inequalities $|P^{ij}| + |\bar P^{ij}| \le 1$, while the usual star product $C_1(y,\bar y) * C_2(y,\bar y) * C_3(y,\bar y)$ yields $|P^{ij}| + |\bar P^{ij}| = 2$. At the moment, moderately non-local vertices are minimally non-local among known vertices in the mixed $\eta\bar\eta$ sector of the HS gauge theory. Note that the usual spin-local vertices of [27] in the (anti)holomorphic sector form a subclass of moderately non-local vertices. Let us also stress that our construction is manifestly invariant under HS gauge symmetries.

The results of this paper raise a number of interesting questions for the future study. The most important one is to understand whether it is possible to improve further the level of non-locality of HS theory by choosing appropriate field variables. Another interesting problem is to compare the level of non-locality of the moderately non-local vertices with that deduced by Sleight and Taronna [18] from the Klebanov-Polyakov holographic conjecture [2].

It is also important to extend the results of this paper to the vertex $\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C)$. Presumably, spin-local $S_2^{\eta\bar\eta}$ and $W_2^{\eta\bar\eta}$ obtained in this paper lead to the special form of the local bilinear $\eta\bar\eta$−current deformation in the one-form sector, originally obtained in [28] using conventional homotopy supplemented by some field redefinitions, that leads to the current contribution to Fronsdal equations [31] in agreement with Metsaev's classification [32,33].

Moreover, the IH approach of this paper makes it possible to obtain the spin-local vertex $\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C)$ such that $[\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C), C]_*$ is MNL. (Note that $[\tilde\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C), C]_*$ is not MNL for the spin-local vertex $\tilde\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C)$ obtained in [12].) This property is important for the analysis of the contribution of the vertices $\Upsilon^{\eta\bar\eta}$ to Fronsdal equations.

The sketch of the calculation is as follows. Equation (2.3) yields

$$\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C) = -\big(\mathrm{d}_x W_1 + W_1 * W_1 + \mathrm{d}_x W_2 + \omega * W_2 + W_2 * \omega\big)\big|_{\eta\bar\eta}. \tag{7.1}$$

Plugging W_1^η, $W_1^{\bar\eta}$ ((A.3), (A.5)), $W_2^{\eta\bar\eta}$ ((B.1), (B.5), (B.11)) into the *r.h.s.* of (7.1) along with Eqs. (C.3)–(C.5) and their conjugated one can make sure that $\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C)$ (7.1) is spin-local and $[\Upsilon^{\eta\bar\eta}(\omega, \omega, C, C), C]_*$ is MNL for every ordering of ω and C's. Hence, to eliminate Z-dependence in a way preserving MNL, one can again use IH of Sect. 4.2. This is work in progress.

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Appendix A: S_1, W_1, B_2

We rewrite S_1,W_1 and B_2 obtained in [11,12] via $\mathcal{E}(...)$ defined in (3.12) and its conjugated $\bar{\mathcal{E}}(...)$ (see also [17])

$$S_1 = -\frac{\eta}{2}\int_\tau l(\tau)\mathcal{E}(\Omega)C*k - \frac{\bar\eta}{2}\int_{\bar\tau} l(\bar\tau)\bar{\mathcal{E}}(\bar\Omega)C*\bar k, \tag{A.1}$$

$$\Omega^\alpha = \tau z^\alpha, \qquad \bar\Omega^\alpha = \bar\tau\bar z^{\dot\alpha}, \tag{A.2}$$

$$W_1|_{\omega C} = \frac{i\eta}{4}\int_{\tau,\sigma} l(\sigma)l(\tau)\mathcal{E}(\Omega)\omega C*k + \frac{i\bar\eta}{4}\int_{\bar\tau,\sigma} l(\sigma)l(\bar\tau)\bar{\mathcal{E}}(\bar\Omega)\omega C*\bar k, \tag{A.3}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)(-\sigma p_0)^\alpha, \quad \bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)(-\sigma\bar p_0)^{\dot\alpha}, \tag{A.4}$$

$$W_1|_{C\omega} = \frac{\eta}{4i}\int_{\bar\tau\sigma} l(\sigma)l(\tau)\mathcal{E}(\Omega)C\omega*k + \frac{\bar\eta}{4i}\int_{\tau,\sigma} l(\sigma)l(\bar\tau)\bar{\mathcal{E}}(\bar\Omega)C\omega*\bar k, \tag{A.5}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)(\sigma p_0)^\alpha, \quad \bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)(\sigma\bar p_0)^{\dot\alpha}, \tag{A.6}$$

$$B_2^\eta = \frac{\eta}{4i}\int_{\tau\sigma(2)} l(\tau)\nabla(\sigma(2))\mathcal{E}(\Omega)\, CC*k, \tag{A.7}$$

$$\Omega^\alpha = \tau z^\alpha - (1-\tau)(\sigma_2 p_2 - \sigma_1 p_1)^\alpha, \tag{A.8}$$

$$B_2^{\bar\eta} = \frac{\bar\eta}{4i}\int_{\bar\tau\sigma(2)} l(\bar\tau)\nabla(\sigma(2))\bar{\mathcal{E}}(\bar\Omega)\, CC*\bar k, \tag{A.9}$$

$$\bar\Omega^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)(\sigma_2\bar p_2 - \sigma_1\bar p_1)^{\dot\alpha}. \tag{A.10}$$

Appendix B: $W_2^{\eta\bar\eta}$

Here we construct an appropriate $W_2^{\eta\bar\eta}$ using $S_2^{\eta\bar\eta}$ (5.10).

One can make sure straightforwardly that if $dW_2^{\eta\bar\eta}$ satisfies (4.26) with S_2 (5.10) then $[dW_2^{\eta\bar\eta}, C]_*$ is MNL by virtue of Eqs (A.1)–(A.9) and (5.10). Hence, using the technique of [17] and IH we obtain such $W_2^{\eta\bar\eta}$ that $[W_2^{\eta\bar\eta}, C]_*$ is MNL,

$$W_2^{\eta\bar\eta}{}_{\omega CC} = \frac{\eta\bar\eta}{16}\int_{\tau,\bar\tau,\sigma(2)\,\nu(2)} \nabla(\sigma(2))\nabla(\nu(2)) \times\Big[\Box(\tau,\bar\tau)\mathbf{E}(\Omega_1|\bar\Omega_1) + \int_{\alpha(2)}\nabla(\alpha(2)) \Big\{-\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_2|\bar\Omega_2) -\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3|\bar\Omega_3)\Big\}\Big]\omega CCk\bar k, \tag{B.1}$$

where

$$\Omega_1{}^\alpha := \tau z^\alpha - (1-\tau)[-(\nu_1+\nu_2\sigma_1)p_0 -\sigma_1 p_1 + \sigma_2 p_2]^\alpha,$$
$$\bar\Omega_1{}^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-(\nu_1+\nu_2\sigma_2)\bar p_0 -\sigma_2\bar p_1 + \sigma_1\bar p_2]^{\dot\alpha}, \tag{B.2}$$

$$\Omega_2{}^\alpha := -[-(\nu_1+\nu_2\sigma_1)p_0 -\sigma_1 p_1 + \sigma_2 p_2]^\alpha,$$
$$\bar\Omega_2{}^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)\alpha_1[-(\nu_1+\nu_2\sigma_2)\bar p_0 -\sigma_2\bar p_1 + \sigma_1\bar p_2]^{\dot\alpha}, \tag{B.3}$$

$$\Omega_3{}^\alpha := \tau z^\alpha - (1-\tau)\alpha_1[-(\nu_1+\nu_2\sigma_1)p_0 - \sigma_1 p_1 + \sigma_2 p_2]^\alpha,$$
$$\bar\Omega_3{}^{\dot\alpha} := -[-(\nu_1+\nu_2\sigma_2)\bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2]^{\dot\alpha}, \tag{B.4}$$

$$W_2^{\eta\bar\eta}{}_{C\omega C} = \frac{\eta\bar\eta}{16}\int_{\tau\,\bar\tau\,\nu(2)\sigma(2)} \nabla(\nu(2))\nabla(\sigma(2)) \Big[\Box(\tau,\bar\tau)\big(\mathbf{E}(\Omega_0|\bar\Omega_0)+\mathbf{E}(\Omega_1|\bar\Omega_1)+\mathbf{E}(\Omega_2|\bar\Omega_2)\big) + \int_{\alpha(2)}\nabla(\alpha(2))\Big\{\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_3|\bar\Omega_3) -\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_4|\bar\Omega_4)\Big\}\Big]C\omega Ck\bar k, \tag{B.5}$$

where

$$\Omega_0{}^\alpha = \tau z^\alpha - (1-\tau)[(-\sigma_1\nu_1+\sigma_2\nu_2)p_0 -\sigma_1 p_1 + \sigma_2 p_2]^\alpha,$$
$$\bar\Omega_0{}^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[(-\sigma_2\nu_1+\sigma_1\nu_2)\bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2]^{\dot\alpha}, \tag{B.6}$$

$$\Omega_1{}^\alpha = \tau z^\alpha - (1-\tau)[-(\sigma_1\nu_2+\nu_1)p_0 - p_1]^\alpha,$$
$$\bar\Omega_1{}^{\dot\alpha} = \bar\tau\bar z_1{}^{\dot\alpha} - (1-\bar\tau)[(\sigma_1\nu_1+\nu_2)\bar p_0 + \bar p_2]^{\dot\alpha}, \tag{B.7}$$

$$\Omega_2{}^\alpha = \tau z^\alpha - (1-\tau)[(\sigma_2\nu_1+\nu_2)p_0 + p_2]^\alpha,$$
$$\bar\Omega_2{}^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-(\sigma_2\nu_2+\nu_1)\bar p_0 - \bar p_1]^{\dot\alpha}, \tag{B.8}$$

$$\Omega_3{}^\alpha = -([\nu_2\sigma_2 - \nu_1\sigma_1]p_0 - \sigma_1 p_1 + \sigma_2 p_2)^\alpha,$$
$$\bar\Omega_3{}^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)\alpha_1([\nu_2\sigma_1 - \nu_1\sigma_2]\bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2)^{\dot\alpha}, \tag{B.9}$$

$$\Omega_4{}^{\alpha} = \tau z^{\alpha} - (1-\tau)\alpha_1([\nu_2\sigma_1 - \nu_1\sigma_2]p_0 - \sigma_2 p_1 + \sigma_1 p_2)^{\alpha},$$
$$\bar{\Omega}_4{}^{\dot\alpha} = -([\nu_2\sigma_2 - \nu_1\sigma_1]\bar{p}_0 - \sigma_1\bar{p}_1 + \sigma_2\bar{p}_2)^{\dot\alpha}, \quad (B.10)$$

$$W_2^{\eta\bar\eta}{}_{CC\omega} = \frac{\eta\bar\eta}{16}\int\limits_{\tau\,\bar\tau\,\sigma(2)\nu(2)} \nabla(\sigma(2))\nabla(\nu(2)) \times\Big[\Box(\tau,\bar\tau)\mathbf{E}(\Omega_1|\bar\Omega_1) - \int\limits_{\alpha(2)}\nabla(\alpha(2))\Big\{\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_2|\bar\Omega_2) + \mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3|\bar\Omega_3)\Big\}\Big] CC\omega k\bar{k}, \quad (B.11)$$

where

$$\Omega_1{}^{\alpha} := \tau z^{\alpha} - (1-\tau)[(\nu_2+\nu_1\sigma_2) p_0 - \sigma_1 p_1 + \sigma_2 p_2]^{\alpha},$$
$$\bar\Omega_1{}^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[(\nu_2+\nu_1\sigma_1) \bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2]^{\dot\alpha}, \quad (B.12)$$
$$\Omega_2{}^{\alpha} := -[(1-\nu_1\sigma_1)p_0 - \sigma_1 p_1 + \sigma_2 p_2]^{\alpha},$$
$$\bar\Omega_2{}^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)\alpha_2[(1-\nu_1\sigma_2) \bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2]^{\dot\alpha}, \quad (B.13)$$
$$\Omega_3{}^{\alpha} := \tau z^{\alpha} - (1-\tau)\alpha_2[(1-\nu_1\sigma_1) p_0 - \sigma_1 p_1 + \sigma_2 p_2]^{\alpha},$$
$$\bar\Omega_3{}^{\dot\alpha} := -[(1-\nu_1\sigma_2)\bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2]^{\dot\alpha} \quad (B.14)$$

with $\mathbf{E}$ (3.29).

Appendix C: $\Upsilon^{\eta}(\omega, C, C)$

Plugging B_2^{η} (A.7) and W_1 from (A.3) and (A.5) into the equation

$$d_x C + [\omega, C]_* = -d_x B_2^{\eta} - [\omega, B_2^{\eta}]_* - [W_1^{\eta}, C]_* + h.c. + \cdots \quad (C.1)$$

after some simple algebra one finds using IH and definitions (3.30), (3.31)

$$d_x C + [\omega, C]_* = \Upsilon^{\eta}_{\omega CC} + \Upsilon^{\eta}_{CC\omega} + \Upsilon^{\eta}_{C\omega C} + h.c. + \cdots, \quad (C.2)$$

where

$$\Upsilon^{\eta}_{\omega CC} = \frac{i\eta}{4}\int\limits_{\tau,\rho(2)}\mathbb{D}(\tau)\nabla(\sigma(2))\nabla(\rho(2))\mathcal{E}(\Omega_1)\omega CCk, \quad (C.3)$$

$$\Upsilon^{\eta}_{C\omega C} = \frac{i\eta}{4}\int\limits_{\tau,\rho(2)}\mathbb{D}(\tau)\nabla(\sigma(2))\nabla(\rho(2))\mathcal{E}(\Omega_2)C\omega Ck, \quad (C.4)$$

$$\Upsilon^{\eta}_{CC\omega} = \frac{i\eta}{4}\int\limits_{\tau,\rho(2)}\mathbb{D}(\tau)\nabla(\sigma(2))\nabla(\rho(2))\mathcal{E}(\Omega_3)CC\omega k \quad (C.5)$$

with

$$\Omega_1 = -(\rho_1 + \sigma_1\rho_2)p_0 - \sigma_1 p_1 + \sigma_2 p_2, \quad (C.6)$$
$$\Omega_2 = (-\sigma_1\rho_1 + \sigma_2\rho_2)p_0 - \sigma_1 p_1 + \sigma_2 p_2, \quad (C.7)$$
$$\Omega_3 = (\rho_2 + \rho_1\sigma_2)p_0 - \sigma_1 p_1 + \sigma_2 p_2. \quad (C.8)$$

Complex conjugated vertices $\Upsilon^{\bar\eta}$ are analogous.

Appendix D: Solving for $\Upsilon^{\eta\bar\eta}_{\omega CCC}$ in detail

Details of extraction of $\Upsilon_{C\omega CC}$ (6.3) from Eq. (6.2) are presented in Sects. D.1–D.6.

D.1 $W_2^{\eta\bar\eta}|_{\omega CC} * C$

Taking into account $W_2|_{\omega CC}$ (B.1) along with (B.2)–(B.4) one obtains

$$-W_2^{\eta\bar\eta}{}_{\omega CC} * C = \frac{\eta\bar\eta}{16}\int\limits_{\tau\,\bar\tau\sigma(2)\nu(2)}\nabla(\sigma(2))\nabla(\nu(2)) \times\Big[-\Box(\tau,\bar\tau)\mathbf{E}(\Omega_1|\bar\Omega_1) + \int\limits_{\alpha(3)}\nabla(\alpha(2))\Big\{\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_3|\bar\Omega_3) + \mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_2|\bar\Omega_2)\Big\}\Big]\omega CCCk\bar k, \quad (D.1)$$

$$\Omega_1{}^{\alpha} = \tau z^{\alpha} - (1-\tau)[-(\nu_1+\nu_2\sigma_1) p_0 - \sigma_1 p_1 + \sigma_2 p_2 + p_3]^{\alpha},$$
$$\bar\Omega_1{}^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-(\nu_1+\nu_2\sigma_2) \bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2 + \bar p_3]^{\dot\alpha}, \quad (D.2)$$
$$\Omega_2{}^{\alpha} := \tau z^{\alpha} - (1-\tau)(\alpha_1[-(\nu_1+\nu_2\sigma_1) p_0 - \sigma_1 p_1 + \sigma_2 p_2] + p_3)^{\alpha},$$
$$\bar\Omega_2{}^{\dot\alpha} := -[-(\nu_1+\nu_2\sigma_2)\bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2 + \bar p_3]^{\dot\alpha}, \quad (D.3)$$
$$\Omega_3{}^{\alpha} := -[-(\nu_1+\nu_2\sigma_1)p_0 - \sigma_1 p_1 + \sigma_2 p_2 + p_3]^{\alpha},$$
$$\bar\Omega_3{}^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)\big\{\alpha_1[-(\nu_1+\nu_2\sigma_2) \bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2] + \bar p_3\big\}^{\dot\alpha}. \quad (D.4)$$

The 'bulk' term of $-W_2|_{\omega CC} * C$ (D.1), that depends on $\Omega_1, \bar\Omega_1$ (D.2) is canceled by the term of $(d_x + \omega*)B_3^{\eta\bar\eta}|_{blk}$ (D.9) with $\Omega, \bar\Omega$ (D.12) generated by $d(\theta(\sigma_1))$.

The 'boundary' terms of $W_2|_{\omega CC} * C$ (D.1) are considered in Sect. D.6.

One can easily make sure that all terms in (D.1) do satisfy (4.4) thus being MNL.

D.2 $(W_1^{\bar\eta} * B_2^{\eta} + W_1^{\eta} * B_2^{\bar\eta})|_{\omega CCC}$

According to Eqs. (A.1)–(A.10), taking into account Eqs. (3.30), (3.31), (3.29), (5.1) one gets

$$-(W_1^{\bar\eta} * B_2^{\eta} + W_1^{\eta} * B_2^{\bar\eta})|_{\omega CCC} = -\frac{\eta\bar\eta}{16}\int\limits_{\tau\bar\tau\nu(2)\sigma(2)} \nabla(\sigma(2))\nabla(\nu(2))\Box(\tau,\bar\tau) \times\Big[\mathbf{E}(\Omega_1|\bar\Omega_1)+\mathbf{E}(\Omega_2|\bar\Omega_2)\Big]\omega CCCk\bar k, \tag{D.5}$$

$$\Omega_1{}^{\alpha} = \tau z - (1-\tau)[-p_0 - p_1 - \sigma_1 p_2 + \sigma_2 p_3]^{\alpha},\quad \bar\Omega_1{}^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)(-\nu_1\bar p_0 + \bar p_2 + \bar p_3)^{\dot\alpha}, \tag{D.6}$$

$$\Omega_2{}^{\alpha} = \tau z^{\alpha} - (1-\tau)(-\nu_1 p_0 + p_2 + p_3)^{\alpha},\quad \bar\Omega_{2\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-\bar p_0 - \bar p_1 - \sigma_2\bar p_2 + \sigma_1\bar p_3]^{\dot\alpha}. \tag{D.7}$$

Note, that the terms on the *r.h.s.* of (D.5) will be canceled below by terms of (D.9) with $\Omega, \bar\Omega$ (D.11) generated by $\mathrm{d}(\theta(\alpha_1)\theta(\alpha_2))$.

D.3 $(\mathrm{d}_x + \omega *)B_3^{\eta\bar\eta}|_{blk}|_{\omega CCC}$

Using that $\mathrm{d}\mathbf{E} = 0$ one can see that $B_3^{\eta\bar\eta}$ (5.45) yields

$$-(\mathrm{d}_x + \omega *)B_3^{\eta\bar\eta}|_{blk}|_{\omega CCC} = -\frac{\eta\bar\eta}{16} \times \int\limits_{\tau\bar\tau\alpha(2)\sigma(2)\nu(2)} \Big(\mathrm{d}\Big\{\Box(\tau,\bar\tau)\nabla(\alpha(2))\nabla(\sigma(2))\nabla(\nu(2))\Big\} \tag{D.8}$$

$$-\mathrm{d}\Big\{\theta(\alpha_1)\theta(\alpha_2)\theta(\sigma_1)\theta(\sigma_2)\Big\}\mathbb{D}(1-\alpha_1-\alpha_2)\,\mathbb{D}(1-\sigma_1-\sigma_2)\Box(\tau,\bar\tau)\nabla(\nu(2)) \tag{D.9}$$

$$-\mathrm{d}\Big\{\Box(\tau,\bar\tau)\Big\}\nabla(\alpha(2))\nabla(\sigma(2))\nabla(\nu(2))\Big) \times\Big[\mathbf{E}(\Omega_1|\bar\Omega_1)-\mathbf{E}(\Omega_2|\bar\Omega_2)\Big]\omega CCCk\bar k, \tag{D.10}$$

where

$$\Omega_1^{\alpha} := \tau z^{\alpha} - (1-\tau)[-(\nu_2+\nu_1\alpha_2)\,p_0 - \alpha_2(p_1+p_2) + (1-\alpha_2\sigma_1)(p_3+p_2)]^{\alpha},$$
$$\bar\Omega_1^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-(\nu_2+\nu_1\alpha_1)\,\bar p_0 - \alpha_1(\bar p_1+\bar p_2) + (1-\alpha_1\sigma_1)(\bar p_3+\bar p_2)]^{\dot\alpha}, \tag{D.11}$$

$$\Omega_2^{\alpha} := \tau z^{\alpha} - (1-\tau)[-(1+\nu_1(-\alpha_2\sigma_2))\,p_0 - (1-\alpha_2\sigma_2)(p_1+p_2) + \alpha_2(p_3+p_2)]^{\alpha},$$
$$\bar\Omega_2^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-(1+\nu_1(-\alpha_1\sigma_2))\,\bar p_0 - (1-\alpha_1\sigma_2)(\bar p_1+\bar p_2) + \alpha_1(\bar p_3+\bar p_2)]^{\dot\alpha}. \tag{D.12}$$

One can see that nontrivial 'bulk' terms of (D.9) either cancel each other or cancel $-W_1^{\bar\eta} * B_2^{\eta}|_{\omega CCC}$ (D.6), $-W_1^{\eta} * B_2^{\bar\eta}$ (D.7) and the 'bulk' term of $-W_2^{\eta\bar\eta} * C|_{\omega CCC}$ (D.2). Hence all 'bulk' terms on the *r.h.s.* of (2.14) in the sector under consideration vanish. The next step is to consider 'boundary' terms.

Note that since $B_3^{\eta\bar\eta}$ satisfies (4.4), $\mathcal{D}B_3^{\eta\bar\eta}(\omega, C, C, C)$ satisfies it as well.

D.4 $(\mathrm{d}_x + \omega *)B_3^{\eta\bar\eta}|_{bnd}|_{\omega CCC}$

From (5.46) along with (5.43), (5.44) it follows

$$-(\mathrm{d}_x + \omega *)B_3^{\eta\bar\eta}|_{bnd}|_{\omega CCC} = \frac{\eta\bar\eta}{16}\int\limits_{\tau\bar\tau\nu(2)\alpha(2)\xi(3)} \mathrm{d}\Big\{\mu_1 \times\Big[\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3|\bar\Omega_3) - \mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_4|\bar\Omega_4)\Big]\Big\}\,\omega CCCk\bar k \tag{D.13}$$

$$-\eta\bar\eta 16\int\limits_{\tau\bar\tau\nu(2)\alpha(2)\xi(3)} \mathbb{D}(1-\alpha_1-\alpha_2)\nabla(\nu(2))\mathbb{D}(1-\xi_1-\xi_2-\xi_3) \times\mathrm{d}\Big\{\theta(\xi_1)\theta(\xi_2)\theta(\xi_3)\theta(\alpha_1)\theta(\alpha_2)\Big\} \times\Big[\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3|\bar\Omega_3) - \mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_4|\bar\Omega_4)\Big]\,\omega CCCk\bar k \tag{D.14}$$

$$-\eta\bar\eta 16\int\limits_{\tau\bar\tau\nu(2)\alpha(2)\xi(3)} \mathbb{D}(\tau)\mathbb{D}(\bar\tau)\mu_1 \times\Big[\mathbf{E}(\Omega_3|\bar\Omega_3)+\mathbf{E}(\Omega_4|\bar\Omega_4)\Big]\omega CCCk\bar k \tag{D.15}$$

with $\mu_1 = \nabla(\alpha(2))\nabla(\nu(2))\nabla(\xi(3))$ and

$$\Omega_3^{\alpha} := \tau z^{\alpha} - (1-\tau)\Big[-(\nu_2+\nu_1\{\alpha_1\xi_2(1-\xi_1)^{-1}+\xi_3\})p_0 - \{\alpha_1\xi_2(1-\xi_1)^{-1}+\xi_3\}(p_1+p_2) + \{\alpha_2\xi_2(1-\xi_3)^{-1}+\xi_1\}(p_3+p_2)\Big]^{\alpha},$$
$$\bar\Omega_3^{\dot\alpha} := -[-(\nu_2+\nu_1(1-\xi_3))\bar p_0 - (1-\xi_3)(\bar p_1+\bar p_2) + (1-\xi_1)(\bar p_3+\bar p_2)]^{\dot\alpha}, \tag{D.16}$$

$$\Omega_4^{\alpha} := -[-(\nu_2+\nu_1(1-\xi_3))p_0 - (1-\xi_3)(p_1+p_2) + (1-\xi_1)(p_3+p_2)]^{\alpha},$$
$$\bar\Omega_4^{\dot\alpha} := \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[-(\nu_2+\nu_1\{\alpha_1\xi_2(1-\xi_1)^{-1}+\xi_3\})\bar p_0 - \{\alpha_1\xi_2(1-\xi_1)^{-1}+\xi_3\}(\bar p_1+\bar p_2) + \Big\{\alpha_2\xi_2(1-\xi_3)^{-1}+\xi_1\Big\}(\bar p_3+\bar p_2)]^{\dot\alpha}. \tag{D.17}$$

One can see that the terms of (D.14) generated by $\mathrm{d}\big\{\theta(\alpha_1)\theta(\alpha_2)\big\}$ cancel against the respective 'boundary' terms of (D.10) by (5.33)-like changes of variables. The rest nontrivial terms of (D.14), namely $\mathrm{d}\theta(\xi_1)$-dependent ones, will be considered in Sect. D.6.

Note, that cohomology terms (D.15) are represented in Eq. (6.3) of Sect. 6.1.

D.5 $\left(\mathrm{d}_x B_2^{\bar\eta} + \mathrm{d}_x B_2^{\eta}\right)|_{\omega CCC}$

By virtue of (C.2), taking into account (C.3) and its conjugated, one obtains from (A.7) and (A.9)

$$-\left(\mathrm{d}_x B_2^{\bar\eta} + \mathrm{d}_x B_2^{\eta}\right)|_{\omega CCC} = \frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\sigma(2)} \nabla(\alpha(2))\nabla(\sigma(2))\nabla(\nu(2)) \Big[-\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_1|\bar\Omega_1) + \mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_2|\bar\Omega_2)\Big] \omega CCCk\bar k, \tag{D.18}$$

$$\begin{aligned}\Omega_1{}^{\alpha} &= -[-(1-\sigma_2\nu_2)p_0 - \sigma_2 p_1 + \sigma_1 p_2 + p_3]^{\alpha}\\ \bar\Omega_1{}^{\dot\alpha} &= \bar\tau\bar z^{\dot\alpha} -(1-\bar\tau)[-\alpha_1(\bar p_0+\bar p_1+\bar p_2)+\alpha_2\bar p_3]^{\dot\alpha},\end{aligned} \tag{D.19}$$

$$\begin{aligned}\Omega_2{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)[-\alpha_1 (p_0+p_1+p_2)+\alpha_2 p_3]^{\alpha}\\ \bar\Omega_2{}^{\dot\alpha} &= [-(1-\sigma_2\nu_2)\bar p_0 - \sigma_2\bar p_1 + \sigma_1\bar p_2 + \bar p_3]^{\dot\alpha}.\end{aligned} \tag{D.20}$$

These terms are considered in Sect. D.6. One can easily make sure that $\left(\mathrm{d}_x B_2^{\bar\eta} + \mathrm{d}_x B_2^{\eta}\right)|_{\omega CCC}$ (D.18) satisfies (4.4), thus being MNL.

D.6 The rest cohomology terms

Here we consider the rest 'boundary' terms at $\bar\tau = 0$ dependent on $\Omega, \bar\Omega$ of the form (D.3), (D.16) at $\xi_1 = 0$ and (D.20), as well as the 'boundary' terms at $\tau = 0$ dependent on $\Omega, \bar\Omega$ of the form (D.4), (D.17) at $\xi_1 = 0$ and (D.19).

To obtain rest cohomology terms from those with $\bar\tau = 0$ consider

$$\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\alpha(2)\beta(2)\sigma(2)} \mathrm{d}\Big\{\mathbb{D}(\bar\tau)l(\tau)\nabla(\alpha(2))\nabla(\sigma(2)) \nabla(\nu(2))\nabla(\beta(2))\mathbf{E}(\Omega|\bar\Omega)\omega CCCk\bar k\Big\} \tag{D.21}$$

with

$$\begin{aligned}\Omega^{\alpha} &= \tau z^{\alpha} - (1-\tau)\Big[\alpha_2\big\{\beta_1[-(1-\nu_2\sigma_2)p_0 -\sigma_1 p_1 + \sigma_2 p_2] + p_3\big\} -\beta_1\alpha_1(p_1+p_2+p_0)+\alpha_1\beta_2 p_3\Big]^{\alpha}\\ \bar\Omega^{\dot\alpha} &= -[-(\nu_1+\nu_2\sigma_2) \bar p_0 - \sigma_2(\bar p_1+\bar p_2)+(\bar p_3+\bar p_2)]^{\dot\alpha}.\end{aligned} \tag{D.22}$$

Expression (D.21) is in the dZ-independent sector and hence gives zero as an integral of an exact form. (Recall, that in this sector we discard dZ-dependent weak terms.)

Differentiation in (D.21) gives the cohomology term (with a sign "–") of (6.3) that depends on $\Omega_3, \bar\Omega_3$ (6.6) along with all the rest 'boundary' terms with $\bar\tau = 0$. Namely, the term ($\sim \mathrm{d}(\theta(\beta_2))$) equals to the term in (D.14) that depends on $\Omega_3, \bar\Omega_3$ (D.16) at $\xi_1 = 0$, the term ($\sim \mathrm{d}(\theta(\alpha_1))$) equals to a part of $-W_2^{\eta\bar\eta} * C|_{\omega CCC}$ (D.1) that depends on $\Omega, \bar\Omega$ of the form (D.3) while that ($\sim \mathrm{d}(\theta(\alpha_2))$) equals to the part of $-d_x B_2^{\eta}|_{\omega CCC}$ (D.18) that depends on $\Omega, \bar\Omega$ (D.20). Note that the term $\sim \mathrm{d}(\theta(\beta_1))$ is weak since $\Omega^{\alpha}|_{\beta_1=0} = \tau z^{\alpha} - (1-\tau)p_3^{\alpha}$.

Analogously, differentiation in the following expression

$$-\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\alpha(2)\beta(2)\sigma(2)} d\Big\{\mathbb{D}(\tau)l(\bar\tau)\nabla(\alpha(2))\nabla(\sigma(2)) \nabla(\nu(2))\nabla(\beta(2))\mathbf{E}(\Omega|\bar\Omega)\omega CCCk\bar k\Big\} \tag{D.23}$$

with

$$\begin{aligned}\Omega^{\alpha} &= -[-(\nu_1+\nu_2\sigma_2)p_0 - \sigma_2(p_1+p_2)+(p_3+p_2)]^{\alpha},\\ \bar\Omega^{\dot\alpha} &= \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)\Big[\alpha_2\big\{\beta_1[-(1-\nu_2\sigma_2) \bar p_0 - \sigma_1\bar p_1 + \sigma_2\bar p_2] + \bar p_3\big\} -\beta_1\alpha_1(\bar p_1+\bar p_2+\bar p_0)+\alpha_1\beta_2\bar p_3\Big]^{\dot\alpha}\end{aligned} \tag{D.24}$$

gives all the rest 'boundary' terms with $\tau = 0$ plus a cohomological one. Namely we obtain the cohomology term (with a sign "-") of (6.3), that depends on $\Omega_4, \bar\Omega_4$ (6.7), along with the term of $(\mathrm{d}_x + \omega *)B_3^{\eta\bar\eta}|_{bnd}|_{\omega CCC}$ that depends on $\Omega_4, \bar\Omega_4$ (D.17) at $\xi_1 = 0$, the term of $W_2 * C|_{\omega CCC}$, that depends on $\Omega_3, \bar\Omega_3$ (D.4) and $\mathrm{d}_x B_2^{\bar\eta}|_{\omega CCC}$, that depends on $\Omega_1, \bar\Omega_1$ (D.19). Note that the expressions (D.24) result from the application of MNL preserving IH to $-W_2 * C|_{\omega CCC}$ and $-\mathrm{d}_x B_2^{\bar\eta}|_{\omega CCC}$, which are MNL (see (D.4), (D.19)).

Thus all cohomological terms in the sector ωCCC are extracted from Eqs. (D.15), (D.21) and (D.23) yielding $\Upsilon^{\eta\bar\eta}|_{\omega CCC}$ (6.3).

Appendix E: Solving for $\Upsilon^{\eta\bar\eta}_{C\omega CC}$ in detail

Details of derivation of $\Upsilon_{C\omega CC}$ (6.9) from Eq. (6.8) are presented in Sects. E.1–E.8.

E.1 $C * (W_2|_{\omega CC})$

Using $W_2|_{\omega CC}$ (B.1) along with (B.2)–(B.4) one obtains

$$C * (W_2|_{\omega CC}) = \frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)\nu(2)} \mu_1\Box(\tau,\bar\tau)\mathbf{E}(\Omega_1|\bar\Omega_1)C\omega CCk\bar k \tag{E.1}$$

$$-\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\sigma(2)\nu(2)\alpha(2)} \mu_1\mu_2\Big\{\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_2|\bar\Omega_2) +\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3|\bar\Omega_3)\Big\}C\omega CCk\bar k \tag{E.2}$$

with $\mu_1 = \nabla(\sigma(2))\nabla(\nu(2))\,, \qquad \mu_2 = \nabla(\alpha(2))\,,$

$$\begin{aligned}
\Omega_1{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)[-(\nu_1+\nu_2\sigma_1)\\
&\quad p_0 - p_1 - \sigma_1 p_2 + \sigma_2 p_3]^{\alpha},\\
\bar{\Omega}_1{}^{\dot\alpha} &= \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[-(\nu_1+\nu_2\sigma_2)\\
&\quad \bar p_0 - \bar p_1 - \sigma_2 \bar p_2 + \sigma_1 \bar p_3]^{\dot\alpha}, \qquad (E.3)\\
\Omega_2{}^{\alpha} &= -[-(\nu_1+\nu_2\sigma_1)p_0 - p_1 - \sigma_1 p_2 + \sigma_2 p_3]^{\alpha},\\
\bar{\Omega}_2{}^{\dot\alpha} &= \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[-\alpha_1(\nu_1+\nu_2\sigma_2)\\
&\quad \bar p_0 - \bar p_1 - \alpha_1\sigma_2 \bar p_2 + \alpha_1\sigma_1 \bar p_3]^{\dot\alpha}, \qquad (E.4)\\
\Omega_3{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)[-\alpha_1(\nu_1+\nu_2\sigma_1)\\
&\quad p_0 - p_1 - \alpha_1\sigma_1 p_2 + \alpha_1\sigma_2 p_3]^{\alpha},\\
\bar{\Omega}_3{}^{\dot\alpha} &= -[-(\nu_1+\nu_2\sigma_2)\bar p_0 - \bar p_1 - \sigma_2 \bar p_2\\
&\quad +\sigma_1 \bar p_3]^{\dot\alpha}. \qquad (E.5)
\end{aligned}$$

Note that the term in $C*(W_2|_{\omega CC})$, that depends on $\Omega_1,\ \bar\Omega_1$ (E.3), is cancelled in Sect. E.4. The 'boundary' terms dependent on $\Omega_2,\ \bar\Omega_2$ (E.4) and $\Omega_3,\ \bar\Omega_3$ (E.5) are considered in Sect. E.8.

E.2 $W_2|_{C\omega C} * C$

Using $W_2|_{C\omega C}$ (B.5) along with (B.6)–(B.10) one obtains

$$-W_{2C\omega C} * C = -\frac{\eta\bar\eta}{16}\int_{\tau,\bar\tau,\rho(2),\sigma(2)} \mu_1\Box(\tau,\bar\tau)\Big\{\mathbf{E}(\Omega_0|\bar\Omega_0)$$
$$+\mathbf{E}(\Omega_1|\bar\Omega_1) + \mathbf{E}(\Omega_2|\bar\Omega_2)\Big\}C\omega CCk\bar k, \qquad (E.6)$$
$$-\frac{\eta\bar\eta}{16}\int_{\tau,\bar\tau,\nu(2),\sigma(2),\xi(2)} \mu_1\mu_2\Big\{\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_3|\bar\Omega_3)$$
$$-\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_4|\bar\Omega_4)\Big\}C\omega CCk\bar k \qquad (E.7)$$

with $\mu_1 = \nabla(\sigma(2))\nabla(\nu(2))\,, \qquad \mu_2 = \nabla(\alpha(2)),$

$$\begin{aligned}
\Omega_0{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)[(-\sigma_1\nu_1+\sigma_2\nu_2)\\
&\quad p_0 - \sigma_1 p_1 + \sigma_2 p_2 + p_3]^{\alpha},\\
\bar\Omega_0{}^{\dot\alpha} &= \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[(-\sigma_2\nu_1+\sigma_1\nu_2)\bar p_0\\
&\quad -\sigma_2 \bar p_1 + \sigma_1 \bar p_2 + \bar p_3]^{\dot\alpha}, \qquad (E.8)\\
\Omega_1{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)[(-\sigma_1\nu_2-\nu_1)p_0 - p_1 + p_3]^{\alpha},\\
\bar\Omega_1{}^{\dot\alpha} &= \bar\tau \bar z_1{}^{\dot\alpha} - (1-\bar\tau)[(\sigma_1\nu_1+\nu_2)\bar p_0 + \bar p_2 + \bar p_3]^{\dot\alpha},\\
&\qquad (E.9)\\
\Omega_2{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)\\
&\quad \times[(\sigma_2\nu_1+\nu_2)p_0 + p_2 + p_3]^{\alpha},\\
\bar\Omega_2{}^{\dot\alpha} &= \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)\\
&\quad \times[-(\sigma_2\nu_2+\nu_1)\bar p_0 - \bar p_1 + \bar p_3]^{\dot\alpha}, \qquad (E.10)\\
\Omega_3{}^{\alpha} &= -([\nu_2\sigma_2-\nu_1\sigma_1]p_0\\
&\quad -\sigma_1 p_1 + \sigma_2 p_2 + p_3)^{\alpha},\\
\bar\Omega_3{}^{\dot\alpha} &= \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)\{\xi_1([\nu_2\sigma_1-\nu_1\sigma_2]\\
&\quad \bar p_0 - \sigma_2 \bar p_1 + \sigma_1 \bar p_2) + \bar p_3\}^{\dot\alpha}, \qquad (E.11)\\
\Omega_4{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)\{\xi_1([\nu_2\sigma_1-\nu_1\sigma_2]\\
&\quad p_0 - \sigma_2 p_1 + \sigma_1 p_2) + p_3\}^{\alpha},\\
\bar\Omega_4{}^{\dot\alpha} &= -([\nu_2\sigma_2-\nu_1\sigma_1]\bar p_0\\
&\quad -\sigma_1 \bar p_1 + \sigma_2 \bar p_2 + \bar p_3)^{\dot\alpha}. \qquad (E.12)
\end{aligned}$$

Note that the $\Omega_0, \bar\Omega_0$ dependent term (E.8) cancels against that proportional to $\mathrm{d}\big[\theta(\sigma_1)\big]$ of $\mathrm{d}B_3^{\eta\bar\eta}|_{blk}|_{C\omega CC}$ (E.17) that depends on $\Omega, \bar\Omega$ (E.23), while the terms dependent on $\Omega, \bar\Omega$(E.9) and (E.10) are considered in Sect. E.7. The terms dependent on $\Omega, \bar\Omega$ (E.11) and (E.12) are considered in Sect. E.8.

E.3 $\left(W_1^{\bar\eta} * B_2^{\eta} + W_1^{\eta} * B_2^{\bar\eta}\right)|_{C\omega CC}$

According to (A.5), (A.7) and (A.9)

$$-\big(W_1^{\bar\eta} * B_2^{\eta} + W_1^{\eta} * B_2^{\bar\eta}\big)|_{C\omega CC}$$
$$= -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\rho(2)\sigma(2)} \mu\ \Box(\tau,\bar\tau)\Big[\mathbf{E}(\Omega_1|\bar\Omega_1) - \mathbf{E}(\Omega_2|\bar\Omega_2)\Big]$$
$$\times C\omega CCk\bar k, \qquad (E.13)$$

with $\mu = \nabla(\sigma(2))\nabla(\rho(2)),$

$$\begin{aligned}
\Omega_1{}^{\alpha} &= \tau z^{\alpha} - (1-\tau)[-(p_0+p_1) - \rho_1 p_2 + \rho_2 p_3]^{\alpha},\\
\bar\Omega_1{}^{\dot\alpha} &= \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)(\sigma_1 \bar p_0 + \bar p_2 + \bar p_3)^{\dot\alpha}, \qquad (E.14)\\
\Omega_2{}^{\alpha} &:= \tau z^{\alpha} - (1-\tau)(\sigma_1 p_0 + p_2 + p_3)^{\alpha},\\
\bar\Omega_{2\,\dot\alpha} &= \bar\tau \bar z - (1-\bar\tau)\\
&\quad \times[-(\bar p_0 + \bar p_2 + \bar p_1) + \rho_1(\bar p_2 + \bar p_3)]^{\dot\alpha}. \qquad (E.15)
\end{aligned}$$

The term (E.13) is considered in Sect. E.7.

E.4 $\mathrm{d}_x B_3|_{blk}|_{C\omega CC}$

Using that $\mathrm{d}\mathbf{E} = 0$ one can see that Eq. (5.45) yields

$$-\mathrm{d}B_3^{\eta\bar\eta}|_{blk}|_{C\omega CC} = -\frac{\eta\bar\eta}{16}$$
$$\int_{\tau\bar\tau\,\nu(2)\sigma(2)\xi(2)} \Big(\mathrm{d}\Big\{\Box(\tau,\bar\tau)\mu_1\mu_2\Big\} \qquad (E.16)$$
$$-\mathrm{d}\Big\{\mu_1\Big\}\Box(\tau,\bar\tau)\mu_2 \qquad (E.17)$$
$$-\mathrm{d}\Big\{\Box(\tau,\bar\tau)\Big\}\mu_1\mu_2\Big)\Big[\mathbf{E}(\Omega_1|\bar\Omega_1) - \mathbf{E}(\Omega_2|\bar\Omega_2)\Big]$$
$$\omega CCCk\bar k, \qquad (E.18)$$

with $\mu_1 = \nabla(\sigma(2))\nabla(\nu(2))\,, \mu_2 = \nabla(\xi(2))\,,$

$$\begin{aligned}
\Omega_1^{\alpha} &:= \tau z^{\alpha} - (1-\tau)[(\xi_1(1-\nu_2\sigma_1) - \nu_2)\\
&\quad p_0 - \nu_2(p_1+p_2) + (1-\nu_2\sigma_1)(p_3+p_2)]^{\alpha},\\
\bar\Omega_1^{\dot\alpha} &:= \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[(\xi_1(1-\nu_1\sigma_1) - \nu_1)\\
&\quad \bar p_0 - \nu_1(\bar p_1+\bar p_2) + (1-\nu_1\sigma_1)(\bar p_3+\bar p_2)]^{\dot\alpha}, \qquad (E.19)
\end{aligned}$$

$$\Omega_2^\alpha := \tau z^\alpha - (1-\tau)[(-(1-\nu_2\sigma_2)+\xi_1\nu_2) p_0 - (1-\nu_2\sigma_2)(p_1+p_2)+\nu_2(p_3+p_2)]^\alpha,$$
$$\bar\Omega_2^{\dot\alpha} := \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[(-(1-\nu_1\sigma_2)+\xi_1\nu_1) \bar p_0 - (1-\nu_1\sigma_2)(\bar p_1+\bar p_2)+\nu_1(\bar p_3+\bar p_2)]^{\dot\alpha}. \tag{E.20}$$

Non-zero terms of (E.17) are those that depend on

$$\Omega_1^\alpha|_{\nu_1=0} = \tau z^\alpha - (1-\tau)[(\xi_1\sigma_2-1)p_0 - (p_1+p_2)+\sigma_2(p_3+p_2)]^\alpha,$$
$$\bar\Omega_1^{\dot\alpha}|_{\nu_1=0} = \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[(\xi_1) \bar p_0 + (\bar p_3+\bar p_2)]^{\dot\alpha}, \tag{E.21}$$
$$\Omega_1^\alpha|_{\nu_2=0} = \tau z^\alpha - (1-\tau)[(\xi_1)p_0 + (p_3+p_2)]^\alpha,$$
$$\bar\Omega_1^{\dot\alpha}|_{\nu_2=0} = \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[(\xi_1\sigma_2-1) \bar p_0 - (\bar p_1+\bar p_2)+\sigma_2(\bar p_3+\bar p_2)]^{\dot\alpha}, \tag{E.22}$$
$$\Omega_1^\alpha|_{\sigma_1=0} = \tau z^\alpha - (1-\tau)[(\xi_1-\nu_2) p_0 - \nu_2(p_1+p_2)+(p_3+p_2)]^\alpha,$$
$$\bar\Omega_1^{\dot\alpha}|_{\sigma_1=0} = \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[(\xi_1-\nu_1) \bar p_0 - \nu_1(\bar p_1+\bar p_2)+(\bar p_3+\bar p_2)]^{\dot\alpha}, \tag{E.23}$$
$$\Omega_2^\alpha|_{\sigma_2=0} = \tau z^\alpha - (1-\tau)[(-1+\xi_1\nu_2)p_0 -(p_1+p_2)+\nu_2(p_3+p_2)]^\alpha,$$
$$\bar\Omega_2^{\dot\alpha}|_{\sigma_2=0} = \bar\tau \bar z^{\dot\alpha} - (1-\bar\tau)[(-1+\xi_1\nu_1) \bar p_0 - (\bar p_1+\bar p_2)+\nu_1(\bar p_3+\bar p_2)]^{\dot\alpha}. \tag{E.24}$$

Note, that the sum of the terms on the *r.h.s.* of (E.17) dependent on $(\Omega_1\,,\bar\Omega_1)|_{\sigma_2=0}$ and $(\Omega_2\,,\bar\Omega_2)|_{\sigma_2=0}$ gives zero. The term of (E.17), that depends on $\Omega_2|_{\sigma_2=0}\,,\ \bar\Omega_2|_{\sigma_2=0}$ (E.24) cancels the term of $C * (W_2|_{\omega CC})$ (E.3), while the term that depends on $\Omega_1|_{\sigma_1=0},\ \bar\Omega_1|_{\sigma_1=0}$ (E.23) cancels the term of $-W_{2C\omega C} * C$ (E.6), that depends on $\Omega,\bar\Omega$ (E.8).

The non-zero 'boundary' terms of (E.18) are cancelled by the respective terms of (E.26) from $\mathrm{d}\{\theta(\alpha_1)\theta(\alpha_2)\}$ as can be seen with the help of the (5.33)-like changes of variables.

The rest non-zero $(\Omega_1,\ \bar\Omega_1)|_{\nu_{1,2}=0}$-dependent terms of (E.17) associated with (E.21), (E.22) are considered in Sect. E.7.

E.5 $\mathrm{d}_x B_3|_{bnd}|_{C\omega CC}$

From (5.46) along with (5.43), (5.44) it follows

$$\mathrm{d}_x B_3|_{bnd}|_{C\omega CC} = -\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\rho(2)\xi(3)} \mathrm{d}\Big\{\nabla(\rho(2))\nabla(\nu(2))\nabla(\xi(3)) \times\Big[\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3|\bar\Omega_3)+\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_4|\bar\Omega_4)\Big]\Big\} \times C\omega CCk\bar k \tag{E.25}$$
$$-\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\rho(2)\xi(3)} \times\mathbb{D}(1-\rho_1-\rho_2)\nabla(\nu(2))\mathbb{D}(1-\xi_1-\xi_2-\xi_3) \times\mathrm{d}\Big\{\theta(\xi_1)\theta(\xi_2)\theta(\xi_3)\theta(\rho_1)\theta(\rho_2)\Big\} \Big[\mathbb{D}(\bar\tau)l(\tau)\mathbf{E}(\Omega_3|\bar\Omega_3)\ +\mathbb{D}(\tau)l(\bar\tau)\mathbf{E}(\Omega_4|\bar\Omega_4)\Big] \times C\omega CCk\bar k \tag{E.26}$$
$$-\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\nu(2)\rho(2)\xi(3)} \mathbb{D}(\tau)\mathbb{D}(\bar\tau)\nabla(\rho(2))\nabla(\nu(2))\nabla(\xi(3)) \Big[\mathbf{E}(\Omega_3|\bar\Omega_3)-\mathbf{E}(\Omega_4|\bar\Omega_4)\Big]C\omega CCk\bar k, \tag{E.27}$$
$$\Omega_3{}^\alpha = \tau z^\alpha - (1-\tau)\Big[(\nu_1\{\rho_2\xi_2(1-\xi_3)^{-1}+\xi_1\} -\{\rho_1\xi_2(1-\xi_1)^{-1}+\xi_3\})p_0 -\{\rho_1\xi_2(1-\xi_1)^{-1}+\xi_3\}(p_1+p_2) +\{\rho_2\xi_2(1-\xi_3)^{-1}+\xi_1\}(p_3+p_2)\Big]^\alpha,$$
$$\bar\Omega_3{}^{\dot\alpha} := -[(\nu_1(1-\xi_1)-(1-\xi_3)) \bar p_0 - (1-\xi_3)(\bar p_1+\bar p_2)+(1-\xi_1)(\bar p_3+\bar p_2)]^{\dot\alpha}, \tag{E.28}$$

$$\Omega_4{}^\alpha := -[(\nu_1(1-\xi_1)-(1-\xi_3)) p_0 - (1-\xi_3)(p_1+p_2)+(1-\xi_1)(p_3+p_2)]^\alpha,$$
$$\bar\Omega_4{}^{\dot\alpha} = \bar\tau\bar z^{\dot\alpha} - (1-\bar\tau)[(\nu_1\{\rho_1\xi_2(1-\xi_3)^{-1} +\xi_1\} - \{\xi_3+\rho_2\xi_2(1-\xi_1)^{-1}\})\bar p_0 -\{\xi_3+\rho_2\xi_2(1-\xi_1)^{-1}\}(\bar p_1+\bar p_2) +\{\rho_1\xi_2(1-\xi_3)^{-1}+\xi_1\}(\bar p_3+\bar p_2)]^{\dot\alpha}. \tag{E.29}$$

Since dZ-dependent terms do not contribute to this sector (are weak), d-exact terms (E.25) do not contribute to the final result as well.

As mentioned above, the terms of (E.26) generated by $\mathrm{d}\{\theta(\rho_1)\theta(\rho_2)\}$ cancel against non-zero 'boundary' terms of (E.18) through (5.33)-like changes of variables. The rest non-zero terms of (E.26) generated by $\mathrm{d}\{\theta(\xi_1)\theta(\xi_3)\}$ are considered in Sect. E.8.

Note that the cohomology terms (E.27) are presented in Sect. 6.2 as those dependent on $\Omega\,,\bar\Omega$ (6.10) and (6.11).

E.6 $\mathrm{d}_x B_2|_{C\omega CC}$

By virtue of (C.2), taking into account (C.3) and (C.4) and their conjugates, one obtains from (A.7) and (A.9)

$$\mathrm{d}_x\big(B_2^{\bar\eta}+B_2^{\eta}\big)|_{C\omega CC} = \frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\xi(2)\sigma(2)\rho(2)} \mu\Big[\mathbb{D}(\bar\tau)l(\tau) \big\{-\mathbf{E}(\Omega_1|\bar\Omega_1)+\mathbf{E}(\Omega_2|\bar\Omega_2)\big\} +\mathbb{D}(\tau)l(\bar\tau)\big\{\mathbf{E}(\Omega_3|\bar\Omega_3)-\mathbf{E}(\Omega_4|\bar\Omega_4)\big\}\Big]C\omega CCk\bar k \tag{E.30}$$

with $\mu=\nabla(\sigma(2))\nabla(\rho(2))\nabla(\xi(2))$ and

$$\Omega^\alpha{}_1 = \tau z^\alpha - (1-\tau)[\sigma_2 p_0 - \sigma_1 p_1 +\sigma_2 p_2+\sigma_2 p_3]^\alpha,$$
$$\bar\Omega^{\dot\alpha}{}_1 = -[-(\xi_1\rho_2+\rho_1)\bar p_0$$

$$-\bar{p}_1 - \xi_1 \bar{p}_2 + \xi_2 \bar{p}_3]^{\dot\alpha}, \qquad \text{(E.31)}$$

$$\Omega^{\alpha}{}_2 = \tau z^{\alpha} - (1-\tau)[-\sigma_1 p_0 - \sigma_1 p_1 - \sigma_1 p_2 + \sigma_2 p_3]^{\alpha},$$
$$\bar\Omega^{\dot\alpha}{}_2 = -[(-\xi_1\rho_1 + \xi_2\rho_2)\bar{p}_0 - \xi_1\bar{p}_1 + \xi_2\bar{p}_2 + \bar{p}_3]^{\dot\alpha}, \qquad \text{(E.32)}$$

$$\Omega^{\alpha}{}_3 = -[-(\xi_1\rho_2 + \rho_1)p_0 - p_1 - \xi_1 p_2 + \xi_2 p_3]^{\alpha},$$
$$\bar\Omega^{\dot\alpha}{}_3 = \bar\tau\bar{z}^{\dot\alpha} - (1-\bar\tau) \times[\sigma_2\bar{p}_0 - \sigma_1\bar{p}_1 + \sigma_2(\bar{p}_2 + \bar{p}_3)]^{\dot\alpha}, \qquad \text{(E.33)}$$

$$\Omega^{\alpha}{}_4 = -[-(\xi_1\rho_1 - \xi_2\rho_2)p_0 - \xi_1 p_1 + \xi_2 p_2 + p_3]^{\alpha},$$
$$\bar\Omega^{\dot\alpha}{}_4 = \bar\tau\bar{z}^{\dot\alpha} - (1-\bar\tau) \times[-\sigma_1\bar{p}_0 - \sigma_1\bar{p}_1 - \sigma_1\bar{p}_2 + \sigma_2\bar{p}_3]^{\dot\alpha}. \qquad \text{(E.34)}$$

These terms are considered in Sect. E.8.

E.7 Rest 'bulk' terms

Taking into account that the leftover non-zero 'bulk' terms of Sects. E.2–E.4 [(namely, those dependent on $\Omega, \bar\Omega$ (E.9), (E.14), (E.21), (E.10), (E.15), (E.22)] are spin-local, one can straightforwardly make sure that the sum of these terms equals to a total differential that gives zero modulo the 'boundary' terms (E.36):

$$\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\rho(2)\xi(2)\sigma(2)} \Big\{\mathrm{d}\Big[\Box(\tau,\bar\tau)\nabla(\sigma(2))\nabla(\rho(2))\nabla(\xi(2))\Big] \qquad \text{(E.35)}$$

$$-\mathrm{d}\Big[\Box(\tau,\bar\tau)\Big]\nabla(\sigma(2))\nabla(\rho(2))\nabla(\xi(2))\Big\}$$
$$\Big\{\mathbf{E}(\Omega_1|\bar\Omega_1) - \mathbf{E}(\Omega_2|\bar\Omega_2)\Big\}C\omega CCk\bar{k}, \qquad \text{(E.36)}$$

$$\Omega_1^{\alpha} := \tau z^{\alpha} - (1-\tau)[(\xi_2\rho_2\sigma_2 - 1) p_0 - p_1 + p_3 - \sigma_1(p_3 + p_2)]^{\alpha},$$
$$\bar\Omega_1^{\dot\alpha} := \bar\tau\bar{z}^{\dot\alpha} - (1-\bar\tau) [(\xi_1\rho_1 + \xi_2)\bar{p}_0 + (\bar{p}_3 + \bar{p}_2)]^{\dot\alpha}, \qquad \text{(E.37)}$$

$$\Omega_2^{\alpha} := \bar\tau z^{\alpha} - (1-\tau)[(\xi_1\rho_1 + \xi_2)p_0 + (p_3 + p_2)^{\alpha}],$$
$$\bar\Omega_2^{\dot\alpha} := \tau z^{\alpha} - (1-\tau) [(\xi_2\rho_2\sigma_2 - 1)\bar{p}_0 - \bar{p}_1 + \bar{p}_3 - \sigma_1(\bar{p}_3 + \bar{p}_2)]^{\dot\alpha}. \qquad \text{(E.38)}$$

Note, that the terms (E.35) and (E.36) are spin-local.

Hence, at this stage, all 'bulk' terms cancel. We are left with the 'boundary' terms of (E.36). The non-zero ones are proportional to $\mathbb{D}(\tau)$ or $\mathbb{D}(\bar\tau)$, namely those dependent on

$$\Omega_1^{\alpha}|_{\tau=0} = -[(\xi_2\rho_2\sigma_2 - 1)p_0 - p_1 + p_3 - \sigma_1(p_3 + p_2)]^{\alpha},$$
$$\bar\Omega_1^{\dot\alpha} = \bar\tau\bar{z}^{\dot\alpha} - (1-\bar\tau)[(1-\xi_1\rho_2)\bar{p}_0 + (\bar{p}_3 + \bar{p}_2)]^{\dot\alpha}, \qquad \text{(E.39)}$$
$$\Omega_2^{\alpha} := \tau z^{\alpha} - (1-\tau)[(1-\xi_1\rho_2)p_0 + (p_3 + p_2)]^{\alpha},$$
$$\bar\Omega_2^{\dot\alpha}|_{\bar\tau=0} = -[(\xi_2\rho_2\sigma_2 - 1)\bar{p}_0 - \bar{p}_1 + \bar{p}_3 - \sigma_1(\bar{p}_3 + \bar{p}_2)]^{\dot\alpha}, \qquad \text{(E.40)}$$

are considered in the next section.

E.8 Rest cohomology terms

Now we are in a position to consider non-zero 'boundary' terms of Sects. E.1, E.2, E.5, E.6 and E.7 contained in Eqs. (E.2), (E.7), (E.26), (E.30) and (E.36).

For instance, consider the terms with $\tau = 0$, i.e.,

1. Eq. (E.2) with $\Omega, \bar\Omega$ (E.4).
2. Eq. (E.7) with $\Omega, \bar\Omega$ (E.11).
3. The term of Eq. (E.26), generated by $\Omega, \bar\Omega$ (E.29) proportional to $\mathbb{D}(1-\xi_1-\xi_2)\mathbb{D}(\xi_3)$ with

$$\Omega^{\alpha} = -[(\sigma_1(1-\xi_1) - (1))p_0 - (p_1 + p_2) + (1-\xi_1)(p_3 + p_2)]^{\alpha},$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar{z}^{\dot\alpha} - (1-\bar\tau)[(\sigma_1\{-\rho_2\xi_2 + 1\} - \rho_2)\bar{p}_0 - \rho_2(\bar{p}_1 + \bar{p}_2) + \{\rho_1\xi_2 + \xi_1\}(\bar{p}_3 + \bar{p}_2)]^{\dot\alpha}. \qquad \text{(E.41)}$$

4. The non-zero term of Eq. (E.26), generated by $\Omega, \bar\Omega$ (E.29) proportional to $\mathbb{D}(1-\sigma_1-\xi_2-\xi_3)\times\mathbb{D}(\xi_1)$ with

$$\Omega^{\alpha} = -[(\sigma_1 - \xi_2)p_0 - \xi_2(p_1 + p_2) + (p_3 + p_2)]^{\alpha},$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar{z}^{\dot\alpha} - (1-\bar\tau)[(\sigma_1\rho_1 - \{\xi_3 + \rho_2\xi_2\})\bar{p}_0 - \{\xi_3 + \rho_2\xi_2\}(\bar{p}_1 + \bar{p}_2) + \rho_1(\bar{p}_3 + \bar{p}_2)]^{\dot\alpha}. \qquad \text{(E.42)}$$

5. Eq. (E.30) with $\Omega, \bar\Omega$ (E.33) .
6. Eq. (E.30) with $\Omega, \bar\Omega$ (E.34).
7. Eq. (E.36) with $\Omega, \bar\Omega$ (E.39).

I. Firstly, we observe that the sum of the terms Eq. (E.7) with $\Omega, \bar\Omega$ (E.11), Eq. (E.26) with $\Omega, \bar\Omega$ (E.42) and Eq. (E.30) with $\Omega, \bar\Omega$ (E.34) acquires the form

$$\frac{\eta\bar\eta}{16}\int_{\tau\bar\tau\beta(2)\alpha(2)\sigma(2)\rho(2)} \Big(\mathrm{d}[\mathbb{D}(\tau)\,\mu\, l(\bar\tau)] - \mathbb{D}(\bar\tau)\mathbb{D}(\tau)\mu\Big)\mathbf{E}(\Omega|\bar\Omega)\Big] C\omega CCk\bar{k}, \qquad \text{(E.43)}$$

where $\mu = \nabla(\beta(2))\nabla(\rho(2))\nabla(\sigma(2))\nabla(\alpha(2))$,

$$\Omega^{\alpha} = -([\rho_2 - \sigma_1]p_0 - \sigma_1 p_1 + \sigma_2 p_2 + p_3)^{\alpha}$$
$$\bar\Omega^{\dot\alpha} = \bar\tau\bar{z}^{\dot\alpha} - (1-\bar\tau)\{\beta_1(\alpha_1[\sigma_1 - \rho_1]\bar{p}_0 - \alpha_1\sigma_2\bar{p}_1 + \alpha_1\sigma_1\bar{p}_2 + \bar{p}_3) + \beta_2(-\alpha_1\bar{p}_0 - \alpha_1\bar{p}_1 - \alpha_1\bar{p}_2 + \alpha_2\bar{p}_3)\}^{\dot\alpha}. \qquad \text{(E.44)}$$

Since (E.44) was obtained by application of IH to (E.34) and (E.11), the cohomology term of (E.43) satisfies the condition (4.4), i.e., is MNL. It is represented on the *r.h.s.* of Eq. (6.9) $\Omega_3, \bar\Omega_3$ (6.12).

II. Secondly, noticing that the following expression is exact, thus giving zero in the dZ-independent sector upon integration,

$$-\frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\beta(2)\sigma(2)\alpha(2)\rho(2)} \mathrm{d}\Big(\mathbb{D}(\tau)\mu\theta(\bar{\tau})\theta(1-\bar{\tau})\mathbf{E}(\Omega|\bar{\Omega})\Big)\,C\omega CCk\bar{k}, \tag{E.45}$$

with $\mu = \nabla(\beta(2))\nabla(\rho(2))\nabla(\sigma(2))\nabla(\alpha(2))$ and

$$\begin{aligned}\Omega^{\alpha} &= [-(-\rho_2\sigma_2+1)p_0 - p_1 - \rho_1 p_2 + \rho_2 p_3]^{\alpha}\\ \bar{\Omega}^{\dot\alpha} &= \bar{\tau}\bar{z}^{\dot\alpha} - (1-\bar{\tau})[\beta_1\alpha_1\bar{p}_0\\ &\quad -\beta_1\alpha_2\bar{p}_1 + \beta_1\alpha_1(\bar{p}_2+\bar{p}_3)\\ &\quad +\beta_2(\sigma_2-\sigma_2\alpha_2\rho_2-\alpha_2)\bar{p}_0 - \{\beta_2\alpha_2\}(\bar{p}_1\\ &\quad +\bar{p}_2) + \beta_2\{-\alpha_2\rho_2+1\}(\bar{p}_3+\bar{p}_2)]^{\dot\alpha},\end{aligned} \tag{E.46}$$

we observe that the differentiation yields a sum of the terms of Eq. (E.2) with $\Omega, \bar{\Omega}$ (E.4), Eq. (E.26) with $\Omega, \bar{\Omega}$ (E.41), Eq. (E.30) with $\Omega, \bar{\Omega}$ (E.33) plus the following one:

$$-\frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\beta(2)\xi(2)\alpha(2)\rho(2)} \mathbb{D}(\tau)\nabla(\beta(2))\nabla(\rho(2)) \nabla(\sigma(2))l(\bar{\tau})\mathbf{E}(\Omega|\bar{\Omega})\Big)\,C\omega CCk\bar{k} \tag{E.47}$$

with

$$\begin{aligned}\Omega^{\alpha} &= [-(-\rho_2\sigma_2+1)p_0 - p_1 - \rho_1 p_2 + \rho_2 p_3]^{\alpha},\\ \bar{\Omega}^{\dot\alpha} &= \bar{\tau}\bar{z}^{\dot\alpha} - (1-\bar{\tau})[(1-\beta_2\sigma_1)\bar{p}_0 + (\bar{p}_3+\bar{p}_2)]^{\dot\alpha}\end{aligned} \tag{E.48}$$

plus the cohomology term represented with the minus sign in Eq. (6.9) with $\Omega, \bar{\Omega}$ (6.14).

Note that to obtain (E.46) we apply IH to (E.33) and (E.41), hence preserving MNL.

III. Finally, applying IH to the terms (E.47) with $\Omega, \bar{\Omega}$ (E.48) and (E.36) with $\Omega, \bar{\Omega}$ (E.39), we obtain the following exact form that does not contribute to the vertex

$$\frac{\eta\bar{\eta}}{16}\int_{\tau\bar{\tau}\beta(2)\alpha(2)\sigma(2)\rho(2)} \mathrm{d}\Big(\mathbb{D}(\tau)\mu\theta(\bar{\tau})\theta(1-\bar{\tau})\mathbf{E}(\Omega|\bar{\Omega})\Big) C\omega CCk\bar{k} \tag{E.49}$$

with $\mu = \nabla(\beta(2))\nabla(\rho(2))\nabla(\sigma(2))\nabla(\alpha(2))$ and

$$\begin{aligned}\Omega^{\alpha} &:= -[(\alpha_1\beta_2\sigma_2\rho_2 + \alpha_2\rho_2\sigma_2 - 1)\\ &\quad p_0 - p_1 + p_3 - \rho_1(p_3+p_2)]^{\alpha},\\ \bar{\Omega}^{\dot\alpha} &:= \bar{\tau}\bar{z}^{\dot\alpha} - (1-\bar{\tau})[(1-\beta_2\sigma_1)\bar{p}_0 + (\bar{p}_3+\bar{p}_2)]^{\dot\alpha}\,.\end{aligned} \tag{E.50}$$

The sector of terms with $\bar{\tau} = 0$ is considered analogously. The final results are presented on the *r.h.s.* of Eq. (6.9) with $\Omega, \bar{\Omega}$ (6.13) and (6.15).

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