



The isospin and compositeness of the $T_{cc}(3875)$ state

L. R. Dai^{1,2,a}, L. M. Abreu^{2,3,b}, A. Feijoo^{2,c}, E. Oset^{2,d}

¹ School of Science, Huzhou University, Huzhou 313000, Zhejiang, China

² Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC Institutos de Investigación de Paterna, Aptdo.22085, 46071 Valencia, Spain

³ Instituto de Física, Universidade Federal da Bahia, Campus Universitário de Ondina, Salvador, Bahia 40170-115, Brazil

Received: 23 May 2023 / Accepted: 18 October 2023 / Published online: 31 October 2023
© The Author(s) 2023

Abstract We perform a fit to the LHCb data on the $T_{cc}(3875)$ state in order to determine its nature. We use a general framework that allows to have the $D^0 D^{*+}$, $D^+ D^{*0}$ components forming a molecular state, as well as a possible nonmolecular state or contributions from missing coupled channels. From the fits to the data we conclude that the state observed is clearly of molecular nature from the $D^0 D^{*+}$, $D^+ D^{*0}$ components and the possible contribution of a nonmolecular state or missing channels is smaller than 3%, compatible with zero. We also determine that the state has isospin $I = 0$ with a minor isospin breaking from the different masses of the channels involved, and the probabilities of the $D^0 D^{*+}$, $D^+ D^{*0}$ channels are of the order of 69% and 29% with uncertainties of 1%. The differences between these probabilities should not be interpreted as a measure of the isospin violation. Due to the short range of the strong interaction where the isospin is manifested, the isospin nature is provided by the couplings of the state found to the $D^0 D^{*+}$, $D^+ D^{*0}$ components, and our results for these couplings indicate that we have an $I = 0$ state with a very small isospin breaking. We also find that the potential obtained provides a repulsive interaction in $I = 1$, preventing the formation of an $I = 1$ state, in agreement with what is observed in the experiment.

1 Introduction

The discovery of the $T_{cc}(3875)$ reported in [1,2] was a turning point in hadron physics, showing the first evidence of a meson state clearly exotic with two open charm quarks. The

state, showing as a very narrow peak in the $D^0 D^0 \pi^+$ mass distribution very close to the $D^0 D^{*+}$ and $D^+ D^{*0}$ thresholds, has spurred lots of work in the hadron community, trying to find out its origin and nature. It should be quoted that prior to this discovery there were many works looking for tetraquark structures which predicted the existence of a state of this nature with two heavy quarks [3–17], yet with a large span in the mass from about 250 MeV below to about 250 MeV above the actual $T_{cc}(3875)$ mass. There were also works predicting such an object as a molecular state stemming from the interaction of the $D^0 D^{*+}$ and $D^+ D^{*0}$ pairs in [18–20]. Actually, the proximity of the mass of the $T_{cc}(3875)$ to the $D^0 D^{*+}$ and $D^+ D^{*0}$ thresholds will guarantee that a molecule is made provided a reasonable attractive force between the pseudoscalar and vector meson components appears [21]. The existence of such a state could also be concluded from the results of the $D^* D^*$ interaction in [22], where an extrapolation of the local hidden gauge approach for meson–meson interaction [23–26] was used and sufficient attraction was found to form a bound state. The extreme proximity of the $T_{cc}(3875)$ state to the $D^0 D^{*+}$ threshold, of about 360 keV, has motivated many works to explain this state as a molecular state of the $D^0 D^{*+}$ and $D^+ D^{*0}$ channels [27–50]. The compact tetraquark structure has also had its supporters in [51–54]. Some works suggest that there should be a molecular state with a small admixture of a compact tetraquark [55]. A comparative study of the tetraquark and molecular structures is done in [56] showing that the compact tetraquarks are about 100 MeV more bound than the molecular ones. Other works suggest to measure magnetic moments to elucidate the nature of the states [49,57]. As mentioned, the proximity of the mass of the $T_{cc}(3875)$ state to the $D^0 D^{*+}$, $D^+ D^{*0}$ thresholds, makes the molecular picture very natural and this can explain the large amount of works supporting this picture, mostly by using effective meson–meson potentials, or quark

^a e-mail: dailianrong@zjhu.edu.cn

^b e-mail: luciano.abreu@ufba.br

^c e-mail: edfeijoo@ific.uv.es (corresponding author)

^d e-mail: oset@ific.uv.es

model interaction [50], QCD sum rules [38,49] or lattice QCD simulations [40,44]. Further discussions concerning the $T_{cc}(3875)$ and related states can be seen in [54,58–61].

While the proximity of the $T_{cc}(3875)$ to the $D^0 D^{*+}$ and $D^+ D^{*0}$ thresholds, together with a moderate attractive interaction, will lead to a molecular state of these components [21], one cannot rule out that there could be a mixture, even if small, of a genuine compact tetraquark structure, as suggested in [55]. The issue of the compositeness and elementariness of states has drawn much attention, starting by the well known work of Weinberg [62] supporting the deuteron as a bound state of a proton and a neutron. Improvements on the original idea of [62] have been done in [41,63–84]. In [84] the effect of the finite range of the interaction, implicitly assumed to be of zero range in coordinate space in [62], is investigated and found to be relevant in the determination of the compositeness.

For bound states the probability to have a certain component in the wave function is given by $-g^2 \partial G / \partial s$, where g is the coupling of the state to the given component and G the loop function of the propagators of the two hadrons of the component, and for a potential that is energy independent the sum rule $\sum -g_i^2 \partial G_i / \partial s = 1$ holds, indicating that the state is fully molecular and the sum of probabilities of the different components is unity [68,78,83]. However, if the potential is energy dependent, the former sum rule can be smaller than 1, accounting for the possibility of having other components in the wave function. In fact, it is possible to eliminate some channels resulting in an energy dependent effective potential for the remaining channels, and the sum rule of these latter channels gives unity minus the probability of the eliminated channel [69,74,83,85]. Similarly, the existence of a contribution from a genuine, or compact tetraquark in this case, can also be taken into account eliminating this channel in favor of an energy effective potential for the remaining channels [86].

The works assuming a molecular picture for the $T_{cc}(3875)$ ordinarily assume energy independent interaction potentials between the components of the molecule, and hence, the sum rule is satisfied and the sum of probabilities of the $D^0 D^{*+}$ and $D^+ D^{*0}$ channels is unity. In the case of [34], the probabilities of these states are 71% and 29% respectively,¹ or the case of [42], where the probabilities are also of the order of 72% and 27% respectively, or in [87] with 69% and 31%,² or in [33] where the probabilities are 80% and 20%. There is a similarity in all these results, and the probability of the first

channel is bigger than the second channel because its threshold is closer to the energy of the state. This difference has not to be interpreted as a large violation of isospin. Indeed the $T_{cc}(3875)$ is assumed to have isospin $I = 0$ and no trace of a peak is seen in the partner isospin $I = 1$ channel $D^+ D^{*+}$. Isospin is a symmetry of strong interactions, which are of short range, and what matters in reactions where isospin is evidenced is the wave function at the origin (for s-wave), $g G$ according to [68]. With values of G similar for the two channels here, the couplings are what determine the isospin nature of the states. From [87] we obtain values of g_i for each channel which are of opposite sign and equal in size within 2–3%, as it corresponds to an $I = 0$ state. Since the probability is given by $-g^2 \partial G / \partial s$ and $\partial G / \partial s$ goes to infinity as we approach the threshold of the channel, it comes natural that, even having good isospin, the probability of the channel whose threshold is closer to the mass of the state will be larger than the one of the other channel. This is also what happens with the $D^0 \bar{D}^{*0}$ and $D^+ D^{*-}$ components in the $X(3872)$ [88]. The other thing to consider is that what determines the molecular nature of the state is the sum of the probabilities of the two channels that form the state.

Very recently, two papers have addressed the issue of the probability of the $D^0 D^{*+}$ and $D^+ D^{*0}$ components. One of them is [89], where only the probability of the first channel is evaluated with a result of about 50–80%. Note that in this case the Weinberg prescription is used, which according to many posterior works, has been shown to be inaccurate in many cases.³ The other work is [90], where a general discussion is done based on the value of the binding energy alone, together with several other assumptions made to validate the results. As we shall see, we will base our conclusions on two types of fits to the data, one fitting the binding, scattering lengths and effective ranges, seven data in our case, and another one making a fit to the $D^0 D^0 \pi^+$ distribution itself. This information obviously contains more dynamical information to make conclusions than the binding energy alone, which is clearly insufficient, since one can easily have a genuine state with a given binding energy, provided there is no interaction between the meson–meson components, or a pure molecular state from the interaction of the meson–meson components, provided there is no contribution from a genuine state, or such a state is very far away from the mass of the observed state (see further discussions along this line in [91]). We must also stress that in [90] the effect of the interaction range is also investigated and it is found that, in addition to the binding energy of the T_{cc} , if the range of the interaction in momentum space is assumed to be of the order of 770 MeV, as it

¹ Actually in Ref. [34] there is a weak energy dependence in the potential stemming from an extra pion exchange mechanism in the interaction, in spite of which the state is still largely molecular.

² In [87] the potentials, stemming from the local hidden gauge approach, have a weak energy dependence which however has a negligible influence in the results of the sum rule.

³ It suffices to mention that this prescription gives a molecular probability for the deuteron of 1.68 and the effective range, r_0 , has opposite sign to the experimental one (see details in [84]). However, as shown in [65], the prescription works better when the effective range is negative.

would correspond to vector exchange, the compositeness of the T_{cc} is much bigger than assuming a range due to pion exchange, and becomes of the order of 94%.

The former discussion shows that the subject is ripe to make a thorough discussion of the issue, which is our purpose here. For this purpose we shall include a general potential for the interaction of the $D^0 D^{*+}$ and $D^+ D^{*0}$ channels, including the necessary terms with energy dependence to account for possible nonmolecular components, and carry fits to data, from where we will evaluate the probabilities of the two channels and Z , the probability of other possible non-molecular components. We shall see that the results strongly support the molecular picture, with a negligible probability of a nonmolecular component.

2 Formalism

The $T_{cc}(3875)$ is found according to Ref. [2] with mass

$$M_{T_{cc}} = M_{D^0 D^{*+}} + \delta m_{\text{exp}}, \tag{1}$$

with $M_{D^0 D^{*+}} = 3875.09$ MeV and

$$\delta m_{\text{exp}} = -360 \pm 40_{-0}^{+4} \text{ keV},$$

and width

$$\Gamma = 48 \pm 2_{-14}^{+0} \text{ keV}. \tag{2}$$

The state is $J^P = 1^+$ and is assumed to be an isospin $I = 0$, since no signal is seen in the $I = 1, D^+ D^{*+}$ channel. The two channels close to the T_{cc} mass are $D^0 D^{*+}$ and $D^+ D^{*0}$, which we consider explicitly in our study. We have the isospin doublets $(D^+, -D^0), (D^{*+}, -D^{*0})$ and hence

$$\begin{aligned} |D^* D, I = 0\rangle &= -\frac{1}{\sqrt{2}}(D^{*+} D^0 - D^{*0} D^+), \\ |D^* D, I = 1\rangle &= -\frac{1}{\sqrt{2}}(D^{*+} D^0 + D^{*0} D^+). \end{aligned} \tag{3}$$

In our study we shall not assume a priori that the T_{cc} is a state of $I = 0$, it will come out from the analysis of the experimental data. We shall only assume that isospin is a good symmetry for the interaction such that the potential can be diagonalized with the isospin basis. The different masses of $D^0 D^{*+}$ and $D^+ D^{*0}$, that go in the kinetic energy of the Hamiltonian, can, and will, produce a small violation of isospin in the final wave function. This said, we take a general potential for the interaction of the two channels given by

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix}, \tag{4}$$

from where the scattering matrix is

$$T = [1 - VG]^{-1} V, \tag{5}$$

with $G = \text{diag}(G_1, G_2)$, where G_i are the loop functions for the two channels which we regularize in the cutoff method, with

$$G = \int_{|q| < q_{\text{max}}} \frac{d^3 q}{(2\pi)^3} \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \frac{1}{s - (\omega_1 + \omega_2)^2 + i\epsilon} \tag{6}$$

where $\omega_i = \sqrt{q^2 + m_i^2}$, m_1 is the mass of the D and m_2 that of D^* . The value of q_{max} reflects the range of the interaction in momentum space [68, 84] and will be obtained from the fits to the data. Since we assume isospin symmetry for the potential we shall have

$$\langle I = 0 | V | I = 1 \rangle = 0,$$

which gives us

$$V_{11} = V_{22}. \tag{7}$$

We also find that

$$\begin{aligned} \langle I = 0 | V | I = 0 \rangle &= V_{11} - V_{12}, \\ \langle I = 1 | V | I = 1 \rangle &= V_{11} + V_{12}. \end{aligned} \tag{8}$$

According to the findings of [2] we should expect that $V_{11} - V_{12}$ is negative, to allow for a bound state, and $V_{11} + V_{12}$ should be positive to prevent a bound state, from where we should have

$$V_{12} > 0; \quad V_{12} > |V_{11}|. \tag{9}$$

However, we could relax these conditions, since, as we shall see, the state that we obtain has $I = 0$ and all we need is that $V_{11} - V_{12}$ is negative and $V_{11} + V_{12}$ relatively smaller in size than $V_{11} - V_{12}$, such that it cannot lead to any bound state.

Then we have three parameters in the formalism, V_{11} , V_{12} and q_{max} . But to take into account the possibility of a genuine component (compact tetraquark) or contribution from other meson–meson channels different than the two explicitly considered, we will add terms in the potential which are energy dependent. According to the discussions in [84, 86] it is a good approximation to retain just the terms linear in s . Thus we write

$$\begin{aligned} V_{11} &= V'_{11} + \frac{\alpha}{m_V^2}(s - s_0), \\ V_{12} &= V'_{12} + \frac{\beta}{m_V^2}(s - s_0), \end{aligned} \tag{10}$$

where α, β are free parameters, $m_V = 800$ MeV is a typical vector mass, chosen to have α, β dimensionless, and s_0 is the mass squared of the T_{cc} . Our formalism has now five free parameters, but demanding that we have a pole at s_0 , they will be reduced to four. For the fits to the experimental data we will follow two different strategies:

- (a) a fit to the a and r_0 parameters of the effective range expansion for the two channels, D^0D^{*+} and D^+D^{*0} , obtained from the experimental analysis of the data in Ref. [2];
- (b) a fit to the unitary amplitudes extracted in the analysis of the experimental data in Ref. [2].

The scattering length a and effective range r_0 correspond to the effective range expansion of the scattering matrix

$$f(s) = \frac{1}{-\frac{1}{a} + \frac{1}{2}r_0k^2 - ik}. \tag{11}$$

Let us also mention here that in our formalism with Eqs. (5), (6) for the scattering amplitude we have the relationship

$$T(s) = -8\pi\sqrt{s}f(s) \tag{12}$$

which one can easily induce by looking at the imaginary parts of $\text{Im } T^{-1}(s) = -\text{Im } G$ (for a real potential) and $f^{-1}(s)$.

To avoid unnecessary complications due to the small width of the D^* , in the strategy (a) we shall disregard the D^* width.⁴ In the direct fit to the unitary amplitude of [2] we shall take into account the D^* width. The two fits are complementary and for fit (a) we rely on the fact that the very small width of the D^* will not change the character of the T_{cc} state. Actually, we are used to call bound state to atoms and nuclei with unstable particles. For strategy (a) we have $a_1, r_{0,1}$ real for the D^0D^{*+} channel and $a_2, r_{0,2}$ complex for the D^+D^{*0} channel. Hence, we have 6 data to be fitted with 4 free parameters, for which we minimize the χ^2 in the fit. The data of [2] neglecting the D^* width are

$$\begin{aligned} a_1 &= 6.134 \pm 0.51 \text{ fm}, \\ r_{0,1} &= -3.516 \pm 0.50 \text{ fm}, \\ a_2 &= (1.707 \pm 0.30) - i(1.07 \pm 0.30) \text{ fm}, \\ r_{0,2} &= (0.259 \pm 0.30) - i(3.769 \pm 0.30) \text{ fm}. \end{aligned} \tag{13}$$

The errors for $a_1, r_{0,1}$ are taken from [2], and for $a_2, r_{0,2}$ are taken relatively of the same order of magnitude. The errors influence the value of χ^2 but not the parameters of the minimum of χ^2 when this reaches values close to zero, as is the case here.

It is interesting to compare the results of Eq. (13) with those induced from [87]

$$\begin{aligned} a_1 &= 6.42 \text{ fm}, \\ r_{0,1} &= -3.49 \text{ fm}, \\ a_2 &= (2.08 - i0.98) \text{ fm}, \\ r_{0,2} &= (-0.06 - i2.27) \text{ fm}. \end{aligned} \tag{14}$$

⁴ We thank Mikhail Mikhasenko for providing us the data in that limit. We also recall that the prescription in Ref. [2] has a with opposite sign, which we change to our convention here.

The results are very similar although not equal. The real parts of $r_{0,2}$ are different and even with opposite sign, but let us note that they are one order of magnitude smaller than their corresponding imaginary parts, and not incompatible within the assumed uncertainties. Note that $a_2, r_{0,2}$ are complex, even neglecting the D^* width because the decay of D^+D^{*0} to D^0D^{*+} is open at the D^+D^{*0} threshold. The magnitudes of Eq. (14) are obtained with a potential

$$V = g^2 \frac{1}{2} \left[3s - (M^2 + m^2 + M'^2 + m'^2) - \frac{1}{s} (M^2 - m^2)(M'^2 - m'^2) \right] \begin{pmatrix} \frac{1}{M_{J/\psi}^2} & \frac{1}{m_\rho^2} \\ \frac{1}{m_\rho^2} & \frac{1}{M_{J/\psi}^2} \end{pmatrix}, \tag{15}$$

where M, M' are the initial, final vector masses and m, m' the initial, final pseudoscalar masses, with s calculated at the threshold of D^0D^{*+} and average masses of D^0, D^+ and D^{*0}, D^{*+} . The actual masses are used in the evaluation of the scattering matrices of Eqs. (5), (6). When we fit the magnitudes of Eq. (14) with the general potential of Eqs. (4), (10) and the T matrix of Eq. (5), we obtain a best fit with $\chi^2 = 0$ with $\alpha = \beta = 0$, certainly the potential of Eq. (15), and the sum rule

$$-\left(g_1^2 \frac{\partial G_1}{\partial s} + g_2^2 \frac{\partial G_2}{\partial s} \right) \Big|_{s=s_0} = 1, \tag{16}$$

is exactly fulfilled indicating that we have a pure molecular state. This is actually a tautology, since we start with a potential that produces a molecule and we get the answer that it is indeed a molecule. However, the test tells us that the inverse problem of getting the potential parameters from the magnitudes $a_1, r_{0,1}, a_2, r_{0,2}$ is feasible and meaningful. The interesting test is to obtain the potential of Eqs. (4), (10) from the experimental data of Eq. (13) where no assumptions concerning the interaction have been done, and then evaluate the probabilities P_1, P_2 for the D^0D^{*+} and D^+D^{*0} channels.

3 Evaluation of the scattering matrix, the couplings and the probabilities of the channels

The evaluation of the T matrix of Eq. (4) with two channels is easy and we show the formula below

$$T = \frac{1}{\text{DET}} \begin{pmatrix} V_{11} + (V_{12}^2 - V_{11}^2)G_2 & V_{12} \\ V_{12} & V_{11} + (V_{12}^2 - V_{11}^2)G_1 \end{pmatrix}, \tag{17}$$

with the determinant of $(1 - VG)$, DET, given by

$$\text{DET} = 1 - V_{11}(G_1 + G_2) - (V_{12}^2 - V_{11}^2)G_1G_2. \tag{18}$$

The bound state appears when $\text{DET} = 0$ at s_0 , hence

$$\text{DET}(s = s_0) = 0, \tag{19}$$

from where we can eliminate V_{12} in terms of V_{11} , as (note that the α, β terms of Eq. (10) vanish at $s = s_0$)

$$V'^2_{12} = \frac{1}{G_1 G_2} \left\{ 1 - V'_{11}(G_1 + G_2) + V'^2_{11} G_1 G_2 \right\} \Big|_{s=s_0}, \tag{20}$$

and we take the positive root of V_{12} according to Eq. (9). The couplings are defined from the residues of the T matrix. Since

$$T_{ij} \simeq \frac{g_i g_j}{s - s_0}, \tag{21}$$

then

$$\begin{aligned} g^2_1 &= \lim_{s \rightarrow s_0} (s - s_0) T_{11} = \frac{V_{11} + (V^2_{12} - V^2_{11}) G_2}{\frac{\partial}{\partial s} \text{DET}} \Big|_{s=s_0}, \\ g^2_2 &= \lim_{s \rightarrow s_0} (s - s_0) T_{22} = \frac{V_{11} + (V^2_{12} - V^2_{11}) G_1}{\frac{\partial}{\partial s} \text{DET}} \Big|_{s=s_0}, \\ g_1 g_2 &= \lim_{s \rightarrow s_0} (s - s_0) T_{12} = \frac{V_{12}}{\frac{\partial}{\partial s} \text{DET}} \Big|_{s=s_0}, \end{aligned} \tag{22}$$

where in the second step in the former equations we have used L'Hôpital rule. Note that the last equation allows us to get the relative sign of g_1 and g_2 . Then, according to [68, 83] we have the probabilities for the $D^0 D^{*+}$ and $D^+ D^{*0}$ channels, respectively, as

$$P_1 = -g^2_1 \frac{\partial G_1}{\partial s} \Big|_{s=s_0}, \quad P_2 = -g^2_2 \frac{\partial G_2}{\partial s} \Big|_{s=s_0}, \tag{23}$$

and consequently, the nonmolecular part, that could also account for possible missing coupled channels, is

$$Z = 1 - P_1 - P_2. \tag{24}$$

With the explicit formulae for the couplings, Eq. (22), it is now easy to prove explicitly the sum rule of Eq. (16) when $\alpha = \beta = 0$, although a general proof, extended to any number of channels, can be found in [68, 83], identifying each term of the sum rule with the probability of the corresponding channel in the wave function of the state.

4 Evaluation of the scattering lengths and effective ranges

From Eqs. (11), (12) we find

$$-\frac{1}{a} + \frac{1}{2} r_0 k^2 - i k \equiv -8\pi \sqrt{s} T^{-1}. \tag{25}$$

Hence, for each of the two channels $D^0 D^{*+}$ and $D^+ D^{*0}$, we have

$$\text{Im } T_{ii}^{-1} = -\text{Im } G_i = \frac{1}{8\pi \sqrt{s}} k_i,$$

with

$$k_i = \frac{\lambda^{1/2}(s, M_i^2, m_i^2)}{2\sqrt{s}},$$

and then we can write

$$\begin{aligned} -\frac{1}{a_1} &= -8\pi \sqrt{s} T_{11}^{-1} \\ &= -8\pi \sqrt{s} \left[\frac{1 - V_{11} G_2}{V_{11} + (V^2_{12} - V^2_{11}) G_2} - \text{Re } G_1 \right] \Big|_{s=s_1}, \\ r_{0,1} &= -\frac{\sqrt{s_1}}{\mu_1} \frac{\partial}{\partial s} \\ &\times \left\{ 16\pi \sqrt{s} \left[\frac{1 - V_{11} G_2}{V_{11} + (V^2_{12} - V^2_{11}) G_2} - \text{Re } G_1 \right] \right\} \Big|_{s=s_1}, \end{aligned} \tag{26}$$

with s_i the square of the threshold energies for the two channels and where to calculate $r_{0,1}$ we have used that

$$\frac{\partial}{\partial k^2} = \frac{\partial s}{\partial k^2} \frac{\partial}{\partial s} = \frac{\sqrt{s}}{\mu} \frac{\partial}{\partial s},$$

with μ the reduced mass of M, m

$$\mu = \frac{Mm}{M + m}.$$

Similarly,

$$\begin{aligned} -\frac{1}{a_2} &= -8\pi \sqrt{s} T_{22}^{-1} \\ &= -8\pi \sqrt{s} \left[\frac{1 - V_{11} G_1}{V_{11} + (V^2_{12} - V^2_{11}) G_1} - \text{Re } G_2 \right] \Big|_{s=s_2}, \\ r_{0,2} &= -\frac{\sqrt{s_2}}{\mu_2} \frac{\partial}{\partial s} \\ &\times \left\{ 16\pi \sqrt{s} \left[\frac{1 - V_{11} G_1}{V_{11} + (V^2_{12} - V^2_{11}) G_1} - \text{Re } G_2 \right] \right\} \Big|_{s=s_2}. \end{aligned} \tag{27}$$

5 Results

5.1 Fit to the data of a and r_0

We first conduct the fit (a) to the data of a, r_0 of Mikhasenko [89] given in Eq. (13). We define

$$\chi^2 = \sum_i \left(\frac{y_i^{th} - y_i^{exp}}{\Delta y_i^{exp}} \right)^2, \tag{28}$$

where y_i^{exp} are $a_1, r_{0,1}$ which are real, and $a_2, r_{0,2}$ which are complex. In the end we have six data (the binding energy, was used in Eq. (20) to eliminate V'_{12}), and four parameters $V'_{11}, \alpha, \beta, q_{max}$. The reduced χ^2_r is $\chi^2/2$. We minimize χ^2_r to obtain these parameters and, once obtained, we calculate P_1, P_2, Z . The first thing we must consider is that there are correlations between $V'_{11} - V'_{12}$ and q_{max} . Indeed, if we use

Table 1 The obtained scattering lengths and effective ranges

a_1 [fm]	$r_{0,1}$ [fm]	a_2 [fm]	$r_{0,2}$ [fm]
6.04 ± 0.11	-3.55 ± 0.37	$(1.72 \pm 0.12) - i(1.07 \pm 0.03)$	$(0.29 \pm 0.08) - i(3.76 \pm 0.14)$

Table 2 The obtained coupling constants and probabilities

g_1 [MeV]	g_2 [MeV]	P_1	P_2	Z
3727 ± 54	-3752 ± 164	0.67 ± 0.02	0.26 ± 0.02	0.07 ± 0.03

a non relativistic G function [68] we observe that in one channel $t = (V^{-1} - G)^{-1}$ with $ReG \approx -8\pi\mu(q_{max} - \frac{k^2}{2q_{max}} + O(k^4))$ (with μ the reduced mass) and there would be relativistic corrections on the k^2 term. We see that what matters in $V^{-1} - G$ is $V^{-1} + 8\pi\mu q_{max}$ at leading order and there is a trade off between V and q_{max} . By the contrary, the effective range depends on q_{max} , the range of the interaction, as shown in the works [65, 83, 84, 92]. However, this is not the only correlation and α and β are also very much correlated since in the $I = 0$ state that we get what matters is the $\alpha - \beta$ combination. We do not want to single out any particular parameter, and deal with the problem using the resampling or bootstrap method, particularly suited for the case that there are strong correlations between the parameters [93–100]. We take the 6 data and generate a random centroid for each of them with a gaussian distribution suited to the error of the datum. With these centroids and the same errors we perform a fit and determine the parameters. With these parameters we evaluate the observables, and we repeat the procedure a sufficiently large number of times, usually larger than 50, and evaluate the average value and the dispersion of each of the observables, as

$$\bar{P}_1 = \frac{1}{N} \sum_i P_{1,i},$$

$$(\Delta P_1)^2 = \frac{1}{N} \sum_i (P_{1,i} - \bar{P}_1)^2. \tag{29}$$

The results that we obtain are shown in Table 1. The agreement of the values obtained for the scattering lengths and effective range with the experimental ones of Eq. (13) is remarkable. As we see in Table 2, we obtain P_1, P_2 of the order of 69%, 28% with uncertainties of the order of 2%, and $Z = 0.026 \pm 0.025$. The numbers obtained qualify the T_{cc} as a clear molecular state made of the $D^0 D^{*+}$ and $D^+ D^{*0}$ components. We also find there that the couplings g_1, g_2 are very close to each other and of opposite sign, indicating, according to Eq. (3) that we have basically a state of $I = 0$.⁵ As we

have already stated before, one does not have to pay much attention to the value of the parameters obtained because they are much correlated. Yet, it is still interesting to see the average values and the errors of these parameters using the resampling method. We get

$$q_{max} = 504 \pm 110 \text{ MeV}, \quad V'_{11} = -157 \pm 53,$$

$$V'_{12} = 223 \pm 118, \quad \alpha = -969 \pm 1499,$$

$$\beta = -713 \pm 1428. \tag{30}$$

The errors are not small, reminding us of the correlations between the parameters. Particularly large are the errors in α and β . Yet, one can see that there is a strong correlation between the values of α and β , always very similar in size and such that $(\alpha - \beta)(s - s_0)/m_V^2$, as it corresponds to the $I = 0$ state that we find, is very small, always smaller than 1% of V'_{12} of Eq. (30) in the range of 5 MeV above the $\sqrt{s_0}$. The smallness of this number is pointing to the tiny nonmolecular component of the $I = 0$ state. The values of the parameters in Eq. (30) contrasts with those of the fit obtained in [87] using the potential of Eq. (15) and $\alpha, \beta = 0$ (shown in Eq. (31)),

$$q_{max} = 420 \text{ MeV}, \quad V'_{11} = 28.37,$$

$$V'_{12} = 459.87, \quad \alpha = \beta = 0. \tag{31}$$

It looks like the solutions of Eqs. (30), (31) are very different. Indeed, the values of q_{max} , although qualitatively similar, are still different, albeit compatible within errors and the strength of the potential is also rather different. However, this only reflects the strong correlations between the parameters. Indeed, if we look only to the binding energy (but we have more information) and assuming we just have the $I = 0$ state that we have induced from the data, we would have in one channel

$$T = [1 - VG]^{-1}V = \frac{1}{V^{-1} - G}, \tag{32}$$

where, according to Eq. (8)

$$V = V_{11} - V_{12}. \tag{33}$$

⁵ Mathematically speaking the formulas for T_{11} and T_{22} depend only on V_{12}^2 and T_{12} is proportional to V_{12} . This means that we could get solutions with V_{12} positive or negative and eventually interpret the state as having $I = 1$ (see discussion in [33]). We reject the cases with

Footnote 5 continued
positive $V_{11} - V_{12}$ based on the results of the local hidden gauge formalism [87], where $V_{11} - V_{12}$ is negative without any ambiguity (Eq. 15).

In order to get the binding at s_0 we have $V^{-1} = G(s_0)$. Hence there is a trade off between V and G , such that making changes with $\delta V^{-1} = \delta G$ we would get the same binding. Then starting with V from [87], let us call it V_{FL} , and G with $q_{max} = 420$ MeV, we would get an equivalent V_{EQ} for $q_{max} \approx 504$ MeV such that

$$V_{EQ}^{-1} - V_{FL}^{-1} = G(q_{max} = 504 \text{ MeV}) - G(q_{max} = 420 \text{ MeV}).$$

With this we obtain

$$V_{EQ} = -350, \tag{34}$$

to be compared to V from Eq. (30), let us call it $V_{Mi} = V'_{11} - V'_{12}$

$$V_{Mi} = -380. \tag{35}$$

The agreement found is remarkable, stressing once more the $I = 0$ character of the T_{cc} state. Considering the probabilities P_1, P_2 we observe that what we obtain in Table 1, is essentially the same result as was found from the analysis of [87] in spite of the apparently different solutions in the fitting parameters. This simply indicates strong correlations between the parameters, yet within a range of natural values, like having the range q_{max} between 420 MeV and 700 MeV, typical of the exchange of a light vector meson. This is also very valuable information.

5.2 Direct fit to the $D^0 D^0 \pi^+$ mass distribution

Now we turn to fit (b). For it we take the $D^0 D^0 \pi^+$ mass distribution obtained in Ref. [2] which corrects the raw data by the experimental resolution and fits the distribution with a unitary amplitude, accounting for the decay of the D^* . We compare then our results for $\Gamma(s)$ in Eqs. (18), (19) of Ref. [87], using the new potential of Eqs. (4), (7), (10), with the results in Fig. 8 of the supplementary information of Ref. [2]. The parameters are $V'_{11}, V'_{12}, \alpha, \beta, q_{max}$ and a normalization constant. There is a strong correlation between V'_{11}, V'_{12} and q_{max} . However, we leave $V'_{11}, V'_{12}, q_{max}$ as free parameters without using the constraint of Eq. (20) on the binding energy, since the mass distribution contains information on the position of the peak. On the other hand, we are using now the G functions accounting for the width of the D^* as done in [101] with

$$G = \int_{|q| < q_{max}} \frac{d^3q}{(2\pi)^3} \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \frac{1}{\sqrt{s} + \omega_1 + \omega_2} \times \frac{1}{\sqrt{s} - \omega_1 - \omega_2 + i \frac{\sqrt{s'}}{2m_{D^*}} \Gamma_{D^*}(s')}, \tag{36}$$

where $s' = (\sqrt{s} - \omega_D)^2 - q^2$ and $\Gamma_{D^*}(s')$ as given in Eqs. (14), (15) of [87].

We take 44 points from Fig. 8 of Ref. [2], with the typical errors obtained from the raw data taking a 10% value of the

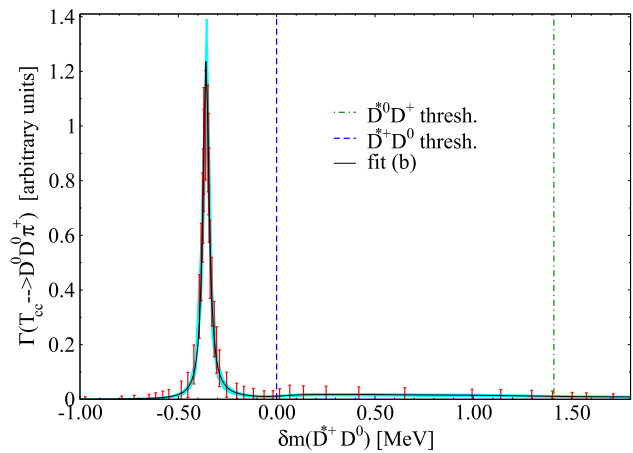


Fig. 1 Fit to the data of Fig. 8 of supplementary material in Ref. [2] with the size of the errors taken from the raw data in the way explained in the text

square root of the number of events, and perform a best fit to the data whose result is shown in Fig. 1. In order to estimate the statistical uncertainties we use the resampling (bootstrap) method [93–100], with a new fit for every random choice of centroids of the data. We show below the average values of the parameters and uncertainties obtained from these fits:

$$q_{max} = 406 \pm 14 \text{ MeV}, \quad V'_{11} = -61 \pm 58, \\ V'_{12} = 388 \pm 65, \quad \alpha = -12 \pm 60, \quad \beta = 8 \pm 55. \tag{37}$$

One should not worry too much about these values and their errors, since we know that there are also correlations between V'_{11} and V'_{12} and α and β (the differences between these parameters is what matters if one has an $I = 0$ state). The relevant thing is what we get for the observables, and from the different fits in the resampling method we obtain the average values of the observables and their dispersion. The results are summarized in Tables 3 and 4.

These results are very similar to those displayed in Table 1. Yet, they should be better compared to the results of Ref. [2] when the D^* width is explicitly taken into account, which are

$$a_1^{exp} = [(7.16 \pm 0.51) - i(1.85 \pm 0.28)] \text{ fm}, \\ a_2^{exp} = (1.76 - i 1.82) \text{ fm}. \tag{38}$$

The agreement is perfect within errors if we assume the relative errors in a_2^{exp} to be similar to those of a_1^{exp} . We see that a_1 is now complex, yet with the imaginary part reasonably smaller than the real one. The real parts obtained for a_1 or a_1^{exp} in this case are also very similar to a_1 of Eqs. (13) and (14) obtained with fit (a) to the data neglecting the width of the D^* .

We have taken advantage of the weak dependence of the obtained magnitudes with the D^* decay width and evaluate

Table 3 The obtained scattering lengths and effective ranges

	a_i [fm]	$r_{0,i}$ [fm]
$i = 1 (D^{*+}D^0)$	$(7.60 \pm 0.14) - i(1.73 \pm 0.09)$	-2.94 ± 0.04
$i = 2 (D^{*0}D^+)$	$(1.99 \pm 0.07) - i(1.25 \pm 0.23)$	$(0.11 \pm 0.17) - i(2.74 \pm 0.22)$

Table 4 The obtained binding energy, width, coupling constants and probabilities

B [KeV]	Γ [KeV]	g_1 [MeV]	g_2 [MeV]	P_1	P_2
360 ± 2	38 ± 1	3875 ± 51	-4077 ± 72	0.697 ± 0.017	0.301 ± 0.009

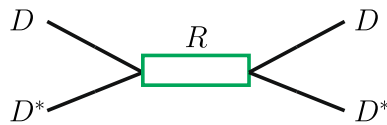


Fig. 2 Amplitude for DD^* from a preexisting nonmolecular state

$P_1, P_2, P_1 + P_2 (P_1 + P_2 = 0.998 \pm 0.024)^6$ and $r_{0,1}, r_{0,2}, g_1, g_2$ in the limit of $\Gamma_{D^*} \rightarrow 0$, using the parameters obtained in Eq. (37) and the formulas described in Sect. 3.

We see that the systematic errors in the effective ranges are similar to the ones obtained for the scattering lengths. The values of the couplings obtained are also very similar to those shown in Table 2 with the fit (a), compatible within errors. Coming now to the compositeness P_1, P_2 , the values obtained are also remarkably similar to those in Table 2, indicating again the molecular nature of the T_{cc} state, with $P_1 + P_2$ essentially 1 in the present fit with an uncertainty of the order of 0.024.

6 Limiting case of a nonmolecular state

We shall assume that the origin of the T_{cc} state corresponds to a state, which has a very small overlap with the D^0D^{*+}, D^+D^{*0} components (a minimum overlap is needed such that the state is observed in the D^0D^{*+} and D^+D^{*0} channels). We shall also assume, to be consistent with experiment, that the state has $I = 0$, although the conclusions are not tied to this fact. The amplitude for the $DD^* (I = 0)$ state will be given by

$$\tilde{T}_{D^*D, D^*D} = \frac{g^2}{s - s_0}$$

depicted in Fig. 2, and g^2 should be small to prevent the overlap of the genuine state with the DD^* components. Consistently with the nonmolecular assumption, we will also assume that the D^0D^{*+}, D^+D^{*0} components have no interaction stemming from a potential, or t -channel exchange of

⁶ $P_1 + P_2$ is calculated in each fit and the average of $P_1 + P_2$ and its dispersion is obtained from the statistical analysis.

other hadrons. Yet, the picture for the T matrix is not complete since, for consistency with the small coupling of the genuine state to the DD^* components, we have to consider the selfenergy of the state due to its coupling to the DD^* components, as depicted in Fig. 3.

By taking into account the $I = 0$ structure of the DD^* state, we find from Fig. 3

$$T_{D^0D^{*+}, D^0D^{*+}} = \frac{1}{2} \frac{g^2}{s - \tilde{s}_0 - \frac{1}{2}g^2G_{D^0D^{*+}} - \frac{1}{2}g^2G_{D^+D^{*0}}}. \tag{39}$$

Once again, by considering Eq. (25) and that $\text{Im } T^{-1} = \frac{1}{8\pi\sqrt{s}}k$, we realize that the amplitude of Eq. (39) satisfies unitarity in the two channels D^0D^{*+}, D^+D^{*0} , reproducing exactly the imaginary part of $f^{-1}(s)$. We now derive the results for a, r_0 in the limit of $g \rightarrow 0$, such that Eq. (39) has a pole at s_0 , where it should be. Let us call D the denominator of Eq. (39),

$$D = s - \tilde{s}_0 - \frac{1}{2}g^2G_{D^0D^{*+}} - \frac{1}{2}g^2G_{D^+D^{*0}}. \tag{40}$$

In order to have a pole of $T_{D^0D^{*+}, D^0D^{*+}}$ at s_0 , we need

$$D|_{s=s_0} = 0. \tag{41}$$

Let us note that if $g \neq 0$, then $\tilde{s}_0 \neq s_0$ and we can make an expansion of T in powers of $s - s_0$, and we would then go back to the previous analysis accounting for the genuine state in terms of the energy dependent terms, the constant parts reabsorbed in the V_{ij} coefficients. Thus, the novel thing should be found in the $g^2 \rightarrow 0$ limit.

In this case it is easy to find a and r_0 from Eq. (39)

$$-\frac{1}{a} = \lim_{g^2 \rightarrow 0} -8\pi\sqrt{s} \frac{2}{g^2} \left\{ s - s_0 - \frac{1}{2}g^2G_{D^0D^{*+}} - \frac{1}{2}g^2G_{D^+D^{*0}} \right\}. \tag{42}$$

Thus, we immediately see that $-\frac{1}{a} \rightarrow \infty$ and hence $a \rightarrow 0; a_1 \rightarrow 0, a_2 \rightarrow 0$.

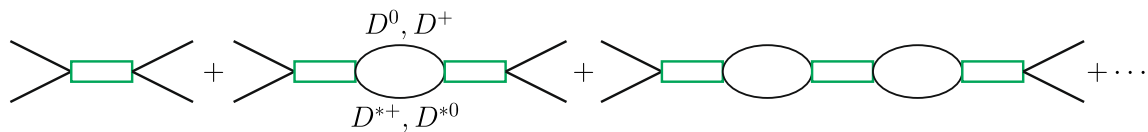


Fig. 3 Diagrammatic representation of the T matrix for the nonmolecular state including the DD^* selfenergy

For the effective range we find

$$\begin{aligned} \frac{1}{2} r_{0,1} &= \lim_{g^2 \rightarrow 0} -\frac{\sqrt{s_1}}{\mu_1} \frac{16\pi}{g^2} \frac{\partial}{\partial s} \left\{ s^{1/2}(s - s_0) - O(g^2) \right\} \\ &= \lim_{g^2 \rightarrow 0} -\frac{\sqrt{s_1}}{\mu_1} \frac{16\pi}{g^2} \frac{1}{2\sqrt{s}} (3s - s_0) \Big|_{s=s_1} \rightarrow -\infty, \\ \frac{1}{2} r_{0,2} &= \lim_{g^2 \rightarrow 0} -\frac{\sqrt{s_2}}{\mu_2} \frac{16\pi}{g^2} \frac{1}{2\sqrt{s}} (3s - s_0) \Big|_{s=s_2} \rightarrow -\infty. \end{aligned} \quad (43)$$

We see that in the strict limit of the nonmolecular state, the scattering length goes to 0 and the effective range to $-\infty$. This contrast with the experimental results of Eq. (13) where a is relatively large for hadronic reactions, and r_0 is smaller than a . A discussion on the effects of the genuine state along the line described here, but not addressing the scattering length and effective range can be seen in [33] (see also section 2 of Ref. [86]).

7 Conclusions

We have performed fits to the data of the LHCb collaboration on the $T_{cc}(3875)$ state with a general framework that contains explicitly the $D^0 D^{*+}$ and $D^+ D^{*0}$ components, which are allowed to interact with a potential containing energy dependent terms by means of which one can account for contributions of missing coupled channels, as well as effects from some genuine state, like a compact tetraquark state. We have conducted two types of fits to the data. One of them is a fit to data from LHCb for the scattering lengths and effective ranges of the channels $D^0 D^{*+}$ and $D^+ D^{*0}$, evaluated ignoring the D^* width. The other fit is done directly to the data of the mass distribution of $D^0 D^0 \pi^+$ corrected by the experimental resolution and parametrized in terms of a unitary amplitude in [2], where the D^* width is explicitly considered. In all fits we have calculated the parameters of the theoretical framework and from them we have looked at the bound state produced and evaluated the probabilities P_1, P_2 of having $D^0 D^{*+}$ and $D^+ D^{*0}$ in the wave function, as well as the couplings g_1, g_2 of the state obtained to these two components. Without any assumption of the nature of the state observed, we find that the couplings obtained are very close to each other and of opposite sign, indicating the isospin $I = 0$ nature of the state observed. The potential obtained produces a repulsive interaction in $I = 1$, justifying the non

observance of any structure with $I = 1$ in the LHCb experiment. On the other hand, from the couplings obtained, we could calculate the probabilities P_1, P_2 of $D^0 D^{*+}, D^+ D^{*0}$ and $Z = 1 - P_1 - P_2$, the non compositeness of the state. We found values for Z of the order of 3% with an uncertainty that makes it compatible with zero or, conversely, in the case of fit (b) $P_1 + P_2 = 1$ with uncertainties of 2.5%, indicating that we have a very clear case of a molecular state made of the $D^0 D^{*+}, D^+ D^{*0}$ components. We found that the probabilities of the two channels are of the order of 69% and 29% respectively, very similar to what has been obtained in other approaches which, however, assume a priori energy independent potentials which do not allow for nonmolecular components to appear. We also clarified that the isospin nature should not be induced by the probabilities P_1, P_2 , but from the couplings g_1, g_2 , or related to them, the wave functions at the origin of these two channels, since in strong interactions, where the isospin symmetry holds, the wave functions at short distances is what matters. We also shortly discussed that using all the available experimental information, apart from the mass of the state, was essential to reach the present conclusions, with the information on the binding being clearly insufficient to reach these conclusions.

Acknowledgements We would like to thank M. Mikhasenko for useful discussions and for providing us with data of his analysis. We also thank M. Albaladejo for useful discussions. This work is partly supported by the National Natural Science Foundation of China under Grants Nos. 12175066, 11975009 and LRD would like to express special thanks to her students of Yuhang Wang and Langning Chen in helping to read the fitting data. LMA has received funding from the Brazilian agencies Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) under contracts 309950/2020-1, 400215/2022-5, 200567/2022-5, and Fundação de Amparo à Pesquisa do Estado da Bahia (FAPESB) under the contract INT0007/2016. This work is also partly supported by the Spanish Ministerio de Economía y Competitividad (MINECO) and European FEDER funds under Contracts No. FIS2017-84038-C2-1-P B, PID2020-112777GB-I00, and by Generalitat Valenciana under contract PROMETEO/2020/023. This project has received funding from the European Union Horizon 2020 research and innovation programme under the program H2020-INFRAIA-2018-1, grant agreement No. 824093 of the STRONG-2020 project. The work of AF was partially supported by the Generalitat Valenciana and European Social Fund APOSTD-2021-112.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical work with no data associated.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation,

distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Funded by SCOAP³. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

- R. Aaij et al. (LHCb Collaboration), *Nat. Phys.* **18**, 751 (2022)
- R. Aaij et al. (LHCb Collaboration), *Nat. Commun.* **13**, 3351 (2022)
- J.P. Ader, J.M. Richard, P. Taxil, *Phys. Rev. D* **25**, 2370 (1982)
- H.J. Lipkin, *Phys. Lett. B* **172**, 242 (1986)
- S. Zouzou, B. Silvestre-Brac, C. Gignoux, J.M. Richard, *Z. Phys. C* **30**, 457 (1986)
- J. Carlson, L. Heller, J.A. Tjon, *Phys. Rev. D* **37**, 744 (1988)
- F.S. Navarra, M. Nielsen, S.H. Lee, *Phys. Lett. B* **649**, 166 (2007)
- C. Semay, B. Silvestre-Brac, *Z. Phys. C* **61**, 271 (1994)
- D. Janc, M. Rosina, *Few Body Syst.* **35**, 175 (2004)
- J. Vijande, E. Weissman, A. Valcarce, N. Barnea, *Phys. Rev. D* **76**, 094027 (2007)
- S.H. Lee, S. Yasui, *Eur. Phys. J. C* **64**, 283 (2009)
- Y. Yang, C. Deng, J. Ping, T. Goldman, *Phys. Rev. D* **80**, 114023 (2009)
- M. Karliner, J.L. Rosner, *Phys. Rev. Lett.* **119**, 202001 (2017)
- E.J. Eichten, C. Quigg, *Phys. Rev. Lett.* **119**, 202002 (2017)
- Z.G. Wang, *Acta Phys. Pol. B* **49**, 1781 (2018)
- P. Jannnarkar, N. Mathur, M. Padmanath, *Phys. Rev. D* **99**, 034507 (2019)
- J.B. Cheng, S.Y. Li, Y.R. Liu, Z.G. Si, T. Yao, *Chin. Phys. C* **45**, 043102 (2021)
- N. Li, Z.F. Sun, X. Liu, S.L. Zhu, *Phys. Rev. D* **88**, 114008 (2013)
- M.Z. Liu, T.W. Wu, M.P. Valderrama, J.J. Xie, L.S. Geng, *Phys. Rev. D* **99**, 094018 (2019)
- Z.M. Ding, H.Y. Jiang, J. He, *Eur. Phys. J. C* **80**, 1179 (2020)
- X.K. Dong, F.K. Guo, B.S. Zou, *Phys. Rev. Lett.* **126**, 152001 (2021)
- R. Molina, T. Branz, E. Oset, *Phys. Rev. D* **82**, 014010 (2010)
- M. Bando, T. Kugo, K. Yamawaki, *Phys. Rep.* **164**, 217 (1988)
- M. Harada, K. Yamawaki, *Phys. Rep.* **381**, 1 (2003)
- U.G. Meissner, *Phys. Rep.* **161**, 213 (1988)
- H. Nagahiro, L. Roca, A. Hosaka, E. Oset, *Phys. Rev. D* **79**, 014015 (2009)
- X.Z. Ling, M.Z. Liu, L.S. Geng, E. Wang, J.J. Xie, *Phys. Lett. B* **826**, 136897 (2022)
- X.K. Dong, F.K. Guo, B.S. Zou, *Commun. Theor. Phys.* **73**, 125201 (2021)
- Q. Xin, Z.G. Wang, *Eur. Phys. J. A* **58**, 110 (2022)
- Y. Huang, H.Q. Zhu, L.S. Geng, R. Wang, *Phys. Rev. D* **104**, 116008 (2021)
- S. Fleming, R. Hodges, T. Mehen, *Phys. Rev. D* **104**, 116010 (2021)
- H. Ren, F. Wu, R. Zhu, *Adv. High Energy Phys.* **2022**, 9103031 (2022)
- M. Albaladejo, *Phys. Lett. B* **829**, 137052 (2022)
- M.L. Du, V. Baru, X.K. Dong, A. Filin, F.K. Guo, C. Hanhart, A. Nefediev, J. Nieves, Q. Wang, *Phys. Rev. D* **105**, 014024 (2022)
- C. Deng, S.L. Zhu, *Phys. Rev. D* **105**, 054015 (2022)
- M.J. Zhao, Z.Y. Wang, C. Wang, X.H. Guo, *Phys. Rev. D* **105**, 096016 (2022)
- H.W. Ke, X.H. Liu, X.Q. Li, *Eur. Phys. J. C* **82**, 144 (2022)
- S.S. Agaev, K. Azizi, H. Sundu, *JHEP* **06**, 057 (2022)
- J. He, X. Liu, *Eur. Phys. J. C* **82**, 387 (2022)
- M. Padmanath, S. Prelovsek, *Phys. Rev. Lett.* **129**, 032002 (2022)
- M. Albaladejo, J. Nieves, *Eur. Phys. J. C* **82**, 724 (2022)
- J.B. Cheng, Z.Y. Lin, S.L. Zhu, *Phys. Rev. D* **106**, 016012 (2022)
- L.M. Abreu, *Nucl. Phys. B* **985**, 115994 (2022)
- S. Chen, C. Shi, Y. Chen, M. Gong, Z. Liu, W. Sun, R. Zhang, *Phys. Lett. B* **833**, 137391 (2022)
- Z.S. Jia, M.J. Yan, Z.H. Zhang, P.P. Shi, G. Li, F.K. Guo, *Phys. Rev. D* **107**(7), 074029 (2023)
- B. Wang, L. Meng, *Phys. Rev. D* **107**(9), 094002 (2023)
- L. Dai, S. Fleming, R. Hodges, T. Mehen, *Phys. Rev. D* **107**(7), 076001 (2023)
- Y. Li, Y.B. He, X.H. Liu, B. Chen, H.W. Ke, *Eur. Phys. J. C* **83**(3), 258 (2023)
- U. Özdem, *Phys. Rev. D* **105**, 054019 (2022)
- K. Chen, R. Chen, L. Meng, B. Wang, S.L. Zhu, *Eur. Phys. J. C* **82**, 581 (2022)
- S.S. Agaev, K. Azizi, H. Sundu, *Nucl. Phys. B* **975**, 115650 (2022)
- Y. Jin, S.Y. Li, Y.R. Liu, Q. Qin, Z.G. Si, F.S. Yu, *Phys. Rev. D* **104**, 114009 (2021)
- T.W. Wu, Y.L. Ma, *Phys. Rev. D* **107**(7), L071501 (2023)
- L. Meng, B. Wang, G.J. Wang, S.L. Zhu, *Phys. Rept.* **1019**, 1–149 (2023)
- M.J. Yan, M.P. Valderrama, *Phys. Rev. D* **105**, 014007 (2022)
- X. Chen, Y. Yang, *Chin. Phys. C* **46**, 054103 (2022)
- K. Azizi, U. Özdem, *Phys. Rev. D* **104**, 114002 (2021)
- H.X. Chen, W. Chen, X. Liu, Y.R. Liu, S.L. Zhu, *Rep. Prog. Phys.* **86**, 026201 (2023)
- X.Z. Weng, W.Z. Deng, S.L. Zhu, *Chin. Phys. C* **46**, 013102 (2022)
- L.M. Abreu, H.P.L. Vieira, F.S. Navarra, *Phys. Rev. D* **105**, 116029 (2022)
- L.M. Abreu, F.S. Navarra, M. Nielsen, H.P.L. Vieira, *Eur. Phys. J. C* **82**, 296 (2022)
- S. Weinberg, *Phys. Rev.* **137**, B672 (1965)
- F.K. Guo, C. Hanhart, U.G. Meissner, Q. Wang, Q. Zhao, B.S. Zou, *Rev. Mod. Phys.* **90**, 015004 (2018)
- C. Garcia-Recio, C. Hidalgo-Duque, J. Nieves, L.L. Salcedo, L. Tolos, *Phys. Rev. D* **92**, 034011 (2015)
- Y. Li, F.K. Guo, J.Y. Pang, J.J. Wu, *Phys. Rev. D* **105**, L071502 (2022)
- V. Baru, X.K. Dong, M.L. Du, A. Filin, F.K. Guo, C. Hanhart, A. Nefediev, J. Nieves, Q. Wang, *Phys. Lett. B* **833**, 137290 (2022)
- V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A. Kudryavtsev, *Phys. Lett. B* **586**, 53 (2004)
- D. Gamermann, J. Nieves, E. Oset, E.R. Arriola, *Phys. Rev. D* **81**, 014029 (2010)
- T. Hyodo, D. Jido, A. Hosaka, *Phys. Rev. C* **85**, 015201 (2012)
- V. Baru, C. Hanhart, Y.S. Kalashnikova, A.E. Kudryavtsev, A.V. Nefediev, *Eur. Phys. J. A* **44**, 93 (2010)
- C. Hanhart, Y.S. Kalashnikova, A.V. Nefediev, *Eur. Phys. J. A* **47**, 101 (2011)
- F. Aceti, E. Oset, *Phys. Rev. D* **86**, 014012 (2012)
- T. Hyodo, *Phys. Rev. Lett.* **111**, 132002 (2013)
- T. Sekihara, T. Hyodo, D. Jido, *PTEP* **2015**, 063D04 (2015)
- C. Hanhart, J.R. Pelaez, G. Rios, *Phys. Lett. B* **739**, 375 (2014)
- Z.H. Guo, J.A. Oller, *Phys. Rev. D* **93**, 096001 (2016)
- T. Sekihara, T. Arai, J. Yamagata-Sekihara, S. Yasui, *Phys. Rev. C* **93**, 035204 (2016)
- Y. Kamiya, T. Hyodo, *Phys. Rev. C* **93**, 035203 (2016)

79. T. Sekihara, Phys. Rev. C **95**, 025206 (2017)
80. Y. Kamiya, T. Hyodo, PTEP **2017**, 023D02 (2017)
81. I. Matuschek, V. Baru, F.K. Guo, C. Hanhart, Eur. Phys. J. A **57**, 101 (2021)
82. T. Kinugawa, T. Hyodo, EPJ Web Conf. **262**, 01019 (2022)
83. T. Hyodo, Structure and compositeness of hadron resonances. Int. J. Mod. Phys. A **28**, 1330045 (2013)
84. J. Song, L.R. Dai, E. Oset, Eur. Phys. J. A **58**, 133 (2022)
85. F. Aceti, L.R. Dai, L.S. Geng, E. Oset, Y. Zhang, Eur. Phys. J. A **50**, 57 (2014)
86. A. Martínez Torres, E. Oset, S. Prelovsek, A. Ramos, JHEP **05**, 153 (2015)
87. A. Feijoo, W.H. Liang, E. Oset, Phys. Rev. D **104**, 114015 (2021)
88. D. Gamermann, E. Oset, Phys. Rev. D **80**, 014003 (2009)
89. M. Mikhasenko, [arXiv:2203.04622](https://arxiv.org/abs/2203.04622) [hep-ph]
90. T. Kinugawa, T. Hyodo, [arXiv:2303.07038](https://arxiv.org/abs/2303.07038) [hep-ph]
91. M. Albaladejo, J. Nieves, L. Tolos, Phys. Rev. C **104**, 035203 (2021)
92. M. Albaladejo, J. Nieves, Eur. Phys. J. C **82**(8), 724 (2022)
93. B. Efron, R. Tibshirani, Stat. Sci. **57**, 54 (1986)
94. M. Albaladejo et al. (JPAC Collaboration), Prog. Part. Nucl. Phys. **127**, 103981 (2021)
95. J. Landay, M. Döring, C. Fernández-Ramírez, B. Hu, R. Molina, Phys. Rev. C **95**, 015203 (2017)
96. R. Navarro Pérez, J.E. Amaro, E. Ruiz Arriola, Phys. Lett. B **738**, 155 (2014)
97. J.J. Xie, W.H. Liang, E. Oset, Eur. Phys. J. A **55**, 6 (2019)
98. J. Brownlee, A gentle introduction to statistical sampling and resampling (2018). <https://machinelearningmastery.com/statistical-sampling-and-resampling/>
99. B. LeBeau, ZAS, statistical reasoning through computation and R (2022). <https://lebebr01.github.io/stat,hinking/>
100. J. Song, A. Feijoo, E. Oset, Phys. Rev. D **106**, 074027 (2022)
101. L.R. Dai, E. Oset, A. Feijoo, R. Molina, L. Roca, A. Martínez Torres, K.P. Khemchandani, Phys. Rev. D **105**, 074017 (2022)