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Practical Dirac Majorana confusion theorem: issues and applicability

C. S. Kim[1,2,a]

[1] Department of Physics and IPAP, Yonsei University, Seoul 03722, Korea
[2] Institute of High Energy Physics, Dongshin University, Naju 58245, Korea

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Abstract We inspect the model-independent study of practical Dirac Majorana confusion theorem (pDMCT) – a wide spread belief that the difference between Dirac and Majorana neutrinos via any kinematical observable would be practically impossible to determine because of the difference only being proportional to the square of neutrino mass – in context of processes that have at least a neutrino antineutrino pair in their final state. We scrutinize the domain of applicability of pDMCT and also highlight those aspects that are often misunderstood. We try to clarify some of the frequently used concepts that are used to assert pDMCT as a generic feature irrespective of the process, or observable, such as the existence of any analytic continuity between Dirac and Majorana neutrinos in the limit $m_\nu \to 0$. In summary, we illustrate that pDMCT is not any fundamental property of neutrinos, instead, it is a phenomenological feature of neutrino non-observation, depending on models and processes.

1 Introduction

Are neutrinos distinct from their antiparticles like the rest of the known fermions of the Standard Model (SM), or are the neutrino and antineutrino quantum mechanically identical to one another? An affirmative response to the first (second) question would imply neutrinos are Dirac (Majorana) fermions. We are yet to have a definite answer to this fundamental question regarding Dirac or Majorana nature of neutrinos. However, the literature is replete with attempts made on both theoretical and experimental fronts, without much success. The situation is such that there is an ossified belief in the community that the difference between Dirac and Majorana neutrinos via any kinematical observable would be practically impossible to determine due to fact that the observable difference between Dirac and Majorana neutrinos is proportional to the tiny neutrino mass. This is often cited as the "practical Dirac Majorana confusion theorem" (pDMCT, in short) [1][1]. While the "theorem" has been verified in some cases, there is no general model-independent and process-independent proof. Also there is a general lack of clarity regarding its domain of validity. It is therefore necessary and important to explore whether there are any SM allowed processes and kinematic observables that can directly probe the Majorana nature of neutrinos avoiding this pDMCT constraint. In this article we discuss the domain of validity of pDMCT as well as its exceptions. This does not invalidate pDMCT but brings more clarity with regard to its applicability as an useful tool.

There are two issues regarding pDMCT to which we would like to draw readers' attention.

1. The pDMCT should not be taken out-of-context of its historical development. Historically, only SM allowed neutral current interaction mediated processes as well as those processes mediated by exchange of massive Majorana neutrinos [4], were analyzed. In processes involving neutral current interaction, there had been no way to gain any information regarding individual neutrino antineutrino energies or 3-momenta. This invariably leads to integration over neutrino antineutrino related kinematic variables while proposing any relevant kinematic observables. If one considers a process which is mediated not through weak neutral current interactions, and if one has access to individual information of neutrino and antineutrino momenta without directly measuring them, then one need not take pDMCT for granted while analyzing the relevant observables.

[a] e-mail: cskim@yonsei.ac.kr (corresponding author)

[1] A few notable precursors to the formulation of pDMCT were the analysis made by Refs. [2,3].

2. The usual approach to validate pDMCT as a general theorem is by alluding to a non-existent correspondence between massive Dirac and Majorana fermions in the massless limit where the neutrinos have specific chirality. This strangely overlooks the well-known mathematical impossibility of having a chiral massless Majorana neutrino. In any case it does not make practical sense since the neutrinos have non-zero mass.

For the first time in Refs. [5,6], we respectively implemented both model-independent and specific process-independent studies of pDMCT. The current work borrows some of the features from both the above papers in a more accessible manner and directly addresses some pedagogical aspects to bring clarity. Thus in Sect. 2 we first consider a generic model-independent analysis of processes that contain a neutrino antineutrino pair in the final state. This structure in the final state allows for application of Pauli exclusion principle through anti-symmetrisation in the case of Majorana neutrinos. The application of the exclusion principle is independent of the size of the non-zero mass of the neutrino or any other dimensional parameter for that matter. We refer to this as model-independent in the sense that our analysis includes both the SM and new physics (NP) contributions. However, the process itself is allowed in the SM. This is exclusively discussed in Sect. 3. In Sect. 4 we highlight the domain of applicability of pDMCT. This is followed by some pedagogical explanations on pDMCT in Sect. 5. Finally we conclude highlighting the important features in Sect. 6.

2 A model-independent analysis of processes containing $\nu\,\overline{\nu}$ in the final state

2.1 Details on the process under consideration

Consider a general process with a neutrino and an antineutrino[2] of the same flavor in the final state, say

$$X(p_X) \to Y(p_Y)\,\nu(p_1)\,\overline{\nu}(p_2),$$

where X, Y can be single or multi-particle states, Y can also be null, the contents of X and Y (if it exists) are visible particle/s and the 4-momenta p_X, p_Y are assumed to be well measured so that one can unambiguously infer the total missing 4-momentum of $\nu\,\overline{\nu}$, $p_{\text{miss}} = p_1 + p_2$. The 4-momentum of X must either be fixed by design of the experiment (e.g. X might be a particle produced at rest in the laboratory or be the constituent of a collimated beam of known energy or it could consist of two colliding particles of known 4-momenta) or the 4-momentum of X be inferred from the fully-tagged partner particle with which it is pair-produced. The final state Y should not contain any additional neutrinos or antineutrinos. The process could be a decay or scattering depending on whether X is a single particle state or two particle state. Some actual processes that satisfy such criteria are $e^+ e^- \to \nu\,\overline{\nu}$, $Z \to \nu\,\overline{\nu}$, $e^+ e^- \to \gamma\,\nu\,\overline{\nu}$, $K \to \pi\,\nu\,\overline{\nu}$, $B \to K\,\nu\,\overline{\nu}$, $R \to \mu^+\mu^-\nu_\mu\,\overline{\nu}_\mu$ with $R = B^0, H, J/\psi, \Upsilon(1s)$, etc.

A word of caution: the process $X \to Y\,\nu\,\overline{\nu}$ is *not necessarily* a neutral current process, and could proceed through other means such as by doubly weak charged currents. To keep our analysis model-independent we allow the process $X \to Y\,\nu\,\overline{\nu}$ to proceed even via NP interactions. We do not consider any specific NP possibility, but simply ensure that whenever explicit NP contributions are needed there are no Lorentz-symmetry violation as well as CPT violation in the underlying effective Lagrangian.

It should be noted that in this work we discuss processes where the effect of measurements does not destroy the identical nature of Majorana neutrino and antineutrino. This is akin to putting the constraint that in a double-slit experiment, meant to observe the interference of light, no measurement should identify the slit through which the photon has passed.

2.2 Origin of observable difference between Dirac and Majorana neutrinos and practical Dirac Majorana Confusion Theorem (pDMCT)

We now recall some features from our earlier work for completeness. The transition amplitude is, in general, dependent on all the 4-momenta. For brevity of expression and without loss of generality, we denote the transition amplitude by only mentioning the p_1, p_2 dependence. For Dirac neutrinos, the transition amplitude for $X \to Y\,\nu\,\overline{\nu}$ can be written as,

$$\mathscr{M}^D = \mathscr{M}(p_1, p_2), \tag{1}$$

while for Majorana case the amplitude is anti-symmetrized[3] with respect to the exchange of the Majorana neutrino and antineutrino which are quantum mechanically identical fermions,

$$\mathscr{M}^M = \frac{1}{\sqrt{2}}\Big(\underbrace{\mathscr{M}(p_1, p_2)}_{\text{Direct amplitude}} - \underbrace{\mathscr{M}(p_2, p_1)}_{\text{Exchange amplitude}} \Big), \tag{2}$$

where $1/\sqrt{2}$ takes care of the symmetry factor. Note that the amplitudes of Eqs. (1) and (2) do not necessarily assume the SM interactions, they can involve NP effects as well, and hence they include the most general structures of the amplitude that are allowed by Lorentz invariance.

[2] Note that although Majorana antineutrino is indistinguishable from Majorana neutrino, we keep using the notation of $\overline{\nu}$ for antineutrino and ν for neutrino simply as a book-keeping device.

[3] For any clarification regarding amplitude anti-symmetrization in case of Majorana neutrino, please see Sect. 5.2.

The difference between Dirac and Majorana cases that can possibly be probed is obtained after squaring the amplitudes (including the usual summation over final spins of ν, $\overline{\nu}$ and averaging over initial spins[4]) and taking their difference, which is given by,

$$\left|\mathscr{M}^D\right|^2 - \left|\mathscr{M}^M\right|^2 = \frac{1}{2}\Big(\underbrace{|\mathscr{M}(p_1, p_2)|^2}_{\text{Direct term}} - \underbrace{|\mathscr{M}(p_2, p_1)|^2}_{\text{Exchange term}}\Big) + \underbrace{\mathrm{Re}\left(\mathscr{M}(p_1, p_2)^* \mathscr{M}(p_2, p_1)\right)}_{\text{Interference term}}. \quad (3)$$

From Eq. (3) it is easy to conclude that there are essentially two major sources of any possible difference between Dirac and Majorana cases:

1. Unequal contributions from "Direct term" and "Exchange term" in general, i.e.

$$\underbrace{|\mathscr{M}(p_1, p_2)|^2}_{\text{Direct term}} \neq \underbrace{|\mathscr{M}(p_2, p_1)|^2}_{\text{Exchange term}}. \quad (4)$$

   For the special cases that satisfy $|\mathscr{M}(p_1, p_2)|^2 = |\mathscr{M}(p_2, p_1)|^2$ see Sect. 2.3.
2. Non-zero contribution from the "Interference term", i.e.

$$\underbrace{\mathrm{Re}\left(\mathscr{M}(p_1, p_2)^* \mathscr{M}(p_2, p_1)\right)}_{\text{Interference term}} \neq 0. \quad (5)$$

   It is interesting to note that in the case of the SM the interference term always depends on the size of the neutrino mass, that is

$$\underbrace{\mathrm{Re}\left(\mathscr{M}_{\mathrm{SM}}(p_1, p_2)^* \mathscr{M}_{\mathrm{SM}}(p_2, p_1)\right)}_{\text{Interference term}} \propto m_\nu^2. \quad (6)$$

   In presence of NP contributions, the full interference term need not follow Eq. (6).

The above sources of difference between Dirac and Majorana cases at the level of amplitude square, may or may not survive at the level of observables[5] which requires appropriate phase space considerations. We note that in the case when *no individual information about $\nu\,\overline{\nu}$ are either known or deducible*, the only difference between Dirac and Majorana cases that can be experimentally accessed is obtained after full integration over p_1 and p_2 which gives,

$$\iint \left(\left|\mathscr{M}^D\right|^2 - \left|\mathscr{M}^M\right|^2\right) \mathrm{d}^4 p_1\, \mathrm{d}^4 p_2 = \iint \underbrace{\mathrm{Re}\left(\mathscr{M}(p_1, p_2)^* \mathscr{M}(p_2, p_1)\right)}_{\text{Interference term}} \mathrm{d}^4 p_1\, \mathrm{d}^4 p_2, \quad (7)$$

which is directly proportional to m_ν^2 if only the SM interactions are considered. Here we have used the fact that although, in general, the "Direct" and "Exchange" terms differ as shown in Eq. (4), when we fully integrate over the 4-momenta of neutrino and antineutrino we get

$$\iint \underbrace{|\mathscr{M}(p_1, p_2)|^2}_{\text{Direct term}} \mathrm{d}^4 p_1\, \mathrm{d}^4 p_2 = \iint \underbrace{|\mathscr{M}(p_2, p_1)|^2}_{\text{Exchange term}} \mathrm{d}^4 p_1\, \mathrm{d}^4 p_2, \quad (8)$$

as p_1 and p_2 act as dummy variables since the range of integration is identical.

In the simple processes with the SM mediated interaction alone (e.g. weak neutral current mediated decay $Z^{(*)} \to \nu\bar{\nu}$) one finds that, (1) the "Direct term" and "Exchange term" are equal (see Sect. 2.3), (2) the "Interference term" is proportional to m_ν^2 and (3) the observable usually requires full phase space integration over $p_{1,2}$. This leads to the conclusion that all kinematical observable differences between Dirac and Majorana cases would be proportional to m_ν^2. This is essentially the statement of the "practical Dirac–Majorana Confusion Theorem" (pDMCT).

Note that our model-independent and process-independent analysis suggests that if (1) the doubly weak charged current processes[6] that *possibly* lead to a non-zero difference between "Direct term" and "Exchange term" of Eq. (3) and (2) some kinematic configurations could be identified where individual information about ν, $\overline{\nu}$ can be accessed so as to avoid doing the full phase space integration in Eq. (7), then one *might* avoid pDMCT constraint.

[4] This is the usual procedure unless one is interested in quantities that depend on the neutrino spin-projections, such as what is intended in Refs. [7,8]. In this work, just like in Refs. [5,6], we do not consider any neutrino spin-dependent observable. Since, active sub-eV neutrinos remain undetected close to their place of production, their spin-projections also remain experimentally inaccessible. Therefore, all our amplitude squares in this work include summation over final spins and average over initial spins. For a generic discussion with spin-dependent amplitudes have a look at Sect. 5.2.

[5] Here by an observable we mean a physical quantity for which we can make certain predictions from theory for Dirac and Majorana nature of neutrinos and which can be accessed experimentally.

[6] For the sequential weak charged current mediated decays that produce neutrino and antineutrino of different flavors, e.g. $\ell^- \to \nu_\ell\, \overline{\nu}_{\ell'}\, \ell'^-$, where $\ell, \ell' \in \{e, \mu, \tau\}$ and one can never have $\ell = \ell'$, the ν_ℓ and $\overline{\nu}_{\ell'}$ can never be considered as identical fermions even if they might indeed be Majorana fermions. We do not consider such processes in our analysis.

2.3 $Z^{(*)} \to \nu\bar{\nu}$ in the SM and special cases of $|\mathscr{M}(p_1, p_2)|^2 = |\mathscr{M}(p_2, p_1)|^2$

There are certain special cases when Eq. (4) is not satisfied. These are (a) collinear case: $p_1 = p_2$, (b) symmetric case: $\mathscr{M}(p_1, p_2) = \mathscr{M}(p_2, p_1)$ and (c) antisymmetric case: $\mathscr{M}(p_1, p_2) = -\mathscr{M}(p_2, p_1)$. As an example, within the SM for the neutral current mediated processes $Z^{(*)} \to \nu\bar{\nu}$, such as $Z \to \nu\bar{\nu}$, $e^+e^- \to \nu\bar{\nu}$, $B \to K\nu\bar{\nu}$ and etc. [5], one gets equal direct and exchange terms,

$$\underbrace{|\mathscr{M}(p_1, p_2)|^2}_{\text{Direct term}} = \underbrace{|\mathscr{M}(p_2, p_1)|^2}_{\text{Exchange term}}. \tag{9}$$

Therefore, for these processes we find that even at the level of amplitude square,

$$|\mathscr{M}^D|^2 - |\mathscr{M}^M|^2 \propto m_\nu^2, \tag{10}$$

which suggests that in such a case pDMCT holds true always without any exception. Since in this case pDMCT holds at the amplitude square level, it naturally holds true for all observables.[7] See Sect. 3 to find out how this conclusion changes in presence of NP.

3 New Physics scenarios and pDMCT in $Z^{(*)} \to \nu_\ell\,\bar{\nu}_\ell$

There is no reason a priori for the "practical DMCT" to hold, if NP contributions in the neutrino interactions are allowed, as in this case Eq. (3) "Direct" and "Exchange" terms in general do not need to cancel each other. To illustrate it more clearly using symmetry properties of the transition amplitude, let us assume that some (yet unknown) NP at high energy modifies the low energy effective neutrino interactions $Z^{(*)} \to \nu_\ell\,\bar{\nu}_\ell$.

Considering Lorentz invariance, CP and CPT conservation, applying Gordon identities as well as neglecting any m_ν dependent terms at the amplitude level, we find that the most general decay amplitude for $Z(p) \to \nu(p_1)\,\bar{\nu}(p_2)$ is as follows (for Dirac neutrinos) [5],

$$\begin{aligned}\mathscr{M}^D &= \mathscr{M}(p_1, p_2)\\ &= -\frac{i\,g_Z}{2}\epsilon_\alpha(p)\left[\overline{u}(p_1)\,\gamma^\alpha\left(C_V^\ell - C_A^\ell\,\gamma^5\right)v(p_2)\right],\end{aligned} \tag{11}$$

where $g_Z = e/(\sin\theta_W\,\cos\theta_W)$ with θ_W being the weak mixing angle and e being the electric charge of positron, and for different lepton family $\ell = e, \mu, \tau$ we have the possibility of having different vector and axial-vector coupling parameters C_V^ℓ, C_A^ℓ. Since we are considering NP possibilities here, we can write the vector and axial-vector coupling parameters as follows,

$$C_{V,A}^\ell = \frac{1}{2} + \varepsilon_{V,A}^\ell, \tag{12}$$

where ε_V^ℓ, ε_A^ℓ parameterise the NP effects, vanishing in the SM case. The amplitude for Majorana case is given[8] by

$$\begin{aligned}\mathscr{M}^M &= \frac{1}{\sqrt{2}}\Big(\mathscr{M}(p_1, p_2) - \mathscr{M}(p_2, p_1)\Big)\\ &= \frac{i\,g_Z\,C_A^\ell}{\sqrt{2}}\epsilon_\alpha(p)\,\left[\overline{u}(p_1)\,\gamma^\alpha\,\gamma^5\,v(p_2)\right].\end{aligned} \tag{13}$$

It is clear that we can combine the direct and exchange amplitudes in this case and effectively redefine the vertex structure for $Z \to \nu_\ell\,\overline{\nu}_\ell$ when Majorana neutrinos are considered.

Keeping neutrino mass dependent terms in the amplitude squares, we get different results for Dirac and Majorana neutrinos:

$$\begin{aligned}\left|\mathscr{M}^D\right|^2 &= \frac{g_Z^2}{3}\bigg(\Big((C_V^\ell)^2 + (C_A^\ell)^2\Big)\Big(m_Z^2 - m_\nu^2\Big)\\ &\qquad + 3\Big((C_V^\ell)^2 - (C_A^\ell)^2\Big)\,m_\nu^2\bigg),\end{aligned} \tag{14}$$

$$\left|\mathscr{M}^M\right|^2 = \frac{2\,g_Z^2\,(C_A^\ell)^2}{3}\Big(m_Z^2 - 4\,m_\nu^2\Big), \tag{15}$$

such that

$$\begin{aligned}&\left|\mathscr{M}^D\right|^2 - \left|\mathscr{M}^M\right|^2\\ &= \frac{g_Z^2}{3}\bigg(\Big((C_V^\ell)^2 - (C_A^\ell)^2\Big)\Big(m_Z^2 + 2\,m_\nu^2\Big) + 6\,(C_A^\ell)^2\,m_\nu^2\bigg)\\ &= \begin{cases}\dfrac{g_Z^2}{2}m_\nu^2, & \left(\begin{array}{c}\text{for the SM}\\ \text{alone}\end{array}\right)\\ \dfrac{g_Z^2}{3}\left(\varepsilon_V^\ell - \varepsilon_A^\ell\right)m_Z^2, & \left(\begin{array}{c}\text{with NP but}\\ \text{neglecting } m_\nu\end{array}\right)\end{cases}\end{aligned} \tag{16}$$

where we have kept only the leading order contributions of $\varepsilon_{V,A}^\ell$ while considering NP effects. It is clear that the SM result is fully in agreement with "practical Dirac Majorana confusion theorem" even at the amplitude-squared level, i.e. in the limit $m_\nu \to 0$ there is no observable difference between Dirac and Majorana cases in the SM via the process $Z \to \nu\,\bar{\nu}$. It implies

$$|\mathscr{M}(p_1, p_2)|^2 = |\mathscr{M}(p_2, p_1)|^2,$$

[7] The equality of direct and exchange terms, either at amplitude square level as shown in Eq. (9) or at the level of experimentally measurable observable that involves full phase space integration shown in Eq. (8), is often generalized as existence of one-to-one correspondence between Dirac and Majorana neutrinos in the massless limit $m_\nu \to 0$. See Sect. 5.1 for more details.

[8] Anti-symmetrized amplitude for Majorana neutrino can have contributions only from scalar, pseudo-scalar and axial-vector interactions (if the process is mediated via a neutral current), as shown in [5,9].

within the SM. However, the difference between Dirac and Majorana neutrinos appears in context of NP contributions even when one neglects m_ν dependent terms (unless, of course, $\varepsilon_V^\ell = \varepsilon_A^\ell$ in which case the additional NP contributions effectively rescale the SM allowed $V-A$ coupling). Possible example of NP effects in this Z boson decay could arise from kinetic mixing of Z with the neutral gauge bosons from extra gauge groups like additional $U(1)$ or $SU(2)_R$. In this work we are not concerned with any specific model of NP to keep our results and discussions very general.

Note that for the neutral current mediated processes in the SM, such as $Z \to \nu\bar{\nu}$, $e^+e^- \to \nu\bar{\nu}$, $B \to K\nu\bar{\nu}$ etc., the deduction of energy and/or momentum of the invisible neutrinos does not help to distinguish between Dirac and Majorana neutrinos because the pDMCT holds true even at the amplitude-squared level. In such a case every conceivable observable gives the same result for Dirac and Majorana neutrino, except showing the tiny difference coming from the interference term which is proportional to the neutrino mass. Therefore, only in the presence of physics beyond the SM [5,10,11] a distinction may be made between Dirac and Majorana nature of neutrinos through such processes.

4 Practical Dirac–Majorana Confusion Theorem and its exceptions

4.1 The general strategy to probe nature of neutrinos and pDMCT

The general formalism discussed in previous sections may be suitably illustrated by the chart shown in Fig. 1. The red (color in online) arrows drawn in the figure present the previous works of the pDMCT, all of which have studied the weak neutral current processes within the SM, such as $\gamma^* \to \nu\,\bar{\nu}$ [1], $Z \to \nu\,\bar{\nu}$ [12], $e^+e^- \to \nu\,\bar{\nu}$ [7], $K^+ \to \pi^+\,\nu\,\bar{\nu}$ [13], $e^+e^- \to \nu\,\bar{\nu}\,\gamma$ [8], $|es\rangle \to |gs\rangle + \nu\,\bar{\nu}\,\gamma$ [14], $e^-\gamma \to e^-\,\nu\,\bar{\nu}$ [15], $\nu + N \to N\,\nu\,\gamma$ [16], etc. As can be easily seen, all of these processes have confirmed the pDMCT because (1) $|\mathscr{M}(p_1, p_2)|^2 = |\mathscr{M}(p_2, p_1)|^2$ due to the weak neutral current processes within the SM and/or (2) there is no way to observe or deduce the 4-momenta of ν and/or $\bar{\nu}$ as these are simple 2- or 3-body processes.

Although there is no general, model-independent, process-independent and observable-independent proof[9] of the "practical Dirac Majorana confusion theorem", it is generally assumed to apply to all the probes of Majorana nature of sub-eV neutrinos. The formalism presented in Sects. 2 and 3 provides a simple model-independent, process-independent and observable independent view of the pathways by which the confusion theorem can be overcome, such as by using the properties of the NP interactions or analysing the "special kinematical scenarios" utilising the chosen parts of neutrino momentum spectra.

In order to illustrate the general strategy within the SM we would like to point out that the 2-body and 3-body $\nu\bar{\nu}$ final state processes are not suitable for the special kinematic scenarios because those decays are weak neutral current processes. On the other hand, 4-body decays such as $B, D, K, H, J/\psi, \Upsilon(1s), \ldots \to \mu^+\,\mu^-\,\nu_\mu\,\overline{\nu}_\mu$, which are doubly charged weak decay processes [6], could be more relevant for utilising the dependence of decay distributions on kinematic variables to distinguish between Dirac and Majorana neutrino in the case of the SM-like interactions. The main advantage of 4-body decays over the 2/3-body decays is the multitude of kinematic configurations and related observables that can be explored for the purpose of distinguishing between Dirac and Majorana neutrinos. The difference between Dirac and Majorana neutrinos that exists at the level of amplitude square, requires a proper observable so that we can access that difference. This once again highlights how crucial an observable is in this context. In the next subsection we summarize our findings on how to *possibly* overcome the pDMCT to distinguish between Dirac and Majorana neutrino using the special kinematics in $B^0 \to \mu^+\,\mu^-\,\nu_\mu\,\overline{\nu}_\mu$ decay in the SM.

4.2 How to overcome pDMCT within the SM by using the special kinematics in $B^0\,(D,\ H,\ J/\psi,\ \Upsilon(1s), \ldots) \to \mu^+\,\mu^-\,\nu_\mu\,\overline{\nu}_\mu$

In most of the experimental scenarios, especially true for processes of the form $X \to Y\,\nu\,\bar{\nu}$, information about individual neutrino momenta is not available. In such a case the difference between the Dirac and the Majorana neutrinos, that may be realised, is given by the integrated interference term in Eq. (7). In such a case the evaluation of the squared Feynman diagram for the "Interference term" in the SM necessarily involves two helicity flips which would make it proportional to m_ν^2. Thus, *if only the SM interactions are considered and one fully integrates over the neutrino and antineutrino 4-momenta, the difference between Dirac and Majorana cases is proportional to* m_ν^2. This may be considered as the most general statement of the "practical Dirac Majorana confusion theorem".

However, there is no reason a priori for the pDMCT to hold, if one can consider special kinematic configurations where the 4-momenta (or some components of the 4-momenta) of the neutrino and antineutrino are known, so that

[9] We use the phrase 'observable-independent proof' to underline the fact that no mathematical proof of pDMCT can be given without referring to any specific observable. So observables play a pivotal role in the discussion on pDMCT. As there is no observable-independent proof of pDMCT, one is free to explore different possible observables to distinguish between Dirac and Majorana nature of neutrinos in the specific context of one's chosen process.

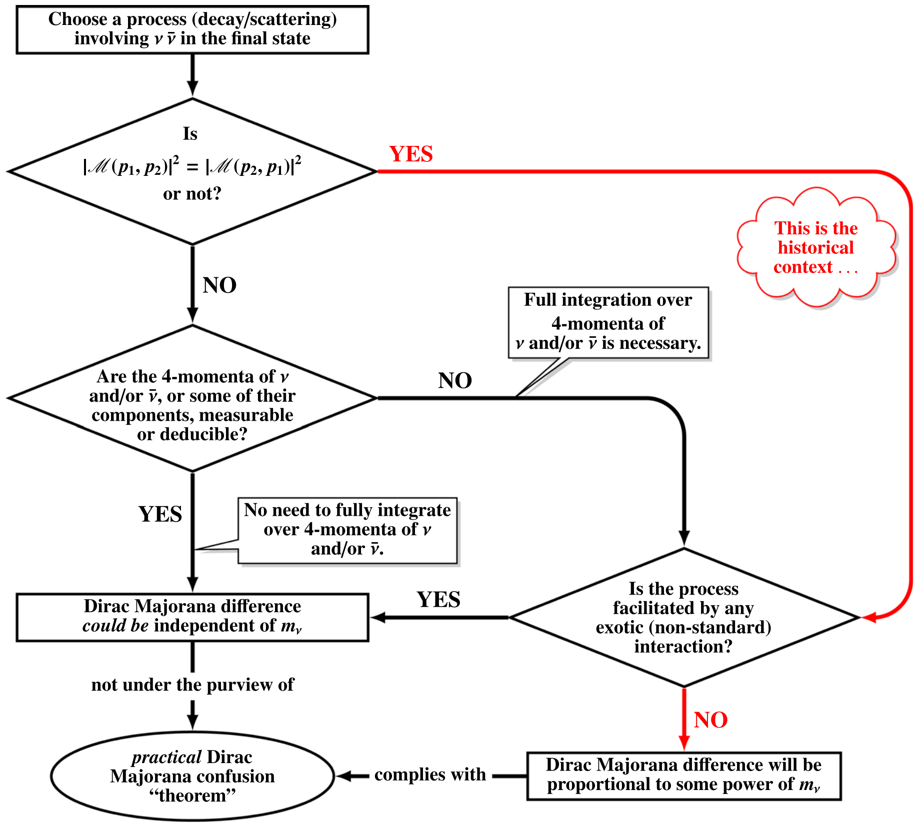

Fig. 1 Distinct pathways that can be explored to probe the Majorana nature of sub-eV neutrinos and overcome the limitations imposed by the "practical Dirac Majorana confusion theorem"

full integration over 4-momenta is not necessary before comparison with the experiment. This is valid even in the SM. In the following sub-subsections we discuss this scenario where such situation may occur.

4.2.1 *pDMCT and the doubly weak charged current decay* $B^0 \to \mu^-\,\mu^+\,\nu_\mu\,\bar{\nu}_\mu$

The decay $B^0 \to \mu^-\,\mu^+\,\nu_\mu\,\bar{\nu}_\mu$ takes place via doubly weak charged currents since flavor changing neutral currents are impossible at tree-level in the SM. The branching ratio of this mode gets substantial contributions from intermediate resonances such as π^- and D^-. Details on the process have been thoroughly investigated in Ref. [6]. Similar 4-body decays such as $B,\ D,\ K,\ H,\ J/\psi,\ \Upsilon(1s),\ldots \to \mu^+\,\mu^-\,\nu_\mu\,\bar{\nu}_\mu$ could be studied in an analogous manner.

The Eqs. (31, 32) in Ref. [6] clearly show the unequal contributions from "Direct term" and "Exchange term", which satisfy our Eq. (4)

$$\underbrace{|\mathscr{M}(p_1,p_2)|^2}_{\text{Direct term}} \neq \underbrace{|\mathscr{M}(p_2,p_1)|^2}_{\text{Exchange term}},$$

unlike the weak neutral current processes. As explained in Sect. 2.2 if we can measure or deduce the individual energy or momentum of the missing neutrino, then the pDMCT constraint will not apply.

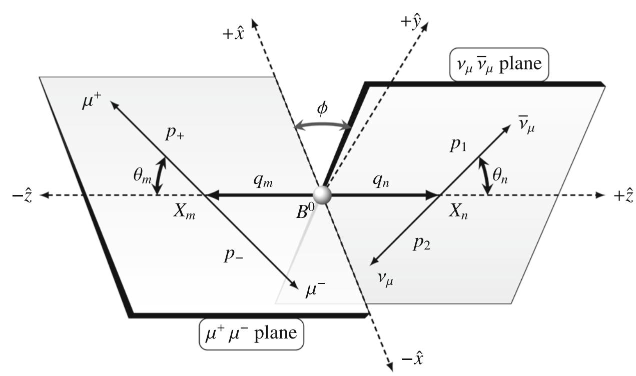

Fig. 2 The general kinematics of $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ in the rest frame of B, showing the polar angles θ_m and θ_n, as well as the azimuthal angle ϕ. Here X_m and X_n denote the muon pair and the neutrino pair

In Fig. 2, we show the general kinematics of $B^0 \to \mu^-\mu^+\nu_\mu\bar{\nu}_\mu$ in the rest frame of B. The angles θ_n and ϕ are indeed inaccessible in general, as the neutrino pair goes missing. Therefore, for a physically useful differential decay rate we must integrate over both θ_n and ϕ, i.e.

$$\frac{\mathrm{d}^3\Gamma^{D/M}}{\mathrm{d}m^2_{\mu\mu}\ \mathrm{d}m^2_{\nu\nu}\ \mathrm{d}\cos\theta_m} = \frac{Y\,Y_m\,Y_n}{(4\,\pi)^6 m_B^2\, m_{\mu\mu}\, m_{\nu\nu}} \times \int_{-1}^{1}\int_0^{2\pi} \left\langle \left|\mathscr{M}^{D/M}\right|^2 \right\rangle \mathrm{d}\cos\theta_n\ \mathrm{d}\phi, \tag{17}$$

where Y is the magnitude of 3-momentum of the di-muon system (muon pair with invariant mass $m_{\mu\mu}$) or di-neutrino system (neutrino pair with invariant mass $m_{\nu\nu}$) in the rest frame of B^0, Y_m is the magnitude of 3-momentum of $\mu^\pm$ in the di-muon rest frame, Y_n is the magnitude of 3-momentum of ν_μ or $\overline{\nu}_\mu$ in the di-neutrino rest frame. It is straightforward to show that the difference between Dirac and Majorana cases, as shown in Eqs. (33, 34) of [6], is given by

$$\frac{\mathrm{d}^3\Gamma^M}{\mathrm{d}m_{\mu\mu}^2\,\mathrm{d}m_{\nu\nu}^2\,\mathrm{d}\cos\theta_m} - \frac{\mathrm{d}^3\Gamma^D}{\mathrm{d}m_{\mu\mu}^2\,\mathrm{d}m_{\nu\nu}^2\,\mathrm{d}\cos\theta_m} \propto m_\nu^2, \quad (18)$$

which agrees with the pDMCT. Therefore, even in the case of a doubly weak charged current mediated decay process if we integrate fully over the available phase space of the invisible neutrino pair, we confirm pDMCT as expected from our discussion in Sect. 2.2. The situation changes, if and when we access a special kinematic scenario where the individual energy or 3-momentum of the invisible neutrinos can be inferred. In the following we discuss such a special kinematic situation.

4.2.2 *Back-to-back muon special kinematic configuration in* $B^0 \to \mu^-\,\mu^+\,\nu_\mu\,\bar{\nu}_\mu$

Consider the decay of the parent B^0 in its rest frame in which the μ^+, μ^- back to back with equal but opposite 3-momenta. Experimentally this is an ideal situation since it is easier to detect muons. The neutrino antineutrino pair must also fly away back-to-back since 3-momentum is conserved. This is a much simpler kinematic configuration than the general kinematics for any 4-body decay. Instead of the usual five independent variables one needs to describe any 4-body decay, we only need two independent variables to describe the back-to-back configuration. In this case, the energies of the two muons are the same and let us denote them by E_μ. Similarly, the energies of the back-to-back neutrino and antineutrino are the same and let us denote them by E_ν. Either E_μ or E_ν is independent, because from conservation of energy we get,

$$E_\mu + E_\nu = m_B/2, \quad (19)$$

where m_B is the mass of the B^0 meson. Let us choose E_μ as one independent variable. The other independent variable would then be the angle, say θ, between the muon direction and the neutrino direction.

For back-to-back case, with $E_1 = E_2 = E_\nu$ (say) and the angle between the two neutrinos $\Theta = \pi$, we get the following,

$$m_{\nu\nu}^2 = 4\,E_\nu^2, \quad (20a)$$
$$m_{\mu\mu}^2 = (m_B - 2\,E_\nu)^2\,. \quad (20b)$$

Moreover, for the back-to-back case we have

$$Y_m = \sqrt{\left(\frac{m_B}{2} - E_\nu\right)^2 - m_\mu^2}, \quad (21a)$$
$$Y_n = \sqrt{E_\nu^2 - m_\nu^2}. \quad (21b)$$

It can be shown that, in general,

$$\cos\theta_n = \frac{m_{\nu\nu}\,(E_1 - E_2)}{2\,Y\,Y_n}. \quad (22)$$

Whenever $E_1 = E_2$ for any value of the angle Θ between the neutrino and antineutrino we get $\cos\theta_n = 0$. By analytic continuation we extend this feature to the back-to-back kinematics for which the $\cos\theta_n$ has a discontinuity otherwise. Moreover, in the back-to-back case we have both the back-to-back muons and the back-to-back neutrino antineutrino pair, in one single plane. This implies that for the back-to-back case we have $\phi = 0$. These choices put the orientation of the coordinate axes in such a way that the back-to-back neutrino and antineutrino fly away defining the x-axis. The xz-plane in Fig. 2 is the one in which the 3-momenta of muons lie, and now the back-to-back neutrino antineutrino define the x-direction. The direction perpendicular to the neutrino direction is the z-direction. If we define the angle between the neutrino and muon directions to be θ, then $\theta_m = \pi/2 - \theta$. This implies that

$$\cos\theta_m = \sin\theta. \quad (23)$$

The differential decay rate in the back-to-back case is therefore given by,

$$\frac{\mathrm{d}^3\Gamma_{\leftrightarrow}^{D/M}}{\mathrm{d}E_\mu^2\,\mathrm{d}\sin\theta} = \frac{2\sqrt{E_\mu^2 - m_\mu^2}}{(4\,\pi)^6\,m_B\,E_\mu}\left(\left(\frac{m_B}{2} - E_\mu\right)^2 - m_\nu^2\right) \left\langle\left|\mathscr{M}_{\leftrightarrow}^{D/M}\right|^2\right\rangle, \quad (24)$$

where $\left\langle\left|\mathscr{M}_{\leftrightarrow}^{D/M}\right|^2\right\rangle$ is same as $\left\langle\left|\mathscr{M}^{D/M}\right|^2\right\rangle$ with the necessary dot product substitutions in the back-to-back case (this is the meaning of the subscript '$\leftrightarrow$'). Please note that the difference between the integrated widths of $\Gamma_{\leftrightarrow}^D$ and $\Gamma_{\leftrightarrow}^M$ can be very large as shown in Eq. (51) of [6], and computable in the SM or any other framework. It is also independent of the magnitude of the unknown neutrino mass for the leading terms.

For simplicity we neglect the masses of muons and neutrinos in comparison with the mass of B^0 as well as the energies. Note again that this does not mean that we consider muons and neutrinos to be massless. With this condition we find that only the non-resonant contributions survive. We consider only the dominant form factor contribution, and assume it to be a constant form factor. The full differential back-to-back decay rates are then given by,

$$\frac{\mathrm{d}^3\Gamma^D_{\leftrightarrow}}{\mathrm{d}E_\mu^2\,\mathrm{d}\sin\theta} = \frac{G_F^4\,|\mathbb{F}_a|^2\,(m_B - 2\,E_\mu)^4\,K_\mu}{512\,\pi^6\,m_B\,E_\mu} \times (E_\mu - K_\mu\,\cos\theta)^2\,, \tag{25a}$$

$$\frac{\mathrm{d}^3\Gamma^M_{\leftrightarrow}}{\mathrm{d}E_\mu^2\,\mathrm{d}\sin\theta} = \frac{G_F^4\,|\mathbb{F}_a|^2\,(m_B - 2\,E_\mu)^4\,K_\mu}{512\,\pi^6\,m_B\,E_\mu} \times \left(E_\mu^2 + K_\mu^2\cos^2\theta\right), \tag{25b}$$

where $K_\mu = \sqrt{E_\mu^2 - m_\mu^2}$ is the magnitude of the 3-momentum of the back-to-back muons. There are no m_ν dependent terms here. The muon energy distribution obtained after integrating over $\sin\theta$ shows that there exists non-zero difference between the Dirac and the Majorana cases. Moreover, the corresponding branching ratio for the back-to-back kinematics for Majorana case is more than 15 times bigger than that for the Dirac case. Thus, these results are not in agreement with pDMCT. Strictly speaking, since we are *not* integrating over the *full* phase space of neutrinos, the pDMCT need not apply in this case. The back-to-back kinematic configuration provides a way of realising this exception. For more details of the back-to-back kinematics and the related issues, please see Appendix A.

This result confirms the discussion in Sect. 2. However, it should be acknowledged that the reader may find the result contradictory or even counter intuitive to the previous understanding of the pDMCT. In the next section we try to clarify some of the frequently used concepts that are used to assert pDMCT as a generic feature irrespective of the process, or observable.

5 Discussions on concepts usually accompanying explanations of pDMCT

It is generally believed that all observable difference between Dirac and Majorana neutrinos must always be proportional to some power of neutrino mass m_ν [1], which is the content of practical Dirac–Majorana Confusion Theorem (pDMCT). However, all processes where the theorem was shown to hold involved either full integration over the 4-momenta of missing neutrinos and/or only for the weak neutral current process within the SM [5,6].

As we have discussed, pDMCT is not a fundamental theorem of neutrinos: pDMCT actually depends on physics models, processes and observables, e.g. even for $Z \to \nu\bar{\nu}$, pDMCT holds within the SM, but can be violated beyond the SM depending on the model parameters. Even within the SM, pDMCT depends on the processes, e.g. $B \to K\nu\bar{\nu}$ confirms pDMCT, but $B \to \mu^+\mu^-\nu\bar{\nu}$ can violate pDMCT. *Therefore, while the quantum statistics of Majorana neutrinos*[10] *is a fundamental property of neutrinos, pDMCT is not. Instead, pDMCT is an emergent phenomenological feature arising out of non-observation of neutrinos.*

In this section we give comments on the existence of any analytic continuity between Dirac and Majorana neutrinos in the limit $m_\nu \to 0$ and the issue on anti-symmetrization of amplitude while dealing with pair of identical Majorana neutrinos. We also address in the appendix some pedagogical issue on massless Majorana neutrino, which can be a fundamental difference between Dirac and Majorana neutrinos.

5.1 Is there any one-to-one correspondence between Dirac and Majorana neutrinos in the massless limit, $m_\nu \to 0$?

The issue of one-to-one correspondence between Dirac and Majorana neutrinos in the massless limit $m_\nu \to 0$ can be analyzed from the context of specific processes and observables. As mentioned in Sect. 2.3 if one considers neutral current mediated processes such as $Z \to \nu\,\overline{\nu}$ (and include summation over neutrino spins while evaluating amplitude squares) within the SM, the direct and exchange terms in amplitude square become equal and the difference between Dirac and Majorana neutrinos becomes proportional to m_ν^2, which vanishes if we were to simply apply the limit $m_\nu \to 0$ at the end.[11] However, when one considers processes that are not facilitated by the SM neutral current interactions, the direct and exchange terms can have non-zero difference even when we neglect m_ν dependent terms (or equivalently put $m_\nu \to 0$). In such a case some specific observable might be able to probe these important non-zero differences. *In these instances, in the context of a specific process and observable, there is indeed no one-to-one correspondence between the Dirac and Majorana neutrino scenarios.* However, in all cases with the SM only interactions if the observable includes full phase space integration over the neutrino and antineutrino, we do find that the direct and exchange terms have equal contribution after integration (see Eq. (8)) which amounts to no observable difference between the two scenarios in the limit $m_\nu \to 0$.

As explained in Appendix B, it is a mathematical impossibility to preserve Majorana nature of a fermion when its mass becomes zero. In fact due to Lorentz invariance and conservation of chirality for massless fermions, such chiral fermions have a distinct nomenclature as being Weyl fermions. To describe Weyl fermions it is sufficient to use 2-component complex spinors instead of 4-component complex spinors.

[10] *The quantum statistics of Majorana neutrino and antineutrino (which are quantum mechanically identical) does not depend on the size of their mass, but only on their spin.*

[11] As shown in Appendix B, we can not start from massless neutrinos as then the neutrinos can not have Majorana nature. At the end of full calculation, one can certainly apply the limit $m_\nu \to 0$ which amounts to neglecting m_ν dependent terms due to their tininess.

Nevertheless, the 2-component Weyl spinors can be used to construct 4-component complex Dirac spinors as well as 4-component Majorana spinors (that are real in the Majorana basis). Both the Dirac and Majorana spinors have both left- and right-chiral components. When one takes the massless limit or when one considers ultra-relativistic fermions one finds that these constructs of Dirac or Majorana spinors prefer specific chirality states. Nevertheless, as long as the mass of the fermion is non-zero both the chiral states are present. However, once the fermion is massless, it becomes fully chiral and it can not have Majorana nature at all. The Dirac nature (meaning its particle and antiparticle states are distinct and distinguishable) survives the massless limit. Therefore, although both Dirac and Majorana 4-component spinors get reduced to 2-component Weyl spinors in the massless limit, the Weyl spinors only show Dirac nature and the Majorana nature is completely lost. *This implies there is really no one-to-one correspondence between Dirac and Majorana nature of neutrinos in the massless limit.*

5.2 Should the amplitude be anti-symmetrized for pair of Majorana neutrinos of the same flavor with $m_\nu > 0$?

It is well known that when two identical particles are present in the final state, the corresponding transition amplitude needs to be symmetrized (or anti-symmetrized) with respect to their exchange if they are bosons (or fermions). Therefore, if a final state has two massive neutrinos or two massive antineutrinos of the same flavor (i.e. $\nu_\ell\, \nu_\ell$ or $\overline{\nu}_\ell\, \overline{\nu}_\ell$, with $\ell = e, \mu, \tau$) then the transition amplitude would *always* be anti-symmetric under their exchange (which involves exchange of 4-momenta and spin) irrespective of whether they are Dirac and Majorana fermions. This is to ensure the Fermi–Dirac statistics. However, if we have a final state that has $\nu_\ell\, \overline{\nu}_\ell$, then it has distinct particles for Dirac neutrinos, but it has identical particles for massive Majorana neutrinos. Thus, when considering Majorana nature of the massive neutrinos, one needs to anti-symmetrize the transition amplitude in this case. This is one of the main differences between Dirac and Majorana neutrinos, and it has been noted by many authors, (see e.g. [7,8,13] and etc.) before us. One simple example of this amplitude anti-symmetrization is, as shown in Sect. 3, the most general amplitude of $Z \to \nu\bar{\nu}$ for Dirac neutrino in Eq. (11) and for Majorana neutrino in Eq. (13).

If the 4-momenta (and spins) of ν_ℓ, $\overline{\nu}_\ell$ be denoted by p_1, p_2 (and s_1, s_2) respectively, then the transition amplitude for Dirac case can be symbolically written as,

$$\mathscr{M}^D \equiv \mathscr{M}^{s_1,s_2}(p_1, p_2), \tag{26}$$

while the amplitude for Majorana case would be,

$$\mathscr{M}^M \equiv \mathscr{M}^{s_1,s_2}(p_1, p_2) - \mathscr{M}^{s_2,s_1}(p_2, p_1). \tag{27}$$

In the calculation of the observable for the specific process in both Dirac and Majorana cases, one takes the square of the amplitude, does the usual trace calculations by summing over the final spins and averages over the initial spins, except when one is interested in an observable that depends on neutrino spins which is practically impossible to do experimentally for sub-eV active neutrinos. Thus the spin information gets wiped out via the spin summation. In context of the SM, we know that the $V - A$ nature of weak interaction ensures that we always get a left-chiral neutrino and a right-chiral antineutrino. However, despite being produced in specific chiral states, their chirality is not conserved due to non-zero mass, following Eq. (38). Due to non-zero mass, chirality is not same as helicity which is the projection of spin along the direction of motion. Thus, left and right helical massive neutrinos get produced from the SM weak interaction. *Thus we consider all spin possibilities of the massive eigenstates in our calculation. Any issues related to relativistic or non-relativistic kinematics are automatically taken care of by the field theoretic calculations for amplitude square with summation over final spins and average over initial spins.*

Another approach to include all spin possibilities in the final calculation leading to correct amplitude square is via splitting the full amplitude into all possible helicity amplitudes, where one specifies the individual helicities, say λ_1 and λ_2 instead of s_1 and s_2. In such a method, one has to exchange the helicities (equivalent to exchange of spins) for the Majorana case, i.e.

$$\mathscr{M}^D \equiv \sum_{\lambda_1,\lambda_2} \mathscr{M}^{\lambda_1,\lambda_2}(p_1, p_2), \tag{28a}$$

$$\mathscr{M}^M \equiv \sum_{\lambda_1,\lambda_2} \left(\mathscr{M}^{\lambda_1,\lambda_2}(p_1, p_2) - \mathscr{M}^{\lambda_2,\lambda_1}(p_2, p_1)\right). \tag{28b}$$

In the Majorana case amplitude square, it thus becomes clear that there will be interference terms that require helicity flip and such terms would be proportional to m_ν^2. If the SM interactions are only taken into account, then all the interference terms always involve helicity flips.

6 Conclusions

In this work, we have revisited the practical Dirac Majorana confusion theorem and studied its domain of applicability. We find that one should always keep the historical context of neutral currents in mind while applying this theorem rigorously. If the process involves doubly weak charged currents, or some new physics contributions, and if one can infer the energy or 3-momentum of neutrino and antineutrino using some special kinematic configurations, then this theorem need not hold true. It might hold true in specific processes, but this theorem does not have any generic, model-independent, process-independent and observable-independent proof. We have also highlighted and addressed some of the most commonly and easily misunderstood concepts that come to mind while thinking of this theorem.

As a final note, it is rather tempting to confirm the pDMCT and/or find out how to overcome the pDMCT from the fundamental Lagrangian level. The effective interaction Lagrangian always respects quantum statistics even though it might not be evident at the level of fundamental interaction Lagrangian. Since the Lagrangian pertaining to neutrino masses have neither a direct bearing on the effective interaction Lagrangian nor carry any signature of the quantum statistical difference we are interested in, the mass generating Lagrangians do not affect our analysis. As is previously explained, the use of basic weak neutral current interaction, $Z \to \nu\bar{\nu}$, will always lead to pDMCT within the SM. And the use of weak charged current processes, such as $W^{\pm} \to \ell^{\pm}\nu_\ell$ and $\ell^- \to \nu_\ell\ \overline{\nu}_{\ell'}\ \ell'^-$, do not introduce any identical Majorana neutrino pair, so no difference between Dirac and Majorana neutrinos can be probed using these. Only doubly weak charged 4-body decay processes, e.g. $B, D, K, H, J/\psi, \Upsilon(1s), \ldots \to \mu^+\mu^-\nu_\mu\overline{\nu}_\mu$, could be considered to find out whether the process gives equal contributions from "Direct term" and "Exchange term" – the only meaningful way to overcome pDMCT within the SM.

The neutrino-less double beta decay ($0\nu\beta\beta$) [17,18] has a limitation that it is dependent on the unknown tiny mass of the neutrino. If it is too small, there is no possibility of establishing the nature of the neutrino through $0\nu\beta\beta$. Our proposal to probe quantum statistics of Majorana neutrinos seems to be the only viable alternative to $0\nu\beta\beta$ as far as probing Majorana nature of sub-eV active neutrinos is concerned.

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Appendix A: Clarifications on back-to-back kinematics

In back-to-back kinematic configuration of Fig. 2, we note the following points.

1. All the final particles fly away in a *single decay plane* in the rest frame of the parent particle, i.e. $\vec{p}_+$, $\vec{p}_-$, $\vec{p}_1$ and $\vec{p}_2$ lie on a single plane, and

$$\vec{p}_1 + \vec{p}_2 = \vec{0}, \qquad \vec{p}_+ + \vec{p}_- = \vec{0}. \tag{29}$$

2. Only by assuming all the final particles in Fig. 2 to be massless, or by neglecting their masses, do we get equal energies for the particles flying back-to-back, i.e.

$$E_1 = E_2 \equiv E_\nu, \quad E_+ = E_- \equiv E_\mu, \tag{30}$$

and from conservation of energy

$$E_\nu + E_\mu = \frac{1}{2} m_B. \tag{31}$$

Thus knowing either E_ν or E_μ is sufficient.

3. The back-to-back kinematics for a measured event would specify E_μ as well as the angle θ shown in Fig. 2. Since ν_μ and $\bar{\nu}_\mu$ are invisible in the detector, the angle θ is experimentally unknown and therefore should be integrated out for the final observable.
4. The back-to-back configuration is a *special case* of the general kinematic configuration, and *not* arrived at by any integration or summation. The general kinematic configuration involves two decay planes (see Fig. 2), and requires five independent variables for complete specification (see Sec.IV.E of [6]). For full specification of the back-to-back kinematics one instead needs to specify, only the energy E_ν or E_μ (here we are making the massless assumption mentioned above) and the angle θ in Fig. 2. To come from general kinematics to the back-to-back kinematics one needs to *fix* certain quantities.

A.1 Important issues to address when coming to back-to-back kinematics from general kinematics

The angle between two decay planes: The usual description of general kinematics for the 4-body final state has two decay planes, with an angle ϕ between them. For back-to-back kinematics the two decay planes coincide to form a single decay plane. So for back-to-back kinematics $\phi = 0$. No integration over ϕ is involved to arrive at the final observables in back-to-back kinematics.

Discontinuity from general kinematics to back-to-back kinematics: The expression

$$\cos\theta_\nu = \frac{\sqrt{s_{\nu\bar{\nu}}}\,(E_\nu - E_{\bar{\nu}})}{2X\beta_\nu}, \tag{32}$$

has a discontinuity when $E_\nu \to E_{\bar{\nu}}$ and the angle $\Theta_{\nu\bar{\nu}}$ between ν_μ and $\bar{\nu}_\mu$ approaches π. In fact this expression yields 0/0 form if we simply substitute $E_\nu = E_{\bar{\nu}}$ and $\Theta_{\nu\bar{\nu}} = \pi$. Thus, one needs to apply L'Hospital's rule to get the limit. The limit $E_\nu \to E_{\bar{\nu}}$ with $\Theta_{\nu\bar{\nu}} = \pi$ yields ± 1, i.e. $\cos\theta_\nu$ has a discontinuity which needs to be resolved. Here we note that for $E_\nu = E_{\bar{\nu}}$ we get $\cos\theta_\nu = 0$, as long as $\Theta_{\nu\bar{\nu}} \neq \pi$. At $\Theta_{\nu\bar{\nu}} = \pi$ this discontinuity appears and it can be resolved by taking average of the two limits, which yields 0 and makes $\cos\theta_\nu$ continuous. The approach is similar to what is done to remove the discontinuity in Heaviside step function at $x = 0$. Once $\cos\theta_\nu$ and ϕ are put to zero, the rest of the results needs to be consistent with expectations from helicity arguments, as is found consistently in Ref. [6].

Inferred neutrino energy distribution: The neutrino energy distribution in back-to-back configuration requires that the neutrino energies be inferred. In the case of back-to-back events we have $E_1 = E_2 \equiv E_\nu = \frac{1}{2} m_B - E_\mu$. In our paper [6] in Eq. (35), we have shown that the difference between Dirac and Majorana neutrinos vanishes when full integration over neutrino phase space is done. Moreover as we have noted before, the back-to-back kinematics can be obtained from general kinematics not via any integration but by taking specific values of the parameters in the general kinematics. This makes the back-to-back kinematics a special case of the general kinematics.

For realistic experimental observation of back-to-back kinematics: Our treatment of back-to-back kinematics in Ref. [6] is purely mathematical, i.e. we have used the following exact conditions,

$$E_1 = E_2 = E_\nu, \quad \Theta = \pi, \quad \phi = 0, \quad \theta_m = \frac{\pi}{2} - \theta.$$

Out of these, the first and second conditions are primary ones, while the rest arise as a consequence of the first two criteria. However, none of these quantities are physically observable. The first two conditions also imply that $E_+ = E_- = E_\mu \equiv m_B/2 - E_\nu$ which is experimentally observable. Any error, on the muon energy measurement would imply that the muon energy distribution as shown in Fig. 5(c) of Ref. [6] would involve energy bins with bin width corresponding to the experimental error within which equality of E_+ and E_- is satisfied. There would also possibly be extremely slight deviation from the 180° angle between the two final muons, in an experimental back-to-back realization. This can lead to small deviation of Θ from π, say $\Theta = \pi \pm \Delta\Theta$. This implies $\cos\Theta = -\cos\Delta\Theta \approx -1 + \Delta\Theta^2/2$, so that the error in $\cos\Theta$ measurement is $\Delta\Theta^2/2$ and it gets multiplied to the back-to-back muon energy distribution of Eq. (50) of Ref. [6] to give the amount of smearing one can expect from the measurement. Thus, a real experimental realization of the back-to-back kinematics would lead to a muon energy distribution curve similar to Fig. 5(c) of Ref. [6] but it would be a histogram plot with bin size determined by the muon energy resolution and there will be some vertical smearing arising from the slight deviation from 180° angle requirement. However, the difference between Dirac and Majorana neutrinos should not get washed away as a result of such a tiny smearing from experimental measurements.

Appendix B: Can a massless neutrino with the SM interactions be a Majorana neutrino?

To properly address this question we need to make a small detour and start from the beginning, the Dirac equation itself,

$$\left(i\,\gamma^\mu\,\partial_\mu - m\right)\psi(x) = 0, \tag{33}$$

where $\psi(x)$ is the 4-component *complex* Dirac spinor field that describes a spin-$\frac{1}{2}$ fermion of mass m, and γ^μ ($\mu = 0, 1, 2, 3$) denote the set of four complex 4×4 matrices which satisfy the anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}$, and also ensure the hermiticity of the corresponding Dirac Hamiltonian via $\gamma^0\,(\gamma^\mu)^\dagger\,\gamma^0 = \gamma^\mu$. The important question to ask here is whether one can have a *real solution* of the Dirac equation. It turns out that, if one works in Majorana basis which has only imaginary γ matrices, say $\gamma^\mu = -i\,\tilde{\gamma}^\mu$ where $\tilde{\gamma}^\mu$ are real 4×4 matrices, then one can have a *real* spinor field $\tilde{\psi}(x)$ which satisfies the equation,

$$\left(\tilde{\gamma}^\mu\,\partial_\mu - m\right)\tilde{\psi}(x) = 0. \tag{34}$$

Such a basis of fully imaginary gamma matrices is called the Majorana basis and the real solution to Dirac equation is said to describe the Majorana fermion. The reality condition in Majorana basis,

$$\tilde{\psi}(x) = \tilde{\psi}^*(x) \tag{35}$$

when viewed from any other basis for the gamma matrices, yields that the Majorana spinor field be identical to its charge conjugate spinor field (more accurately it is the Lorentz-covariant conjugate). This implies that a Majorana fermion is one which is indistinguishable from its antiparticle state. From Eq. (34) it seems clear that one could have a massless Majorana fermion as well. However, in context of the fermion being neutrino, which gets produced only by the weak interaction in the SM, the answer is slightly more involved.

Before we address the issue with massless Majorana neutrino in the context of the SM weak interactions we need to take another detour. Using the four gamma matrices, one can always define a fifth gamma matrix, $\gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3$, called the *chirality matrix* and it commutes with other gamma matrices, it is its own Hermitian adjoint and its own inverse. We note that the matrix γ^5 is also fully imaginary in the Majorana basis. The usefulness of γ^5 is that it allows us to split the complex 4-component spinor field ψ into two distinct parts,

$$\psi = \psi_L + \psi_R, \tag{36}$$

where ψ_L, ψ_R are two distinct eigenfunctions of γ^5,

$$\gamma^5\psi_L = -\psi_L, \quad \gamma^5\psi_R = +\psi_R. \tag{37}$$

The two parts of ψ, namely ψ_L and ψ_R are called the *left-chiral* and *right-chiral* spinor fields. In terms of these chiral parts, the Dirac equation gets split into two equations,

$$i\,\gamma^\mu\,\partial_\mu\,\psi_R = m\,\psi_L, \tag{38a}$$
$$i\,\gamma^\mu\,\partial_\mu\,\psi_L = m\,\psi_R. \tag{38b}$$

The space-time evolution of either of the chiral spinor fields is dependent on the mass of the fermion as well as the other chiral spinor field. Thus, when the fermion has non-zero mass ($m \neq 0$) the chirality is not conserved. However, when $m = 0$, the chirality is not only conserved, but it also has the same physical meaning as helicity (which is the projection of the fermion spin along its direction of flight). Because a massless particle always travels with speed of light, it has the same chirality or helicity in all frames of reference. Therefore, massless one-half spin fermions of definite chirality are distinct particles. Upon charge conjugation (or Lorentz-covariant conjugation) the chirality of the particle gets reversed, implying that the chirality of the antiparticle is opposite to that of its particle. Thus, for a massless chiral spin-$\frac{1}{2}$ fermion chirality (or helicity) distinguish between particle and antiparticle which is against the requirement one has for it to qualify as a Majorana fermion. In the SM, the weak interaction always produces left-chiral neutrinos and right-chiral antineutrinos. If neutrino is taken to be a *massless* fermion ($m_\nu = 0$), then its chirality would be conserved and remain Lorentz invariant. Thus one can distinguish a massless SM neutrino from the corresponding antineutrino by its chirality. Another way to realize the impossibility of having a massless chiral fermion with Majorana nature is by asking the mathematical question whether a chiral spinor field can ever be real in the Majorana basis. Since γ^5 in Majorana basis is purely imaginary, its eigenfunctions $\psi_{R/L}$ with eigenvalues ± 1 in Eq. (37) can never be real. This is a mathematical impossibility. Therefore, one can never have massless chiral Majorana fermions or neutrinos.[12]

[12] The conclusion of this section is inspired from Ref. [19].

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