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Dixon-Rosenfeld lines and the Standard Model

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Abstract We present three new coset manifolds named Dixon-Rosenfeld lines that are similar to Rosenfeld projective lines except over the Dixon algebra $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. Three different Lie groups are found as isometry groups of these coset manifolds using Tits' formula. We demonstrate how Standard Model interactions with the Dixon algebra in recent work from Furey and Hughes can be uplifted to tensor products of division algebras and Jordan algebras for a single generation of fermions. The Freudenthal–Tits construction clarifies how the three Dixon-Rosenfeld projective lines are contained within $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$, $\mathbb{O}\otimes J_2(\mathbb{C}\otimes\mathbb{H})$, and $\mathbb{C}\otimes\mathbb{O}\otimes J_2(\mathbb{H})$.

Contents

1 Introduction and motivation

We focus on the definition of three coset manifolds of dimension 64 that we call *Dixon-Rosenfeld lines*. Each contains an isometry group whose Lie algebra is obtained from Tits' magic formula. These three constructions are obtained similarly to how projective lines are obtained over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$; therefore, they can be thought of as "generalized" projective lines over the Dixon algebra $\mathbb{T}\equiv\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ in the sense presented by Rosenfeld in [41–43].

The division algebras have been used for a wide variety of applications in physics [5,10,32,37,39,47]. In 1973, Gürsey and Günaydin discussed the relationship of octonions and split octonions to QCD [32,33]. Later, Dixon introduced the algebra $\mathbb{T}\equiv\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ for a single generation of fermions in the Standard Model [13–17]. This line of investigation was revived when Furey further explored the Standard Model with the Dixon algebra [22–28] and Castro introduced gravi-

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tational models involving the Dixon algebra [7–9].[1] Recently, Furey and Hughes focused on Weyl spinors for one generation of the Standard Model fermions with $\mathbb{T}$ [29,30].

Our work on Dixon-Rosenfeld lines defines three homogeneous spaces that locally embed a representation of $\mathbb{T}$ to encode one generation of fermions in the Standard Model. Section 2 shows that three coset manifolds of real dimension 64 are possible, giving three non-simple Lie algebras as isometry groups that are obtained from Tits formula. Section 3 analyzes the relationship between the new Dixon-Rosenfeld lines with the Rosenfeld lines. Section 4 uplifts scalar, spinor, vector, and 2-form representations of the Lorentz group representations with $\mathbb{C} \otimes \mathbb{H}$ from Furey [22] to $\mathbb{C} \otimes J_2(\mathbb{O})$. Section 5 uplifts the Standard Model fermionic charge sector described by Furey with $\mathbb{C} \otimes \mathbb{O}$ [24] to $\mathbb{C} \otimes J_2(\mathbb{O})$. Section 6 uplifts recent work by Furey and Hughes for encoding Standard Model interactions with $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ [29] to the three different realizations of the Dixon-Rosenfeld lines via $\mathbb{C} \otimes \mathbb{H} \otimes J_2(\mathbb{O})$, $\mathbb{O} \otimes J_2(\mathbb{C} \otimes \mathbb{H})$, and $\mathbb{C} \otimes \mathbb{O} \otimes J_2(\mathbb{H})$. Section 7 concludes with a summary of our work and outlines prospects for future work.

1.1 Tensor products on unital composition algebras

An *algebra* is a vector space X with a bilinear multiplication. Different properties of the multiplication give rise to numerous kind of algebras. Indeed, for what it will be used in the following sections, an algebra X is said to be *commutative* if $xy = yx$ for every $x, y \in X$, is *associative* if satisfies $x(yz) = (xy)z$, is *alternative* if $x(yx) = (xy)x$, *flexible* if $x(yy) = (xy)y$ and, finally, *power-associative* if $x(xx) = (xx)x$ and $(xx)(xx) = ((xx)x)x$. It is worth noting that the last four proprieties are progressive and proper refinements of associativity, i.e.

associative ⇒ alternative ⇒ flexible ⇒ power-associative.

Every algebra has a zero element $0 \in X$, since X has to be a group in respect to the sum, but if it also does not have zero divisors, then X is called a *division* algebra, i.e. if $xy = 0$ then or $x = 0$ or $y = 0$. While the zero element is always present in any algebra, if it exists an element $1 \in X$ such that $1x = x1 = x$ for all $x \in X$ then the algebra is *unital*. Finally, if we can define over X an involution, called *conjugation*, and a quadratic form N, called *norm*, such that

$$N(x) = x\overline{x}, \tag{1}$$

$$N(xy) = N(x)N(y), \tag{2}$$

with $x, y \in X$ and $\overline{x}$ as the conjugate of x, then the algebra is called a *composition* algebra.

A well-known theorem due to Hurwitz [34] states that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only four normed division algebras that are also unital and composition [4,19]. More specifically, $\mathbb{R}$ is also totally ordered, commutative and associative, $\mathbb{C}$ is just commutative and associative, $\mathbb{H}$ is only associative and, finally, $\mathbb{O}$ is only alternative, as summarized in Table 1.

Table 1 Ordinality, commutativity, associativity, alternativity, flexibility, and power associativity are summarized for the division algebras

Algebra	Ord	Comm	Ass	Alter	Flex	Pow. Ass
$\mathbb{R}$	Yes	Yes	Yes	Yes	Yes	Yes
$\mathbb{C}$	No	Yes	Yes	Yes	Yes	Yes
$\mathbb{H}$	No	No	Yes	Yes	Yes	Yes
$\mathbb{O}$	No	No	No	Yes	Yes	Yes

Since all four normed division algebras are vector spaces over the field of reals $\mathbb{R}$ we are able to define a tensor product $\mathbb{A} \otimes \mathbb{B}$ of two normed division algebras, with a bilinear product defined by

$$(a \otimes b)(c \otimes d) = ac \otimes bd, \tag{3}$$

where $a, c \in \mathbb{A}$ and $b, d \in \mathbb{B}$. The resulting tensor products are well known tensor algebras called $\mathbb{C}\otimes\mathbb{C}$ *Bicomplex*, $\mathbb{C}\otimes\mathbb{H}$ *Biquaternions*, $\mathbb{H}\otimes\mathbb{H}$ *Quaterquaternions*, $\mathbb{C}\otimes\mathbb{O}$ *Bioctonions*, $\mathbb{H} \otimes \mathbb{O}$ *Quateroctonions* and $\mathbb{O} \otimes \mathbb{O}$ *Octooctonions*. By the definition of the product, it is clear that all algebras involving the Octonions are not associative. Moreover, while Bioctonions $\mathbb{C}\otimes\mathbb{O}$ is an alternative algebra, Quateroctonions $\mathbb{H} \otimes \mathbb{O}$ and Octooctonions $\mathbb{O} \otimes \mathbb{O}$ are not alternative nor power-associative. Every alternative algebra tensor a commutative algebra yields again to an alternative algebra, so that with few additional efforts we can easily find all properties for triple tensor products listed in Table 2.

1.2 The Dixon algebra

The *Dixon Algebra* $\mathbb{T}$ is the $\mathbb{R}$-linear tensor product of the four normed division algebras, i.e. $\mathbb{R}\otimes\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ or equivalently $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$, with linear product defined by

$$(z \otimes q \otimes w)\left(z' \otimes q' \otimes w'\right) = zz' \otimes qq' \otimes ww', \tag{4}$$

with $z, z' \in \mathbb{C}$, $q, q' \in \mathbb{H}$ and $w, w' \in \mathbb{O}$. From the previous formula it is evident that $\mathbb{T}$ is unital with unit element $\mathbf{1} = 1 \otimes 1 \otimes 1$. As a real vector space, the Dixon Algebra has an $\mathbb{R}^{64}$ decomposition for which every element t is of the form

$$t = \sum_{\alpha=0}^{63} t^{\alpha} \; z \otimes q \otimes w, \tag{5}$$

[1] While Gürsey and Günaydin explored the use of split octonions $\mathbb{O}_s$ for quarks, Furey clarified how $\mathbb{C}\otimes\mathbb{O}$ leads to quarks and leptons [22], which contains both $\mathbb{O}$ and $\mathbb{O}_s$ as subalgebras. QCD gauges the compact $SU(3)$, which is a maximal subgroup of the automorphism group over the octonions $\mathrm{Aut}(\mathbb{O}) = G_2$.

Table 2 Commutativity, associativity, alternativity, flexibility and power associativity of two and three tensor products of normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are shown. The split version of the algebras obeys the same property of the division version

Algebra	Comm	Ass	Alter	Flex	Pow. Ass
$\mathbb{C}\otimes\mathbb{C}$	Yes	Yes	Yes	Yes	Yes
$\mathbb{C}\otimes\mathbb{H}$	No	Yes	Yes	Yes	Yes
$\mathbb{H}\otimes\mathbb{H}$	No	Yes	Yes	Yes	Yes
$\mathbb{C}\otimes\mathbb{O}$	No	No	Yes	Yes	Yes
$\mathbb{H}\otimes\mathbb{O}$	No	No	No	No	No
$\mathbb{O}\otimes\mathbb{O}$	No	No	No	No	No
$\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{C}$	Yes	Yes	Yes	Yes	Yes
$\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{H}$	Yes	Yes	Yes	Yes	Yes
$\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{H}$	No	Yes	Yes	Yes	Yes
$\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$	No	Yes	Yes	Yes	Yes
$\mathbb{C}\otimes\mathbb{C}\otimes\mathbb{O}$	No	No	Yes	Yes	Yes
$\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$	No	No	No	No	No

where $t^{\alpha} \in \mathbb{R}$, and z, q, w are elements of a basis for $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ respectively, i.e. $z \in \{1, I\}$, $q \in \{1, i, j, k\}$ and $w \in \{1, e_1, ..., e_7\}$ with

$$I^2 = i^2 = j^2 = k^2 = e_{\alpha}^2 = -1, \tag{6}$$

$$[I\,, i] = [I\,, j] = [I\,, k] = [I\,, e_{\alpha}] = 0, \tag{7}$$

$$[e_{\alpha}, i] = [e_{\alpha}, j] = [e_{\alpha}, k] = 0, \tag{8}$$

and the other rules of multiplication given in Fig. 1.

It is straightforward to see that every element in the set

$$D = \{(I\,q \pm 1)\,, (I\,e_{\alpha} \pm 1)\,,\ (qe_{\alpha} \pm 1) : q \in \{i, j, k\}\}\,, \tag{9}$$

is a *zero divisor* and therefore $\mathbb{T}$ is not a division algebra. Moreover, the Dixon algebra is not commutative, neither associative, nor alternative or flexible and, finally, not even power-associative, i.e. in general $x\,(xx) \neq (xx)\,x$.

Nevertheless, it is possible to define a quadratic norm N over $\mathbb{T}$, starting from the decomposition in Eq. (5), i.e.

$$N\,(t) = \sum_{\alpha=0}^{63} \left(t^{\alpha}\right)^2\,, \tag{10}$$

with an associated *polar form* $\langle\cdot,\cdot\rangle$ given by the symmetric bilinear form

$$2\,\langle t_1, t_2\rangle = N\,(t_1 + t_2) - N\,(t_1) - N\,(t_2)\,. \tag{11}$$

2 Dixon-Rosenfeld lines

The geometrical motivation for defining Dixon-Rosenfeld lines as coset manifolds relies on the study of the octonionic planes explored by Tits, Freudenthal and Rosenfeld in a series of seminal works [21,41,46,50] that led to a geometric interpretation of Lie algebras and to the construction of the Tits-Freudenthal Magic Square. While Freudenthal interpreted the entries of the Magic Square as different forms of automorphisms of the projective plane such as isometries, collineations, homography etc., Rosenfeld thought of every row of the magic square as the Lie algebra of the isometry groups of a "generalized" projective plane over a tensorial product of Hurwitz algebras [41] (see also [38] for a recent systematic review). In fact, tensor products over Hurwitz algebras are not division algebras, which therefore do not allow the definition of a projective plane in a strict sense. Nevertheless, later works of Atsuyama proved the insight of Rosenfeld to be correct and that it is possible to use these algebras to define projective planes in a "wider sense" [1,3,36]. A similar analysis was then carried out for generalized projective lines making use the Tits-Freudenthal Magic Square of order two instead of three, thus relating the resulting Lie algebras with isometries of generalized projective lines, instead of planes (see [38], for more details).

2.1 Dixon lines as coset manifold

Coset manifolds arise from coset spaces over a Lie group G given by an equivalence relation of the type

$$g \sim g' \Longleftrightarrow gh = g', \tag{12}$$

where $g, g' \in G$ and $h \in H$ and H is a closed subgroup of G. In this case, the coset space G/H, obtained from the equivalence classes gH, inherits a manifold structure from G and is therefore a manifold of dimension

$$\dim\,(G/H) = \dim\,(G) - \dim\,(H)\,. \tag{13}$$

Moreover, G/H can be endowed with invariant metrics such that all elements of the original group G are isometries of the constructed metric [20,38]. More specifically, the structure constants of the Lie algebra $\mathfrak{g}$ of the Lie group G define completely the metric and therefore all the metric-dependent tensors, such as the curvature tensor, the Ricci tensor, etc. Finally, the coset space G/H is a homogeneous manifold by construction, i.e. the group G acts transitively, and its *isotropy subgroup* is precisely H, i.e. the group H is such that for any given point p in the manifold $hp = p$. Therefore, for our purposes in the definition of the Dixon-Rosenfeld lines, it will be sufficient to define the isometry group and the isotropy

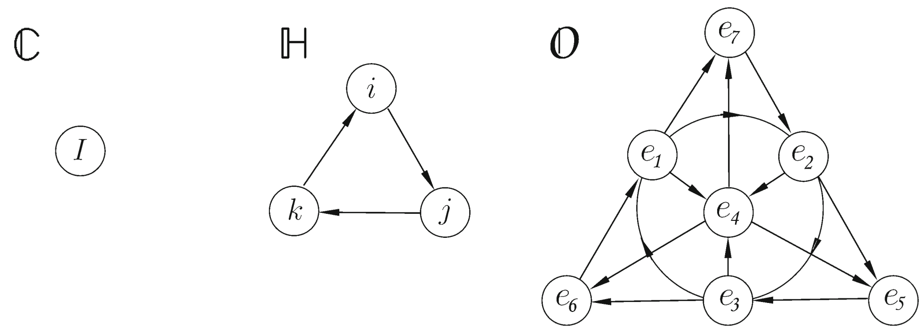

Fig. 1 Multiplication rule of Octonions $\mathbb{O}$ (*right*), Quaternions $\mathbb{H}$ (*middle*) and Complex $\mathbb{C}$ (*left*)

group of the coset manifold to have them completely defined in its topological and metrical descriptions.

2.2 Tits' magic formula

We now proceed defining three Dixon projective lines as three different coset spaces of real dimension 64 obtained from three isometry algebras $\mathfrak{a}_I$, $\mathfrak{a}_{II}$ and $\mathfrak{a}_{III}$ making the use of Tits' magic formula [46] for $n = 2$, i.e.

$$\mathcal{L}_2(\mathbb{A}, \mathbb{B}) = \mathfrak{Der}(\mathbb{A}) \oplus \mathfrak{Der}(J_2(\mathbb{B})) \oplus \left(\mathbb{A}' \otimes J_2'(\mathbb{B})\right), \quad (14)$$

where $\mathbb{A}, \mathbb{B}$ are alternative algebras and $J_2(\mathbb{B})$ is a Jordan algebra over Hermitian two by two matrices.[2] Brackets on $\mathcal{L}_2(\mathbb{A}, \mathbb{B})$ can be defined following notation in [6, sec. 3] for which, given the an algebra $\mathbb{A}$, we define

$$X' = X - \frac{1}{2}\mathrm{Tr}(X)\,\mathbf{1}, \quad (15)$$

as the projection of an element of the algebra in the subspace orthogonal to the identity denoted as $\mathbf{1}$. We then define $J_2'(\mathbb{B})$ the algebra obtained by such elements with the product $\bullet$ given by the projection back on the subspace orthogonal to the identity of the Jordan product, i.e.

$$X' \bullet Y' = X' \cdot Y' - 2\left\langle X', Y'\right\rangle \mathbf{1}, \quad (16)$$

where as usual, we intended $X \cdot Y = XY + YX$ and $\langle X, Y\rangle = \frac{1}{2}\mathrm{Tr}(X \cdot Y)$ for every $X, Y \in J_2(\mathbb{B})$. With this notation, the vector space (14) is endowed with the following brackets [6]

1. The usual brackets on the Lie subalgebra $\mathfrak{Der}(\mathbb{A}) \oplus \mathfrak{Der}(J_2(\mathbb{B}))$.
2. When $a \in \mathfrak{Der}(\mathbb{A}) \oplus \mathfrak{Der}(J_2(\mathbb{B}))$ and $A \in \mathbb{A}' \otimes J_2'(\mathbb{B})$ then

$$[a, A] = a(A). \quad (17)$$

3. When $a \otimes A, b \otimes B \in \mathbb{A}' \otimes J_2'(\mathbb{B})$ then

$$[a \otimes A, b \otimes B] = \frac{1}{2}\langle A, B\rangle D_{a,b} - \langle a, b\rangle [L_A, L_B] + \frac{1}{2}[a, b] \otimes (A \bullet B), \quad (18)$$

where L_x and R_x are the left and right action on the algebra and $D_{x,y}$ is given by

$$D_{x,y} = \left[L_x, L_y\right] + \left[L_x, R_y\right] + \left[R_x, R_y\right]. \quad (19)$$

Applying now formula (19) to the Jordan algebra with left and right Jordan product, the left and right products are the same and

$$D_{X,Y} = 3[L_X, R_Y]. \quad (20)$$

and

$$D_{X,Y}(Z) = 3[X, Z, Y] = 3[[X, Y], Z]. \quad (21)$$

2.3 Three isometry groups

Tits' formula is the most general formula compared to those of Vinberg [50], Atsuyama [2], Santander and Herranz [45], Barton and Sudbery [6], and Elduque [18] since it does not require the use of two composition algebras, but only the use of an alternative algebra and a Jordan algebra obtained from another alternative algebra.

Next, we consider all tensor products of the form $\mathbb{A} \otimes J_2(\mathbb{B})$ with $\mathbb{A}$ and $\mathbb{B}$ alternative such that $\mathbb{A} \otimes \mathbb{B}$ corresponds to the Dixon algebra $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Since $\mathbb{H} \otimes \mathbb{O}$ is not alternative, the possible candidates can be *a priori* only related with the following four different $\mathbb{A}$ and $\mathbb{B}$, i.e.

$$I : \mathbb{A} = (\mathbb{C} \otimes \mathbb{H}), \quad \mathbb{B} = \mathbb{O}, \quad (22)$$

$$II : \mathbb{A} = \mathbb{O}, \quad \mathbb{B} = (\mathbb{C} \otimes \mathbb{H}), \quad (23)$$

$$III : \mathbb{A} = (\mathbb{C} \otimes \mathbb{O}), \quad \mathbb{B} = \mathbb{H}, \quad (24)$$

and, finally, $\mathbb{A} = \mathbb{H}, \mathbb{B} = (\mathbb{C} \otimes \mathbb{O})$. However the latter case, i.e. $\mathbb{A} = \mathbb{H}, \mathbb{B} = (\mathbb{C} \otimes \mathbb{O})$, would need the existence of a

[2] After [6], when $\mathbb{A} = \mathbb{O}$ (see case **II** below), the formula (14) has $\mathfrak{Der}(\mathbb{A})$ replaced by $\mathfrak{so}(\mathbb{A}')$.

Table 3 Isometry and isotropy Lie algebras of the three Dixon-Rosenfeld lines. For $\mathbb{T}P^1_{II}$, the "minimal" enhancement (51) is considered

	$\mathbb{T}P^1_I$	$\mathbb{T}P^1_{II}$	$\mathbb{T}P^1_{III}$
$\mathfrak{isom}$	$\mathfrak{so}_9 \oplus \mathfrak{su}_2 \oplus (\mathbf{9}, 2\cdot\mathbf{3}+\mathbf{1})$	$\mathfrak{so}_7 \oplus \mathfrak{so}_6 \oplus 3\cdot(\mathbf{7},\mathbf{4}) \oplus (\mathbf{1},\mathbf{1})$	$\mathfrak{g}_2 \oplus \mathfrak{so}_5 \oplus (2\cdot\mathbf{7}+\mathbf{1},\mathbf{5})$
$\mathfrak{isot}$	$\mathfrak{so}_8 \oplus \mathfrak{su}_2 \oplus (\mathbf{1}, 2\cdot\mathbf{3}+\mathbf{1})$	$\mathfrak{so}_7 \oplus \mathfrak{su}_2 \oplus (\mathbf{7},\mathbf{3}+\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})$	$\mathfrak{g}_2 \oplus \mathfrak{su}_2 \oplus (2\cdot\mathbf{7}+\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$

Jordan algebra $J_2(\mathbb{C}\otimes\mathbb{O})$ over bioctonions $\mathbb{C}\otimes\mathbb{O}$, which is not possible.[3]

We are therefore left with only three different possibilities, i.e.

$$\begin{aligned}\mathfrak{a}_I &= \mathcal{L}_2(\mathbb{C}\otimes\mathbb{H},\mathbb{O}),\\ \mathfrak{a}_{II} &= \mathcal{L}_2(\mathbb{O},\mathbb{C}\otimes\mathbb{H}),\\ \mathfrak{a}_{III} &= \mathcal{L}_2(\mathbb{C}\otimes\mathbb{O},\mathbb{H}).\end{aligned} \tag{25}$$

We will now discuss how, due to the three possible cases in Eq. (25), there exist three "homogeneous realizations" of the Dixon-Rosenfeld projective line $\mathbb{T}P^1$, which will be distinguished by the subscript I, II and III, respectively.

We start and observe that

$$\mathfrak{der}(\mathbb{C}\otimes\mathbb{H}) \simeq \mathfrak{der}(\mathbb{C}) \oplus \mathfrak{der}(\mathbb{H}) \simeq \mathfrak{der}(\mathbb{H}) \simeq \mathfrak{su}_2, \tag{26}$$

such that

$$\mathbb{C}\otimes\mathbb{H} \simeq (2\cdot\mathbf{1})\otimes(\mathbf{1}+\mathbf{3}) = 2\cdot(\mathbf{1}\oplus\mathbf{3}) \text{ of } \mathfrak{su}_2, \tag{27}$$

implying that the imaginary biquaternions are

$$(\mathbb{C}\otimes\mathbb{H})' \simeq \mathbf{1}\oplus 2\cdot\mathbf{3} \text{ of } \mathfrak{su}_2. \tag{28}$$

This can be understood by observing that

$$\left.\begin{array}{l}\mathbb{C}\simeq\{1,I\}\\ \mathbb{H}\simeq\{1,i,j,k\}\end{array}\right\} \Rightarrow \mathbb{C}\otimes\mathbb{H} \simeq \left\{\underbrace{1,I}_{\mathbf{1}\oplus\mathbf{1}},\underbrace{i,j,k}_{\mathbf{3}},\underbrace{Ii,Ij,Ik}_{\mathbf{3}}\right\}. \tag{29}$$

Note that $\mathfrak{der}(\mathbb{C}\otimes\mathbb{H})$ is next-to-maximal into $\mathfrak{der}(\mathbb{O}) \simeq \mathfrak{g}_2$, because it can be obtained by a chain of two maximal (and symmetric) embeddings,

$$\begin{aligned}&\mathfrak{g}_2 \supset \mathfrak{su}_2\oplus\mathfrak{su}_2 \supset \mathfrak{su}_{2,d},\\ &\mathbf{7} = (\mathbf{1},\mathbf{3})+(\mathbf{2},\mathbf{2}) = \mathbf{3}+\mathbf{3}+\mathbf{1},\\ &\mathbf{14} = (\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})+(\mathbf{4},\mathbf{2}) = 3\cdot\mathbf{3}+\mathbf{5},\end{aligned} \tag{30}$$

or equivalently by a chain of two maximal (one non-symmetric and one symmetric) embeddings,

$$\begin{aligned}&\mathfrak{g}_2 \supset \mathfrak{su}_3 \supset \mathfrak{su}_{2,P},\\ &\mathbf{7} = \mathbf{3}+\overline{\mathbf{3}}+\mathbf{1} = \mathbf{3}+\mathbf{3}+\mathbf{1},\\ &\mathbf{14} = \mathbf{3}+\overline{\mathbf{3}}+\mathbf{8} = 3\cdot\mathbf{3}+\mathbf{5}.\end{aligned} \tag{31}$$

In all cases, the Dixon algebra $\mathbb{T}$ will have the same covariant realization in terms of

$$\begin{aligned}\mathfrak{der}(\mathbb{T}) &\simeq \mathfrak{der}(\mathbb{C}\otimes\mathbb{H}) \oplus \mathfrak{der}(\mathbb{O}) \simeq \mathfrak{der}(\mathbb{H}) \oplus \mathfrak{der}(\mathbb{O})\\ &\simeq \mathfrak{su}_2 \oplus \mathfrak{g}_2,\end{aligned} \tag{32}$$

i.e.

$$\begin{aligned}\mathbb{T} &\simeq T\left(\mathbb{T}P^1_I\right) \simeq T\left(\mathbb{T}P^1_{II}\right) \simeq T\left(\mathbb{T}P^1_{III}\right)\\ &\simeq 2\cdot(\mathbf{3}+\mathbf{1},\mathbf{7}+\mathbf{1}) \text{ of } \mathfrak{su}_2\oplus\mathfrak{g}_2,\end{aligned} \tag{33}$$

which can enjoy the following enhancements of (manifest) covariance,

$$\mathbb{T} \simeq 2\cdot(\mathbf{1}+\mathbf{3},\mathbf{7}+\mathbf{1}) \text{ of } \mathfrak{su}_2\oplus\mathfrak{so}_7 \tag{34}$$

$$\simeq 2\cdot(\mathbf{1}+\mathbf{3},\mathbf{8}_v) \text{ of } \mathfrak{su}_2\oplus\mathfrak{so}_8. \tag{35}$$

I. In the case $\mathbb{A} = \mathbb{C}\otimes\mathbb{H}$ and $\mathbb{B} = \mathbb{O}$, Tits' formula (14) yields (cf.(26))

$$\begin{aligned}\mathfrak{a}_I &= \mathcal{L}_2(\mathbb{C}\otimes\mathbb{H},\mathbb{O}) = \mathfrak{isom}\left(\mathbb{T}P^1_I\right)\\ &:= \mathfrak{der}(\mathbb{C}\otimes\mathbb{H}) \oplus \mathfrak{der}(J_2(\mathbb{O})) \oplus (\mathbb{C}\otimes\mathbb{H})'\otimes J_2'(\mathbb{O})\\ &= \mathfrak{su}_2\oplus\mathfrak{so}_9\oplus(\mathbf{1}+2\cdot\mathbf{3},\mathbf{9}),\end{aligned} \tag{36}$$

because

$$\mathfrak{der}(J_2(\mathbb{O})) = \mathfrak{so}_9, \tag{37}$$

$$J_2'(\mathbb{O}) \simeq \mathbf{9}, \tag{38}$$

The Lie algebra $\mathfrak{isom}\left(\mathbb{T}P^1_I\right)$ has therefore dimension $3+36+63=102$.

II. In the case $\mathbb{A} = \mathbb{O}$ and $\mathbb{B} = \mathbb{C}\otimes\mathbb{H}$, after the treatment given in Sec. 8 of [6], Tits' formula (14) gets $\mathfrak{der}(\mathbb{O})$ replaced by $\mathfrak{so}(\mathbb{O}')$, and thus one obtains

$$\mathfrak{a}_{II} = \mathcal{L}_2(\mathbb{O},\mathbb{C}\otimes\mathbb{H}) = \mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)$$

[3] We excluded on purpose the cases where the involution on Hermitian matrices does not involve whole blocks of imaginary units in the algebra, giving rise to $\mathbb{C}\otimes J_2(\mathbb{O})$ and $\mathbb{O}\otimes J_2(\mathbb{C})$, for which the analogy with Rosenfeld construction does not hold. Alternatively, one might consider an involution given by the composition of the conjugation of $\mathbb{C}$ and the conjugation of $\mathbb{O}$, or also an involution flipping all imaginary units of $\mathbb{C}\otimes\mathbb{O}$; while the former yields to non-real diagonal elements of the corresponding Hermitian 2×2 matrices over $\mathbb{C}\otimes\mathbb{O}$, the latter has real diagonal elements. However, we checked that in both cases the resulting rank-2 Hermitian matrices do not form a Jordan algebra satisfying the Jordan identity.

$$:= \mathfrak{so}\left(\mathbb{O}'\right) \oplus \mathfrak{Der}\left(J_2(\mathbb{C}\otimes\mathbb{H})\right) \oplus \mathbb{O}' \otimes J_2'(\mathbb{C}\otimes\mathbb{H}). \quad (39)$$

$J_2(\mathbb{C}\otimes\mathbb{H})$ is a rank-2 Jordan algebra, defined as the algebra of 2×2 matrices over $\mathbb{C}\otimes\mathbb{H}$ (cf. (27) and (29)) which are Hermitian with respect to the involution $\imath$ given by the composition of the conjugation of $\mathbb{C}$ and of the conjugation of $\mathbb{H}$:

$$\imath : \begin{cases} I & \to -I; \\ i,j,k & \to -i,-j,-k; \\ Ii,Ij,Ik \to Ii,Ij,Ik. \end{cases} \quad (40)$$

Interestingly, this implies that the diagonal elements of the matrices of $J_2(\mathbb{C}\otimes\mathbb{H})$ are non-real, being of the form $d = d_1 + Iid_2 + Ijd_3 + Ikd_4$, with $d_1, d_2, d_3, d_4 \in \mathbb{R}$, and[4] $(Ii)^2 = (Ij)^2 = (Ik)^2 = 1$. Thus, with respect to $\mathfrak{Der}(\mathbb{C}\otimes\mathbb{H}) \simeq \mathfrak{su}_2$ (26), $J_2(\mathbb{C}\otimes\mathbb{H})$ fit into the following representations:

$$J_2(\mathbb{C}\otimes\mathbb{H}) \simeq \begin{pmatrix} \mathbf{1}\oplus\mathbf{3} & 2\cdot(\mathbf{1}\oplus\mathbf{3}) \\ * & \mathbf{1}\oplus\mathbf{3} \end{pmatrix} \simeq 4\cdot(\mathbf{1}\oplus\mathbf{3}) \text{ of } \mathfrak{su}_2, \quad (41)$$

yielding for the traceless part that

$$J_2'(\mathbb{C}\otimes\mathbb{H}) \simeq 3\cdot(\mathbf{1}\oplus\mathbf{3}) \text{ of } \mathfrak{su}_2. \quad (42)$$

On the other hand, as proved in Appendix B, it holds that

$$\mathfrak{Der}\left(J_2(\mathbb{C}\otimes\mathbb{H})\right) \simeq \mathfrak{so}_6, \quad (43)$$

and thus (41) and (42) respectively enjoy the following enhancements[5]:

$$J_2(\mathbb{C}\otimes\mathbb{H}) \simeq \begin{pmatrix} \mathbf{4} & \mathbf{4}\oplus\mathbf{4} \\ * & \mathbf{4} \end{pmatrix} \simeq 4\cdot\mathbf{4} \text{ of } \mathfrak{so}_6; \quad (44)$$

$$J_2'(\mathbb{C}\otimes\mathbb{H}) \simeq 3\cdot\mathbf{4} \text{ of } \mathfrak{so}_6. \quad (45)$$

Therefore, since

$$\mathfrak{so}\left(\mathbb{O}'\right) = \mathfrak{so}_7 = \mathfrak{g}_2 \oplus \mathbf{7}; \quad (46)$$

$$\mathbb{O}' \simeq \mathbf{7} \text{ of } \mathfrak{so}_7 = \mathbf{7} \text{ of } \mathfrak{g}_2, \quad (47)$$

formula (39) can be made explicit as follows:

$$\begin{aligned} \mathfrak{a}_{II} &= \mathcal{L}_2(\mathbb{O}, \mathbb{C}\otimes\mathbb{H}) = \mathfrak{isom}\left(\mathbb{T}P^1_{II}\right) \\ &= \mathfrak{so}_7 \oplus \mathfrak{so}_6 \oplus 3\cdot(\mathbf{7},\mathbf{4}) \quad (48) \\ &= \mathfrak{g}_2 \oplus \mathfrak{so}_6 \oplus 3\cdot(\mathbf{7},\mathbf{4}) \oplus (\mathbf{7},\mathbf{1}) \\ &= \mathfrak{g}_2 \oplus \mathfrak{so}_4 \oplus 3\cdot(\mathbf{7},\mathbf{2},\mathbf{2}) \oplus (\mathbf{7},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3},\mathbf{3}) \\ &= \mathfrak{g}_2 \oplus \mathfrak{su}_2 \oplus 3\cdot(\mathbf{7},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}) \oplus 2 \\ &\quad \cdot(\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1}). \quad (49) \end{aligned}$$

The last line (49) has a manifest $(\mathfrak{g}_2 \oplus \mathfrak{su}_2)$-covariance, which is the natural one for the Dixon algebra $\mathbb{T}$ (cf. (33)), giving

$$\mathbb{T} \not\subseteq \mathfrak{a}_{II}, \quad (50)$$

because there is only one singlet $(\mathbf{1},\mathbf{1})$ in (49). See Appendix C for a more exhaustive treatment of all $\mathfrak{su}_2$'s inside $\mathfrak{so}_6$.

However, it is anticipated that $\mathbb{T} \in \mathfrak{a}_{II}$. Therefore, there are two possibilities to resolve this issue. First, it was asserted above that $\mathbb{H}$ corresponds to $\mathbf{1}\oplus\mathbf{3}$ of $\mathfrak{su}_2$, which led to $\mathbb{T} \in \mathfrak{a}_I$. However, $\mathbb{C}\otimes\mathbb{H}$ is known to allow for three different representations of $\mathfrak{sl}_{2,\mathbb{C}}$ [22]. If the spinor representations were chosen instead, then $\mathbb{T} \in \mathfrak{a}_{II}$. Second, one may claim that the 2×2 Freudenthal–Tits formula does not apply to the case where $\mathbb{A} = \mathbb{O}$ and $\mathbb{B} = \mathbb{C}\otimes\mathbb{H}$. Freudenthal and Tits' formula was designed for 3×3, but the 2×2 case already has a precedent of the formula depending on the algebras chosen, as $\mathbb{A} = \mathbb{O}$ leads to a difference from $\mathbb{A} = \mathbb{C}$ or $\mathbb{H}$ in the 2×2 case. In this work, we merely claim that a non-simple Lie algebra $\mathfrak{a}_{II}$ exists, but we do not fully determine its precise structure.

Assuming that the Freudenthal–Tits formula does not apply to The "minimal" enhancement of $\mathfrak{a}_{II}$ such that it contains $\mathbb{T}$ with $\mathbf{1}\oplus\mathbf{3}$ of $\mathfrak{su}_2$ amounts to adding a $(\mathfrak{g}_2 \oplus \mathfrak{su}_2)$ -singlet generator:

$$\begin{aligned} \mathfrak{a}_{II} &\longrightarrow \mathfrak{a}_{II,\text{enh.}} := \mathfrak{a}_{II} \oplus (\mathbf{1},\mathbf{1}) \quad (51) \\ &= \mathfrak{g}_2 \oplus \mathfrak{su}_2 \oplus 3\cdot(\mathbf{7},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}) \oplus 2 \\ &\quad \cdot(\mathbf{1},\mathbf{3}) \oplus 2\cdot(\mathbf{1},\mathbf{1}) \\ &= \mathfrak{g}_2 \oplus \mathfrak{su}_2 \oplus (\mathbf{7},\mathbf{3}) \oplus 2\cdot(\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}) \oplus \mathbb{T}. \quad (52) \end{aligned}$$

Thanks to (34), the last line (52) enjoys the further symmetry enhancement

[4] Despite this, such diagonal elements do not belong to $\mathbb{H}_s$, because the composite imaginary units Ii, Ij, Ik are not a basis for $\mathbb{H}_s'$.

[5] (41) resp. (42) can be obtained from (44) resp. (45) by observing that $\mathfrak{Der}(\mathbb{C}\otimes\mathbb{H}) \simeq \mathfrak{su}_2$ (26) is a diagonal subalgebra of $\mathfrak{Der}(J_2(\mathbb{C}\otimes\mathbb{H})) \simeq \mathfrak{so}_6$ (43) obtained by the following chain of maximal and symmetric embeddings: $\mathfrak{so}_6 \simeq \mathfrak{su}_4 \to \mathfrak{so}_4 \simeq \mathfrak{su}_2 \oplus \mathfrak{su}_2 \overset{d}{\to} \mathfrak{su}_2$.

$$\mathfrak{a}_{II,\text{enh.}} = \mathfrak{so}_7 \oplus \mathfrak{su}_2 \oplus (\mathbf{7},\mathbf{3}) \oplus (\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}) \oplus \mathbb{T}. \tag{53}$$

Note however that a further symmetry enhancement to $\mathfrak{so}_8 \oplus \mathfrak{su}_2$ is not possible without breaking $\mathbb{T}$ itself.[6] If the Lie algebra $\mathfrak{a}_{II}$ should be enhanced, then $\mathfrak{a}_{II,\text{enh.}} \equiv \mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}$ given by (52) has dimension $21+15+3\cdot 28+1 = 121$. Alternatively, it is possible that $\mathfrak{a}_{II}$ is 120-dimensional such that $\mathbb{H} \in \mathbb{T}$ contains spinor representations, such as $\mathbf{2}\oplus\mathbf{2}$ of $\mathfrak{der}(\mathbb{H}) = \mathfrak{su}_2$.

III. Finally, in the case $\mathbb{A} = \mathbb{C}\otimes\mathbb{O}$ (non-associative) and $\mathbb{B} = \mathbb{H}$, Tits' formula (14) yields

$$\begin{aligned}\mathfrak{a}_{III} &= \mathcal{L}_2(\mathbb{C}\otimes\mathbb{O},\mathbb{H}) = \mathfrak{isom}\left(\mathbb{T}P^1_{III}\right)\\ &:= \mathfrak{der}(\mathbb{C}\otimes\mathbb{O}) \oplus \mathfrak{der}(J_2(\mathbb{H})) \oplus (\mathbb{C}\otimes\mathbb{O})' \otimes J_2'(\mathbb{H})\\ &= \mathfrak{g}_2 \oplus \mathfrak{so}_5 \oplus (2\cdot\mathbf{7}+\mathbf{1},\mathbf{5}),\end{aligned} \tag{54}$$

because

$$\begin{aligned}\mathfrak{der}(J_2(\mathbb{H})) &\simeq \mathfrak{so}_5,\\ J_2'(\mathbb{H}) &\simeq \mathbf{5},\end{aligned} \tag{55}$$

and

$$\mathfrak{der}(\mathbb{C}\otimes\mathbb{O}) \simeq \mathfrak{der}(\mathbb{O}) \simeq \mathfrak{g}_2, \tag{56}$$

$$(\mathbb{C}\otimes\mathbb{O})' \simeq 2\cdot\mathbf{7}+\mathbf{1}. \tag{57}$$

The Lie algebra $\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right)$ has dimension $14+10+75 = 99$.

2.4 Three Dixon lines

A Dixon-Rosenfeld projective line $\mathbb{T}P^1$ can be realized as an homogeneous space of dimension $\dim_{\mathbb{R}}\mathbb{T}P^1 = \dim_{\mathbb{R}}\mathbb{T} = 64$, whose corresponding Lie algebra generators $\mathfrak{Lie}\left(\mathbb{T}P^1\right)$ relate to the isometry and isotropy Lie algebras as follows:

$$\mathfrak{Lie}\left(\mathbb{T}P^1\right) \simeq \mathfrak{isom}\left(\mathbb{T}P^1\right) \ominus \mathfrak{isot}\left(\mathbb{T}P^1\right), \tag{58}$$

and whose tangent space $T\left(\mathbb{T}P^1\right)$ carries a $\mathfrak{isot}\left(\mathbb{T}P^1\right)$-covariant realization of $\mathbb{T}$ itself.

I By iterated branchings of $\mathfrak{isom}\left(\mathbb{T}P^1_I\right)$ given by (36), one obtains

$$\begin{aligned}\mathfrak{isom}\left(\mathbb{T}P^1_I\right) &\simeq \mathfrak{su}_2 \oplus \mathfrak{so}_8 \oplus (\mathbf{1}\oplus 2\cdot\mathbf{3},\mathbf{8}_v+\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}_v)\\ &= \mathfrak{su}_2 \oplus \mathfrak{so}_7 \oplus (2\cdot\mathbf{3},\mathbf{7}+2\cdot\mathbf{1}) \oplus 3\cdot(\mathbf{1},\mathbf{7}+\mathbf{1})\\ &= \mathfrak{su}_2 \oplus \mathfrak{g}_2 \oplus (2\cdot\mathbf{3},\mathbf{7}+2\cdot\mathbf{1}) \oplus (\mathbf{1},4\cdot\mathbf{7}+3\cdot\mathbf{1})\\ &= \mathfrak{su}_2 \oplus \mathfrak{g}_2 \oplus (2\cdot\mathbf{3},\mathbf{7}+\mathbf{1}) \oplus (2\cdot\mathbf{3},\mathbf{1}) \oplus (2\cdot\mathbf{1},\mathbf{7}+\mathbf{1})\\ &\quad\oplus (\mathbf{1},2\cdot\mathbf{7}+\mathbf{1})\end{aligned} \tag{59}$$

$$=: \mathfrak{isot}\left(\mathbb{T}P^1_I\right) \oplus \mathfrak{c}\left(\mathbb{T}P^1_I\right), \tag{60}$$

thus implying that

$$\begin{aligned}\mathfrak{isot}\left(\mathbb{T}P^1_I\right) &:= \mathfrak{su}_2 \oplus \mathfrak{g}_2 \oplus (\mathbf{1}+2\cdot\mathbf{3},\mathbf{1}) \oplus 2\cdot(\mathbf{1},\mathbf{7})\\ &= \mathfrak{su}_2 \oplus \mathfrak{so}_7 \oplus (\mathbf{1}+2\cdot\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{7})\\ &= \mathfrak{su}_2 \oplus \mathfrak{so}_8 \oplus (\mathbf{1}+2\cdot\mathbf{3},\mathbf{1}),\end{aligned} \tag{61}$$

$$\mathfrak{c}\left(\mathbb{T}P^1_I\right) \simeq T\left(\mathbb{T}P^1_I\right) \overset{(33)-(35)}{\simeq} 2\cdot(\mathbf{1}+\mathbf{3},\mathbf{8}_v) \text{ of } \mathfrak{su}_2\oplus\mathfrak{so}_8. \tag{62}$$

Therefore, one obtains the following (non-symmetric) presentation of the Dixon projective line $\mathbb{T}P^1_I$ as a homogeneous space:

$$\mathbb{T}P^1_I \simeq \frac{SO_9 \times SU_2 \ltimes (\mathbf{9},\mathbf{1}+2\cdot\mathbf{3})}{SO_8 \times SU_2 \ltimes (2\cdot(\mathbf{1},\mathbf{3})+(\mathbf{1},\mathbf{1}))}, \tag{63}$$

with

$$\dim\left(\mathbb{T}P^1_I\right) = 64 = \dim\mathbb{T}. \tag{64}$$

The coset (63) is not symmetric, because it can be checked that

$$\begin{aligned}&\left[\mathfrak{c}\left(\mathbb{T}P^1_I\right),\mathfrak{c}\left(\mathbb{T}P^1_I\right)\right]\\ &\simeq 4\cdot(\mathbf{1}+\mathbf{3},\mathbf{8}_v)\otimes_a(\mathbf{1}+\mathbf{3},\mathbf{8}_v) \not\subseteq \mathfrak{isot}\left(\mathbb{T}P^1_I\right),\end{aligned} \tag{65}$$

where subscript "a" denotes anti-symmetrization of the tensor product throughout.

II From (48) and (52), the "minimally" enhanced $\mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}$ reads

$$\mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}} =: \mathfrak{isot}\left(\mathbb{T}P^1_{II}\right) \oplus \mathfrak{c}\left(\mathbb{T}P^1_{II}\right), \tag{66}$$

with

$$\mathfrak{isot}\left(\mathbb{T}P^1_{II}\right) := \mathfrak{so}_7 \oplus \mathfrak{su}_2 \oplus (\mathbf{7},\mathbf{3}) \oplus (\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}), \tag{67}$$

$$\mathfrak{c}\left(\mathbb{T}P^1_{II}\right) \simeq T\left(\mathbb{T}P^1_{II}\right) \overset{(34)}{\simeq} T\left(\mathbb{T}P^1_I\right). \tag{68}$$

Thence, one obtains the following (non-symmetric) presentation of the Dixon projective line $\mathbb{T}P^1_{II}$ as a homogeneous space:

$$\mathbb{T}P^1_{II} \simeq \frac{SO_7 \times SO_6 \ltimes (3\cdot(\mathbf{7},\mathbf{4})\oplus(\mathbf{1},\mathbf{1}))}{SO_7 \times SU_2 \ltimes ((\mathbf{7},\mathbf{3}+\mathbf{1})\oplus(\mathbf{1},\mathbf{5}))}, \tag{69}$$

[6] A further $(\mathbf{13})$-generator would be needed in the r.h.s. of (51).

once again with

$$\dim\left(\mathbb{T}P^1_{II}\right) = 64 = \dim\mathbb{T}. \tag{70}$$

The coset (69) is not symmetric, because it can be checked that

$$\left[\mathfrak{c}\left(\mathbb{T}P^1_{II}\right), \mathfrak{c}\left(\mathbb{T}P^1_{II}\right)\right] \simeq 4\cdot(\mathbf{1}+\mathbf{3},\mathbf{7}+\mathbf{1})\otimes_a(\mathbf{1}+\mathbf{3},\mathbf{7}+\mathbf{1}) \nsubseteq \mathfrak{isot}\left(\mathbb{T}P^1_{II}\right). \tag{71}$$

III By iterated branchings of $\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right)$, given by (54), one obtains

$$\begin{aligned}\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right) &= \mathfrak{g}_2\oplus\mathfrak{su}_2\oplus\mathfrak{su}_2\oplus(\mathbf{1},\mathbf{2},\mathbf{2})\\ &\quad\oplus(2\cdot\mathbf{7}+\mathbf{1},\mathbf{2},\mathbf{2})\oplus(2\cdot\mathbf{7}+\mathbf{1},\mathbf{1},\mathbf{1})\\ &= \mathfrak{g}_2\oplus\mathfrak{su}_{2,d}\oplus 2\cdot(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{1})\oplus(2\cdot\mathbf{7}+\mathbf{1},\mathbf{3})\\ &\quad\oplus 2\cdot(2\cdot\mathbf{7}+\mathbf{1},\mathbf{1})\\ &\simeq: \mathfrak{isot}\left(\mathbb{T}P^1_{III}\right)\oplus\mathfrak{c}\left(\mathbb{T}P^1_{III}\right),\end{aligned} \tag{72}$$

thus implying that

$$\mathfrak{isot}\left(\mathbb{T}P^1_{III}\right) :\simeq \quad \mathfrak{g}_2\oplus\mathfrak{su}_2\oplus(2\cdot\mathbf{7}+\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3}), \tag{73}$$

$$\mathfrak{c}\left(\mathbb{T}P^1_{III}\right)\simeq T\left(\mathbb{T}P^1_{III}\right)\overset{(33),(68)}{\simeq} T\left(\mathbb{T}P^1_{I}\right), \tag{74}$$

and therefore leading to the following (non-symmetric) presentation of the Dixon projective line $\mathbb{T}P^1_{III}$ as a homogeneous space:

$$\mathbb{T}P^1_{III}\simeq\frac{G_2\times SO_5\ltimes(2\cdot\mathbf{7}+\mathbf{1},\mathbf{5})}{G_2\times SU_2\ltimes((2\cdot\mathbf{7}+\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3}))}, \tag{75}$$

once again with

$$\dim\left(\mathbb{T}P^1_{III}\right) = 64 = \dim\mathbb{T}. \tag{76}$$

The coset (75) is not symmetric, because it can be checked that

$$\left[\mathfrak{c}\left(\mathbb{T}P^1_{III}\right), \mathfrak{c}\left(\mathbb{T}P^1_{III}\right)\right] \simeq 4\cdot(\mathbf{7}+\mathbf{1},\mathbf{3}+\mathbf{1})\otimes_a(\mathbf{7}+\mathbf{1},\mathbf{3}+\mathbf{1}) \nsubseteq \mathfrak{isot}\left(\mathbb{T}P^1_{III}\right). \tag{77}$$

Remark The above analysis yields the following isometry algebras:

$$\underset{\dim_{\mathbb{R}}=102}{\mathfrak{isom}\left(\mathbb{T}P^1_{I}\right)}\simeq\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus 2\cdot(\mathbf{7},\mathbf{3})\oplus 4\cdot(\mathbf{1},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1})\oplus 3\cdot(\mathbf{1},\mathbf{1}), \tag{78}$$

$$\underset{\dim_{\mathbb{R}}=121}{\mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}}\simeq\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus 3\cdot(\mathbf{7},\mathbf{3})\oplus 2\cdot(\mathbf{1},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1})\oplus 2\cdot(\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{5}), \tag{79}$$

$$\underset{\dim_{\mathbb{R}}=99}{\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right)}=\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus 2\cdot(\mathbf{7},\mathbf{3})\oplus 3\cdot(\mathbf{1},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1})\oplus 3\cdot(\mathbf{1},\mathbf{1}), \tag{80}$$

as well as the following isotropy algebras:

$$\underset{\dim_{\mathbb{R}}=38}{\mathfrak{isot}\left(\mathbb{T}P^1_{I}\right)}\simeq\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus 2\cdot(\mathbf{1},\mathbf{3})\oplus 2\cdot(\mathbf{7},\mathbf{1})\oplus(\mathbf{1},\mathbf{1}), \tag{81}$$

$$\underset{\dim_{\mathbb{R}}=57}{\mathfrak{isot}\left(\mathbb{T}P^1_{II}\right)}\simeq\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus(\mathbf{7},\mathbf{3})\oplus 2\cdot(\mathbf{7},\mathbf{1})\oplus(\mathbf{1},\mathbf{5}), \tag{82}$$

$$\underset{\dim_{\mathbb{R}}=35}{\mathfrak{isot}\left(\mathbb{T}P^1_{III}\right)}=\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus(\mathbf{1},\mathbf{3})\oplus 2\cdot(\mathbf{7},\mathbf{1})\oplus(\mathbf{1},\mathbf{1}), \tag{83}$$

which all imply the same coset Lie algebra locally on the tangent space, providing a manifestly $(\mathfrak{g}_2\oplus\mathfrak{su}_2)$-covariant (or, equivalently, $(\mathfrak{so}_7\oplus\mathfrak{su}_2)$-covariant) realization of the Dixon algebra $\mathbb{T}$, as given by (33) and (34):

$$\mathfrak{c}\left(\mathbb{T}P^1_{I}\right)\simeq\mathfrak{c}\left(\mathbb{T}P^1_{I}\right)\simeq\mathfrak{c}\left(\mathbb{T}P^1_{I}\right)\simeq(\mathbf{7}+\mathbf{1},2\cdot\mathbf{3}+2\cdot\mathbf{1}), \tag{84}$$

$$\Updownarrow$$

$$T\left(\mathbb{T}P^1_{I}\right)\simeq T\left(\mathbb{T}P^1_{I}\right)\simeq T\left(\mathbb{T}P^1_{I}\right)\simeq(\mathbf{7}+\mathbf{1},2\cdot\mathbf{3}+2\cdot\mathbf{1}). \tag{85}$$

Thus, the three Dixon-Rosenfeld projective lines $\mathbb{T}P^1_{I}$, $\mathbb{T}P^1_{II}$ and $\mathbb{T}P^1_{III}$ have slightly different isometry and isotropy Lie algebras; from the formulæ above, it follows that

$$\mathfrak{isom}\left(\mathbb{T}P^1_{I}\right)\simeq\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right)\oplus(\mathbf{1},\mathbf{3}); \tag{86}$$

$$\mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}\nsubseteq\nsupseteq\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right); \tag{87}$$

$$\mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}\cap\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right) \simeq\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus 2\cdot(\mathbf{7},\mathbf{3})\oplus 2\cdot(\mathbf{1},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1})\oplus 2\cdot(\mathbf{1},\mathbf{1}), \tag{88}$$

and

$$\mathfrak{isot}\left(\mathbb{T}P^1_{I}\right)\simeq\mathfrak{isot}\left(\mathbb{T}P^1_{III}\right)\oplus(\mathbf{1},\mathbf{3}); \tag{89}$$

$$\mathfrak{isot}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}\nsubseteq\nsupseteq\mathfrak{isot}\left(\mathbb{T}P^1_{III}\right); \tag{90}$$

$$\mathfrak{isot}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}\cap\mathfrak{isot}\left(\mathbb{T}P^1_{III}\right)\simeq\mathfrak{g}_2\oplus\mathfrak{su}_2\oplus 2\cdot(\mathbf{7},\mathbf{1}). \tag{91}$$

However, the set of generators of the isometry Lie group whose non-linear realization gives rise to the Dixon-Rosenfeld projective line is the same for $\mathbb{T}P_I^1$, $\mathbb{T}P_{II}^1$ and $\mathbb{T}P_{III}^1$; since such a set of generators also provide a local realization of the tangent space, one can conclude that $\mathbb{T}P_I^1$, $\mathbb{T}P_{II}^1$ and $\mathbb{T}P_{III}^1$ are *locally isomorphic* as homogeneous (non-symmetric) spaces.

3 Relationship with octonionic Rosenfeld lines

It is interesting to point out the relationship between the Dixon-Rosenfeld lines and the other octonionic Rosenfeld lines, whose definition can be found in from an historical point of view in [42,43] and in a more rigorous definition in [38]. Let us just recall here the homogeneous space realization of Rosenfeld lines over $\mathbb{A}\otimes\mathbb{O}$, with $\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$ (see [11,12,38,42,43]), i.e. for the *octonionic projective line* $(\mathbb{R}\otimes\mathbb{O})\,P^1$, the *bioctonionic Rosenfeld line* $(\mathbb{C}\otimes\mathbb{O})\,P^1$, the *quateroctonionic Rosenfeld line*$(\mathbb{H}\otimes\mathbb{O})\,P^1$ and, finally, for the *octooctonionic Rosenfeld line* $(\mathbb{O}\otimes\mathbb{O})\,P^1$:

$$
\begin{aligned}
(\mathbb{R}\otimes\mathbb{O})\,P^1 &= \frac{SO_9}{SO_8}\simeq S^8,\\
(\mathbb{C}\otimes\mathbb{O})\,P^1 &= \frac{SO_{10}\times U_1}{SO_8\times U_1\times U_1}\simeq\frac{SO_{10}}{SO_8\times U_1},\\
(\mathbb{H}\otimes\mathbb{O})\,P^1 &= \frac{SO_{12}\times Sp_2}{SO_8\times SU_2\times SU_2\times Sp_2}\\
&\simeq\frac{SO_{12}}{SO_8\times SU_2\times SU_2},\\
(\mathbb{O}\otimes\mathbb{O})\,P^1 &= \frac{SO_{16}}{SO_8\times SO_8},
\end{aligned}
$$

from which it consistently follows that

$$
\begin{aligned}
T\left(\mathbb{O}P^1\right) &\simeq \mathbf{8}_v \text{ of } \mathfrak{so}_8\\
&\simeq \mathbf{7}+\mathbf{1} \text{ of } \mathfrak{so}_7\\
&\simeq \mathbf{7}+\mathbf{1} \text{ of } \mathfrak{g}_2,
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
T\left((\mathbb{C}\otimes\mathbb{O})\,P^1\right) &\simeq \mathbf{8}_{v,+}\oplus\mathbf{8}_{v,-} \text{ of } \mathfrak{so}_8\oplus\mathfrak{u}_1\\
&\simeq 2\cdot(\mathbf{7}+\mathbf{1}) \text{ of } \mathfrak{so}_7\\
&\simeq 2\cdot(\mathbf{7}+\mathbf{1}) \text{ of } \mathfrak{g}_2,
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
T\left((\mathbb{H}\otimes\mathbb{O})\,P^1\right) &\simeq (\mathbf{8}_v,\mathbf{2},\mathbf{2}) \text{ of } \mathfrak{so}_8\oplus\mathfrak{su}_2\oplus\mathfrak{su}_2\\
&\simeq (\mathbf{8}_v,\mathbf{3}+\mathbf{1}) \text{ of } \mathfrak{so}_8\oplus\mathfrak{su}_{2,d}\\
&\simeq (\mathbf{7}+\mathbf{1},\mathbf{3}+\mathbf{1}) \text{ of } \mathfrak{g}_2\oplus\mathfrak{su}_2,
\end{aligned}
\tag{94}
$$

$$
\begin{aligned}
T\left((\mathbb{O}\otimes\mathbb{O})\,P^1\right) &\simeq (\mathbf{8}_v,\mathbf{8}_v) \text{ of } \mathfrak{so}_8\oplus\mathfrak{so}_8\\
&\simeq (\mathbf{7}+\mathbf{1},\mathbf{7}+\mathbf{1}) \text{ of } \mathfrak{so}_7\oplus\mathfrak{so}_7\\
&\simeq (\mathbf{7}+\mathbf{1},\mathbf{7}+\mathbf{1}) \text{ of } \mathfrak{g}_2\oplus\mathfrak{g}_2.
\end{aligned}
\tag{95}
$$

which illustrates how the tangent spaces of octonionic projective lines generally carry an enhancement of the symmetry with respect to the Lie algebra $\mathfrak{der}\,(\mathbb{A}\otimes\mathbb{O})\simeq\mathfrak{der}\,(\mathbb{A})\oplus\mathfrak{g}_2$.

Geometrically, the octonionic projective lines $(\mathbb{A}\otimes\mathbb{O})\,P^1$ can be regarded as $\mathbb{A}\otimes\mathbb{O}$ together with a point at infinity, and thus as a $8\dim_{\mathbb{R}}\mathbb{A}$-sphere, namely as a maximal totally geodesic sphere in the corresponding octonionic Rosenfeld projective plane $(\mathbb{A}\otimes\mathbb{O})\,P^2$ [43]. In the case $\mathbb{A}=\mathbb{R}$, such a "spherical characterization" of octonionic projective lines is well known, whereas for the other cases (the "genuinely Rosenfeld" ones) it is less trivial (see e.g. [40]).

We can now study the relations among the Dixon-Rosenfeld lines discussed above and the octonionic Rosenfeld lines. Of course,

$$
\begin{aligned}
\dim\left((\mathbb{O}\otimes\mathbb{O})\,P^1\right) &= \dim\left(\mathbb{T}P_I^1\right)\\
&= \dim\left(\mathbb{T}P_{II}^1\right) = \dim\left(\mathbb{T}P_{III}^1\right) = 64.
\end{aligned}
\tag{96}
$$

By recalling (33) and considering the $(\mathfrak{g}_2\oplus\mathfrak{g}_2)$-covariant representation of the tensor algebra

$$
\mathbb{O}\otimes\mathbb{O}\simeq(\mathbf{7}+\mathbf{1},\mathbf{7}+\mathbf{1}) \text{ of } \underset{\mathfrak{der}(\mathbb{O})\oplus\mathfrak{der}(\mathbb{O})}{\mathfrak{g}_2\oplus\mathfrak{g}_2},
\tag{97}
$$

one observes that, when restricting the first (or, equivalently, the second) $\mathfrak{g}_2$ to a $\mathfrak{su}_2$ subalgebra defined by (30) (or, equivalently, by (31)), the irrepr. $\mathbf{7}$ of $\mathfrak{g}_2$ breaks into $2\cdot\mathbf{3}+\mathbf{1}$ of $\mathfrak{su}_2$, and therefore it holds that

$$
\begin{aligned}
\mathbb{O}\otimes\mathbb{O} &\simeq \underset{\mathfrak{g}_2\oplus\mathfrak{g}_2\simeq\mathfrak{der}(\mathbb{O})\oplus\mathfrak{der}(\mathbb{O})}{(\mathbf{7}+\mathbf{1},\mathbf{7}+\mathbf{1})}\\
&\overset{\mathfrak{g}_2\to\mathfrak{su}_2}{\simeq} \underset{\mathfrak{su}_2\oplus\mathfrak{g}_2\simeq\mathfrak{der}(\mathbb{C}\otimes\mathbb{H})\oplus\mathfrak{der}(\mathbb{O})}{2\cdot(\mathbf{3}+\mathbf{1},\mathbf{7}+\mathbf{1})} \overset{(33)}{\simeq}\mathbb{T}.
\end{aligned}
\tag{98}
$$

In other words, as resulting from the treatment below, $\mathbb{O}\otimes\mathbb{O}$ and $\mathbb{T}$ are isomorphic as vector spaces (but not as algebras), with $\mathfrak{der}\,(\mathbb{O}\otimes\mathbb{O})\supsetneq\mathfrak{der}\,(\mathbb{T})$: thus, octo-octonions have a larger derivation algebra than the Dixon algebra, with an enhancement/restriction expressed by (30) or, equivalently, by (31).

I From (36) and (61), one respectively obtains

$$
\begin{aligned}
\mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\,P^1\right) &\simeq \mathfrak{so}_{16}=\mathfrak{so}_9\oplus\mathfrak{so}_7\oplus(\mathbf{9},\mathbf{7})\\
&= \mathfrak{so}_9\oplus\mathfrak{g}_2\oplus(\mathbf{9},\mathbf{7})\oplus(\mathbf{1},\mathbf{7})\\
&\overset{(30)\text{ or }(31)}{=} \mathfrak{so}_9\oplus\mathfrak{su}_2\oplus(\mathbf{9},\mathbf{1}+2\cdot\mathbf{3})\\
&\quad\oplus(\mathbf{1},2\cdot\mathbf{3}+\mathbf{1})\oplus2\cdot(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{5})\\
&= \mathfrak{so}_8\oplus\mathfrak{su}_2\oplus(\mathbf{8}_v+\mathbf{1},\mathbf{1}+2\cdot\mathbf{3})\oplus(\mathbf{1},2\cdot\mathbf{3}+\mathbf{1})\\
&\quad\oplus2\cdot(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{5})\oplus(\mathbf{8}_v,\mathbf{1})\\
&= \mathfrak{so}_7\oplus\mathfrak{su}_2\oplus(\mathbf{7}+2\cdot\mathbf{1},\mathbf{1}+2\cdot\mathbf{3})\oplus(\mathbf{1},2\cdot\mathbf{3}+\mathbf{1})\\
&\quad\oplus2\cdot(\mathbf{1},\mathbf{3})\oplus(\mathbf{1},\mathbf{5})\oplus(2\cdot\mathbf{7}+\mathbf{1},\mathbf{1})\\
&= \mathfrak{so}_7\oplus\mathfrak{su}_2\oplus2\cdot(\mathbf{7},\mathbf{3})\oplus8\cdot(\mathbf{1},\mathbf{3})
\end{aligned}
$$

$$\oplus 3 \cdot (\mathbf{7},\mathbf{1}) \oplus 4 \cdot (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})$$
$$\simeq \mathfrak{isom}\left(\mathbb{T}P^1_I\right) \oplus 4 \cdot (\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}), \quad (99)$$

and

$$\begin{aligned}\mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) &\simeq \mathfrak{so}_8 \oplus \mathfrak{so}_8 = \mathfrak{so}_8 \oplus \mathfrak{so}_7 \oplus (\mathbf{1},\mathbf{7}) \\ &= \mathfrak{so}_8 \oplus \mathfrak{g}_2 \oplus 2\cdot(\mathbf{1},\mathbf{7}) \\ &\overset{(30)\text{ or }(31)}{=} \mathfrak{so}_8 \oplus \mathfrak{su}_2 \oplus 2\cdot(\mathbf{1},\mathbf{1}+2\cdot\mathbf{3}) \\ &\oplus 2\cdot(\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{5}) \\ &= \mathfrak{so}_7 \oplus \mathfrak{su}_2 \oplus 2\cdot(\mathbf{1},\mathbf{1}) \oplus 6\cdot(\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{5}) \oplus (\mathbf{7},\mathbf{1}) \\ &\simeq \mathfrak{isot}\left(\mathbb{T}P^1_I\right) \oplus 4\cdot(\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}). \quad (100)\end{aligned}$$

Thus, it holds that

$$\mathfrak{isom}\left(\mathbb{T}P^1_I\right) \subsetneq \mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right), \quad (101)$$
$$\mathfrak{isot}\left(\mathbb{T}P^1_I\right) \subsetneq \mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right), \quad (102)$$

and

$$\begin{aligned}\mathfrak{c}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) &:\simeq \mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \\ &\ominus \mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \\ &= \left(\mathfrak{isom}\left(\mathbb{T}P^1_I\right) \oplus (\mathbf{5}+4\cdot\mathbf{3}+\mathbf{1},\mathbf{1})\right) \\ &\ominus \left(\mathfrak{isot}\left(\mathbb{T}P^1_I\right) \oplus (\mathbf{5}+4\cdot\mathbf{3}+\mathbf{1},\mathbf{1})\right) \\ &\simeq \mathfrak{isom}\left(\mathbb{T}P^1_I\right) \ominus \mathfrak{isot}\left(\mathbb{T}P^1_I\right) \simeq: \mathfrak{c}\left(\mathbb{T}P^1_I\right). \quad (103)\end{aligned}$$

II Analogously, from (53) and (67), one respectively obtains

$$\mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \not\subseteq\not\supseteq \mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}}; \quad (104)$$
$$\begin{aligned}&\mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \cap \mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}} \\ &\simeq \mathfrak{so}_7 \oplus \mathfrak{su}_2 \oplus 2\cdot(\mathbf{7},\mathbf{3}) \oplus 2\cdot(\mathbf{1},\mathbf{3}) \\ &\oplus 3\cdot(\mathbf{7},\mathbf{1}) \oplus 2\cdot(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}), \quad (105)\end{aligned}$$

and

$$\mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \not\subseteq\not\supseteq \mathfrak{isot}\left(\mathbb{T}P^1_{II}\right); \quad (106)$$
$$\begin{aligned}&\mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \cap \mathfrak{isot}\left(\mathbb{T}P^1_{II}\right) \\ &\simeq \mathfrak{so}_7 \oplus \mathfrak{su}_2 \oplus (\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}). \quad (107)\end{aligned}$$

However,

$$\begin{aligned}\mathfrak{c}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) &:\simeq \mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \\ &\ominus \mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \\ &\simeq \mathfrak{isom}\left(\mathbb{T}P^1_{II}\right)_{\text{enh.}} \ominus \mathfrak{isot}\left(\mathbb{T}P^1_{II}\right) \\ &\simeq: \mathfrak{c}\left(\mathbb{T}P^1_{II}\right). \quad (108)\end{aligned}$$

III Again, from (54) and (73), one respectively obtains

$$\begin{aligned}&\mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \\ &\simeq \mathfrak{isom}\left(\mathbb{T}P^1_{III}\right) \oplus 5\cdot(\mathbf{1},\mathbf{3}) \oplus 2\cdot(\mathbf{7},\mathbf{1}) \\ &\oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}), \quad (109)\end{aligned}$$

and

$$\begin{aligned}&\mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) = \mathfrak{so}_7 \oplus \mathfrak{su}_2 \oplus 2\cdot(\mathbf{1},\mathbf{1}) \\ &\oplus 6\cdot(\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\mathbf{5}) \oplus (\mathbf{7},\mathbf{1}) \\ &\simeq \mathfrak{isot}\left(\mathbb{T}P^1_{III}\right) \oplus 5\cdot(\mathbf{1},\mathbf{3}) \oplus 2\cdot(\mathbf{7},\mathbf{1}) \\ &\oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5}). \quad (110)\end{aligned}$$

Thus, it holds that

$$\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right) \subsetneq \mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right), \quad (111)$$
$$\mathfrak{isot}\left(\mathbb{T}P^1_{III}\right) \subsetneq \mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right), \quad (112)$$

and

$$\begin{aligned}&\mathfrak{c}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) :\simeq \mathfrak{isom}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \ominus \mathfrak{isot}\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right) \\ &= \left(\mathfrak{isom}\left(\mathbb{T}P^1_{III}\right) \oplus 5\cdot(\mathbf{1},\mathbf{3}) \oplus 2\cdot(\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})\right) \\ &\ominus \left(\mathfrak{isot}\left(\mathbb{T}P^1_I\right) \oplus 5\cdot(\mathbf{1},\mathbf{3}) \oplus 2\cdot(\mathbf{7},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{5})\right) \\ &\simeq \mathfrak{isom}\left(\mathbb{T}P^1_{III}\right) \ominus \mathfrak{isot}\left(\mathbb{T}P^1_{III}\right) \simeq: \mathfrak{c}\left(\mathbb{T}P^1_{III}\right). \quad (113)\end{aligned}$$

In other words, the Dixon-Rosenfeld projective lines $\mathbb{T}P^1_I$ and $\mathbb{T}P^1_{III}$ have the isometry resp. isotropy Lie algebra strictly contained in the isometry resp. isotropy Lie algebra of the octo-octonionic projective line $(\mathbb{O}\otimes\mathbb{O})\, P^1$, whereas the Dixon-Rosenfeld projective line $\mathbb{T}P^1_{II}$ does not contain nor is contained into $(\mathbb{O}\otimes\mathbb{O})\, P^1$. Nonetheless, as pointed out above, the set of generators of the isometry Lie group whose non-linear realization gives rise to the Dixon-Rosenfeld projective line is the same for $\mathbb{T}P^1_I$, $\mathbb{T}P^1_{II}$ and $\mathbb{T}P^1_{III}$; thus, one can conclude that all such spaces are *locally isomorphic* as homogeneous spaces:

$$T\left(\mathbb{T}P^1_I\right) \simeq T\left(\mathbb{T}P^1_{II}\right) \simeq T\left(\mathbb{T}P^1_{III}\right) \simeq T\left((\mathbb{O}\otimes\mathbb{O})\, P^1\right). \quad (114)$$

It is interesting to remark that this holds notwithstanding the fact that, while the three Dixon-Rosenfeld projective lines have *non-symmetric* presentations, the octo-octonionic Rosenfeld projective line is a *symmetric* space.

4 Projective lines over $\mathbb{C}\otimes\mathbb{H}$ via $\mathbb{C}\otimes J_2(\mathbb{H})$

4.1 Generalized minimal left ideals of $\mathbb{C}\otimes\mathbb{H}$

In pursuing the Standard Model physics of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$, Furey started by considering generalized minimal left ideals of $\mathbb{C}\otimes\mathbb{H}$ and demonstrated how scalar, chiral spinors, vector, and 2-form representations of the Lorentz spacetime group may be identified [22]. Given some algebra $\mathfrak{g}$, a (generalized) minimal ideal $\mathfrak{i}\subset\mathfrak{g}$ is a subalgebra where $m(a,v)\in\mathfrak{i}$ for all $a\in\mathfrak{g}$ and $v\in\mathfrak{i}$ with m as a (generalized) multiplication. The generalized minimal left ideal that Furey considered for spinors from $\mathfrak{g}=\mathbb{C}\otimes\mathbb{H}$ is

$$m_1(a,v)=v'=avP+a^*vP^* \tag{115}$$

where $P=(1+Ik)/2$ such that $P^*=(1-Ik)/2$ are projectors satisfying $P^2=P$, $P^{*2}=P^*$, and $PP^*=P^*P=0$. The 4-vectors (1-forms) were found as generalized minimal ideals via the the following generalized multiplication,

$$m_2(a,v)=v'=ava^\dagger, \tag{116}$$

where $a^\dagger=\hat{a}^*$ is used just for this subsection when $a\in\mathbb{C}\otimes\mathbb{H}$, with $\hat{}$ and * denoting the quaternionic and complex conjugate, respectively. The symbol † is used throughout as a Hermitian conjugate of the algebra, but the explicit mathematical operation will differ depending on the algebra under consideration. The scalars and field strength (2-forms) were found as generalized minimal ideals via the generalized multiplication below,

$$m_3(a,v)=v'=av\hat{a}. \tag{117}$$

Focusing on the spinors, a Dirac spinor ψ_D as an element of $\mathbb{C}\otimes\mathbb{H}$ is decomposed into left- and right-chiral (Weyl) spinors ψ_L and ψ_R as minimal left ideals with respect to Eq. (115),

$$\begin{aligned}\psi_L=v_1&=(c_1+c_3j)\,P\\&=\frac{1}{2}\Big(\big(c_{1,1}+c_{1,2}I\big)-\big(c_{3,2}-c_{3,1}I\big)\,i\\&\quad+\big(c_{3,1}+c_{3,2}I\big)\,j-\big(c_{1,2}-c_{1,1}I\big)\,k\Big),\\\psi_R=v_2&=(c_2-c_4j)\,P^*\\&=\frac{1}{2}\Big(\big(c_{2,1}+c_{2,2}I\big)-\big(c_{4,2}-c_{4,1}I\big)\,i\\&\quad-\big(c_{4,1}+c_{4,2}I\big)\,j+\big(c_{2,2}-c_{2,1}I\big)\,k\Big),\end{aligned} \tag{118}$$

where c_i for $i=1,\ldots 4$ are complex coefficients $c_i=c_{i,1}+c_{i,2}I$. Since $\mathbb{C}\otimes\mathbb{H}$ is associative, it is straightforward to verify that $\psi_L P=\psi_L$, $\psi_R P^*=\psi_R$, and $\psi_L P^*=\psi_R P=0$. Additionally, the Lorentz transformations can be found as the exponentiation of linear combinations of vectors and bivectors of $Cl(3)$.

The basis of minimal ideals is less clear with $\mathbb{C}\otimes\mathbb{H}$ and improved with reference to another basis spanned by $\left\{P,P^*,jP,\hat{j}P^*,IP,IP^*,IjP,I\hat{j}P^*\right\}$. To provide a dictionary of various representations used by Furey for the spinor minimal ideal bases [22–24], consider

$$\begin{aligned}P=[\uparrow L]=|\uparrow\rangle\langle\uparrow|=\epsilon_{\uparrow\uparrow}&=\frac{1+Ik}{2},\\P^*=[\downarrow R]=|\downarrow\rangle\langle\downarrow|=\epsilon_{\downarrow\downarrow}&=\frac{1-Ik}{2},\\jP=[\downarrow L]=|\downarrow\rangle\langle\uparrow|=\epsilon_{\downarrow\uparrow}&=\frac{j+Ii}{2}=\alpha,\\\hat{j}P^*=-jP^*=[\uparrow R]=|\uparrow\rangle\langle\downarrow|=\epsilon_{\uparrow\downarrow}&=\frac{-j+Ii}{2}=\alpha^\dagger.\end{aligned} \tag{119}$$

We found it convenient to confirm that ψ_L and ψ_R are minimal left ideals in Mathematica when converting to the basis above (along with the four elements multiplied by I). The following anti-commutation relations can be found,

$$\begin{aligned}\left\{\alpha,\alpha^\dagger\right\}&=1,\\\{\alpha,\alpha\}&=0,\\\left\{\alpha^\dagger,\alpha^\dagger\right\}&=0.\end{aligned} \tag{120}$$

Note that Ii and Ij act as bases of $\mathbb{C}l(2)$.

4.2 Generalized minimal left ideals of $\mathbb{C}\otimes J_2(\mathbb{H})$

To build up to projective lines of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$, the physics of spinors for $\mathbb{C}\otimes\mathbb{H}$ are uplifted to $\mathbb{C}\otimes J_2(\mathbb{H})$. The $\mathbb{C}\otimes\mathbb{H}$ spinors are also embedded into $\mathbb{C}\otimes J_2(\mathbb{H})$ by placing ψ_D in the upper-right component and adding by its quaternionic Hermitian conjugate to obtain an element of $\mathbb{C}\otimes J_2(\mathbb{H})$,

$$\psi_D\to J\left(\psi_D\right)\equiv\begin{pmatrix}0&\psi_D\\0&0\end{pmatrix}+\begin{pmatrix}0&\psi_D\\0&0\end{pmatrix}^\dagger=\begin{pmatrix}0&\psi_D\\\hat{\psi}_D&0\end{pmatrix}. \tag{121}$$

Note that here † denotes matrix transpose and quaternionic conjugation.

This brings in a complication for generalizing P, as 2×2 matrices admit two projectors as idempotents, yet $\mathbb{C}\otimes J_2(\mathbb{H})$ does not contain $P=(1+Ik)/2$ on any diagonal elements. The action of $\mathbb{C}\otimes\mathbb{H}$ must occur on the off-diagonals. Despite not giving projectors, the bases are embedded as follows

$$P\to J_P\equiv J(P)=\begin{pmatrix}0&P\\\hat{P}&0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0&1+Ik\\1-Ik&0\end{pmatrix},$$

$$
\begin{aligned}
P^* &\rightarrow J_{P^*} \equiv J\left(P^*\right)=\begin{pmatrix} 0 & P^* \\ P^{\dagger} & 0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 0 & 1-Ik \\ 1+Ik & 0 \end{pmatrix}, \\
jP &\rightarrow J_{jP} \equiv J(jP)=\begin{pmatrix} 0 & jP \\ -jP & 0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 0 & j+Ii \\ -j-Ii & 0 \end{pmatrix}, \\
\hat{j}P^* &\rightarrow J_{\hat{j}P^*} \equiv J\left(\hat{j}P^*\right)=\begin{pmatrix} 0 & \hat{j}P^* \\ jP^* & 0 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 0 & -j+Ii \\ j-Ii & 0 \end{pmatrix}.
\end{aligned} \tag{122}
$$

A new generalized multiplication was identified for spinors as elements of $\mathbb{C}\otimes J_2(\mathbb{H})$ by taking the Jordan product with two matrices from the right to replace P and P^* in Eq. (115),

$$
\begin{aligned}
m_4(a,v) = 2\big[&((A\circ v)\circ J_{P^*})\circ J_P - \big((A\circ v)\circ J_{\hat{j}P^*}\big)\circ J_{jP} \\
&+((A\circ v)\circ J_P)\circ J_{P^*} - \big((A\circ v)\circ J_{jP}\big)\circ J_{\hat{j}^*}\big].
\end{aligned} \tag{123}
$$

where $a \in \mathbb{C}\otimes J_2(\mathbb{H})$ and $a\circ b = (ab+ba)/2$ is the Jordan product. We verified in Mathematica that $m_4(a,v)$ gives spinorial ideals for arbitrary $a \in \mathbb{C}\otimes J_2(\mathbb{H})$. Since $\mathbb{C}\otimes J_2(\mathbb{H})$ is larger than the piece of $\mathbb{C}\otimes\mathbb{H}$ embedded in $\mathbb{C}\otimes J_2(\mathbb{H})$, the existence of such a generalized ideal may hold for the entire algebra constructed from the Dixon-Rosenfeld line via the Freudenthal–Tits construction.

For Hermitian and anti-Hermitian vectors, the following generalized multiplication rule is found,

$$
m_5(a,v) = (a\circ v)\circ\hat{a}^* + a\circ\left(v\circ\hat{a}^*\right), \tag{124}
$$

where m_5 is identified as a Jordan anti-associator. If a is chosen to be a purely off-diagonal element of $\mathbb{C}\otimes J_2(\mathbb{H})$, then m_5 leads to an element of i for v as a Hermitian or anti-Hermitian vector. If a is chosen as an arbitrary element of $\mathbb{C}\otimes J_2(\mathbb{H})$, then the Hermitian vector uplifted to $\mathbb{C}\otimes J_2(\mathbb{H})$ develops a purely real diagonal term, while the antiHermitian vector uplifted develops a purely imaginary diagonal term. It is also anticipated that diagonals of $\mathbb{C}\otimes J_2(\mathbb{H})$ not found in $\mathbb{C}\otimes\mathbb{H}$ should be purely bosonic, which motivates a higher-dimensional Hermitian and anti-Hermitian vector to be found as ideals of $\mathbb{C}\otimes J_2(\mathbb{H})$.

For scalars and two-forms, the following generalized multiplication rule is found with a Jordan anti-associator and slightly different conjugation,

$$
m_6(a,v) = (a\circ v)\circ\hat{a} + a\circ(v\circ\hat{a}). \tag{125}
$$

It turns out that the 2-form uplifted to $\mathbb{C}\otimes J_2(\mathbb{H})$ is a minimal ideal, while the scalar uplifted must be generalized to include a complex diagonal.

For concreteness, the left- and right-chiral spinors embedded in $\mathbb{C}\otimes J_2(\mathbb{H})$ as minimal ideals of m_4 in Eq. (123) are

$$
J_{\psi_L} = \begin{pmatrix} 0 & (c_1+c_3 j)\,P \\ c_1P^* - c_3P & 0 \end{pmatrix}
$$

$$
J_{\psi_R} = \begin{pmatrix} 0 & (c_2-c_4 j)\,P^* \\ c_2P + c_4P^* & 0 \end{pmatrix}. \tag{126}
$$

The vectors h^μ and pseudo-vectors g^μ for $\mu = 0,1,2,3$ represented as elements of $\mathbb{C}\otimes\mathbb{H}$ to be used with Eq. (124) are generalized to the following minimal ideals of $\mathbb{C}\otimes J_2(\mathbb{H})$ with diagonal components

$$
\begin{aligned}
J_h &= \begin{pmatrix} h_4 & h_0I+h_1i+h_2j+h_3k \\ h_0I-h_1i-h_2j-h_3k & h_5 \end{pmatrix}, \\
J_g &= \begin{pmatrix} g_4I & g_0+g_1iI+g_2jI+g_3kI \\ g_0-g_1iI-g_2jI-g_3kI & g_5I \end{pmatrix},
\end{aligned} \tag{127}
$$

where h_4, h_5, g_4, and g_5 are scalar degrees of freedom found on the diagonals of the minimal ideals that extend the 4-vector and 4-pseudo-vector. The scalars ϕ and 2-forms F embedded in $\mathbb{C}\otimes J_2(\mathbb{H})$ with Eq. (125) are found as minimal ideals when a complex diagonal is added to the scalars

$$
\begin{aligned}
J_\phi &= \begin{pmatrix} \phi_3+\phi_4I & \phi_1+\phi_2I \\ \phi_1-\phi_2I & \phi_5+\phi_6I \end{pmatrix}, \\
J_F &= \begin{pmatrix} 0 & J_{F,12} \\ J_{F,21} & 0 \end{pmatrix}, \\
J_{F,12} &= F^{32}i+F^{13}j+F^{21}k+F^{01}iI+F^{02}jI+F^{03}kI, \\
J_{F,21} &= -F^{32}i-F^{13}j-F^{21}k-F^{01}iI-F^{02}jI-F^{03}kI.
\end{aligned} \tag{128}
$$

One may anticipate that the vector, spinor, and conjugate spinor representations can be embedded in the three independent off-diagonal components of $\mathbb{C}\otimes J_3(\mathbb{H})$, but this is left for future work.

5 Projective lines over $\mathbb{C}\otimes\mathbb{O}$ via $\mathbb{C}\otimes J_2(\mathbb{O})$

5.1 Minimal left ideals of $\mathbb{C}l(6)$ via chain algebra $\mathbb{C}\otimes\overleftarrow{\mathbb{O}}$

To establish our conventions for octonions, we review the complexification of the octonionic chain algebra applied to raising and lowering operators for $SU(3)_c \times U(1)_{em}$ fermionic charge states [22,24]. For $\mathbb{C}\otimes\mathbb{O}$, we use I and e_i for $i = 1,\ldots,7$ as the imaginary units. To convert from Furey's octonionic basis to ours, take $\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\} \rightarrow \{e_2,e_3,e_6,e_1,e_5,e_7,-e_4\}$. A system of ladder operators was constructed from the complexification of the octonionic chain algebra $\mathbb{C}\otimes\overleftarrow{\mathbb{O}} \cong \mathbb{C}l(6)$, which allows contact with $SU(3)_c\times U(1)_{em}$ [25]. Due to the nonassociative nature of the octonions, the following association of multiplication is always assumed, where an arbitrary element $f \in \mathbb{C}\otimes\mathbb{O}$ must be considered,

$$
\{\alpha_i,\alpha_j\} := \{\alpha_i,\alpha_j\}\,f = \alpha_i\left(\alpha_j f\right) + \alpha_j\left(\alpha_i f\right). \tag{129}
$$

If a^* refers to complex conjugation and $\tilde{a}$ refers to octonionic conjugation, denote $a^\dagger = \tilde{a}^*$ as the Hermitian conjugate only when acting on $a \in \mathbb{C} \otimes \mathbb{O}$.

Our basis of raising and lowering operators is chosen as

$$\begin{aligned} \alpha_1 &= q_1 = \frac{1}{2}\left(-e_5 + I e_1\right), & \alpha_1^\dagger &= -q_1^* = \frac{1}{2}\left(e_5 + I e_1\right), \\ \alpha_2 &= q_2 = \frac{1}{2}\left(-e_6 + I e_2\right), & \alpha_2^\dagger &= -q_2^* = \frac{1}{2}\left(e_6 + I e_2\right), \\ \alpha_3 &= q_3 = \frac{1}{2}\left(-e_7 + I e_3\right), & \alpha_3^\dagger &= -q_3^* = \frac{1}{2}\left(e_7 + I e_3\right). \end{aligned} \tag{130}$$

With this basis, we explicitly confirmed in Mathematica that the following relations hold,

$$\begin{aligned} &\left\{\alpha_i, \alpha_j^\dagger\right\} f = \delta_{ij} f, \\ &\left\{\alpha_i, \alpha_j\right\} f = 0, \\ &\left\{\alpha_i^\dagger, \alpha_j^\dagger\right\} f = 0 \end{aligned} \tag{131}$$

It was also confirmed that $\left\{\alpha_i^*, \tilde{\alpha}_j\right\} = \delta_{ij}$. For later convenience, a leptonic sector of operators is also introduced as

$$\alpha_0 = I l^* = \frac{1}{2}\left(-e_4 + I\right), \qquad \tilde{\alpha}_0 = I l = \frac{1}{2}\left(e_4 + I\right). \tag{132}$$

Due to the non-associativity of octonions, acting from the left once does not span all of the possible transformations, which motivates nested multiplication. This naturally motivates $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$ as the octonionic chain algebra corresponding to $\mathbb{C}l(6)$. This chooses $-e_4$ as a pseudoscalar, such that the k-vector decomposition of $\mathbb{C}l(6)$ is spanned by 1-vectors $\{Ie_2, Ie_3, Ie_6, Ie_1, Ie_5, Ie_7\}$.

Next, a nilpotent object $\omega = \alpha_1\alpha_2\alpha_3$ is introduced, where the parentheses of the chain algebra mentioned above is assumed below. The Hermitian conjugate is $\omega^\dagger = \alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger$. The state $v_c = \omega\omega^\dagger$ is considered roughly as a vacuum state (perhaps renormalized with weak isospin up), since $\alpha_i\omega\omega^\dagger = 0$. Fermionic charge states of isospin up are identified as minimal left ideals via

$$\begin{aligned} S^u \equiv\ & \nu\omega\omega^\dagger + \bar{d}^r\alpha_1^\dagger\omega\omega^\dagger + \bar{d}^g\alpha_2^\dagger\omega\omega^\dagger + \bar{d}^b\alpha_3^\dagger\omega\omega^\dagger \\ & + u^r\alpha_3^\dagger\alpha_2^\dagger\omega\omega^\dagger + u^g\alpha_1^\dagger\alpha_3^\dagger\omega\omega^\dagger + u^b\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger \\ & + \bar{e}\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger, \end{aligned} \tag{133}$$

where $\nu, \bar{d}^i, u^i$, and $\bar{e}$ are complex coefficients. The weak isospin down states are found by building off of $v_c^* = \omega^\dagger\omega$, giving

$$\begin{aligned} S^d \equiv\ & \bar{\nu}\omega^\dagger\omega - d^r\alpha_1\omega^\dagger\omega - d^g\alpha_2\omega^\dagger\omega - d^b\alpha_3\omega^\dagger\omega \\ & + \bar{u}^r\alpha_3\alpha_2\omega^\dagger\omega + \bar{u}^g\alpha_1\alpha_3\omega^\dagger\omega + \bar{u}^b\alpha_2\alpha_1\omega^\dagger\omega \\ & + e\alpha_1\alpha_2\alpha_3\omega^\dagger\omega. \end{aligned} \tag{134}$$

These algebraic operators represent charge states associated with one generation of the Standard Model with reference to $SU(3)_c \times U(1)_{em}$.

A notion of Pauli's exclusion principle is found, since the following relations hold,

$$\begin{aligned} \omega\omega^\dagger\omega\omega^\dagger &= \omega\omega^\dagger, \\ \alpha_i^\dagger\omega\omega^\dagger\omega\omega^\dagger &= \alpha_i^\dagger\omega\omega^\dagger \\ \alpha_i^\dagger\omega\omega^\dagger\alpha_i^\dagger\omega\omega^\dagger &= \alpha_i^\dagger\alpha_j^\dagger\omega\omega^\dagger\alpha_i^\dagger\alpha_j^\dagger\omega\omega^\dagger \\ &= \alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger\alpha_3^\dagger\alpha_2^\dagger\alpha_1^\dagger\omega\omega^\dagger = 0. \end{aligned} \tag{135}$$

The above equations imply that it is impossible to create two identical fermionic states.

As implied, the three raising/lowering operators are associated with three color charges. Furey also demonstrated that the electric charge is associated with the mean of the number operators $N_i = \alpha_i^\dagger\alpha_i$ [25]. To obtain spinors associated with these charge configurations, Furey advocates for $(\mathbb{C} \otimes \mathbb{H}) \otimes_{\mathbb{C}} (\mathbb{C} \otimes \mathbb{O}) = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Before reviewing this procedure, we first generalize the results of $\mathbb{C} \otimes \mathbb{O}$ to $\mathbb{C} \otimes J_2(\mathbb{O})$.

5.2 Uplift of $\mathbb{C}l(6)$ in $\mathbb{C} \otimes \overleftarrow{J_2(\mathbb{O})}$

Next, the analogous raising and lowering operators associated with one generation of the Standard Model are constructed with elements of $\mathbb{C} \otimes \overleftarrow{J_2(\mathbb{O})}$. Our guiding principle is to take elements of $\mathbb{C} \otimes \mathbb{O}$, place them on the upper off-diagonal component of $\mathbb{C} \otimes \overleftarrow{J_2(\mathbb{O})}$, and add the Hermitian octonionic conjugate. We seek a new generalized multiplication that implements the same particle dynamics as $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$. For concreteness, consider J_f as an arbitrary element of $\mathbb{C} \otimes J_2(\mathbb{O})$,

$$J_f = \begin{pmatrix} f_8 & f \\ \tilde{f} & f_9 \end{pmatrix} = \begin{pmatrix} f_8 & f_0 + \sum_{i=1}^{7} e_i f_i \\ f_0 - \sum_{i=1}^{7} e_i f_i & f_9 \end{pmatrix}, \tag{136}$$

where $f_i = f_{i,0} + I f_{i,1}$ for $i = 0, 1, \ldots, 9$.

The Jordan product is utilized to restore elements of $\mathbb{C} \otimes J_2(\mathbb{O})$. However, this conflicts with left multiplication utilized in the chain algebra $\mathbb{C} \otimes \overleftarrow{\mathbb{O}}$. The natural multiplication for $\mathbb{C} \otimes \overleftarrow{J_2(\mathbb{O})}$ used throughout uses a nested commutator of Jordan products,

$$m_7\left(J_1, J_2, J_f\right) \equiv J_1 \circ \left(J_2 \circ J_f\right) - J_2 \circ \left(J_1 \circ J_f\right), \tag{137}$$

where $J_1, J_2 \in \mathbb{C} \otimes J_2(\mathbb{O})$ as arbitrary elements. Rather than having a single element of $\mathbb{C} \otimes J_2(\mathbb{O})$ to implement α_i and

$\alpha_j^\dagger$, the multiplication above is utilized. The following $\mathbb{C}\otimes\mathbb{O}$ variables are first uplifted to elements of $\mathbb{C}\otimes J_2(\mathbb{O})$,

$$\begin{aligned}
J_{\alpha_0} &\equiv \begin{pmatrix} 0 & \alpha_0 \\ \tilde{\alpha}_0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -e_4+I \\ e_4+I & 0 \end{pmatrix}, \\
J_{\alpha_1} &\equiv \begin{pmatrix} 0 & \alpha_1 \\ \tilde{\alpha}_1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -e_5+e_1 I \\ e_5-e_1 I & 0 \end{pmatrix}, \\
J_{\alpha_2} &\equiv \begin{pmatrix} 0 & \alpha_2 \\ \tilde{\alpha}_2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -e_6+e_2 I \\ e_6-e_2 I & 0 \end{pmatrix}, \\
J_{\alpha_3} &\equiv \begin{pmatrix} 0 & \alpha_3 \\ \tilde{\alpha}_3 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -e_7+e_3 I \\ e_7-e_3 I & 0 \end{pmatrix}, \\
J_{\tilde{\alpha}_0} &\equiv \begin{pmatrix} 0 & \tilde{\alpha}_0 \\ \alpha_0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & e_4+I \\ -e_4+I & 0 \end{pmatrix}, \\
J_{\alpha_1^\dagger} &\equiv \begin{pmatrix} 0 & \alpha_1^\dagger \\ \tilde{\alpha}_1^\dagger & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & e_5+e_1 I \\ -e_5-e_1 I & 0 \end{pmatrix}, \\
J_{\alpha_2^\dagger} &\equiv \begin{pmatrix} 0 & \alpha_2^\dagger \\ \tilde{\alpha}_2^\dagger & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & e_6+e_2 I \\ -e_6-e_2 I & 0 \end{pmatrix}, \\
J_{\alpha_3^\dagger} &\equiv \begin{pmatrix} 0 & \alpha_3^\dagger \\ \tilde{\alpha}_3^\dagger & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & e_7+e_3 I \\ -e_7-e_3 I & 0 \end{pmatrix},
\end{aligned} \tag{138}$$

where $\alpha_0 = -e_4 + I$ was introduced for later convenience. We also introduce $J_{I\alpha_i} = I J_{\alpha_i}$ as a shorthand.

These matrices allow for the following nested multiplications to mimic the action of α_i and $\alpha_j^\dagger$,

$$\begin{aligned}
m_{\alpha_1}(J_f) &= 2\left(m_7\left(J_{I\tilde{\alpha}_0}, J_{\alpha_1}, J_f\right) + m_7\left(J_{I\alpha_2^\dagger}, J_{\alpha_3^\dagger}, J_f\right)\right), \\
m_{\alpha_2}(J_f) &= 2\left(m_7\left(J_{I\tilde{\alpha}_0}, J_{\alpha_2}, J_f\right) + m_7\left(J_{I\alpha_3^\dagger}, J_{\alpha_1^\dagger}, J_f\right)\right), \\
m_{\alpha_3}(J_f) &= 2\left(m_7\left(J_{I\tilde{\alpha}_0}, J_{\alpha_3}, J_f\right) + m_7\left(J_{I\alpha_1^\dagger}, J_{\alpha_2^\dagger}, J_f\right)\right), \\
m_{\alpha_1^\dagger}(J_f) &= 2\left(m_7\left(J_{I\alpha_0}, J_{\alpha_1^\dagger}, J_f\right) + m_7\left(J_{I\alpha_2}, J_{\alpha_3}, J_f\right)\right), \\
m_{\alpha_2^\dagger}(J_f) &= 2\left(m_7\left(J_{I\alpha_0}, J_{\alpha_2^\dagger}, J_f\right) + m_7\left(J_{I\alpha_3}, J_{\alpha_1}, J_f\right)\right), \\
m_{\alpha_3^\dagger}(J_f) &= 2\left(m_7\left(J_{I\alpha_0}, J_{\alpha_3^\dagger}, J_f\right) + m_7\left(J_{I\alpha_1}, J_{\alpha_2}, J_f\right)\right).
\end{aligned} \tag{139}$$

The following anticommutation relations were explicitly verified,

$$\begin{aligned}
\left\{m_{\alpha_i}, m_{\alpha_j^\dagger}\right\} J_f &\equiv m_{\alpha_i}\left(m_{\alpha_j^\dagger}(J_f)\right) + m_{\alpha_j^\dagger}\left(m_{\alpha_i}(J_f)\right) = \delta_{ij} J_f^{\text{off}}, \\
\left\{m_{\alpha_i}, m_{\alpha_j}\right\} J_f &\equiv m_{\alpha_i}\left(m_{\alpha_j}(J_f)\right) + m_{\alpha_j}\left(m_{\alpha_i}(J_f)\right) = 0, \\
\left\{m_{\alpha_i^\dagger}, m_{\alpha_j^\dagger}\right\} J_f &\equiv m_{\alpha_i^\dagger}\left(m_{\alpha_j^\dagger}(J_f)\right) + m_{\alpha_j^\dagger}\left(m_{\alpha_i^\dagger}(J_f)\right) = 0,
\end{aligned} \tag{140}$$

where J_f^{off} contains only the off-diagonal components of J_f. This suffices to generalize the fermionic degrees of freedom from $\mathbb{C}\otimes\mathbb{O}$ since they are uplifted to the off-diagonals of $\mathbb{C}\otimes J_2(\mathbb{O})$.

As an abuse of notation, $m_{\alpha_i} m_{\alpha_j}$ is shorthand for $m_{\alpha_i}\left(m_{\alpha_j}(J_f)\right)$. The nilpotent operator of $\mathbb{C}\otimes\overleftarrow{J_2(\mathbb{O})}$ is given by m_ω,

$$m_\omega = m_{\alpha_1} m_{\alpha_2} m_{\alpha_3}, \quad m_{\omega^\dagger} = m_{\alpha_3^\dagger} m_{\alpha_2^\dagger} m_{\alpha_1^\dagger} \tag{141}$$

One may verify that $m_\omega m_\omega = m_{\omega^\dagger} m_{\omega^\dagger} = 0$, while $m_\omega m_{\omega^\dagger}$ acts on J_f to give a generalized minimal ideal of $\mathbb{C}\otimes\overleftarrow{J_2(\mathbb{O})}$,

$$\begin{aligned}
m_\omega m_{\omega^\dagger} J_f &= \begin{pmatrix} 0 & \omega\omega^\dagger f \\ \left(\omega\omega^\dagger f\right)^{*\dagger} & 0 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & \begin{array}{l} f_0(1-e_4 I) \\ +f_4(e_4+I) \end{array} \\ \begin{array}{l} f_0(1+e_4 I) \\ +f_4(-e_4+I) \end{array} & 0 \end{pmatrix},
\end{aligned} \tag{142}$$

where f is the upper-right component of J_f and $\left(\omega\omega^\dagger f\right)^{*\dagger}$ is a shorthand for the octonionic conjugate. This allows for the assignment of a neutrino "vacuum" state, which allows for the following assignments of particles,

$$\begin{aligned}
& m_\nu = m_\omega m_{\omega^\dagger}, \\
& m_{\bar{d}^r} = m_{\alpha_1^\dagger} m_\omega m_{\omega^\dagger}, m_{\bar{d}^g} = m_{\alpha_2^\dagger} m_\omega m_{\omega^\dagger}, \\
& m_{\bar{d}^b} = m_{\alpha_3^\dagger} m_\omega m_{\omega^\dagger} \\
& m_{u^r} = m_{\alpha_3^\dagger} m_{\alpha_2^\dagger} m_\omega m_{\omega^\dagger}, m_{u^g} = m_{\alpha_1^\dagger} m_{\alpha_3^\dagger} m_\omega m_{\omega^\dagger}, \\
& m_{u^b} = m_{\alpha_2^\dagger} m_{\alpha_1^\dagger} m_\omega m_{\omega^\dagger}, \\
& m_{\bar{e}} = m_{\alpha_3^\dagger} m_{\alpha_2^\dagger} m_{\alpha_1^\dagger} m_\omega m_{\omega^\dagger},
\end{aligned} \tag{143}$$

and

$$\begin{aligned}
& m_{\bar{\nu}} = m_{\omega^\dagger} m_\omega, \\
& m_{d^r} = -m_{\alpha_1} m_{\omega^\dagger} m_\omega, m_{d^g} = -m_{\alpha_2} m_{\omega^\dagger} m_\omega, \\
& m_{d^b} = -m_{\alpha_3} m_{\omega^\dagger} m_\omega m_{\bar{u}^r} = m_{\alpha_3} m_{\alpha_2} m_{\omega^\dagger} m_\omega, \\
& m_{\bar{u}^g} = m_{\alpha_1} m_{\alpha_3} m_{\omega^\dagger} m_\omega, m_{\bar{u}^b} = m_{\alpha_2} m_{\alpha_1} m_{\omega^\dagger} m_\omega, \\
& m_e = m_{\alpha_3} m_{\alpha_2} m_{\alpha_1} m_{\omega^\dagger} m_\omega.
\end{aligned} \tag{144}$$

In summary, the collection of weak-isospin up and down states are

$$\begin{aligned}
& m^u\left(\nu, \bar{d}^r, \bar{d}^g, \bar{d}^b, u^r, u^g, u^b, \bar{e}\right) \\
& = \nu m_\nu + \bar{d}^r m_{\bar{d}^r} + \bar{d}^g m_{\bar{d}^g} + \bar{d}^b m_{\bar{d}^b} \\
& + u^r m_{u^r} + u^g m_{u^g} + u^b m_{u^b} + \bar{e} m_{\bar{e}}, \\
& m^d\left(\bar{\nu}, d^r, d^g, d^b, \bar{u}^r, \bar{u}^g, \bar{u}^b, e\right) \\
& = \bar{\nu} m_{\bar{\nu}} + d^r m_{d^r} + d^g m_{d^g} + d^b m_{d^b} \\
& + \bar{u}^r m_{\bar{u}^r} + \bar{u}^g m_{\bar{u}^g} + \bar{u}^b m_{\bar{u}^b} + e m_e,
\end{aligned} \tag{145}$$

where $\nu, \bar{d}^r$, etc. are complex coefficients.

6 Projective lines over $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$

6.1 One generation from $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$

Furey provided a formulation of the electroweak sector [26], which led to the Standard Model embedded in $SU(5)$ and allows for $U(1)_{B-L}$ symmetry [28,31]. The construction relies on identifying $\mathbb{C}l(10)=\mathbb{C}l(6)\otimes_{\mathbb{C}}\mathbb{C}l(4)$, which can be found from a double-sided chain algebra over $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. For instance, the left- and right-chiral spinors can be brought together via $\psi_D=\psi_R+\psi_L$ with the gamma matrices implemented as

$$\gamma^0=1\Big|Ii,\quad \gamma^1=Ii\Big|\,j,\quad \gamma^2=Ij\Big|j,\quad \gamma^3=Ik\Big|\,j, \tag{146}$$

where $a|b$ acting on z is azb, which is well-defined when $(az)b=a(zb)$. This allows for left and right action of $\mathbb{C}\otimes\mathbb{H}$ to give $\mathbb{C}l(4)=\mathbb{C}l(2)\otimes_{\mathbb{C}}\mathbb{C}l(2)$. This idea can be taken further to give $\mathbb{C}l(10)$ to identify $Spin(10)$ and make contact with $SU(3)\times SU(2)\times U(1)$ for the Standard Model. In this manner, $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ allows for $Spin(10)$ to act from the left. While the full $\mathbb{C}l(4)$ spacetime algebra cannot be found, the remaining right action remarkably picks out $SL(2,\mathbb{C})$ as $SU(2)_{\mathbb{C}}$.

A collection of left-chiral Weyl spinors in the $(\mathbf{2},\mathbf{1},\mathbf{16})$ representation of $SL(2,\mathbb{C})\times Spin(10)$ also contains degrees of freedom for right-chiral antiparticles with opposite charges via $(\mathbf{1},\mathbf{2},\overline{\mathbf{16}})$, which leads to a physicist's convention to ignore writing down the conjugate representation. Each of the 16 Weyl spinors is an element of $\mathbb{C}^2$. When working with $\mathbb{C}\otimes\mathbb{H}\otimes\overleftarrow{\mathbb{O}}$, there are no two-component vectors, so it is necessary to find two copies of $\mathbf{16}$. When Furey explored $\mathbb{C}l(10)$ from $\mathbb{C}\otimes\mathbb{H}\otimes\overleftarrow{\mathbb{O}}$, a $\mathbf{16}$ with its conjugate representation was found, instead of two $\mathbf{16}$'s to give $(\mathbf{2},\mathbf{1},\mathbf{16})$ for a single generation of Standard Model fermions. This led to the so-called fermion doubling problem.

Recent work by Furey and Hughes introduced fermions in the non-associative $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ algebra to solve this fermion doubling problem, which can be resolved by taking a slightly different route to $Spin(10)$, rather than taking bivectors of $\mathbb{C}l(10)$ [29]. Instead, consider the following generalization of Pauli matrices,

$$\sigma_i=-e_ij|1,\quad \sigma_8=-Ii|1,\quad \sigma_9=-Ik|1,\quad \sigma_{10}=-I|1, \tag{147}$$

where $i=1,\ldots,7$ and $\{\sigma_i,\sigma_8,\sigma_9\}$ allow for a basis of $\mathbb{C}l(9)$. The ten "generators" σ_I for $I=1,\ldots,10$ lead to transformations on f via

$$\frac{1}{2}\sigma_{[I}\bar{\sigma}_{J]}\psi=\frac{1}{4}\left(\sigma_I\left(\bar{\sigma}_Jf\right)-\sigma_J\left(\bar{\sigma}_If\right)\right), \tag{148}$$

where $\bar{\sigma}_a=-\sigma_a$ for $a=1,\ldots,9$ and $\bar{\sigma}_{10}=\sigma_{10}$. This allows for $Spin(10)$ to act on a Weyl spinor in the $\mathbf{16}$ representation instead of two 1-component objects of $\mathbf{16}\oplus\overline{\mathbf{16}}$ to resolve the fermion doubling problem.

With $\alpha_\mu=(Il^*,q_1,q_2,q_3)$ and $\alpha^*_\mu=(-Il,q_1^*,q_2^*,q_3^*)$ for $\mu=0,1,2,3$ as an electrostrong sector and $\epsilon_{\alpha\beta}$ with $\alpha=\uparrow,\downarrow$ as an electroweak sector, the non-associative algebra $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ can be used to implement particle states for a single generation of the Standard Model fermions. We specify the particle states by using the notation and assignments recently introduced by Furey and Hughes in their solution to the fermion doubling problem [29], namely

$$\begin{aligned}\psi=&\left(\mathcal{V}_L^{\uparrow}\epsilon_{\uparrow\uparrow}+\mathcal{V}_L^{\downarrow}\epsilon_{\uparrow\downarrow}+\mathcal{E}_L^{\uparrow}\epsilon_{\downarrow\uparrow}+\mathcal{E}_L^{\downarrow}\epsilon_{\downarrow\downarrow}\right)l\\&+\left(\mathcal{E}_R^{\downarrow*}\epsilon_{\uparrow\uparrow}-\mathcal{E}_R^{\uparrow*}\epsilon_{\uparrow\downarrow}-\mathcal{V}_R^{\downarrow*}\epsilon_{\downarrow\uparrow}+\mathcal{V}_R^{\uparrow*}\epsilon_{\downarrow\downarrow}\right)l^*\\&-I\left(\mathcal{U}_L^{a\uparrow}\epsilon_{\uparrow\uparrow}+\mathcal{U}_L^{a\downarrow}\epsilon_{\uparrow\downarrow}+\mathcal{D}_L^{a\uparrow}\epsilon_{\downarrow\uparrow}+\mathcal{D}_L^{a\downarrow}\epsilon_{\downarrow\downarrow}\right)q_a\\&+I\left(\mathcal{D}_R^{a\downarrow*}\epsilon_{\uparrow\uparrow}-\mathcal{D}_R^{a\uparrow*}\epsilon_{\uparrow\downarrow}-\mathcal{U}_R^{a\downarrow*}\epsilon_{\downarrow\uparrow}+\mathcal{U}_R^{a\uparrow*}\epsilon_{\downarrow\downarrow}\right)q_a^*.\end{aligned} \tag{149}$$

The coefficients such as $\mathcal{V}_L^{\uparrow}$ are complex.

In our conventions, the $SU(3)$ Gell-Mann matrices are represented as elements of $\mathbb{C}\otimes\overleftarrow{\mathbb{O}}$ given by

$$\begin{aligned}&\Lambda_1=-\frac{I}{2}(e_{61}-e_{25}),\qquad \Lambda_2=-\frac{I}{2}(e_{21}+e_{65}),\\&\Lambda_3=-\frac{I}{2}(e_{26}-e_{15}),\qquad \Lambda_4=\frac{I}{2}(e_{35}-e_{17}),\\&\Lambda_5=-\frac{I}{2}(e_{31}-e_{57}),\qquad \Lambda_6=\frac{I}{2}(e_{27}+e_{36}),\\&\Lambda_7=\frac{I}{2}(e_{23}+e_{67}),\qquad \Lambda_8=\frac{I}{2\sqrt{3}}(e_{26}+e_{15}-e_{37}),\end{aligned} \tag{150}$$

where $e_{ij}f$ stands for $e_i(e_jf)$. For the electroweak sector with $SU(2)\times U(1)$ symmetry, the $SU(2)$ generators are represented in terms of imaginary quaternions and a weak isospin projector $s=(1-Ie_4)/2$,

$$\tau_9=\frac{I}{2}si,\qquad \tau_{10}=\frac{I}{2}sj,\qquad \tau_{11}=\frac{I}{2}sk. \tag{151}$$

The weak hypercharge is given by

$$Y=-\frac{I}{2}\left(\frac{1}{3}\left(e_{15}+e_{26}+e_{37}\right)-s^*k\right). \tag{152}$$

Note that all operators from $SU(3)\times SU(2)\times U(1)$ are elements of $\mathbb{C}\otimes\mathbb{H}\otimes\overleftarrow{\mathbb{O}}$ and act from the left. The electric charge operator Q is

$$Q=\tau_{11}+Y=-\frac{I}{2}\left(\frac{1}{3}\left(e_{15}+e_{26}+e_{37}\right)-k\right). \tag{153}$$

By separating ψ into $\psi_l + \psi_q + \psi^c_\nu + \psi^c_e + \psi^c_u + \psi^c_d$, the following fields are found to correspond to the appropriate representations of the Standard Model,

$$\begin{aligned}
&(\mathbf{1},\mathbf{2})_{-1/2} : \psi_l = \left(\mathcal{V}^{\uparrow}_L \epsilon_{\uparrow\uparrow} + \mathcal{V}^{\downarrow}_L \epsilon_{\uparrow\downarrow} + \mathcal{E}^{\uparrow}_L \epsilon_{\downarrow\uparrow} + \mathcal{E}^{\downarrow}_L \epsilon_{\downarrow\downarrow}\right) l,\\
&(\mathbf{3},\mathbf{2})_{1/6} : \psi_q = -I\left(\mathcal{U}^{a\uparrow}_L \epsilon_{\uparrow\uparrow} + \mathcal{U}^{a\downarrow}_L \epsilon_{\uparrow\downarrow} + \mathcal{D}^{a\uparrow}_L \epsilon_{\downarrow\uparrow} + \mathcal{D}^{a\downarrow}_L \epsilon_{\downarrow\downarrow}\right) q_a,\\
&(\mathbf{1},\mathbf{1})_0 : \psi^c_\nu = \left(-\mathcal{V}^{\downarrow *}_R \epsilon_{\downarrow\uparrow} + \mathcal{V}^{\uparrow *}_R \epsilon_{\downarrow\downarrow}\right) l^*,\\
&(\mathbf{1},\mathbf{1})_1 : \psi^c_e = \left(\mathcal{E}^{\downarrow *}_R \epsilon_{\uparrow\uparrow} - \mathcal{E}^{\uparrow *}_R \epsilon_{\uparrow\downarrow}\right) l^*\\
&(\bar{\mathbf{3}},\mathbf{1})_{-2/3} : \psi^c_u = I\left(-\mathcal{U}^{a\downarrow *}_R \epsilon_{\downarrow\uparrow} + \mathcal{U}^{a\uparrow *}_R \epsilon_{\downarrow\downarrow}\right) q^*_a,\\
&(\bar{\mathbf{3}},\mathbf{1})_{1/3} : \psi^c_d = I\left(\mathcal{D}^{a\downarrow *}_R \epsilon_{\uparrow\uparrow} - \mathcal{D}^{a\uparrow *}_R \epsilon_{\uparrow\downarrow}\right) q^*_a,
\end{aligned} \tag{154}$$

where we confirmed that the above states have the appropriate weak hypercharge values as well as weak isospin and electric charges. Note that complex conjugation leads to the appropriate conjugate states, which turns left(right)-chiral particles into right(left)-chiral anti-particles. Finally, the largest algebra commuting with $\mathfrak{so}_{10}$ derived from $\mathbb{C}\otimes\mathbb{H}\otimes\overleftrightarrow{\mathbb{O}}$ when considering action from the left and right is given by $\mathfrak{sl}_{2,\mathbb{C}}$, which are generated by $\{1|i, 1|j, 1|k, 1|Ii, 1|Ij, 1|Ik\}$.

6.2 Uplift to $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$

To uplift the physics of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ to $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$, we start by considering $f \in \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ uplifted to an off-diagonal matrix $J^{\text{off}}_f \in \mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$. Our first goal is to understand how to implement left multiplication of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ basis elements on f by the analogous construction in $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$ acting on J^{off}_f, where

$$J^{\text{off}}_f = \begin{pmatrix} 0 & f \\ \tilde{f} & 0 \end{pmatrix}. \tag{155}$$

For $\mathbb{C}\otimes\mathbb{H}$ bases, these can be implemented by mapping the basis elements to the same elements times the identity matrix. The same cannot be done for $\mathbb{O}$, as the elements e_i must map to $J_2(\mathbb{O})$ via the eight off-diagonal octonionic Pauli matrices J_{e_i},

$$J_{e_i} = \begin{pmatrix} 0 & e_i \\ \tilde{e}_i & 0 \end{pmatrix}. \tag{156}$$

To understand how to multiply f from the left by e_i generalized to $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$, the Fano plane is crucial. A single octonionic unit can always be implemented by multiplying by two units in four different ways. For instance, $e_1 = e_1 1 = e_2 e_3 = e_4 e_5 = e_7 e_6$. If $e_1 f$ is uplifted to $J^{\text{off}}_{e_1 f}$, by recalling the definition (137) of nested commutator of Jordan products, a generalized multiplication rule can be found to give $J^{\text{off}}_{e_1 f}$ from J^{off}_f,

$$\begin{aligned}
J^{\text{off}}_{e_1 f} &= m_{e_1}(J^{\text{off}}_f) \equiv \left\{J_{e_1}, J_1, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_3}, J_{e_2}, J^{\text{off}}_f\right\}_\circ\\
&\quad + \left\{J_{e_5}, J_{e_4}, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_6}, J_{e_7}, J^{\text{off}}_f\right\}_\circ,\\
J^{\text{off}}_{e_2 f} &= m_{e_2}(J^{\text{off}}_f) \equiv \left\{J_{e_2}, J_1, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_1}, J_{e_3}, J^{\text{off}}_f\right\}_\circ\\
&\quad + \left\{J_{e_6}, J_{e_4}, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_7}, J_{e_5}, J^{\text{off}}_f\right\}_\circ,\\
J^{\text{off}}_{e_3 f} &= m_{e_3}(J^{\text{off}}_f) \equiv \left\{J_{e_3}, J_1, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_2}, J_{e_1}, J^{\text{off}}_f\right\}_\circ\\
&\quad + \left\{J_{e_7}, J_{e_4}, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_5}, J_{e_6}, J^{\text{off}}_f\right\}_\circ,\\
J^{\text{off}}_{e_4 f} &= m_{e_4}(J^{\text{off}}_f) \equiv \left\{J_{e_4}, J_1, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_1}, J_{e_5}, J^{\text{off}}_f\right\}_\circ\\
&\quad + \left\{J_{e_2}, J_{e_6}, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_3}, J_{e_7}, J^{\text{off}}_f\right\}_\circ,\\
J^{\text{off}}_{e_5 f} &= m_{e_5}(J^{\text{off}}_f) \equiv \left\{J_{e_5}, J_1, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_4}, J_{e_1}, J^{\text{off}}_f\right\}_\circ\\
&\quad + \left\{J_{e_2}, J_{e_7}, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_3}, J_{e_6}, J^{\text{off}}_f\right\}_\circ,\\
J^{\text{off}}_{e_6 f} &= m_{e_6}(J^{\text{off}}_f) \equiv \left\{J_{e_6}, J_1, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_4}, J_{e_2}, J^{\text{off}}_f\right\}_\circ\\
&\quad + \left\{J_{e_7}, J_{e_1}, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_3}, J_{e_5}, J^{\text{off}}_f\right\}_\circ,\\
J^{\text{off}}_{e_7 f} &= m_{e_7}(J^{\text{off}}_f) \equiv \left\{J_{e_7}, J_1, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_4}, J_{e_3}, J^{\text{off}}_f\right\}_\circ\\
&\quad + \left\{J_{e_1}, J_{e_6}, J^{\text{off}}_f\right\}_\circ + \left\{J_{e_5}, J_{e_2}, J^{\text{off}}_f\right\}_\circ.
\end{aligned} \tag{157}$$

Above, J_1 represents the uplift of 1 to the real traceless symmetric 2×2 matrix, not an arbitrary element. Even though we are implementing octonionic multiplication, the above relations hold for $f \in \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. This allows for a representation of the Gell-Mann matrices in terms of elements of $\mathbb{C}\otimes\mathbb{H}\otimes\overleftarrow{J_2(\mathbb{O})}$,

$$\begin{aligned}
m_{\Lambda_1} &= -\frac{I}{2}(m_{e_6} m_{e_1} - m_{e_2} m_{e_5}),\\
m_{\Lambda_2} &= -\frac{I}{2}(m_{e_2} m_{e_1} + m_{e_6} m_{e_5}),\\
m_{\Lambda_3} &= -\frac{I}{2}(m_{e_2} m_{e_6} - m_{e_1} m_{e_5}),\\
m_{\Lambda_4} &= \frac{I}{2}(m_{e_3} m_{e_5} - m_{e_1} m_{e_7}),\\
m_{\Lambda_5} &= -\frac{I}{2}(m_{e_3} m_{e_1} - m_{e_5} m_{e_7}),\\
m_{\Lambda_6} &= \frac{I}{2}(m_{e_2} m_{e_7} + m_{e_3} m_{e_6}),\\
m_{\Lambda_7} &= \frac{I}{2}(m_{e_2} m_{e_3} + m_{e_6} m_{e_7}),\\
m_{\Lambda_8} &= \frac{I}{2\sqrt{3}}(m_{e_2} m_{e_6} + m_{e_1} m_{e_5} - 2 m_{e_3} m_{e_7}),
\end{aligned} \tag{158}$$

where $m_{\Lambda_1}(J^{\text{off}}_f) = -\frac{I}{2}(m_{e_6}(m_{e_1}(J^{\text{off}}_f)) - m_{e_2}(m_{e_5}(J^{\text{off}}_f)))$ more precisely. From here, particle states associated with elements of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ can be uplifted to $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$. It was confirmed that the $SU(3)$ generators above annihilate leptons and apply color rotations to the quarks in the appropriate manner.

The same relations found in $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ for $SU(2)\times U(1)$ generators are also found by the appropriate uplift to $\mathbb{C}\otimes\mathbb{H}\otimes\overleftarrow{J_2(\mathbb{O})}$. The appropriate left action of $g\in\mathbb{C}\otimes\mathbb{H}$ on f uplifted to J_f^{off} can be found simply by taking gJ_f^{off}, since the diagonal elements of $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$ can contain $\mathbb{C}\otimes\mathbb{H}$. Uplifting the generators of $SU(2)\times U(1)$ therefore gives

$$
\begin{aligned}
m_{\tau_9} &= \frac{i}{4}\left(I+m_{e_4}\right), \qquad m_{\tau_{10}} = \frac{j}{4}\left(I+m_{e_4}\right),\\
m_{\tau_{11}} &= \frac{k}{4}\left(I+m_{e_4}\right),\\
m_Y &= -\frac{1}{2}\left(\frac{I}{3}\left(m_{e_1}m_{e_5}+m_{e_2}m_{e_6}+m_{e_3}m_{e_7}\right)-\frac{k}{2}\left(I-m_{e_4}\right)\right),
\end{aligned}
\tag{159}
$$

where all multiplication is assumed to act from the left. Similarly, the electric charge operator becomes

$$
\begin{aligned}
m_Q &= m_{\tau_{11}}+m_Y\\
&= -\frac{I}{2}\left(\frac{1}{3}\left(m_{e_1}m_{e_5}+m_{e_2}m_{e_6}+m_{e_3}m_{e_7}\right)-k\right).
\end{aligned}
\tag{160}
$$

The fermionic states in the $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$ are identified as

$$
\begin{aligned}
(\mathbf{1},\mathbf{2})_{-1/2} &: J_{\psi_l} = \left(\mathcal{V}_L^{\uparrow}\epsilon_{\uparrow\uparrow}+\mathcal{V}_L^{\downarrow}\epsilon_{\uparrow\downarrow}+\mathcal{E}_L^{\uparrow}\epsilon_{\downarrow\uparrow}+\mathcal{E}_L^{\downarrow}\epsilon_{\downarrow\downarrow}\right)J_l,\\
(\mathbf{3},\mathbf{2})_{1/6} &: J_{\psi_q} = -I\left(\mathcal{U}_L^{a\uparrow}\epsilon_{\uparrow\uparrow}+\mathcal{U}_L^{a\downarrow}\epsilon_{\uparrow\downarrow}+\mathcal{D}_L^{a\uparrow}\epsilon_{\downarrow\uparrow}\right.\\
&\qquad\left.+\mathcal{D}_L^{a\downarrow}\epsilon_{\downarrow\downarrow}\right)J_{q_a},\\
(\mathbf{1},\mathbf{1})_0 &: J_{\psi_\nu^c} = \left(-\mathcal{V}_R^{\downarrow *}\epsilon_{\downarrow\uparrow}+\mathcal{V}_R^{\uparrow *}\epsilon_{\downarrow\downarrow}\right)J_{l^*},\\
(\mathbf{1},\mathbf{1})_1 &: J_{\psi_e^c} = \left(\mathcal{E}_R^{\downarrow *}\epsilon_{\uparrow\uparrow}-\mathcal{E}_R^{\uparrow *}\epsilon_{\uparrow\downarrow}\right)J_{l^*}\\
(\bar{\mathbf{3}},\mathbf{1})_{-2/3} &: J_{\psi_u^c} = I\left(-\mathcal{U}_R^{a\downarrow *}\epsilon_{\downarrow\uparrow}+\mathcal{U}_R^{a\uparrow *}\epsilon_{\downarrow\downarrow}\right)J_{q_a^*},\\
(\bar{\mathbf{3}},\mathbf{1})_{1/3} &: J_{\psi_d^c} = I\left(\mathcal{D}_R^{a\downarrow *}\epsilon_{\uparrow\uparrow}-\mathcal{D}_R^{a\uparrow *}\epsilon_{\uparrow\downarrow}\right)J_{q_a^*},
\end{aligned}
\tag{161}
$$

where in our conventions, the $\mathbb{C}\otimes\mathbb{O}$ quantities such as l and q_a are uplifted explicitly to give

$$
\begin{aligned}
J_l &= \frac{1}{2}\begin{pmatrix}0 & 1-e_4I\\ 1+e_4I & 0\end{pmatrix},\\
J_{l^*} &= \frac{1}{2}\begin{pmatrix}0 & 1+e_4I\\ 1-e_4I & 0\end{pmatrix},\\
J_{q_1} &= \frac{1}{2}\begin{pmatrix}0 & -e_5+e_1I\\ e_5-e_1I & 0\end{pmatrix},\\
J_{q_1^*} &= \frac{1}{2}\begin{pmatrix}0 & -e_5-e_1I\\ e_5+e_1I & 0\end{pmatrix},\\
J_{q_2} &= \frac{1}{2}\begin{pmatrix}0 & -e_6+e_2I\\ e_6-e_2I & 0\end{pmatrix},\\
J_{q_2^*} &= \frac{1}{2}\begin{pmatrix}0 & -e_6-e_2I\\ e_6+e_2I & 0\end{pmatrix},\\
J_{q_3} &= \frac{1}{2}\begin{pmatrix}0 & -e_7+e_3I\\ e_7-e_3I & 0\end{pmatrix},\\
J_{q_3^*} &= \frac{1}{2}\begin{pmatrix}0 & -e_7-e_3I\\ e_7+e_3I & 0\end{pmatrix}.
\end{aligned}
\tag{162}
$$

It was confirmed that $m_{\tau_{11}}$, m_Y, and m_Q give the appropriate eigenvalues for these states.

6.3 Uplift to $\mathbb{O}\otimes J_2(\mathbb{C}\otimes\mathbb{H})$

Next, we seek to obtain the physics of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ by uplifting to $\mathbb{O}\otimes J_2(\mathbb{C}\otimes\mathbb{H})$. The Hermitian conjugate of $\mathbb{O}\otimes J_2(\mathbb{C}\otimes\mathbb{H})$ takes conjugation with respect to both $\mathbb{C}$ and $\mathbb{H}$. Uplifting an element $f\in\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ to $J_f^{\text{off}}\in\mathbb{O}\otimes J_2(\mathbb{C}\otimes\mathbb{H})$ is given by

$$
J_f^{\text{off}} = \begin{pmatrix}0 & f\\ \hat{f}^* & 0\end{pmatrix},
\tag{163}
$$

where f^* is the complex conjugate and $\hat{f}$ is the quaternionic conjugate. Finding the corresponding left action of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ within $\mathbb{O}\otimes J_2(\mathbb{C}\otimes\mathbb{H})$ is straightforward for $\mathbb{O}$, yet requires care with $\mathbb{C}\otimes\mathbb{H}$.

Left multiplication of I on f uplifted to J_{If}^{off} must be implemented with the nested Jordan commutator product (137),

$$
J_{If}^{\text{off}} = m_I(J_f^{\text{off}}) \equiv \{J_I, J_1, J_f^{\text{off}}\}_\circ.
\tag{164}
$$

This holds for arbitrary elements $f\in\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. The analogous relationship for imaginary quaternionic units are

$$
\begin{aligned}
J_{if}^{\text{off}} &= m_i(J_f^{\text{off}}) \equiv \{J_i, J_1, J_f^{\text{off}}\}_\circ + \{J_k, J_j, J_f^{\text{off}}\}_\circ,\\
J_{jf}^{\text{off}} &= m_j(J_f^{\text{off}}) \equiv \{J_j, J_1, J_f^{\text{off}}\}_\circ + \{J_i, J_k, J_f^{\text{off}}\}_\circ,\\
J_{kf}^{\text{off}} &= m_k(J_f^{\text{off}}) \equiv \{J_k, J_1, J_f^{\text{off}}\}_\circ + \{J_j, J_i, J_f^{\text{off}}\}_\circ.
\end{aligned}
\tag{165}
$$

The corresponding uplift of left multiplication by imaginary octonions is given by left multiplication, such that $J_{e_if}^{\text{off}} = e_iJ_f^{\text{off}}$.

From here, the uplift of the fermionic states and the action of bosonic operators on the fermions is similar to the previous discussion on $\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})$. To highlight this uplift with more detail and for a specific example, consider ψ_e^c as a left-chiral positron and weak isospin singlet,

$$
\begin{aligned}
\psi_\nu^c &= \left(\mathcal{E}_R^{\downarrow *}\epsilon_{\uparrow\uparrow}-\mathcal{E}_R^{\uparrow *}\epsilon_{\uparrow\downarrow}\right)l^*\\
&= \frac{1}{4}\Big(\mathcal{E}_R^{\downarrow *}\left(1+Ik+e_4I-e_4k\right)+\mathcal{E}_R^{\uparrow *}\\
&\quad\times\left(-j+Ii-e_4i-e_4Ij\right)\Big).
\end{aligned}
\tag{166}
$$

Uplifting to $\mathbb{O} \otimes J_2(\mathbb{C} \otimes \mathbb{H})$ explicitly gives

$$J_{\psi_\nu^c} = \begin{pmatrix} 0 & \left(\mathcal{E}_R^{\downarrow *}\epsilon_{\uparrow\uparrow} - \mathcal{E}_R^{\uparrow *}\epsilon_{\uparrow\downarrow}\right) l^* \\ \left(\mathcal{E}_R^{\downarrow}\epsilon_{\uparrow\uparrow} - \mathcal{E}_R^{\uparrow}\epsilon_{\downarrow\uparrow}\right) l & 0 \end{pmatrix}$$

$$\left(\mathcal{E}_R^{\downarrow *}\epsilon_{\uparrow\uparrow} - \mathcal{E}_R^{\uparrow *}\epsilon_{\uparrow\downarrow}\right) l^* = \frac{1}{4}\Big(\mathcal{E}_R^{\downarrow *}(1 + Ik + e_4 I - e_4 k) + \mathcal{E}_R^{\uparrow *} \times (-j + Ii - e_4 i - e_4 I j)\Big)$$

$$\left(\mathcal{E}_R^{\downarrow}\epsilon_{\uparrow\uparrow} - \mathcal{E}_R^{\uparrow}\epsilon_{\downarrow\uparrow}\right) l = \frac{1}{4}\Big(\mathcal{E}_R^{\downarrow}(1 + Ik - e_4 I + e_4 k) + \mathcal{E}_R^{\uparrow}(j + Ii + e_4 i - e_4 I j)\Big). \tag{167}$$

The action of the Gell–Mann generators uplifted to $\mathbb{O} \otimes J_2(\mathbb{C} \otimes \mathbb{H})$ is

$$\begin{aligned} m_{\Lambda_1} &= -\frac{m_I}{2}(e_{61} - e_{25}), & m_{\Lambda_2} &= -\frac{m_I}{2}(e_{21} + e_{65}), \\ m_{\Lambda_3} &= -\frac{m_I}{2}(e_{26} - e_{15}), & m_{\Lambda_4} &= \frac{m_I}{2}(e_{35} - e_{17}), \\ m_{\Lambda_5} &= -\frac{m_I}{2}(e_{31} - e_{57}), & m_{\Lambda_6} &= \frac{m_I}{2}(e_{27} + e_{36}), \\ m_{\Lambda_7} &= \frac{m_I}{2}(e_{23} + e_{67}), & m_{\Lambda_8} &= \frac{m_I}{2\sqrt{3}}(e_{26} + e_{15} - e_{37}). \end{aligned} \tag{168}$$

The electroweak generators are given by

$$\begin{aligned} m_{\tau_9} &= \frac{m_I}{2} m_s m_i, \qquad m_{\tau_{10}} = \frac{m_I}{2} m_s m_j, \\ m_{\tau_{11}} &= \frac{m_I}{2} m_s m_k, \\ m_Y &= -\frac{m_I}{2}\left(\frac{1}{3}(e_{15} + e_{26} + e_{37}) - m_{s^*} m_k\right), \end{aligned} \tag{169}$$

where

$$m_s(J_f) = \frac{1}{2}(1 - e_4 m_I) J_f, \qquad m_{s^*}(J_f) = \frac{1}{2}(1 + e_4 m_I) J_f. \tag{170}$$

The electric charge operator is given by

$$m_Q = m_{\tau_{11}} + m_Y = -\frac{m_I}{2}\left(\frac{1}{3}(e_{15} + e_{26} + e_{37}) - m_k\right). \tag{171}$$

The action of these generators leads to the expected results when acting on $J_{\psi_\nu^c}$. For instance, all of the $SU(3)$ generators vanish and $J_{\psi_\nu^c}$ is an eigenstate of $m_{\tau_{11}}$ and m_Y,

$$\begin{aligned} m_{\Lambda_i}(J_{\psi_\nu^c}) &= 0, \\ m_{\tau_{11}}(J_{\psi_\nu^c}) &= 0, \\ m_Y(J_{\psi_\nu^c}) &= 1 J_{\psi_\nu^c}, \\ m_Q(J_{\psi_\nu^c}) &= 1 J_{\psi_\nu^c}, \end{aligned} \tag{172}$$

where 1 is found as an eigenvalue for electric charge and weak hypercharge with the left-chiral positron.

6.4 Uplift to $\mathbb{C} \otimes \mathbb{O} \otimes J_2(\mathbb{H})$

Finally, the physics of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ is uplifted to $\mathbb{C}\otimes\mathbb{O}\otimes J_2(\mathbb{H})$. Uplifting an element $f \in \mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ to $J_f^{\text{off}} \in \mathbb{C}\otimes\mathbb{O}\otimes J_2(\mathbb{H})$ is given by

$$J_f^{\text{off}} = \begin{pmatrix} 0 & f \\ \hat{f} & 0 \end{pmatrix}, \tag{173}$$

where, as above, $\hat{f}$ denotes the quaternionic conjugation of f. From here, it is clear that the uplift of left multiplication by imaginary quaternionic units is identical to Eq. (165). Less care is needed with the complex numbers and octonions, as they are on the diagonals of $\mathbb{C} \otimes \mathbb{O} \otimes J_2(\mathbb{H})$.

The action of the Gell–Mann generators uplifted to $\mathbb{C} \otimes \mathbb{O} \otimes J_2(\mathbb{H})$ is identical to Eq. (150). The electroweak generators are given by

$$\begin{aligned} m_{\tau_9} &= \frac{I}{2} s m_i, \qquad m_{\tau_{10}} = \frac{I}{2} s m_j, \qquad m_{\tau_{11}} = \frac{I}{2} s m_k, \\ m_Y &= -\frac{I}{2}\left(\frac{1}{3}(e_{15} + e_{26} + e_{37}) - s^* m_k\right). \end{aligned} \tag{174}$$

The electric charge operator is given by

$$m_Q = m_{\tau_{11}} + m_Y = -\frac{I}{2}\left(\frac{1}{3}(e_{15} + e_{26} + e_{37}) - m_k\right). \tag{175}$$

The fermions of $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$ can be uplifted to $\mathbb{C}\otimes\mathbb{O}\otimes J_2(\mathbb{H})$ via Eq. (173) and the generators shown above can be found to act appropriately on the fermionic states.

7 Conclusions

In this work, we showed how to construct three homogeneous spaces that, following Rosenfeld's interpretation of the Magic Square, correspond to his "generalized" projective lines over the Dixon algebra, $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$. Such spaces are obtained from three non-simple Lie algebras obtained from Tits' construction for the Freudenthal Magic Square. The quotient space of these isometry groups modded out by derivations lead to $\mathbb{C} \otimes \mathbb{H} \otimes J_2(\mathbb{O})$, $\mathbb{O} \otimes J_2(\mathbb{C} \otimes \mathbb{H})$, and $\mathbb{C} \otimes \mathbb{O} \otimes J_2(\mathbb{H})$, which contains the three newly found Dixon-Rosenfeld projective lines. The physics of $\mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ can be uplifted to each of these extended Jordan algebras and the generators of $SU(3) \times SU(2) \times U(1)$ for the Standard Model can be uplifted into a (nested) chain algebra over $\overleftarrow{\mathbb{C}\otimes\mathbb{H}\otimes J_2(\mathbb{O})}$, $\overleftarrow{\mathbb{O}\otimes J_2(\mathbb{C}\otimes\mathbb{H})}$, and $\overleftarrow{\mathbb{C}\otimes\mathbb{O}\otimes J_2(\mathbb{H})}$. We provided explicit states for one generation of fermions in the standard model within these projective lines, including operators for gauge boson interactions and identification of charges.

While non-simple Lie algebras were found from the Dixon-Rosenfeld projective lines and one generation of the Standard Model fermions were embedded into these projective lines, further work is needed to see if the appropriate representations of the Standard Model are contained within the corresponding isometry groups. For instance, while the bosonic interactions with fermions were demonstrated to be in the chain algebras over division algebras tensored with Jordan algebras and various $SU(3)\times SU(2)\times U(1)$ groups can be found in the derivation groups, the representations with respect to these groups do not isolate the Standard Model fermionic representations and charges. This is similar to how $Spin(9)$, $SU(3)\times SU(3)$, and F_4 are not GUT groups, but the octonions and F_4 have been used to encode Standard Model fermions [35,48,49].

It appears that the Freudenthal–Tits formula should work for $\mathbb{A}=\mathbb{O}$ and $\mathbb{B}=\mathbb{C}\otimes\mathbb{H}$ to give a Lie algebra $\mathfrak{a}_{II}$. However, there is not a single formula for the 2×2 case, as setting $\mathbb{A}=\mathbb{O}$ already leads to a difference. Here, we articulated the structure of $J_2(\mathbb{C}\otimes\mathbb{H})$ and found $\mathfrak{Der}(J_2(\mathbb{C}\otimes\mathbb{H}))$. However, applying the 2×2 analogue of the Freudenthal–Tits construction did not lead to the anticipated representations of $\mathbb{T}$ with respect to $\mathfrak{Der}(\mathbb{T})$. To further complicate matters, it is known that $\mathbb{C}\otimes\mathbb{H}$ can lead to multiple representations. For now, we merely claim that some non-simple Lie algebra $\mathfrak{a}_{II}$ exists that contains at least 120 dimensions. By exploring the 3×3 case in future work, we hope to gain a further understanding of the true definition of $\mathfrak{a}_{II}$.

Additional work is needed to see if other subalgebras of these non-simple Lie algebras exist that can isolate the appropriate representation theory for the Standard Model. Otherwise, chain algebras such as $\overleftarrow{\mathbb{A}\otimes J_2(\mathbb{B})}$ may lead to Clifford algebras that would be large enough to contain the Standard Model gauge group, just as $\mathbb{C}\otimes\mathbb{H}\otimes\overleftarrow{\mathbb{O}}$ can lead to $Cl(10)$. In future work, we seek to investigate the notion of Dixon-Rosenfeld projective planes to see if this may provide applications for three generations of the Standard Model fermions with $\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{O}$. Interactions with the Higgs boson would also be worth exploring, which has been discussed recently [30].

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Appendix A: $J_2(\mathbb{C}\otimes\mathbb{H})$ as 4×4 complex matrices

The Jordan algebra $J_2(\mathbb{C}\otimes\mathbb{H})$ is 16-dimensional and can be expressed in terms of a set of matrices in $M_4(\mathbb{C})$. We review this isomorphism and determine the action of the double conjugation with respect to $\mathbb{C}$ and $\mathbb{H}$ in the language of 4×4 complex matrices. Before introducing $J_2(\mathbb{C}\otimes\mathbb{H})$, we first clarify how the double conjugation of $\mathbb{C}\otimes\mathbb{H}$ with respect to $\mathbb{C}$ and $\mathbb{H}$ leads to a 4-dimensional element and specify how this maps into the isomorphism with $M_2(\mathbb{C})$.

First, $f\in\mathbb{C}\otimes\mathbb{H}$ is recast in $\mathcal{M}(f)\in M_2(\mathbb{C})$ by the following isomorphism, using our notation of (29) and (118):

$$\begin{aligned} f &= \left(c_{1,1}+c_{1,2}I\right)+\left(c_{2,1}+c_{2,2}I\right)i \\ &\quad +\left(c_{3,1}+c_{3,2}I\right)j+\left(c_{4,1}+c_{4,2}I\right)k \\ &= \Big\{\underbrace{1,I}_{\mathbf{1}\oplus\mathbf{1}},\underbrace{i,j,k}_{\mathbf{3}},\underbrace{Ii,Ij,Ik}_{\mathbf{3}}\Big\}. \\ &\quad \times\left\{c_{1,1},c_{1,2},c_{2,1},c_{2,2},c_{3,1},c_{3,2},c_{4,1},c_{4,2}\right\}^T \\ &\simeq \mathcal{M}(f)=\begin{pmatrix} c_{1,1}-c_{2,2}+i\left(c_{1,2}+c_{2,1}\right) & c_{3,1}-c_{4,2}+i\left(c_{3,2}+c_{4,1}\right) \\ -c_{3,1}-c_{4,2}+i\left(c_{4,1}-c_{3,2}\right) & c_{1,1}+c_{2,2}+i\left(c_{1,2}-c_{2,1}\right) \end{pmatrix}. \end{aligned} \tag{176}$$

Next, we clarify how double conjugation of $\mathbb{C}\otimes\mathbb{H}$ maps into $M_2(\mathbb{C})$,

$$\begin{aligned} \hat{f}^* &= \left(c_{1,1}-c_{1,2}I\right)+\left(-c_{2,1}+c_{2,2}I\right)i \\ &\quad +\left(-c_{3,1}+c_{3,2}I\right)j+\left(-c_{4,1}+c_{4,2}I\right)k \\ &\simeq \overline{\mathcal{M}(f)}^{\top}=\begin{pmatrix} c_{1,1}-c_{2,2}-i\left(c_{1,2}+c_{2,1}\right) & -c_{3,1}-c_{4,2}-i\left(c_{4,1}-c_{3,2}\right) \\ c_{3,1}-c_{4,2}-i\left(c_{3,2}+c_{4,1}\right) & c_{1,1}+c_{2,2}-i\left(c_{1,2}-c_{2,1}\right) \end{pmatrix}. \end{aligned}$$

(177)

As shown above, $\hat{f}^* \cong \overline{\mathcal{M}(f)}^\top$.

By using the conventions of conjugation as shown in Eqs. (115) and (116), we find a "real" element of 4 dimensions by

$$r = \frac{1}{2}\left(f + \hat{f}^*\right) = c_{1,1} + c_{2,2}Ii + c_{3,2}Ij + c_{4,2}Ik,$$
$$\mathcal{M}(r) = \mathcal{M}\left(\frac{1}{2}\left(f + \hat{f}^*\right)\right) = \frac{1}{2}\left(\mathcal{M}(f) + \overline{\mathcal{M}(f)}^\top\right). \tag{178}$$

A 16-dim representation of $X \in J_2(\mathbb{C}\otimes\mathbb{H})$ is built as a Hermitian block-matrix in $\mathcal{M}(X) \in M_4(\mathbb{C})$,

$$X = \begin{pmatrix} r & f \\ \hat{f}^* & s \end{pmatrix},$$
$$\cong \mathcal{M}(X) = \begin{pmatrix} \mathcal{M}(r) & \mathcal{M}(f) \\ \overline{\mathcal{M}(f)}^\top & \mathcal{M}(s) \end{pmatrix}. \tag{179}$$

where r and s real with respect to the double conjugation on $\mathbb{C}\otimes\mathbb{H}$, leading to r and s as 4-dimensional elements spanning $\{1, Ii, Ij, Ik\}$ with $\mathcal{M}(r) = \begin{pmatrix} r_{1,1} - r_{2,2} & -r_{4,2} + ir_{3,2} \\ -r_{4,2} - ir_{3,2} & r_{1,1} + r_{2,2} \end{pmatrix}$ $\in H_2(\mathbb{C})$, while $\mathcal{M}(f) \in M_2(\mathbb{C})$. As such, we refer to X as $X(r, s, f)$.

Appendix B: Demonstration of $\mathfrak{der}\,(J_2\,(\mathbb{C}\otimes\mathbb{H})) \cong \mathfrak{su}(4)$

The derivation of an alternative algebra is defined ([44] page 77) as a Bracket algebra satisfying the Leibniz rule, and a theorem shows it is of the form

$$D_{X,Y} = [L_X, L_Y] + [L_X, R_Y] + [R_X, R_Y]. \tag{180}$$

When the Jordan algebra is not only alternative but is commutative, $L_X = R_X$ and the inner derivations are ([44] page 92)

$$D_{X,Y} = [L_X, L_Y]. \tag{181}$$

A derivation parameterized by X and Y applied to an element $Z \in J_2(\mathbb{C}\otimes\mathbb{H})$ gives

$$\begin{aligned} D_{X,Y}(Z) &= [L_X, L_Y](Z) = L_X(L_Y(Z)) - L_Y(L_X(Z)) \\ &= X.(Y.Z) - Y.(X.Z) = X.(Z.Y) - (X.Z).Y \\ &= -[X, Z, Y], \end{aligned} \tag{182}$$

where $[X, Z, Y]$ is the Jordan associator, sandwiching Z between X and Y.

A pair $X,Y \in J_2(\mathbb{C}\otimes\mathbb{H})$ elements: $X(r, s, f) \cong \mathcal{M}\left(\begin{pmatrix} r & f \\ \hat{f}^* & s \end{pmatrix}\right)$ and $Y(R, S, F) \cong \mathcal{M}\left(\begin{pmatrix} R & F \\ \hat{F}^* & S \end{pmatrix}\right)$ is bracketed by a commutator to give an anti-hermitian matrix in $M_4(\mathbb{C})$. Though, (182) is rewritten from note (2) page 7 and introducing i two times, the derivation becomes a bracket commutator between Hermitian matrices, and $\mathfrak{der}\,(J_2\,(\mathbb{C}\otimes\mathbb{H}))$ is represented by the Hermitian matrix δ,

$$\delta = i[X, Y] = \delta(\rho, \sigma, \phi) \cong \begin{pmatrix} \mathcal{M}(\rho) & \mathcal{M}(\phi) \\ \overline{\mathcal{M}(\phi)}^\top & \mathcal{M}(\sigma) \end{pmatrix},$$
$$D_{X,Y}(Z) = -[X, Z, Y] = i[i[X, Y], Z] = i[\delta, Z]. \tag{183}$$

Next, consider the embedding of $J_2(\mathbb{C}\otimes\mathbb{H})$ in $M_4(\mathbb{C})$ to find the commutator of two elements, defined below as block matrices using (179):

$$[\mathcal{M}(X), \mathcal{M}(Y)]$$
$$= \left[\begin{pmatrix} \mathcal{M}(r) & \mathcal{M}(f) \\ \overline{\mathcal{M}(f)}^\top & \mathcal{M}(s) \end{pmatrix}, \begin{pmatrix} \mathcal{M}(R) & \mathcal{M}(F) \\ \overline{\mathcal{M}(F)}^\top & \mathcal{M}(S) \end{pmatrix}\right] \tag{184}$$

The commutator of these two matrices leads to

$$[\mathcal{M}(X), \mathcal{M}(Y)] = -i\begin{pmatrix} \mathcal{M}(\rho) & \mathcal{M}(\phi) \\ \overline{\mathcal{M}(\phi)}^\top & \mathcal{M}(\sigma) \end{pmatrix}. \tag{185}$$

The block 2×2 matrices are given by

$$\mathcal{M}(\rho) = \begin{pmatrix} \rho_{1,1} - \rho_{2,2} & -\rho_{4,2} + i\rho_{3,2} \\ -\rho_{4,2} - i\rho_{3,2} & \rho_{1,1} + \rho_{2,2} \end{pmatrix},$$
$$\mathcal{M}(\sigma) = \begin{pmatrix} \sigma_{1,1} - \sigma_{2,2} & -\sigma_{4,2} + i\sigma_{3,2} \\ -\sigma_{4,2} - i\sigma_{3,2} & \sigma_{1,1} + \sigma_{2,2} \end{pmatrix},$$
$$\mathcal{M}(\phi) = \begin{pmatrix} \phi_{1,1} - \phi_{2,2} + i\,(\phi_{1,2} + \phi_{2,1}) & \phi_{3,1} - \phi_{4,2} + i\,(\phi_{3,2} + \phi_{4,1}) \\ -\phi_{3,1} - \phi_{4,2} + i\,(\phi_{4,1} - \phi_{3,2}) & \phi_{1,1} + \phi_{2,2} + i\,(\phi_{1,2} - \phi_{2,1}) \end{pmatrix}. \tag{186}$$

The solution for the matrix components above are found by plugging Eq. (184) into Eq. (185) to give

$$\begin{aligned} \rho_{1,1} &= -f_{1,2}F_{1,1} + f_{1,1}F_{1,2} - f_{2,2}F_{2,1} + f_{2,1}F_{2,2} \\ &\quad - f_{3,2}F_{3,1} + f_{3,1}F_{3,2} - f_{4,2}F_{4,1} + f_{4,1}F_{4,2}, \\ \rho_{2,2} &= f_{2,1}F_{1,1} + f_{2,2}F_{1,2} - f_{1,1}F_{2,1} - f_{1,2}F_{2,2} \\ &\quad + f_{4,1}F_{3,1} + f_{4,2}F_{3,2} - f_{3,1}F_{4,1} - f_{3,2}F_{4,2} \\ &\quad + r_{4,2}R_{3,2} - r_{3,2}R_{4,2}, \\ \rho_{3,2} &= f_{3,1}F_{1,1} + f_{3,2}F_{1,2} - f_{4,1}F_{2,1} - f_{4,2}F_{2,2} \\ &\quad - f_{1,1}F_{3,1} - f_{1,2}F_{3,2} + f_{2,1}F_{4,1} + f_{2,2}F_{4,2} \\ &\quad - r_{4,2}R_{2,2} + r_{2,2}R_{4,2}, \\ \rho_{4,2} &= f_{4,1}F_{1,1} + f_{4,2}F_{1,2} + f_{3,1}F_{2,1} + f_{3,2}F_{2,2} \\ &\quad - f_{2,1}F_{3,1} - f_{2,2}F_{3,2} - f_{1,1}F_{4,1} - f_{1,2}F_{4,2} \\ &\quad + r_{3,2}R_{2,2} - r_{2,2}R_{3,2}, \\ \sigma_{1,1} &= f_{1,2}F_{1,1} - f_{1,1}F_{1,2} + f_{2,2}F_{2,1} - f_{2,1}F_{2,2} \\ &\quad + f_{3,2}F_{3,1} - f_{3,1}F_{3,2} + f_{4,2}F_{4,1} - f_{4,1}F_{4,2}, \end{aligned}$$

$$\begin{aligned}
\sigma_{2,2} = &-f_{2,1}F_{1,1} - f_{2,2}F_{1,2} + f_{1,1}F_{2,1} + f_{1,2}F_{2,2}\\
&+f_{4,1}F_{3,1} + f_{4,2}F_{3,2} - f_{3,1}F_{4,1} - f_{3,2}F_{4,2}\\
&+s_{4,2}S_{3,2} - s_{3,2}S_{4,2},\\
\sigma_{3,2} = &-f_{3,1}F_{1,1} - f_{3,2}F_{1,2} - f_{4,1}F_{2,1} - f_{4,2}F_{2,2}\\
&+f_{1,1}F_{3,1} + f_{1,2}F_{3,2} + f_{2,1}F_{4,1} + f_{2,2}F_{4,2}\\
&-s_{4,2}S_{2,2} + s_{2,2}S_{4,2},\\
\sigma_{4,2} = &-f_{4,1}F_{1,1} - f_{4,2}F_{1,2} + f_{3,1}F_{2,1} + f_{3,2}F_{2,2}\\
&-f_{2,1}F_{3,1} - f_{2,2}F_{3,2} + f_{1,1}F_{4,1} + f_{1,2}F_{4,2}\\
&+s_{3,2}S_{2,2} - s_{2,2}S_{3,2},\\
\phi_{1,1} = &-\frac{1}{2}(r_{1,1}F_{1,2} - r_{2,2}F_{2,1} - r_{3,2}F_{3,1} - r_{4,2}F_{4,1}\\
&-R_{1,1}f_{1,2} + R_{2,2}f_{2,1} + R_{3,2}f_{3,1} + R_{4,2}f_{4,1}\\
&. - s_{1,1}F_{1,2} + s_{2,2}F_{2,1} + s_{3,2}F_{3,1} + s_{4,2}F_{4,1}\\
&+S_{1,1}f_{1,2} - S_{2,2}f_{2,1} - S_{3,2}f_{3,1} - S_{4,2}f_{4,1}),\\
\phi_{1,2} = &\frac{1}{2}\big(r_{1,1}F_{1,1} + r_{2,2}F_{2,2} + r_{3,2}F_{3,2} + r_{4,2}F_{4,2}\\
&-R_{1,1}f_{1,1} - R_{2,2}f_{2,2} - R_{3,2}f_{3,2} - R_{4,2}f_{4,2}\\
&-s_{1,1}F_{1,1} - s_{2,2}F_{2,2} - s_{3,2}F_{3,2} - s_{4,2}F_{4,2}\\
&+S_{1,1}f_{1,1} + S_{2,2}f_{2,2} + S_{3,2}f_{3,2} + S_{4,2}f_{4,2}\big), \quad (187)
\end{aligned}$$

where the solutions to the other six $\phi_{i,j}$ can be found similarly.

Next, consider the trace of this 4×4 matrix $\mathcal{M}(\delta)$ by considering the traces of $\mathcal{M}(\rho)$ and $\mathcal{M}(\sigma)$,

$$\mathrm{Tr}(\mathcal{M}(\rho)) = -\mathrm{Tr}(\mathcal{M}(\sigma)) = 2\rho_{1,1}. \quad (188)$$

Since $\mathrm{Tr}(\mathcal{M}(\delta)) = \mathrm{Tr}(\mathcal{M}(\rho)) + \mathrm{Tr}(\mathcal{M}(\sigma)) = 0$, it is derived that the 4×4 matrices are traceless. While $\rho_{i,j}$, $\sigma_{i,j}$, and $\tau_{i,j}$ lead to 16 degrees of freedom, since $\rho_{1,1} = -\sigma_{1,1}$, there are 15 linearly independent elements, which form a basis for the Hermitian traceless generators of $\mathfrak{su}_4$,

$$L_1 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}, L_2 = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&1 \end{pmatrix}, L_4 = \begin{pmatrix} 0&0&0&1\\ 0&0&-1&0\\ 0&-1&0&0\\ 1&0&0&0 \end{pmatrix},$$

$$L_5 = \begin{pmatrix} 0&0&0&-1\\ 0&0&-1&0\\ 0&-1&0&0\\ -1&0&0&0 \end{pmatrix}, L_6 = \begin{pmatrix} 0&0&0&i\\ 0&0&i&0\\ 0&-i&0&0\\ -i&0&0&0 \end{pmatrix},$$

$$L_7 = \begin{pmatrix} 0&0&0&i\\ 0&0&-i&0\\ 0&i&0&0\\ -i&0&0&0 \end{pmatrix}, L_8 = \begin{pmatrix} 0&0&-1&0\\ 0&0&0&1\\ -1&0&0&0\\ 0&1&0&0 \end{pmatrix},$$

$$L_9 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}, L_{10} = \begin{pmatrix} 0&0&-i&0\\ 0&0&0&-i\\ i&0&0&0\\ 0&i&0&0 \end{pmatrix},$$

$$L_{11} = \begin{pmatrix} 0&0&i&0\\ 0&0&0&-i\\ -i&0&0&0\\ 0&i&0&0 \end{pmatrix}, L_{12} = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&-1\\ 0&0&-1&0 \end{pmatrix},$$

$$L_{13} = \begin{pmatrix} 0&-1&0&0\\ -1&0&0&0\\ 0&0&0&-1\\ 0&0&-1&0 \end{pmatrix}, L_{14} = \begin{pmatrix} 0&i&0&0\\ -i&0&0&0\\ 0&0&0&i\\ 0&0&-i&0 \end{pmatrix},$$

$$L_{15} = \begin{pmatrix} 0&i&0&0\\ -i&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix}. \quad (189)$$

Next, we demonstrate that a collection of matrices X_a and Y_b lead to $L_A \to L_{a,b} = \frac{i}{2}[X_a, Y_b]$. While there is not a unique set of matrices, we found a small collection of X_a for $a = 1, \ldots, 4$ and Y_b for $b = 1, \ldots 12$ that lead to the 15 generators L_A. X_a are given by

$$\begin{aligned}
X_1 &= X(0, 0, 1) = X(0, 0, f_{1,1} = 1),\\
X_2 &= X(0, 0, I) = X(0, 0, f_{1,2} = 1),\\
X_3 &= X(jI, -jI, 0) = X(r_{3,2} = -1, s_{3,2} = 1, 0),\\
X_4 &= X(-kI, kI, 0) = X(r_{4,2} = -1, s_{4,2} = 1, 0). \quad (190)
\end{aligned}$$

12 elements Y_1 to Y_{12} in $J_2\,(\mathbb{C} \otimes \mathbb{H})$ are given by

$$\begin{aligned}
Y_1 &= Y(0, 0, 1) = Y(0, 0, F_{1,1} = 1),\\
Y_2 &= Y(0, 0, I) = Y(0, 0, F_{1,2} = 1),\\
Y_3 &= Y(jI, -jI, 0) = Y(R_{3,2} = -S_{3,2} = 1),\\
Y_4 &= Y(-kI, kI, 0) = Y(-R_{4,2} = S_{4,2} = 1),\\
Y_5 &= Y(0, 0, i) = Y(0, 0, F_{2,1} = 1),\\
Y_6 &= Y(0, 0, j) = Y(0, 0, F_{3,1} = -1), \quad (191)\\
Y_7 &= Y(0, 0, k) = Y(0, 0, F_{4,1} = 1),\\
Y_8 &= Y(iI, -iI, 0) = Y(R_{2,2} = -S_{2,2} = 1),\\
Y_9 &= Y(2, 0, 0) = Y(R_{1,1} = 2, 0, 0),\\
Y_{10} &= Y(2iI, 0, 0) = Y(R_{2,2} = 2, 0, 0),\\
Y_{11} &= Y(2jI, 0, 0) = Y(R_{3,2} = 2, 0, 0),\\
Y_{12} &= Y(2kI, 0, 0) = Y(R_{4,2} = 2, 0, 0). \quad (192)
\end{aligned}$$

These Y_b can be combined with X_a to give the generators of $\mathfrak{su}_4$. we build from them the 15 pairs $L_{a,b} = \frac{i}{2}[X_a, Y_b]$ from the following index pairs $\{a, b\}$:

$$\begin{aligned}
&L_1 = L_{1,2}, \quad L_2 = L_{4,3}, \quad L_3 = L_{1,5}, \quad L_4 = L_{1,11},\\
&L_5 = L_{2,12}, L_6 = L_{1,12}, \quad L_7 = L_{2,11}, \quad L_8 = L_{2,10},\\
&L_9 = L_{2,9}, \quad L_{10} = L_{1,9},\\
&L_{11} = L_{1,10}, \quad L_{12} = L_{1,7}, \quad L_{13} = L_{3,8}\\
&L_{14} = L_{4,8} \quad L_{15} = L_{1,6}. \quad (193)
\end{aligned}$$

The three Cartan generators of $\mathfrak{su}_4$ are the three first expressed above, diagonal and commuting. By construction as commutators of Hermitian matrices scaled by the imaginary factor i, the matrices $L_{a,b}$ are all Hermitian, and span at most the 16-dimensional space $\mathfrak{u}_4$, but from the property that $\rho_{1,1} + \sigma_{1,1} = 0$ and that $\mathcal{M}(\delta)$ is traceless, the non-traceless generator of $\mathfrak{u}_4$ can not be obtained as a derivation $L_{a,b}$, and therefore the derivation of $J_2\,(\mathbb{C} \otimes \mathbb{H})$ is $\mathfrak{su}_4$.

Appendix C: Demonstration that the algebra $\mathfrak{a}_{II}$ given by Tits' formula does not contain the Dixon algebra $\mathbb{T}$ with a 3 representation of $\mathfrak{der}(\mathbb{H})$

Theorem *The algebra $\mathfrak{a}_{II}$ given by Tits' formula does not contain the Dixon algebra $\mathbb{T} = \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ when $\mathbb{H} \subset \mathbb{T}$ corresponds to the representations $\mathbf{3} \oplus \mathbf{1}$ with respect to $\mathfrak{der}(\mathbb{H}) = \mathfrak{su}_2$.*

Proof By applying Tits' formula (with Barton-Sudbery's modification), one obtains

$$\begin{aligned} \mathfrak{a}_{II} &= \mathcal{L}_2(\mathbb{O}, \mathbb{C} \otimes \mathbb{H}) = \mathfrak{isom}\left(\mathbb{T}P^1_{II}\right) \\ &:= \mathfrak{so}\left(\mathbb{O}'\right) \oplus \mathfrak{der}\left(J_2(\mathbb{C} \otimes \mathbb{H})\right) \oplus \mathbb{O}' \otimes J_2'(\mathbb{C} \otimes \mathbb{H}) \\ &= \mathfrak{so}_7 \oplus \mathfrak{so}_6 \oplus 3 \cdot (\mathbf{7}, \mathbf{4}) \\ &= \mathfrak{g}_2 \oplus \mathfrak{so}_6 \oplus 3 \cdot (\mathbf{7}, \mathbf{4}) \oplus (\mathbf{7}, \mathbf{1}). \end{aligned} \tag{194}$$

We also recall that

$$\begin{aligned} \mathbb{T} &:= \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O} \simeq 2 \cdot (\mathbf{7} + \mathbf{1}, \mathbf{3} + \mathbf{1}) \text{ of } \mathfrak{der}(\mathbb{T}) \\ &= \mathfrak{g}_2 \oplus \mathfrak{su}_2. \end{aligned} \tag{195}$$

Let us now find all $\mathfrak{su}_2$ subalgebras of $\mathfrak{so}_6$ in (194):

1.

$$\mathfrak{so}_6 \to \underset{\text{symm } I \leftrightarrow II}{\mathfrak{su}_{2,I} \oplus \mathfrak{su}_{2,II}} \to \begin{cases} \mathfrak{su}_{2,I \text{ or } II}; \\ \mathfrak{su}_{2,d}; \end{cases} \tag{196}$$

$$\mathbf{15} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{3}, \mathbf{3}) = \begin{cases} 4 \cdot \mathbf{3} + 3 \cdot \mathbf{1}; \\ \mathbf{5} + 3 \cdot \mathbf{3} + \mathbf{1}; \end{cases} \tag{197}$$

$$\mathbf{4} = (\mathbf{2}, \mathbf{2}) = \begin{cases} 2 \cdot \mathbf{2}; \\ \mathbf{3} + \mathbf{1}, \end{cases} \tag{198}$$

such that (194) can be further branched as

$$\begin{aligned} \mathfrak{a}_{II} &= \mathfrak{g}_2 \oplus \mathfrak{su}_{2,I} \oplus \mathfrak{su}_{2,II} \oplus (\mathbf{1}, \mathbf{3}, \mathbf{3}) \\ &\quad \oplus 3 \cdot (\mathbf{7}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{1}, \mathbf{1}) \\ &= \begin{cases} \mathfrak{g}_2 \oplus \mathfrak{su}_{2,I \text{ or } II} \oplus 3 \cdot (\mathbf{1}, \mathbf{3}) \oplus 3 \\ \cdot (\mathbf{1}, \mathbf{1}) \oplus 6 \\ \cdot (\mathbf{7}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}; \\ \mathfrak{g}_2 \oplus \mathfrak{su}_{2,d} \oplus 2 \cdot (\mathbf{1}, \mathbf{3}) \\ \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{5}) \oplus 3 \cdot (\mathbf{7}, \mathbf{3}) \\ \oplus 4 \cdot (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}. \end{cases} \end{aligned} \tag{199}$$

2.

$$\mathfrak{so}_6 \to \mathfrak{so}_5 \to \begin{cases} \underset{\text{symm } I \leftrightarrow II}{\mathfrak{su}_{2,I} \oplus \mathfrak{su}_{2,II}} \to \begin{cases} \mathfrak{su}_{2,I \text{ or } II}; \\ \mathfrak{su}_{2,d}; \end{cases} \\ \mathfrak{su}_2 \, (\oplus u_1); \\ \mathfrak{su}_{2,P}; \end{cases} \tag{200}$$

$$\mathbf{15} = \mathbf{10} + \mathbf{5} = \begin{cases} (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + 2 \cdot (\mathbf{2}, \mathbf{2}) + (\mathbf{1}, \mathbf{1}) \\ \quad = \begin{cases} \mathbf{3} + 4 \cdot \mathbf{1} + 4 \cdot \mathbf{2}; \\ 4 \cdot \mathbf{3} + 3 \cdot \mathbf{1}; \end{cases} \\ 4 \cdot \mathbf{3} + 3 \cdot \mathbf{1}; \\ \mathbf{3} + \mathbf{5} + \mathbf{7}; \end{cases} \tag{201}$$

$$\mathbf{4} = \mathbf{4} = \begin{cases} (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) = \begin{cases} \mathbf{2} + 2 \cdot \mathbf{1}; \\ 2 \cdot \mathbf{2}; \end{cases} \\ 2 \cdot \mathbf{2}; \\ \mathbf{4}, \end{cases} \tag{202}$$

such that (194) can be further branched as

$$\begin{aligned} \mathfrak{a}_{II} &= \mathfrak{g}_2 \oplus \mathfrak{so}_5 \oplus (\mathbf{1}, \mathbf{5}) \oplus 3 \cdot (\mathbf{7}, \mathbf{4}) \oplus (\mathbf{7}, \mathbf{1}) \\ &= \begin{cases} \mathfrak{g}_2 \oplus \mathfrak{su}_{2,I} \oplus \mathfrak{su}_{2,II} \\ \oplus 2 \cdot (\mathbf{1}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1}) \oplus 3 \\ \cdot (\mathbf{7}, \mathbf{2}, \mathbf{1}) \oplus 3 \cdot (\mathbf{7}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{1}) \\ \quad = \begin{cases} \mathfrak{g}_2 \oplus \mathfrak{su}_{2,I \text{ or } II} \oplus 3 \cdot (\mathbf{1}, \mathbf{1}) \oplus 4 \\ \cdot (\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \\ \oplus 3 \cdot (\mathbf{7}, \mathbf{2}) \oplus 7 \cdot (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}; \\ \mathfrak{g}_2 \oplus \mathfrak{su}_{2,d} \oplus 3 \cdot (\mathbf{1}, \mathbf{3}) \oplus 3 \cdot (\mathbf{1}, \mathbf{1}) \oplus 6 \\ \cdot (\mathbf{7}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}; \end{cases} \\ \mathfrak{g}_2 \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus 3 \cdot (\mathbf{1}, \mathbf{3}) \oplus 2 \cdot (\mathbf{1}, \mathbf{1}) \oplus 6 \\ \cdot (\mathbf{7}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}; \\ \mathfrak{g}_2 \oplus \mathfrak{su}_{2,P} \oplus (\mathbf{1}, \mathbf{5}) \oplus (\mathbf{1}, \mathbf{7}) \oplus 3 \\ \cdot (\mathbf{7}, \mathbf{4}) \oplus (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}. \end{cases} \end{aligned} \tag{203}$$

3.

$$\mathfrak{so}_6 \to \underset{\text{symm } I \leftrightarrow II}{\mathfrak{su}_{2,I} \oplus \mathfrak{su}_{2,II}} (\oplus \mathfrak{u}_1) \to \begin{cases} \mathfrak{su}_{2,I \text{ or } II} \, (\oplus \mathfrak{u}_1); \\ \mathfrak{su}_{2,d} \, (\oplus \mathfrak{u}_1); \end{cases} \tag{204}$$

$$\begin{aligned} \mathbf{15} &= (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + 2 \cdot (\mathbf{2}, \mathbf{2}) \\ &= \begin{cases} \mathbf{3} + 4 \cdot \mathbf{1} + 4 \cdot \mathbf{2}; \\ 4 \cdot \mathbf{3} + 3 \cdot \mathbf{1}; \end{cases} \end{aligned} \tag{205}$$

$$\mathbf{4} = (\mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}) = \begin{cases} \mathbf{2} + 2 \cdot \mathbf{1}; \\ 2 \cdot \mathbf{2}, \end{cases} \tag{206}$$

such that (194) can be further branched as

$$\begin{aligned} \mathfrak{a}_{II} &= \mathfrak{g}_2 \oplus \mathfrak{su}_{2,I} \oplus \mathfrak{su}_{2,II} (\oplus \mathfrak{u}_1) \oplus 2 \cdot (\mathbf{1}, \mathbf{2}, \mathbf{2}) \\ &\quad \oplus 3 \cdot (\mathbf{7}, \mathbf{2}, \mathbf{1}) + 3 \cdot (\mathbf{7}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{1}, \mathbf{1}) \\ &= \begin{cases} \mathfrak{g}_2 \oplus \mathfrak{su}_{2,I \text{ or } II} (\oplus \mathfrak{u}_1) \oplus 3 \cdot (\mathbf{1}, \mathbf{1}) \oplus 4 \\ \cdot (\mathbf{1}, \mathbf{2}) \oplus 3 \cdot (\mathbf{7}, \mathbf{2}) + 7 \cdot (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}; \\ \mathfrak{g}_2 \oplus \mathfrak{su}_{2,d} (\oplus \mathfrak{u}_1) \oplus 3 \cdot (\mathbf{1}, \mathbf{3}) \oplus 2 \cdot (\mathbf{1}, \mathbf{1}) \oplus 6 \\ \cdot (\mathbf{7}, \mathbf{2}) \oplus (\mathbf{7}, \mathbf{1}) \not\supseteq \mathbb{T}. \end{cases} \end{aligned} \tag{207}$$

4.

$$\mathfrak{so}_6 \to \mathfrak{su}_3\,(\oplus\mathfrak{u}_1) \to \begin{cases} \mathfrak{su}_2\,(\oplus 2\mathfrak{u}_1)\,; \\ \mathfrak{su}_{2,P}\,(\oplus\mathfrak{u}_1)\,; \end{cases} \quad (208)$$

$$\mathbf{15} = \mathbf{8} + \mathbf{1} + 2\cdot\mathbf{3} = \begin{cases} \mathbf{3} + 4\cdot\mathbf{1} + 4\cdot\mathbf{2}; \\ \mathbf{5} + 3\cdot\mathbf{3} + \mathbf{1}; \end{cases} \quad (209)$$

$$\mathbf{4} = \mathbf{3} + \mathbf{1} = \begin{cases} \mathbf{2} + 2\cdot\mathbf{1}; \\ \mathbf{3} + \mathbf{1}, \end{cases} \quad (210)$$

such that (194) can be further branched as

$$\mathfrak{a}_{II} = \mathfrak{g}_2 \oplus \mathfrak{su}_3\,(\oplus\mathfrak{u}_1) \oplus 2\cdot(\mathbf{1},\mathbf{3}) \oplus 3\cdot(\mathbf{7},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1})$$

$$= \begin{cases} \mathfrak{g}_2 \oplus \mathfrak{su}_2\,(\oplus 2\mathfrak{u}_1) \oplus 4\cdot(\mathbf{1},\mathbf{2}) \oplus 2\cdot(\mathbf{1},\mathbf{1}) \oplus 3 \\ \cdot(\mathbf{7},\mathbf{2}) \oplus 7\cdot(\mathbf{7},\mathbf{1}) \nsupseteq \mathbb{T}; \\ \mathfrak{g}_2 \oplus \mathfrak{su}_{2,P}\,(\oplus\mathfrak{u}_1) \oplus (\mathbf{1},\mathbf{5}) \oplus 2\cdot(\mathbf{1},\mathbf{3}) \oplus 3 \\ \cdot(\mathbf{7},\mathbf{3}) \oplus 4\cdot(\mathbf{7},\mathbf{1}) \nsupseteq \mathbb{T}. \end{cases} \quad (211)$$

This concludes the proof that there is no $\mathfrak{su}_2 \simeq \mathfrak{Der}\,(\mathbb{C}\otimes\mathbb{H})$ subalgebra of $\mathfrak{so}_6 \simeq \mathfrak{Der}\,(J_2\,(\mathbb{C}\otimes\mathbb{H}))$ such that $\mathfrak{a}_{II}$ given by (194) contains the Dixon algebra $\mathbb{T}$ (195), presuming that $\mathbb{H} \subset \mathbb{T}$ contains a $\mathbf{1}\oplus\mathbf{3}$ representation of $\mathfrak{Der}(\mathbb{H}) = \mathfrak{su}_2$. ■

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