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Flow of shear response functions in hyperscaling violating Lifshitz theories

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Abstract We study the flow equations of the shear response functions for hyperscaling violating Lifshitz (hvLif) theories, with Lifshitz and hyperscaling violating exponents z and θ. Adapting the membrane paradigm approach of analysing response functions as developed by Iqbal and Liu, we focus specifically on the shear gravitational modes which now are coupled to the perturbations of the background gauge field. Restricting to the zero momenta sector, we make further simplistic assumptions regarding the hydrodynamic expansion of the perturbations. Analysing the flow equations shows that the shear viscosity at leading order saturates the Kovtun–Son–Starinets (KSS) bound of $\frac{1}{4\pi}$. When $z = d_i - \theta$, (d_i being the number of spatial dimension in the dual field theory) the first-order correction to shear viscosity exhibits logarithmic scaling, signalling the emergence of a scale in the UV regime for this class of hvLif theories. We further show that the response function associated to the gauge field perturbations diverge near the boundary when $z > d_i + 2 - \theta$. This provides a holographic understanding of the origin of such a constraint and further vindicates results obtained in previous works that were obtained through near horizon and quasinormal mode analysis.

Contents

1 Introduction

The framework of gauge/gravity duality [1–4] has been generalized and applied to understand strongly coupled non-relativistic field theories. In particular, a certain class of non-relativistic field theories, dubbed as hyperscaling violating Lifshitz (hvLif) theories (which are conformal to Lifshitz theories) has been extensively explored in previous works [5–37]. In fact, there are concrete examples of realizable condensed matter systems where certain correlators exhibit similar scaling behaviour as that of hvLif theories [35]. Interested readers can see [25,35] for a comprehensive review of these class of non-relativistic field theories.

The gravity dual of hvLif theories can be realized as solutions to effective Einstein–Maxwell-dilaton theories [5–19]. hvLif solutions may be embedded in string theory as null reductions of boosted black branes [38,39] (Lifshitz spacetimes which are conformal to hvLif spacetimes also admit gauge/string realizations [40–45]). For a better understanding of this class of non-relativistic field theories, it is crucial to understand their infrared (IR) behaviour, in particular, hydrodynamics and various response functions that emerges

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in the low-energy limit. In previous works, the shear diffusion constant and the shear viscosity bound for hvLif theories were analysed using the membrane paradigm approach [26] as well as quasi-normal modes of the dual gravity theory [34]. It was found that for a $d_i + 1$-dimensional hvLif theory with Lifshitz exponent z and hyperscaling violating exponent θ, one must have $z \leq d_i + 2 - \theta$ for a consistent hydrodynamic expansion. When $z = d_i + 2 - \theta$, the shear diffusion constant exhibits a novel logarithmic scaling while the Kovtun–Starinets–Son (KSS) shear viscosity bound is saturated [46]. For $z > d_i + 2 - \theta$, the first order solution diverges at the boundary presumably hinting towards a breakdown of the hydrodynamic expansion for this parameter regime.

In this paper, we take the approach as pioneered by Iqbal and Liu [47]. The gauge/gravity duality maps the strongly coupled field theory on the boundary to the weakly coupled black hole spacetime in the bulk. However, the membrane paradigm approach to black holes endows hydrodynamic properties such as viscosity, entropy, conductivity etc. to a fictitious *stretched horizon* which is hovering very close to the real event horizon. Using the UV/IR point of view, Iqbal and Liu essentially attempted to relate this horizon fluid to the hydrodynamic regime of the strongly coupled field theory living on the boundary in the context of AdS gravity. It turned out that in the low-frequency, long wavelength limit (i.e. hydrodynamic limit) the evolution of retarded Green's function of the boundary with respect to energy scale is trivial. To be more precise, one can think of the radial direction of the bulk gravity theory as the energy scale of the boundary theory. Thus, the perturbed bulk Einstein's equations at linearized order can be thought of as a RG flow equation for a certain generalized response function which turns out to be independent of the radial direction at leading order. The *triviality* of flow of the response function implies that the corresponding transport coefficient can be expressed in terms of geometric quantities over any constant r hypersurface of the bulk theory and hence can be shown to be *universal*.

The aim of this work is to adapt the above approach and study the RG flow of response functions in the context of hvLif theories. The analysis is significantly more complicated due to nontrivial coupling between the shear perturbative modes with the gauge field perturbations. This is to be contrasted with previous works such as [22,48] where such flow equations were studied in the context of anisotropic gravity duals or the background resulted from higher derivative corrected action. In such cases, the holographic duals interpolate between Lifshitz or hvLif in the deep IR while it asymptotes to pure AdS near the boundary.

The starting point of our analysis is a $(d+1)$-dimensional gravity dual of hvLif theory. Turning on perturbations of the form $e^{-i\omega t+iqx}h_{\mu\nu}(r)$ and $e^{-i\omega t+iqx}a_\mu(r)$ respectively for the metric and gauge field, we notice the shear sector modes h_{xi}, h_{ti} and a_i (where i runs over all boundary direction except t and x) forms a coupled set of differential equations. We associate a conjugate momenta to each of these perturbation modes and correspondingly define appropriate response functions. As one would expect, the radial flow equations for each of these response functions also follow complicated coupled non-linear differential equations. However, one must note that our principal aim is to extract the transport coefficient out of these response function in the hydrodynamic limit which one does in the language of linear response theory, adapted to the context of gauge/gravity duality. Consider a generic field theory containing an operator $\mathcal{O}$ which is coupled to a source φ. At the level of linear response they are related as

$$\langle \mathcal{O}(\omega, q)\rangle = -G^R(\omega, q)\varphi(\omega, q) \tag{1.1}$$

where ω and q are very small frequency and momenta respectively, while G^R denotes the retarded correlator for the operator $\mathcal{O}$. The corresponding transport coefficient is defined as

$$\chi = \lim_{\omega\to 0}\frac{G^R(\omega, q=0)}{i\omega}, \tag{1.2}$$

which is known as Kubo's formula. In particular, when $\mathcal{O} \equiv T^{xy}$, the corresponding transport coefficient is the shear viscosity η while for a charge current i.e. $\mathcal{O} \equiv J^x$, the analogous transport coefficient is the DC conductivity. Since in the above we essentially require to find the response function at zero momenta, we focus on that regime and analyse the flow equations. Interestingly, we see that indeed for $q = 0$, the flow equation for χ_{xi} i.e. the response function corresponding to h_{xi} follows a Riccati equation which leads to a constant χ_{xi} at leading order for all values of z and θ. This behaviour is identical to that encountered in pure AdS gravity [47]. However, when $z = d_i - \theta$, the first order correction to the response function has a logarithmic scaling which diverges at the boundary $r \to 0$. This necessitates the introduction of a cut-off presumably signifying the UV scale beyond which the hydrodynamic expansion breaks down.

The analysis for the response function associated with a_i i.e. χ_{a_i} is more involved due to the complicated nature of the flow equation. In fact at $q = 0$, it turns out the variable $\zeta_{a_i} = \omega\chi_{a_i}$ seems to admit a hydrodynamic expansion. In order to analyse the behaviour of ζ_{a_i}, we focus on the near-horizon region and the near-boundary region separately which somewhat simplifies the analysis. At leading order itself, we see the solutions for ζ_{a_i} are different for the two different regimes. This is different qualitatively from the behaviour of χ_{xi} which followed a *trivial flow* equation allowing one to write the response function at any point along the radial direction. Interestingly, we see close to the boundary, the leading piece of χ_{a_i} diverges when $z > d_i + 2 - \theta$ which is identical to results obtained in earlier works [26,28,34]. To further vindicate our result, one can look at the *Markovianity* index of the fluctuating modes in the spirit of [49]. Interestingly, we observe that for $z \leq d_i + 2 - \theta$, the fluctuations starts to behave like a non-Markovian probe.

The paper is organized as follows: In Sect. 2, we describe our setup and define appropriate response functions corresponding to the shear gravitational modes. The general flow equations are worked out which describes the non-perturbative evolution of response function for arbitrary frequency and momenta. Section 3 focuses on the zero momenta sector and look at the transport coefficient associated with the modes h_{xi}, h_{ti} and a_i. Finally, keeping some details of the calculations in three appendices, we end with a discussions of our main results with possible future directions along with a simple analysis of the *Markovianity index* results in Sect. 4.

2 Flow equations of response functions

We are considering a hvLif theory living in $d = d_i + 1$ spacetime dimensions with Lifshitz exponent z and hyperscaling violating exponent θ. This field theory has a $(d+1)$-dimensional gravity dual given by

$$ds^2 = r^{\frac{2\theta}{d_i}} \left(-\frac{f(r)}{r^{2z}} dt^2 + \frac{dr^2}{r^2 f(r)} + \sum_{i=1}^{d_i} \frac{dx_i^2}{r^2} \right); \quad d = d_i + 1; \quad f(r) = 1 - (r_0 r)^{d_i + z - \theta}. \tag{2.1}$$

The above metric is a solution to Einstein–Maxwell-dilaton theory (details of background solution in Appendix A). The temperature of the field theory dual to the hvLif theory (2.1) is the Hawking temperature of the black brane

$$T = \frac{d_i + z - \theta}{4\pi} r_0^z, \tag{2.2}$$

where the event horizon is located at $r = \frac{1}{r_0}$.

As per the holographic dictionary, the radial coordinate r can be thought of as the energy scale in the bulk theory. Our central goal in this section is to essentially set up the RG flow equations governing the response functions that we want to study. In order to obtain the RG flow equations, we turn on linearized perturbations in the bulk theory, which in general is given as,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}; \quad A_\mu = \bar{A}_\mu + a_\mu; \quad \phi = \bar{\phi} + \varphi, \tag{2.3}$$

where quantities $\bar{g}_{\mu\nu}$, $\bar{A}_\mu$ and $\bar{\phi}$ denote background fields as given in Appendix A. We turn on perturbations of the form $e^{-i\omega t + iqx} h_{\mu\nu}(r)$ and $e^{-i\omega t + iqx} a_\mu(r)$ and restrict ourselves to the radial gauge ($h_{\mu r} = a_r = 0$). The shear gravitation modes h_{xi} now couples to h_{ti} and a_i where the index i runs over all boundary coordinates except t and x. For convenience, we define the following field variables

$$H_{xi} = g^{ii} h_{ix} = r^{2 - \frac{2\theta}{d_i}} h_{xi}; \quad H_{ti} = g^{ii} h_{ti} = r^{2 - \frac{2\theta}{d_i}} h_{ti}. \tag{2.4}$$

In terms of these modes the equations of motion take the form

$$\partial_r (r^{z+\theta-(d_i+1)} H'_{ti}) - k a'_i - \frac{r^{z+\theta-(d_i+1)}}{f} q(\omega H_{xi} + q H_{ti}) = 0, \tag{2.5}$$

$$\partial_r (r^{\theta - z - d_i + 1} f H'_{xi}) + \frac{r^{z+\theta-(d_i+1)}}{f} \omega(\omega H_{xi} + q H_{ti}) = 0, \tag{2.6}$$

$$q r^{2-2z} H'_{xi} + \frac{\omega}{f} (H'_{ti} - k r^{(d_i+1)-z-\theta} a_i) = 0, \tag{2.7}$$

$$\partial_r (r^{d_i+3-z-\theta} f a'_i) + \frac{r^{d_i+1+z-\theta}}{f} \omega^2 a_i - r^{d_i+3-z-\theta} q^2 a_i - k H'_{ti} = 0, \tag{2.8}$$

where $k = (d_i + z - \theta)\alpha$. The above linearized equations of motion follow from the perturbed second order action, the details of which are provided in Appendix B. In terms of the variables defined in (2.4), the action (B.1) can be recast in a more 'canonical' form as

$$\begin{aligned} S^{(2)} = \frac{-1}{16\pi G_N} \int dr\, d^d k \times \Bigg[& -\frac{1}{2} r^{1-d_i-z+\theta} f H'_{xi}{}^2 + \frac{1}{2} r^{-1-d_i+z+\theta} H'_{ti}{}^2 \\ & + k H_{ti} a'_i + \frac{\omega^2}{2} \frac{r^{-1-d_i+z+\theta}}{f} H_{xi}^2 \\ & + \frac{q^2}{2} \frac{r^{-1-d_i+z+\theta}}{f} H_{ti}^2 + \frac{q\omega}{f} r^{-1-d_i+z+\theta} H_{xi} H_{ti} \\ & - \frac{1}{2} f r^{d_i+3-z-\theta} a'^2_i \\ & + \left(\frac{\omega^2}{2} \frac{r^{d_i+1+z-\theta}}{f} - \frac{q^2}{2} r^{d_i+3-z-\theta} \right) a_i^2 \Bigg] + S^{(2)}_{bdy}, \end{aligned} \tag{2.9}$$

which yields (2.5)–(2.8) as the equations of motion. For completion, we state the boundary action i.e. $S^{(2)}_{bdy}$ is given by

$$\begin{aligned} S^{(2)}_{bdy} = -\frac{1}{16\pi G_N} \int d^d k \times \Bigg[& 2 r^{1-d_i-z+\theta} f H'_{xi} H_{xi} - 2 r^{-1-d_i+z+\theta} H'_{ti} H_{ti} \\ & + \frac{(\theta - d_i) f}{d_i} r^{-d_i-z+\theta} H_{xi}^2 \\ & + \frac{r^{-d_i-2+\theta+z} \left((d_i^2 - d_i(\theta+z-4) - 2\theta) f - d_i(d_i - \theta + z) \right)}{2 d_i f} H_{ti}^2 \Bigg]. \end{aligned} \tag{2.10}$$

Motivated by the equations of motion appearing in (2.5)–(2.8), we observe that the coupling term between H_{ti} and a_i

appearing in the above action (2.9), namely $+kH_{ti}a_i'$ can be rewritten as $-kH_{ti}'a_i$ along with a boundary term. Thus, the effective Lagrangian reads as

$$S^{(2)} = \frac{1}{16\pi G_N}\int dr\, d^dk \times\Bigg[-\frac{1}{2}r^{1-d_i-z+\theta} f {H_{xi}'}^2 + \frac{1}{2}r^{-1-d_i+z+\theta}{H_{ti}'}^2 -kH_{ti}'a_i + \frac{\omega^2}{2}\frac{r^{-1-d_i+z+\theta}}{f}H_{xi}^2 +\frac{q^2}{2}\frac{r^{-1-d_i+z+\theta}}{f}H_{ti}^2 + \frac{q\omega}{f}r^{-1-d_i+z+\theta}H_{xi}H_{ti} -\frac{1}{2}fr^{d_i+3-z-\theta}{a_i'}^2 +\left(\frac{\omega^2}{2}\frac{r^{d_i+1+z-\theta}}{f} - \frac{q^2}{2}r^{d_i+3-z-\theta}\right)a_i^2\Bigg] + S^{(2)}_{bdy}. \tag{2.11}$$

The conjugate momenta for the modes H_{xi}, H_{ti} and a_i are defined respectively as,

$$16\pi G_N \Pi_{xi} = \frac{\partial\mathcal{L}}{\partial H_{xi}'},\quad 16\pi G_N \Pi_{ti} = \frac{\partial\mathcal{L}}{\partial H_{ti}'},\quad 16\pi G_N \Pi_{a_i} = \frac{\partial\mathcal{L}}{\partial a_i'}. \tag{2.12}$$

The above definitions immediately yield

$$16\pi G_N.\,\Pi_{xi} = -fr^{1-d_i-z+\theta}H_{xi}', \tag{2.13}$$
$$16\pi G_N.\,\Pi_{ti} = r^{-1-d_i+z+\theta}H_{ti}' - ka_i, \tag{2.14}$$
$$16\pi G_N.\,\Pi_{a_i} = -fr^{d_i+3-z-\theta}a_i'. \tag{2.15}$$

Corresponding to each of the modes H_{ti}, H_{xi} and a_i, we associate a response function given by,

$$\chi(r,q,\omega) = \frac{\Pi(r,q,\omega)}{i\omega\phi(r,q,\omega)};\quad \phi = \{H_{xi}, H_{ti}, a_i\}. \tag{2.16}$$

In terms of the response functions, the constraint equation (2.7) takes the form

$$\frac{\chi_{ti}}{\chi_{xi}} = \frac{qH_{xi}}{\omega H_{ti}}. \tag{2.17}$$

Using the Eqs. (2.12)–(2.17), we can eventually write down the generalized flow equations for the response functions χ_{xi}, χ_{ti} and χ_{a_i} which takes the form

$$\partial_r\chi_{xi} = i\omega\left[\frac{16\pi G_N\,\chi_{xi}^2}{fr^{1-d_i-z+\theta}} - \frac{r^{z+\theta-d_i-1}}{16\pi G_N\, f}\left(1+\frac{q^2}{\omega^2}\frac{\chi_{xi}}{\chi_{ti}}\right)\right], \tag{2.18}$$

$$\partial_r\chi_{ti} = -i\omega\Bigg[\frac{16\pi G_N}{r^{-1-d_i+z+\theta}}\chi_{ti}^2 + \frac{ka_i}{i\omega r^{-1-d_i+z+\theta}H_{ti}}\chi_{ti} +\frac{r^{z+\theta-(d_i+1)}}{16\pi G_N\, f}\left(\frac{\chi_{ti}}{\chi_{xi}}+\frac{q^2}{\omega^2}\right)\Bigg], \tag{2.19}$$

$$\partial_r\chi_{a_i} = i\omega\Bigg[16\pi G_N\frac{r^{z+\theta-d_i-3}}{f}\chi_{a_i}^2 - \frac{r^{d_i+1+z-\theta}}{16\pi G_N f} +\frac{k^2}{16\pi G_N\omega^2}r^{d_i+1-z-\theta} +\frac{q^2}{\omega^2}\frac{r^{d_i+3-z-\theta}}{16\pi G_N} - \frac{q\,k}{\omega^3}\frac{fr^{2-2z}}{16\pi G_N}\frac{H_{xi}'}{a_i}\Bigg]. \tag{2.20}$$

Note that the above set of coupled differential equations are *exact* i.e. they describe the flow of response functions for generic values of frequency and momentum. Although, they are complicated and analytically intractable, we are however interested in the hydrodynamic regime which is essentially the limit where the frequency and momenta ω and q are much smaller than the temperature scale i.e. $q \ll T^{1/z} \sim r_0$ and $\omega \ll T \sim r_0^z$. Further, it is evident from (1.2) that the $q = 0$ sector is relevant for evaluating shear transport coefficient. Thus, we will focus exclusively on the $q = 0$ sector of the flow equations (2.18)–(2.20).

3 Zero momentum response functions

Before we proceed to study the $q \to 0$ limit of the flow equations we derived in the preceding section, it is imperative to talk about solutions of the field equations in the $q \to 0$ limit. An earlier work [34] analysed the field equations assuming a hydrodynamic expansion in the dimensionless parameters $\Omega = \frac{\omega}{2\pi T}$ and $Q = \frac{q}{(2\pi T)^{1/z}}$. One can however reabsorb the constant temperature factor in each term of the hydrodynamic expansion and simply write the fields as an expansion in ω and q.

Starting with the equations of motion (2.5)–(2.8), a gauge invariant combination $\mathcal{H}_i$ was defined as

$$\mathcal{H}_i = \omega H_{xi} + qH_{ti} - kq\int_{r_c}^{r} s^{d_i+1-z-\theta}a_i(s)\,ds. \tag{3.1}$$

The fields $\mathcal{H}_i$ and a_i formed a system of coupled differential equations which were solved up to first order in the hydrodynamic expansion. For the redefined field $\mathcal{H}$, it was observed that for $z < d_i+2-\theta$ the terms in the hydrodynamic expansion of the field variables can be solved order-by-order. When $z = d_i+2-\theta$, the first order correction to $\mathcal{H}_i$ scales logarithmically and seems to diverge close to the boundary. The logarithmic scaling is suggestive of the emergence of a new scale in the UV limit. In the regime when $z > d_i+2-\theta$, the first order correction to $\mathcal{H}_i$ diverges suggesting a breakdown of the methodology for parameters in this regime. The solution to the combination $\mathcal{H}_i$ up to first order in the hydrodynamic expansion is given by

$$\mathcal{H}_i = C_0 f(r)^{-\frac{i\Omega}{2}}\left[1 + \frac{iq^2}{(d_i+2-z-\theta)\omega}r_0^{z-2}\right.$$

$$\cdot(1-(r_0 r)^{d_i+2-z-\theta})\Big]. \quad (3.2)$$

The gauge field fluctuations, a_i satisfy a second order non-homogeneous differential equation. Upon imposing regularity on a_i, the leading solution takes the form

$$a_i = -iC_0 k\frac{q}{\omega}\cdot\frac{r_0^{d_i-\theta}}{(d_i+z-\theta)^2} \times f(r)^{1-\frac{i\omega}{4\pi T}}(r_0 r)^{-(d_i+z-\theta)}, \quad (3.3)$$

where C_0 is an arbitrary non-zero constant. The first order piece does not have a closed form solution but can be written as an integral and thus cannot give us further insight into its behaviour. The reader can find details and methodology of solving for a_i up to first order in [34]. Since in the current context, our interest is to explore the flow equations, we will not further concern ourselves with solutions to the fields H_{xi}, H_{ti}. We will however make certain assumptions about them which will help us in dealing with the complicated flow equations we derived in the preceding section.

Motivated from the form of $\mathcal{H}_i$ and a_i as given in (3.2) and (3.3), we will assume that the perturbations H_{xi}, H_{ti} and a_i for $\omega \neq 0$ and $q \neq 0$ admit a hydrodynamic expansion of the form

$$\begin{aligned}\phi(r,\omega,q) &= \phi^{(-1)}(r,\omega,q)+\phi^{(0)}(r,\omega,q) \\ &\quad +\phi^{(1)}(r,\omega,q)+\cdots \\ &= \sum_{n=-1}^{\infty}\phi^{(n)}(r,\omega,q), \end{aligned} \quad (3.4)$$

where $\phi(r,\omega,q)$ represents any one of the perturbative modes H_{xi}, H_{ti} or a_i. The leading term $\phi^{(-1)}(r,\omega,q)$ is parametrically an $\mathcal{O}(\frac{1}{\omega})$ quantity while $\phi^{(n)}(r,\omega,q)\sim\mathcal{O}(\omega^n)\sim\mathcal{O}(q^n)$. The first term in the above expression can be generically of the form

$$\phi^{(-1)}(r,\omega,q)\sim\sum_{a\geq 0}\frac{q^a}{\omega^{a+1}}\mathfrak{b}_a(r) \quad (3.5)$$

while the $\mathcal{O}(1)$ term and the n-th order term in the hydrodynamic expansion will take the general schematic form

$$\phi^{(0)}(r,\omega,q)\sim\sum_{a\geq 0}\frac{q^a}{\omega^a}\mathfrak{g}_a(r) \quad \text{and}$$

$$\phi^{(n)}(r,\omega,q)\sim\sum_{\substack{a,b\\ k>0}}{}' q^a\omega^b\frac{q^k}{\omega^k}\mathfrak{h}_{(a,b,k)}(r), \quad (3.6)$$

respectively. In (3.5) and (3.6), all the exponents a, b and k are strictly positive. The functions $\mathfrak{b}_a(r)$, $\mathfrak{g}_a(r)$ and $\mathfrak{h}_{(a,b,k)}(r)$ are all regular in the interval $0<r<\frac{1}{r_0}$. The sum in the n-th order term has a 'prime' to denote that it is a *constrained* sum such that $a+b=n\geq 1$. The above schematic forms of each term in the hydrodynamic expansion of the field variables is well behaved in the limit $q\to 0$.

A comparison of (3.3) with the schematic forms as given in (3.5) and (3.6) tells us that for a_i, the $\mathcal{O}(\frac{1}{\omega})$ term is identically zero; the $\mathcal{O}(1)$ term consists of a single term with $a=1$ while $\mathfrak{g}_1(r)\sim f(r)^{-\frac{i\Omega}{2}}$. A comparison of the above schematic expansion with $\mathcal{H}_i$ as given in (3.2) is difficult, since it appears as a linear combination of H_{xi}, H_{ti} and an integral over a_i. We can still comment on the heuristic behaviour of the response functions that follows from the above assumptions regarding the hydrodynamic expansion of the field variables.

The structure of (2.13)–(2.16) along with (3.5)–(3.6) implies that the response function can be written schematically as

$$\chi\simeq\frac{F(r)}{i\omega}\left(\chi^{(0)}(r,\omega,q)+\chi^{(1)}(r,\omega,q)+\chi^{(2)}(r,\omega,q)+\cdots\right) \quad (3.7)$$

where $\chi^{(n)}(r,\omega,q)$ denotes a term which is $\mathcal{O}(q^n)\sim\mathcal{O}(\omega^n)$ in the hydrodynamic expansion but is determined by the explicit forms of $\mathfrak{b}_a(r)$, $\mathfrak{g}_a(r)$ and $\mathfrak{h}_{(a,b,k)}(r)$ while $F(r)$ is some specific function depending on which mode is under consideration. More specifically,

$$\begin{aligned}\chi^{(0)}(r,\omega,q) &= \frac{\sum_{a\geq 0}\frac{q^a}{\omega^{a+1}}\partial_r\mathfrak{b}_a(r)}{\sum_{a\geq 0}\frac{q^a}{\omega^{a+1}}\mathfrak{b}_a(r)}, \\ \chi^{(1)}(r,\omega,q) &= \frac{\sum_{a,b\geq 0}\frac{q^{a+b}}{\omega^{a+b+1}}\mathcal{W}[\mathfrak{b}_a(r),\mathfrak{g}_b(r)]}{\sum_{a,b\geq 0}\frac{q^{a+b}}{\omega^{a+b+2}}\mathfrak{b}_a(r)\mathfrak{b}_b(r)}\end{aligned} \quad (3.8)$$

where $\mathcal{W}[f,g]=fg'-f'g$ denotes the Wronskian for the pair of functions f and g. In the case, neither of these are linear combinations of various powers of the ratio $\frac{q}{\omega}$, we simply recover

$$\begin{aligned}\chi^{(0)}(r,\omega,q) &= \partial_r\ln\mathfrak{b}_a(r), \\ \chi^{(1)}(r,\omega,q) &= \omega\frac{\mathcal{W}[\mathfrak{b}_a(r),\mathfrak{g}_b(r)]}{\mathfrak{b}_a(r)\mathfrak{b}_b(r)}.\end{aligned} \quad (3.9)$$

Armed with the above heuristic analysis, we further closely look at the following terms appearing in (2.18)–(2.20).

- The last term appearing in (2.18) can be written as

$$H(r)\frac{q^2}{\omega}\frac{\chi_{xi}}{\chi_{ti}}, \quad (3.10)$$

where $H(r)$ is a function of r whose details we are not concerned with for the purpose of this analysis. For the sake of simplicity, if we assume (3.5) is not in fact a linear combination of various powers of the ratio $\frac{q}{\omega}$, the leading behavior of this term *for non-zero q and ω* is given by

$$H(r)\frac{q^2}{\omega}\frac{\chi_{xi}}{\chi_{ti}} \simeq \frac{q^2}{\omega}\left(\tilde{H}_1(r)+\omega\tilde{H}_2(r)+\cdots\right), \tag{3.11}$$

where the "$\cdots$" represents terms that are higher order in q or ω while $\tilde{H}_n(r)$ represents various functions of r. Now, every expression $\tilde{H}_n(r)$ involve ratios of derivatives of the family of functions $\mathfrak{b}_a(r)$, $\mathfrak{g}_a(r)$ and $\mathfrak{h}_{(a,b,k)}(r)$ that appear in the hydrodynamic expansion of the field variables. To be more explicit,

$$\tilde{H}_1(r) \sim \frac{\partial_r \mathfrak{b}_a^{(xi)}(r)}{\partial_r \mathfrak{b}_a^{(ti)}(r)}. \tag{3.12}$$

At this point, we further make the assumption that for non-zero q and ω, each of these functions i.e. $\mathfrak{b}_a(r)$, $\mathfrak{g}_a(r)$ and $\mathfrak{h}_{(a,b,k)}(r)$ appearing in the field expansion of H_{ti} and H_{xi} are *non-constant, non-trivial functions of r*. Clearly, under the above assumption, this term vanishes when $q \to 0$. We will subsequently infer from the equations of motion at $q = 0$ that $\chi_{ti} = 0$ for this sector however, the term that we just discussed does not have any singularity or does not go to any constant as $q \to 0$. Physically speaking, χ_{ti} presumably contains a leading $O(q)$ piece which ensures that the ratio $\frac{q^2}{\chi_{ti}} \to 0$ as $q \to 0$ while presence of higher powers of the momenta q in subsequent higher order terms ensure it vanishes identically as $q \to 0$.

- In (2.19), we see the last two terms can be schematically written as

$$\omega G(r)\left(\frac{\chi_{ti}}{\chi_{xi}}+\frac{q^2}{\omega^2}\right) \xrightarrow{q\to 0} \omega G(r)\frac{\chi_{ti}}{\chi_{xi}}. \tag{3.13}$$

Although this term will indeed vanish at $q = 0$, since $\chi_{ti} = 0$ in this sector, our assumptions up to this point dictates a possible $O(\omega)$ contribution as to the flow equations as $q \to 0$. Hence, we keep this term in the limit of vanishing momenta.

- Finally, we need to study the final term in (2.20) which we schematically write as

$$\frac{q}{\omega^2}P(r)\frac{H'_{xi}}{a_i}. \tag{3.14}$$

Note that from (3.3) and comparing with the expansion (3.4), it follows that for the field a_i, the family of functions $\mathfrak{b}_a(r) = 0$ identically. Again, for simplicity, assuming H_{xi} does not have a linear combination of terms at $\mathcal{O}(1/\omega)$, we get the leading behaviour of the last term as

$$\frac{q}{\omega^2}P(r)\frac{H'_{xi}}{a_i} \simeq \frac{q^a}{\omega^{a+2}}P(r)\frac{\partial_r \mathfrak{b}_a^{(xi)}(r)}{g_1^{(a_i)}(r)} \xrightarrow{q\to 0} 0. \tag{3.15}$$

Thus, in the $q \to 0$ limit, we recover the simplified flow equations as

$$\partial_r\chi_{xi} = i\omega\left[\frac{16\pi G_N\,\chi_{xi}^2}{f r^{1-d_i-z+\theta}} - \frac{r^{z+\theta-d_i-1}}{16\pi G_N\, f}\right], \tag{3.16}$$

$$\begin{aligned}\partial_r\chi_{ti} = -i\omega\Bigg[&\frac{16\pi G_N}{r^{-1-d_i+z+\theta}}\chi_{ti}^2 \\ &+\frac{k\,a_i}{i\omega r^{-1-d_i+z+\theta}H_{ti}}\chi_{ti} + \frac{r^{z+\theta-(d_i+1)}}{16\pi G_N\, f}\frac{\chi_{ti}}{\chi_{xi}}\Bigg],\end{aligned} \tag{3.17}$$

$$\begin{aligned}\partial_r\chi_{a_i} = i\omega\Bigg[&16\pi G_N\frac{r^{z+\theta-d_i-3}}{f}\chi_{a_i}^2 \\ &-\frac{r^{d_i+1+z-\theta}}{16\pi G_N f} + \frac{k^2}{16\pi G_N\omega^2}r^{d_i+1-z-\theta}\Bigg].\end{aligned} \tag{3.18}$$

The above equations can also be derived by turning on perturbations of the form $e^{-i\omega t}h_{\mu\nu}(r)$ and choosing the radial gauge $h_{\mu r} = 0$. Before we proceed with the detailed analysis of the flow of response functions, the equations of motion in the $q = 0$ sector simplifies significantly to give

$$\partial_r(r^{z+\theta-(d_i+1)}H'_{ti} - ka_i) = 0, \tag{3.19}$$

$$\partial_r(r^{\theta-z-d_i+1}fH'_{xi}) + \frac{r^{z+\theta-(d_i+1)}}{f}\omega^2 H_{xi} = 0, \tag{3.20}$$

$$H'_{ti} - kr^{(d_i+1)-z-\theta}a_i = 0, \tag{3.21}$$

$$\partial_r(r^{d_i+3-z-\theta}fa'_i) + \frac{r^{d_i+1+z-\theta}}{f}\omega^2 a_i - kH'_{ti} = 0. \tag{3.22}$$

Thus, in the $q = 0$ sector, the mode H_{xi} further decouples from H_{ti} and a_i. The constraint equation (3.21) clearly implies

$$\Pi_{ti}|_{q=0} = 0. \tag{3.23}$$

By the assumptions we made in (3.4)–(3.6), we see that

$$\lim_{q\to 0}\chi_{ti} = \lim_{q\to 0}\frac{\Pi_{ti}}{i\omega H_{ti}} = 0. \tag{3.24}$$

This indeed is consistent with (3.17) and renders the equation trivial. Thus the $q = 0$ sector requires us to analyse two *independent* equations governing the flow of H_{xi} and a_i given by (3.16) and (3.18) respectively.

3.1 Response function χ_{xi} at $q = 0$

As argued in the previous section, in the $q = 0$ sector, the χ_{xi} flow equation decouples from the χ_{ti} flow equation and

(2.18) simplifies to

$$\partial_r \chi_{xi} = \frac{i\omega}{f}\left[\frac{16\pi G_N}{r^{1-d_i-z+\theta}}\chi_{xi}^2 - \frac{r^{z+\theta-d_i-1}}{16\pi G_N}\right]. \tag{3.25}$$

If we demand regularity of χ_{xi} at the horizon, we clearly see the RHS of the above is singular at $r = \frac{1}{r_0}$. This forces us to choose

$$\left[\frac{16\pi G_N}{r^{1-d_i-z+\theta}}\chi_{xi}^2 - \frac{r^{z+\theta-d_i-1}}{16\pi G_N}\right]_{r=\frac{1}{r_0}} = 0, \tag{3.26}$$

leading to the boundary condition

$$\chi_{xi}\left(\frac{1}{r_0}, \omega\right) = \frac{r_0^{d_i-\theta}}{16\pi G_N}. \tag{3.27}$$

In the hydrodynamic regime, we are allowed to write a perturbative expansion for $\chi_{xi}(r, \omega)$ as

$$\chi_{xi}(r, \omega) = \chi_{xi}^{(0)}(r) + \omega\chi_{xi}^{(1)}(r) + \mathcal{O}(\omega^2), \tag{3.28}$$

where $\mathcal{O}(\omega^2)$ represents higher order terms beyond the linear one. Plugging in the above expansion, in (3.25), the leading order piece follows

$$\partial_r \chi_{xi}^{(0)}(r) = 0. \tag{3.29}$$

Physically, the above equation tells us that the RG flow of the χ_{xi} is trivial at leading order remaining unchanged as we go along the radial direction. Along with the boundary condition (3.27) that we just derived, we have

$$\chi_{xi}^{(0)}(r) = \frac{r_0^{d_i-\theta}}{16\pi G_N}. \tag{3.30}$$

The $\mathcal{O}(\omega)$ equation which gives the flow of $\chi_{xi}^{(1)}$, is given by,

$$\partial_r \chi_{xi}^{(1)} = -i\frac{r^{-d_i-1+z+\theta}}{16\pi G_N\, f(r)}\left(1 - (r_0 r)^{2(d_i-\theta)}\right). \tag{3.31}$$

The solution to the above equation is

$$\begin{aligned}\chi_{xi}^{(1)}(r) = &-\frac{i r_0^{d_i-z-\theta}}{16\pi G_N}\left[\frac{(r_0 r)^{-d_i+z+\theta}}{z+\theta-d_i}\, {}_2F_1\right.\\ &\times\left[1, 1-\frac{2(d_i-\theta)}{d_i+z-\theta}, 2\right.\\ &\left.-\frac{2(d_i-\theta)}{d_i+z-\theta}; (r_0 r)^{d_i+z-\theta}\right]\\ &\left.+\frac{\log f(r)}{d_i+z-\theta} + C\right] \text{ when } z \neq d_i - \theta,\\ = &-\frac{i}{16\pi G_N}\log\frac{r}{C'} \quad \text{when } z = d_i - \theta,\end{aligned} \tag{3.32}$$

where C and C' are integration constants for the two cases of the Lifshitz exponent z while ${}_2F_1[a, b, c; r]$ represents the hypergeometric function. We then come across the following two cases,

Case I • $z \neq d_i - \theta$: Using the boundary condition (3.27), we can fix the constant of integration to be

$$C = \frac{(\gamma + \psi(\frac{z-d_i+\theta}{d_i+z-\theta}))}{d_i+z-\theta} \tag{3.33}$$

where γ is the Euler–Mascheroni constant and $\psi(x)$ is the polygamma function which is singular over the set non-positive definite integers. Taking into account the null energy condition (A.5), we focus when $d_i - \theta > 0$ and $z > 1$. Since, this solution is true when $z \neq d_i - \theta$, the argument in the polygamma function cannot be 0. However, $\frac{z-d_i+\theta}{d_i+z-\theta} = -1$ gives $z = 0$ which violates our the assumption of $z \geq 1$. For all other parameter values of (z, θ) the null energy condition ensures that $\psi(\frac{z-d_i+\theta}{d_i+z-\theta})$ is non-singular.

Case II • $z = d_i - \theta$: In this case too, plugging in the boundary condition (3.27), we get,

$$C' = \frac{1}{r_0} \tag{3.34}$$

which then gives the full solution

$$\chi_{xi}(r) = \frac{r_0^{d_i-1}}{16\pi G_N} - \frac{i\omega}{16\pi G_N}\log(r_0 r). \tag{3.35}$$

Clearly the divergent nature of the solutions as $r \to 0$, hints at a possible breakdown of the analysis when $z = d_i - \theta$ near the boundary.

Earlier works [28,34] used perturbative techniques to evaluate 2-point correlator of the stress-energy tensor which seemingly broke down when $z > d_i + 2 - \theta$. However, an analysis of the response function corresponding to H_{xi} i.e. χ_{xi} seems to carry through for all values of the Lifshitz exponent. As mentioned earlier in (1.2), shear viscosity up to leading order is thus given by

$$\eta = \chi_{xi} = \frac{r_0^{d_i-\theta}}{16\pi G_N} \tag{3.36}$$

which inturn saturates the KSS bound of $\frac{\eta}{s} = \frac{1}{4\pi}$. Also, note that the first order correction for either cases, namely $z = d_i - \theta$ and $z \neq d_i - \theta$ is positive since $r_0 r < 1$ thus following the bound. However, when $z = d_i - \theta$, we see the first order correction to be logarithmic and is actually divergent at the boundary when $r \to 0$. This enforces us to put a cut-off suggesting the emergence of a new scale.

Interestingly, earlier works [40,41] constructed families of Lifshitz geometries as dimensional reduction of AdS null deformations. Specifically, starting with AdS_5 null deformation, one can perform a reduction along one of the light-cone coordinates, namely x^+ which results in a 4-dimensional metric of the form (2.1) with $z = d_i = 2$ and $\theta = 0$. Thus, dimensional reduction of null deformed AdS_5 results in a metric which falls in the family of hvLif solutions constrained by $z = d_i - \theta$. In light of this observation, it will be interesting to understand the logarithmic scaling of the first order contribution to χ_{xi} from the perspective of the deformed higher dimensional theory.

3.2 Response function χ_{a_i} at $q = 0$

Recall from our earlier definition (2.16), that the response function χ_{a_i} associated to a_i is defined as

$$\chi_{a_i} = \frac{\Pi_{a_i}}{i\omega a_i}. \tag{3.37}$$

To reiterate, the flow equation for the response function χ_{a_i} decouples from that of χ_{xi} and χ_{ti} in the limit $q \to 0$ to yield,

$$\partial_r \chi_{a_i} = i\omega \left(16\pi G_N \frac{r^{z+\theta-d_i-3}}{f} \chi_{a_i}^2 - \frac{r^{d_i+1+z-\theta}}{16\pi G_N f} + \frac{k^2}{16\pi G_N \omega^2} r^{d_i+1-z-\theta} \right). \tag{3.38}$$

The structure of (3.38) is significantly different from the flow equation of χ_{xi}. Assuming a Laurent expansion in ω for the function $\chi_{a_i}(r,\omega)$, we see that in general it must have a term which goes as $\frac{1}{\omega}$ along with regular terms. Thus, like the earlier case of χ_{xi}, it does not make sense to naively perform a hydrodynamic expansion of containing only positive powers of ω. However, we define the new field

$$\zeta_{a_i} = \omega \chi_{a_i}, \tag{3.39}$$

in terms of which (3.38) becomes

$$\partial_r \zeta_{a_i}(r,\omega) = i \left[16\pi G_N \frac{r^{z+\theta-d_i-3}}{f} \zeta_{a_i}^2(r,\omega) + \frac{k^2 r^{d_i-z-\theta+1}}{16\pi G_N} - \omega^2 \frac{r^{d_i+z-\theta+1}}{16\pi G_N f} \right]. \tag{3.40}$$

Imposing regularity for ζ_{a_i} along the radial direction demands us to write the boundary condition as

$$\left[16\pi G_N r^{z+\theta-d_i-3} \zeta_{a_i}^2(r,\omega) - \frac{\omega^2 r^{d_i+1+z-\theta}}{16\pi G_N} \right]_{r=\frac{1}{r_0}} = 0, \tag{3.41}$$

which eventually yields,

$$\zeta_{a_i}\left(\frac{1}{r_0}, \omega\right) = \omega \frac{r_0^{\theta-d_i-2}}{16\pi G_N}. \tag{3.42}$$

One must note that (3.40) is exact in ω and consistent with a hydrodynamic expansion of the form

$$\zeta_{a_i}(r,\omega) = \zeta_{a_i}^{(0)}(r) + \omega \zeta_{a_i}^{(1)}(r) + \omega^2 \zeta_{a_i}^{(2)}(r) + \cdots. \tag{3.43}$$

Also, the demand of regularity gives us the ζ_{a_i} at the horizon which depends explicitly on the frequency ω. Thus, regularity in the context of the above hydrodynamic expansion implies $\zeta_{a_i}^{(m)}(1/r_0) = 0$ for all $m \neq 1$ while $\zeta_{a_i}^{(1)}(1/r_0) = \frac{r_0^{\theta-d_i-2}}{16\pi G_N}$. Unlike the earlier case of χ_{xi}, we see here that at leading order $\partial_r \zeta_{a_i}$ follows a nontrivial flow equation given by

$$\partial_r \zeta_{a_i}^{(0)}(r) = i \left[16\pi G_N \frac{r^{z+\theta-d_i-3}}{f} \zeta_{a_i}^{(0)}(r)^2 + \frac{k^2 r^{d_i-z-\theta+1}}{16\pi G_N} \right]. \tag{3.44}$$

Thus, we see for this response function, the RG flow is not trivial and it actually changes along the radial direction. Solving the above equation yields complicated solutions which one cannot use easily to construct further subleading contributions that are higher order in ω.

To circumvent the issue, we follow a different strategy. We will analyse the flow equation successively in the *near horizon* and the *near boundary* region.

Near horizon region : In order to analyse the flow near the horizon, we define a new radial coordinate ρ given by

$$\rho = \frac{1}{r_0} - r. \tag{3.45}$$

In turn, the blackening factor can be written as

$$f = (d_i + z - \theta) r_0 \rho + \mathcal{O}(\rho^2). \tag{3.46}$$

Thus in the near horizon region, the flow equation can be approximated as

$$\partial_\rho \zeta_{\rm nh}(\rho,\omega) = -i \left[16\pi G_N \frac{r_0^{d_i+2-z-\theta}}{d_i+z-\theta} \times \frac{(1+(d_i+3-z-\theta)r_0\rho)}{\rho} \zeta_{\rm nh}(\rho,\omega)^2 - \frac{\omega^2}{16\pi G_N} \frac{r_0^{-d_i-z+\theta-2}}{d_i+z-\theta} \times \frac{(1-(d_i+z-\theta+1)r_0\rho)}{\rho} \right]. \tag{3.47}$$

An ansatz consistent with a hydrodynamics description may be written as

$$\zeta_{\rm nh}(\rho,\omega) = \zeta_{\rm nh}^{(0)}(\rho) + \omega \zeta_{\rm nh}^{(1)}(\rho) + \omega^2 \zeta_{\rm nh}^{(2)} + \mathcal{O}(\omega^3). \tag{3.48}$$

It is clear from (3.47) that the second term on the RHS affects only at $\mathcal{O}(\omega^2)$. Also, the boundary condition (3.42) implies that $\zeta_{\rm nh}^{(0)}(0) = \zeta_{\rm nh}^{(2)}(0) = 0$. The resulting equation for $\zeta_{\rm nh}^{(0)}(\rho)$ is given by,

$$\partial_\rho \zeta_{\rm nh}^{(0)}(\rho) = -16 i\pi G_N \frac{r_0^{d_i+2-z-\theta}}{d_i+z-\theta} \times \frac{(1+(d_i+3-z-\theta)r_0\rho)}{\rho} \zeta_{\rm nh}^{(0)}(\rho)^2. \tag{3.49}$$

Which on solving naively yields a solution of the form $-\frac{i}{c_1+\mathcal{A}\rho+\mathcal{B}\log\rho}$ where $\mathcal{A}$ and $\mathcal{B}$ are constants depending on r_0, d_i, z and θ while c_1 is an arbitrary constant which remains unfixed even after imposing the relevant boundary condition for $\zeta^{(0)}_{\rm nh}(\rho)$. This is because the very boundary condition (3.42) is specified at a singular point of the equation. We can however choose a cutoff surface at $\rho = \epsilon$ (which can be thought of as a *stretched membrane*) hovering at a distance ϵ outside the real horizon at $\frac{1}{r_0}$ where $\zeta^{(0)}_{\rm nh}(\epsilon) = 0$ which then implies

$$\zeta^{(0)}_{\rm nh}(\rho) = 0, \tag{3.50}$$

identically in the *near horizon region*. This in turn leads to the simple equation at $\mathcal{O}(\omega)$ i.e.

$$\partial_\rho\zeta^{(1)}_{\rm nh}(\rho) = 0. \tag{3.51}$$

The above along with (3.42) implies

$$\zeta^{(1)}_{\rm nh}(\rho) = \frac{r_0^{\theta-d_i-2}}{16\pi G_N}. \tag{3.52}$$

Eventually, the equation at $\mathcal{O}(\omega^2)$ is given by

$$\partial_\rho\zeta^{(2)}_{\rm nh}(\rho) = -\frac{i(d_i+2-\theta)}{8\pi G_N(d_i+z-\theta)}r_0^{-d_i-z+\theta-1}. \tag{3.53}$$

The solution to the above equation consistent with the boundary condition (3.42) is

$$\zeta^{(2)}_{\rm nh}(\rho) = -\frac{i(d_i+2-\theta)}{8\pi G_N(d_i+z-\theta)}r_0^{-d_i-z+\theta-1}\rho. \tag{3.54}$$

Thus in the *near horizon region*, we have,

$$\zeta_{\rm nh}(\rho,\omega) \approx \omega\frac{r_0^{\theta-d_i-2}}{16\pi G_N} - \frac{i\omega^2(d_i+2-\theta)}{8\pi G_N(d_i+z-\theta)}r_0^{-d_i-z+\theta-1}\rho. \tag{3.55}$$

Using (3.39) and (3.45), we see that in the *near horizon region*, we can write,

$$\chi_{a_i}(r,\omega) \approx \frac{r_0^{\theta-d_i-2}}{16\pi G_N} - \frac{i\omega(d_i+2-\theta)}{8\pi G_N(d_i+z-\theta)}r_0^{-d_i-z+\theta-1}\left(\frac{1}{r_0}-r\right). \tag{3.56}$$

Clearly, χ_{a_i} being a constant at leading order exhibits trivial RG flow and is thus qualitatively similar to χ_{xi}. However, one must note that this is true only in the near horizon region.

Near boundary region : In this regime, we can approximate the blackening factor $f(r) \approx 1$ which simplifies (3.40) to

$$\partial_r\zeta_{a_i}(r,\omega) = i\left[16\pi G_N r^{z+\theta-d_i-3}\zeta^2_{a_i}(r,\omega) + \frac{k^2 r^{d_i-z-\theta+1}}{16\pi G_N} - \omega^2\frac{r^{d_i+z-\theta+1}}{16\pi G_N}\right]. \tag{3.57}$$

Assuming a series expansion in ω of the form

$$\zeta_{a_i}(r,\omega) \xrightarrow{r\to 0} \zeta^{(0)}_{\rm bdy}(r) + \omega\zeta^{(1)}_{\rm bdy}(r) + \omega^2\zeta^{(2)}_{\rm bdy}(r) + \mathcal{O}(\omega^3), \tag{3.58}$$

we see that the leading order satisfies an equation of the form

$$\partial_r\zeta^{(0)}_{\rm bdy}(r) = i\left[16\pi G_N r^{z+\theta-d_i-3}\zeta^{(0)}_{\rm bdy}(r)^2 + \frac{k^2 r^{d_i-z-\theta+1}}{16\pi G_N}\right], \tag{3.59}$$

whose solution is given by

$$\zeta^{(0)}_{\rm bdy}(r) = \frac{i r^{d_i+2-z-\theta}}{32\pi G_N}\left[(d_i+3z-\theta-2)\frac{c_1 - r^{-(d_i+3z-\theta-2)}}{c_1 + r^{-(d_i+3z-\theta-2)}} - (d_i+2-z-\theta)\right]. \tag{3.60}$$

Assuming reality of the gauge field (A.3) i.e. $z > 1$ the null energy condition (A.5) implies $d_i + z - \theta > 0$ which in turn implies $d_i + 3z - \theta - 2 = (d_i + z - \theta) + 2(z-1) > 0$. Thus, near the boundary,

$$\lim_{r\to 0}(d_i+3z-\theta-2)\frac{c_1 - r^{-(d_i+3z-\theta-2)}}{c_1 + r^{-(d_i+3z-\theta-2)}} - (d_i+2-z-\theta) = -2(d_i+z-\theta). \tag{3.61}$$

Clearly, when $d_i + 2 - z - \theta > 0$, $\zeta^{(0)}_{\rm bdy}(r) \to 0$ as $r \to 0$, however for $d_i + 2 - z - \theta < 0$, we see a divergent solution as $r \to 0$ while it goes to a constant as $r \to 0$ when $z = d_i + 2 - \theta$. In fact, due to the functional form of the solution (3.60), its limit as $r \to 0$ will be independent of the constant c_1 which will remain unfixed for any Dirichlet condition imposed at the boundary. Hence, for $z > d_i + 2 - \theta$ it seems such a *hydrodynamic* description for the gauge field response function will simply breakdown near the boundary.

Starting with a AdS_{d_i+3} dimensional boosted black brane, performing a boost and taking an appropriate double scaling limit involving the boost parameter and horizon radius yields the so-called AdS_{d_i+3} plane wave. Subsequently reducing along x^+ and identifying $x^- \equiv t$ yields (2.1) where the Lifshitz exponent z and hyperscaling violating exponent θ are related by [38]

$$z = \frac{d_i+4}{2} \quad \text{and} \quad \theta = \frac{d_i}{2}. \tag{3.62}$$

Clearly, from the above expressions it follows that $z = d_i + 2 - \theta$. This is precisely the point in the (z,θ) parameter space where we see the leading behaviour of ζ_{a_i} near the boundary is a constant. From the viewpoint of the AdS_{d_i+3} boosted black brane, this is suggestive that the hydrodynamic analysis for such effective theories obtained as null reductions break down. However, a concrete understanding of this breakdown would require further detailed analysis concerning the stability of such spacetimes which we plan to carry out in subsequent works.

It is interesting to notice that this condition was recovered in earlier works [28,34]. In particular, [34] studied QNM modes in the black brane background given by (2.1). As described earlier in Sect. 3, the gauge invariant combination (3.1) has a solution given by (3.2) up to first order in the hydrodynamic expansion for $z < d_i + 2 - \theta$. For $z = d_i + 2 - \theta$, the first order term develops a logarithmic scaling while it diverges near the boundary when $z > d_i + 2 - \theta$. Our current analysis suggests it is the behaviour of the perturbations in the background gauge field i.e. a_i near the boundary which is presumably the cause of this divergence. Thus, the RG analysis seems to be suggestive of the fact that it is the hydrodynamic expansion of a_i which breaks down causing its response function to yield an *unphysical* answer when $z > d_i + 2 - \theta$. Further, we should contrast this with [26,28] which were near-horizon analysis, also led to the same restriction on the Lifshitz exponent z. In our current analysis, the divergence seem to occur in the boundary theory as $r \to 0$. Earlier work [22] studied hvLif solutions as solutions to theories with higher derivative corrections. Null energy conditions and stability criteria led to certain regions in the (z, θ) parameter space that were identified as *physically allowed*. The criteria that we obtain above i.e. $z < d_i + 2 - \theta$ seems to be an independent bound which cannot be obtained by NECs or stability criteria.

The first order equation is given by

$$\partial_r \zeta^{(1)}_{\text{bdy}}(r) = 32 i\pi G_N r^{z+\theta-d_i-3} \zeta^{(0)}_{\text{bdy}}(r) \zeta^{(1)}_{\text{bdy}}(r), \tag{3.63}$$

which has a solution of the form

$$\zeta^{(1)}_{\text{bdy}}(r) = c_2 \frac{r^{2(d_i+z-\theta)}}{(1 + c_1 r^{d_i+3z-\theta-2})^2}. \tag{3.64}$$

Owing to the null energy condition (A.5) and reality of the gauge fields, which implies $z > 1$, we see that

$$\lim_{r\to 0} \zeta^{(1)}_{\text{bdy}}(r) = 0, \tag{3.65}$$

which leaves the constant c_2 which remains unfixed. Finally, the equation governing the second order contribution is given by

$$\partial_r \zeta^{(2)}_{\text{bdy}}(r) + \frac{2[2(z-1)c_1 r^{d_i+3z-\theta-2} - (d_i + z - \theta)]}{r(c_1 r^{d_i+3z-\theta-2} + 1)} \zeta^{(2)}_{\text{bdy}}(r) + i\left(\frac{r^{2z-1}}{16\pi G_N} - 16\pi G_N c_2^2 \frac{r^{3d_i+5z-3\theta-3}}{(c_1 r^{d_i+3z-\theta-2} + 1)^4}\right) = 0, \tag{3.66}$$

whose solutions are listed in Appendix C. From (3.39), it follows that the response function associated to a_i near the boundary is given by

$$\chi_{a_i}(r,\omega) \approx \frac{i r^{d_i+2-z-\theta}}{32\pi G_N \omega} \times\left[(d_i + 3z - \theta - 2)\frac{c_1 - r^{-(d_i+3z-\theta-2)}}{c_1 + r^{-(d_i+3z-\theta-2)}} - (d_i + 2 - z - \theta)\right] + c_2 \frac{r^{2(d_i+z-\theta)}}{(1 + c_1 r^{d_i+3z-\theta-2})^2} + \mathcal{O}(\omega). \tag{3.67}$$

The response function χ_{a_i} at leading order exhibits non-trivial dependence on the radial coordinate and thus shows a very distinct behaviour compared to the response function χ_{xi}.

4 Discussion and conclusion

In this paper, we have studied and analysed the RG flow equations governing the shear response in hvLif theories from the holographic viewpoint. The presence of $U(1)$ gauge fields along with a dilaton complicate the analysis significantly since certain gauge field perturbations i.e. a_i couples to the shear tensor modes h_{xi} and h_{ti}. Focusing on the $q = 0$ sector, our central observations are:

- The shear viscosity at the leading order seems to saturate the KSS bound for all values of z and θ. Earlier works failed to make any statement about shear viscosity for $z > d_i + 2 - \theta$. This analysis gets around that issue of breakdown of hydrodynamic expansion for $z > d_i + 2 - \theta$. However, for the special value of $z = d_i - \theta$, we see a very interesting logarithmic correction at the first order. This does not violate the KSS bound but, necessitates the introduction of a UV cutoff to control potential divergences at the boundary. This particular logarithmic behaviour of the subleading correction to shear viscosity for $z = d_i - \theta$ seems to be a novel feature. Further, as discussed in previous works [40,41], dimensional reduction of null deformed AdS_5 results in $z = 2$ Lifshitz theories (they have $\theta = 0$) and is consistent with $z = d_i - \theta = 2$. Given this observation, it is natural to ask if such logarithmic correction for $z = d_i - \theta$ can be explained from the perspective of the higher dimensional null deformed AdS_5 theory. It will be interesting to further explore the hydrodynamics of theories dual to such null deformed background.
- In the response function for a_i, we observe non-trivial flow even at leading order in χ_{a_i}. We have performed the analysis in the near horizon and the near boundary region with appropriate approximations. In the near horizon region, the qualitative behaviour of χ_{a_i} seems to mimic

that of χ_{xi}. However, the near boundary analysis reveals a leading behaviour which scales as $\chi_{a_i} \sim r^{d_i+2-z-\theta}$. The response function happens to be convergent provided $z < d_i + 2 - \theta$. Thus, it seems this bound obtained in earlier works [26,28,34] can be interpreted as a regularity condition on the response function of the gauge field perturbations a_i. Earlier works constructed a linear combination involving all the perturbation modes which obfuscated the source of this constraint. Our present analysis seems to suggest that it is the gauge field perturbations exclusively which are responsible for the constraint $z < d_i + 2 - \theta$.

An aside on Markovianity index: At this point one can ask for a more *physical origin* for the constraints observed in this paper. In other words, we want to understand if the breakdown of the hydrodynamic expansion for a certain parameter range, namely $z > d_i + 2 - \theta$ has a more deeper origin or is simply a bug of these non-relativistic gravity duals. Towards that vein one perform a *Markovianity index* analysis of the perturbations in the probe limit in the spirit of [49]. To be more elaborate, [49] studied probes couples to conserved currents in an AdS-Schwarzschild background. The effective coupling of the probe field is characterized by a single parameter, namely the *Markovianity index* $\mathcal{M}$. Probe fields with $\mathcal{M} > -1$ exhibits short-lived memory and behave analogous to the massive scalar probes. Probes with $\mathcal{M} \leq -1$, however, retain long-term memory. In the current context, the metric perturbations we study are coupled to conserved current i.e. the stress tensor.

More precisely, [49] starts from the effective action of a probe scalar of the form

$$\mathcal{S}_{eff} = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}r^{d-1-\mathcal{M}}\nabla^A\phi_{\mathcal{M}}\nabla_A\phi_{\mathcal{M}} + S_{bdy}, \tag{4.1}$$

describing a massless Klein-Gordon field minimally $\phi_{\mathcal{M}}$ minimally coupled to gravity with metric being same as (2.1) and $\mathcal{M} \in \mathbb{R}$ being some designer parameter modulating the coupling. To reiterate more concretely, With this designer scalar the central observation of [49] is that the scalar probe field $\phi_{\mathcal{M}}$ is Markovian if $\mathcal{M} > -1$ or else its non-Markovian. Written in terms of the usual Fourier modes the scalar field equation takes the form in the zero momentum limit[1]

$$\phi''_{\mathcal{M}} + \left(-\frac{\mathcal{M}}{r} + \frac{f'}{f} + i\frac{2\omega}{rf}\right)\phi'_{\mathcal{M}} - i\frac{\omega\mathcal{M}}{r\,f}\phi_{\mathcal{M}} = 0. \tag{4.2}$$

Comparing (4.2) with (2.6) in the limit $\omega \to 0$ one can check that in this case the designer parameter turns out to be

$$\mathcal{M} = d_i + z - \theta - 1.$$

Interestingly, this implies that constraining the perturbations to be Markovian also forces the probe to obey the null energy condition (A.5). In other words

$$\mathcal{M} > -1 \quad \Longrightarrow \quad d_i + z - \theta \geq 0.$$

The situation with (2.5) or (2.8) is much more complicated due to the coupling between the fields. One can simplify the situation by considering the near boundary region for (2.5) where $f(r) \sim 1$. In this regime, a comparison between (4.2) and (2.5) in the limit $\omega \to 0$ reveals[2]

$$\mathcal{M} = d_i - z - \theta + 1.$$

Again imposing the Markovianity condition $\mathcal{M} > -1$, we interestingly have

$$z < d_i + 2 - \theta,$$

which is exactly the limit that we have obtained through the gauge field perturbations. Therefore the regularity condition of $z < d_i + 2 - \theta$., analysed explicitly in the present analysis and observed earlier in [26,28,34] can also be attributed to the fact of the probes being Markovian. The above calculations although rudimentary seems to be hinting towards a connection between Markovianity index of the fluctuations and the breakdown of hydrodynamic expansion. An elaborate investigation of this issue is beyond the scope of this paper, which we hope to address in future works.

Our strategy of analysing the near horizon and near boundary regions separately opens up some possible new directions in the hydrodynamics of hvLif theories. Our analysis is restrictive in the sense that we analysed the $q = 0$ sector only. A natural extension will be to understand the $q \neq 0$ sector and check if one recovers any new transport coefficient at linear order in q. Another interesting question will be to explore if regularity conditions imposed on response functions of higher order transport coefficients leads to any further constraint on the Lifshitz exponent z. We are looking forward to analysing the flow equations both analytically and numerically to comment on higher order transport coefficients which are yet unexplored in the literature. We subsequently plan on studying the RG flow of response functions that arise in the sound channel and scalar channel.

[1] With $q \neq 0$ the equation becomes

$$\phi''_{\mathcal{M}} + \left(-\frac{\mathcal{M}}{r} + \frac{f'}{f} + i\frac{2\omega}{rf}\right)\phi'_{\mathcal{M}} - \left(\frac{q^2}{f} + i\frac{\omega\mathcal{M}}{rf}\right)\phi_{\mathcal{M}} = 0.$$

[2] Near the boundary with $\omega \to 0$ and $q \to 0$ Eq. (2.5) becomes

$$H''_{ti} + \frac{z + \theta - d_i - 1}{r}H'_{ti} = 0,$$

where the term containing a'_i gets dropped due to (3.3).

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Appendix A: Reviewing hyperscaling violating Lifshitz spacetimes

The metric (2.1) is a solution to the Einstein–Maxwell-dilaton action

$$S = -\frac{1}{16\pi G_N^{(d+1)}}\int d^{d+1}x\,\sqrt{-G}\times\left[R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{Z(\phi)}{4}F_{\mu\nu}F^{\mu\nu}+V(\phi)\right], \quad (A.1)$$

where the various fields and parameters appearing in the action are listed as follows:

$$\phi = \sqrt{2(d_i-\theta)(z-\theta/d_i-1)}\,\log r, \quad (A.2)$$

$$A_t = \frac{\alpha f(r)}{r^{d_i+z-\theta}}, \quad \alpha = -\sqrt{\frac{2(z-1)}{d_i+z-\theta}}, \quad A_i = 0. \quad (A.3)$$

$$V(\phi) = (d_i+z-\theta)(d_i+z-\theta-1)r^{-\frac{2\theta}{d_i}};$$
$$Z(\phi) = r^{\frac{2\theta}{d_i}+2d_i-2\theta} = e^{\lambda\phi}. \quad (A.4)$$

The null energy conditions following from (2.1) give constraints on the Lifshitz z and hyperscaling violating θ exponents

$$(z-1)(d_i+z-\theta)\geq 0, \quad (d_i-\theta)(d_i(z-1)-\theta)\geq 0. \quad (A.5)$$

Varying with $\bar{g}_{\mu\nu}$, $\bar{A}_\mu$ and $\bar{\phi}$, we obtain the following equations of motion,

$$\bar{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - \bar{g}_{\mu\nu}\frac{V(\bar{\phi})}{d-1} + \frac{Z(\bar{\phi})}{2}\bar{g}^{\rho\sigma}\bar{F}_{\rho\mu}\bar{F}_{\sigma\nu} - \frac{Z(\bar{\phi})}{4(d-1)}\bar{g}_{\mu\nu}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}, \quad (A.6)$$

$$\nabla_\mu(Z(\bar{\phi})\bar{F}^{\mu\nu}) = 0, \quad (A.7)$$

$$\frac{1}{\sqrt{-\bar{g}}}\partial_\mu(\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\partial_\nu\bar{\phi}) + \frac{\partial V(\bar{\phi})}{\partial\bar{\phi}} - \frac{1}{4}\frac{\partial Z(\bar{\phi})}{\partial\bar{\phi}}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma} = 0. \quad (A.8)$$

Note that from (A.6) it follows that:

$$\bar{R} = \bar{R}_{\mu\nu}\bar{g}^{\mu\nu} = \frac{1}{2}\partial_\rho\bar{\phi}\partial^\rho\bar{\phi} - \frac{d+1}{d-1}V(\bar{\phi}) + \frac{Z(\bar{\phi})}{2}\bar{g}^{\rho\sigma}\bar{F}_\rho{}^\lambda\bar{F}_{\sigma\lambda} - \frac{Z(\bar{\phi})(d+1)}{4(d-1)}\bar{F}^2. \quad (A.9)$$

Alternatively, we can write (A.6) as:

$$\begin{aligned}\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} &= \frac{1}{2}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - \bar{g}_{\mu\nu}\frac{V(\bar{\phi})}{d-1} + \frac{Z(\bar{\phi})}{2}\bar{g}^{\rho\sigma}\bar{F}_{\rho\mu}\bar{F}_{\sigma\nu} - \frac{Z(\bar{\phi})}{4(d-1)}\bar{g}_{\mu\nu}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}\\ &\quad -\frac{1}{2}\bar{g}_{\mu\nu}\left[\frac{1}{2}\partial_\rho\bar{\phi}\partial^\rho\bar{\phi} - \frac{d+1}{d-1}V(\bar{\phi}) + \frac{Z(\bar{\phi})}{2}\bar{g}^{\rho\sigma}\bar{F}_\rho{}^\lambda\bar{F}_{\sigma\lambda} - \frac{Z(\bar{\phi})(d+1)}{4(d-1)}\bar{F}^2\right]\\ \Rightarrow\ \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} &= \frac{1}{2}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} - \frac{1}{4}\bar{g}_{\mu\nu}\partial_\rho\bar{\phi}\partial^\rho\bar{\phi} + \frac{Z(\bar{\phi})}{2}\bar{F}_{\rho\mu}\bar{F}^\rho_\nu - \frac{Z(\bar{\phi})}{8}\bar{g}_{\mu\nu}\bar{F}^2 + \frac{V(\bar{\phi})}{2}\bar{g}_{\mu\nu}. \quad (A.10)\end{aligned}$$

Appendix B: Perturbations to hvLif spacetimes

The perturbed action up to second order terms is given by

$$\begin{aligned}S^{(2)} = \frac{-1}{16\pi G_N}\int dr\, d^dk\Big[&\mathcal{A}(r)h''_{ti}h_{ti} + \tilde{\mathcal{A}}(r)h''_{xi}h_{xi} + \mathcal{B}(r)h'^2_{ti} + \tilde{\mathcal{B}}(r)h'^2_{xi} + \mathcal{C}(r)h_{ti}h'_{ti}\\ &+\tilde{\mathcal{C}}(r)h_{xi}h'_{xi} + \mathcal{D}(r)h^2_{ti} + \tilde{\mathcal{D}}(r)h^2_{xi} - g(r)h_{ti}a'_i + \mathcal{H}(r,q)h^2_{ti} + \tilde{\mathcal{H}}(r,\omega)h^2_{xi}\\ &+2\mathcal{J}(r,q,\omega)h_{ti}h_{xi} + \mathcal{M}(r)a'^2_i + \mathcal{N}(r,q,\omega)a^2_i\Big],\end{aligned} \quad (B.1)$$

where the various functions appearing in the action is given by:

$$\mathcal{A}(r) = -2r^{3-d_i+z+\frac{\theta}{d_i}(d_i-4)}; \quad \tilde{\mathcal{A}}(r) = 2fr^{5-d_i-z+\frac{\theta}{d_i}(d_i-4)}$$
$$\mathcal{B}(r) = -\frac{3}{2}r^{3-d_i+z+\frac{\theta}{d_i}(d_i-4)}; \quad \tilde{\mathcal{B}}(r) = \frac{3}{2}fr^{5-d_i-z+\frac{\theta}{d_i}(d_i-4)}$$
$$\mathcal{C}(r) = \left[3d_i - 8 - 3z + \frac{12\theta}{d_i} - 3\theta - \frac{d_i+z-\theta}{f}\right]$$

$$\times r^{2-d_i+z+\frac{\theta}{d_i}(d_i-4)}$$

$$\tilde{\mathcal{C}}(r) = \left[-2d_i - 2z + 2\theta + 14f - \frac{12\theta}{d_i} f\right] r^{4-d_i-z+\frac{\theta}{d_i}(d_i-4)}$$

$$\mathcal{D}(r) = \Bigg[-2 + 3d_i - \frac{d_i^2}{2} + (d_i - 3)z - \frac{z^2}{2} + \frac{5-d_i}{d_i} z\theta + \left(-8 + \frac{10}{d_i} + d_i\right)\theta + \left(\frac{5}{d_i} - \frac{10}{d_i^2} - \frac{1}{2}\right)\theta^2 + \frac{1}{f}\Bigg((d_i - 1)(d_i + z) + \frac{2-d_i}{d_i} z\theta - \frac{2-d_i}{d_i}\theta^2 + (3 - 2d_i)\theta\Bigg) + \frac{1}{f^2}\left(-\frac{d_i^2}{2} - zd_i - \frac{z^2}{2} + z\theta + d_i\theta - \frac{\theta^2}{2}\right)\Bigg]$$

$$\times r^{1-d_i+z+\frac{\theta}{d_i}(d_i-4)}$$

$$\tilde{\mathcal{D}}(r) = \left[-3d_i - 3z + 6\theta + \frac{3z\theta}{d_i} - \frac{3\theta^2}{d_i} + 10f - \frac{20\theta}{d_i} f + \frac{10\theta^2}{d_i^2} f\right] r^{3-d_i-z+\frac{\theta}{d_i}(d_i-4)}$$

$$g(r) = (d_i + z - \theta)\alpha r^{2-\frac{2\theta}{d_i}};$$

$$\mathcal{H}(r, q, \omega) = \frac{q^2}{2} \frac{r^{3-d_i+z+\frac{\theta}{d_i}(d_i-4)}}{f};$$

$$\tilde{\mathcal{H}}(r, q, \omega) = \frac{\omega^2}{2} \frac{r^{3-d_i+z+\frac{\theta}{d_i}(d_i-4)}}{f}$$

$$\mathcal{J}(r, q, \omega) = \frac{\omega q}{2} \frac{r^{3-d_i+z+\frac{\theta}{d_i}(d_i-4)}}{f} \quad ;$$

$$\mathcal{M}(r) = -\frac{1}{2} f r^{d_i+3-z-\theta}$$

$$\mathcal{N}(r, q, \omega) = \frac{\omega^2}{2} \frac{r^{d_i+1+z-\theta}}{f} - \frac{q^2}{2} r^{d_i+3-z-\theta}.$$

The modes $h_{ti}(t, r, x)$, $h_{xi}(t, r, x)$ and $a_i(t, r, x)$ form a decoupled set of equations along with a constraint equation which can be solved perturbatively for every $x_i \in \{x_2, \ldots, x_{d_i}\}$ and $x \equiv x_1$.

Appendix C: Solution of ζ_{a_i} at second order near boundary

As demonstrated earlier, the equation governing the second order correction to ζ_{a_i} near the boundary is given by (3.66). The generic solution to this equation is given by

$$\begin{aligned}
\zeta^{(2)}_{\text{bdy}}(r) &= c_3 \frac{r^{2(d_i+z-\theta)}}{(1 + c_1 r^{d_i+3z-\theta-2})^2} - i \frac{5r^{d_i+z-\theta+2}}{16\pi G_N (d_i + z - \theta - 2)(1 + c_1 r^{d_i+3z-\theta-2})^2} \\
&\quad - i c_1 \frac{r^{2(d_i+2z-\theta)}((d_i + 5z - \theta - 2) + c_1 z r^{d_i+3z-\theta-2})}{16\pi G_N\, z(d_i + 5z - \theta - 2)(1 + c_1 r^{d_i+3z-\theta-2})^2} + i \frac{r^{2(d_i+z-\theta)}\left[3r^2 - 3c_1^2 r^{2(d_i+3z-\theta-1)} + 128\pi^2 G_N^2\, c_2^2 r^{2(d_i+2z-\theta-1)}\right]}{8\pi G_N (d_i + 3z - \theta - 2)(1 + c_1 r^{d_i+3z-\theta-2})^3} \\
&\quad + \frac{6i}{8\pi G_N} \frac{z r^{d_i+z-\theta+2}\, {}_2F_1[1, \frac{2+\theta-d_i-z}{d_i+3z-\theta-2}; \frac{2z}{d_i+3z-\theta-2}; -c_1 r^{d_i+3z-\theta-2}]}{(d_i + z - \theta - 2)(d_i + 3z - \theta - 2)(1 + c_2 r^{d_i+3z-\theta-2})^2} \\
&\quad + \frac{3i}{8\pi G_N} \frac{2zc_1^2}{(d_i + 3z - \theta - 2)(d_i + 5z - \theta - 2)} \frac{r^{3d_i+7z-3\theta-2}}{(1 + c_1 r^{d_i+3z-\theta-2})^2} {}_2F_1\left[1, \frac{d_i + 5z - \theta - 2}{d_i + 3z - \theta - 2}; \frac{2(d_i + 4z - \theta - 2)}{d_i + 3z - \theta - 2}; -c_1 r^{d_i+3z-\theta-2}\right],
\end{aligned} \tag{C.1}$$

which is valid when $d_i + z - \theta \neq 2$.

When $d_i + z - \theta = 2$ and $z \neq 2$, we have the solution

$$\zeta^{(2)}_{\text{bdy}}(r) = \frac{r^4\left(-\frac{256 i\pi G_N c_2^2}{c_1^2 z r^{2z} + c_1 z} - \frac{i r^{2z-4}\left(c_1 r^{2z}\left(\frac{c_1 r^{2z}}{3z-2} + \frac{1}{z-1}\right) + \frac{1}{z-2}\right)}{\pi G_N} + 32\kappa_1\right)}{32\left(\text{c1} r^{2z} + 1\right)^2}. \tag{C.2}$$

When $z = 2$ and $d_i = \theta$, we recover the solution

$$\zeta^{(2)}_{\text{bdy}}(r) = \frac{r^4\left(16\kappa_2 - \frac{i\left(\frac{64\pi^2 G_N^2 c_2^2}{c_1^2 r^4 + c_1} + \frac{c_1^2 r^8}{8} + \frac{c_1 r^4}{2} + \log(r)\right)}{\pi G_N}\right)}{16\left(c_1 r^4 + 1\right)^2}. \tag{C.3}$$

All of them vanish in the limit $r \to 0$ (near boundary) thus leaving the constants c_3, κ_1 and κ_2 unfixed.

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