



On particle dynamics near the singularity inside the Schwarzschild black hole and T-spheres

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Abstract The problem of the speed of the objects inside the Schwarzschild black hole is considered. The general result is that the value of the relative speed of the objects following their non-zero angular momentum trajectories, both of geodesic and non-geodesic character, when approaching the ultimate singularity, tends to the value of speed of light. There is only one exception when both objects move in the same plane and have parallel angular momenta. This outcome appears to have a deeper sense: it reflects the anisotropic character of the dynamics of interior of this particular black hole. The result in question means that near the singularity, collisions of two particles lead to an indefinitely large energy in the center of mass frame. Aforementioned properties have their counterpart in the phenomenon of an indefinitely large blueshift near the singularity. Thus the angular momentum of a particle turns out to be an important feature that affects the final behavior of particle near the singularity. Motivated by this fact, we generalize the Lemaître frame under the horizon in such a way that reference particles themselves have nonzero angular momentum. Our results apply not only to the Schwarzschild singularity but also to other space-like ones for which the scale factor $g \rightarrow \infty$. We also analyze another type of singularity for which the circumference radius vanishes but g remains finite.

1 Introduction

The studies of the properties of the strong gravitational fields and in particular the properties of the black holes (BH) have

got in a recent decade a significant not only theoretical but also experimental impact. The first ever picture of the BH namely, supermassive M-87 BH [1], gravitational waves emission following two BHs merger [2] and the intriguing temporarily varying radiation emission of the accretion disk of the sources of the strong gravitational field [3–6] are the most important recent experimental aspects of the presence of the strong gravitational fields. The theoretical studies of black holes have been also continuing. Among other interesting things, a special role is played by the so-called Bañados-Silk-West (BSW) effect: under certain conditions, two-particle collisions undergoing in the vicinity of the BH's horizon would lead to the unbounded energy release [7].

Meanwhile, there exists another singled out space-time region. This is the vicinity of the singularity. For the naked singularity it was shown that high energy collisions are indeed possible there (see, e.g. [8]). However, in the case of black holes it is hidden beyond the horizon. We show that a similar effect near the singularity happens even without rotation, i.e. for the Schwarzschild black hole. Such collisions can, in principle, change the fate of a black hole and potentially lead to a new type of the horizon instability due to backreaction of particles on metric.

For spherically symmetric space-times, there exists also another type of space-like singularity, apart from the Schwarzschild one. It occurs in the metric of so-called T-spheres [9]. The geometry near the singularity is highly anisotropic both for the Schwarzschild metric and T-sphere but in the first case one of two scale factors tends to infinity whereas the second one vanishes. For T-spheres one of two scale factors remains finite. Collisions of massive particles is the counterpart of red/blue shift inside a black hole when corresponding particles are massless (photons). It leads to a bright ring around a singularity (see page B-25 of [10]). A

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careful analytical study of red/blue shift for such massless particles have been recently done in [11]. In doing so, the crucial role of angular momenta reveals itself that changes overall picture radically as compared to a pure radial motion (see e.g. [12, 13]).

In recent paper [14] a general radially freely falling frames are described and the formula for 3-velocity with respect to these frames have been derived. If one particle is comoving with the frame in question, the considered 3-velocity of the second particle is just a relative velocity of such particles. If we are going further and describe a mutual motion of two particles, both having non-zero angular momentum, we can proceed in two ways. First, we can use known formulae for a radially falling frame and apply them to both particles, after that extract their mutual motion from this information. In such a way it is reasonable to use the simplest possible choice for the frame. A natural choice is the static frame. We should however mention that a velocity with respect to this frame has particular properties, for example, the radial velocity of a particle following geodesic and approaching the horizon tends to that of light, $V \rightarrow 1$, but after crossing the horizon the speed turns out to be decreasing to zero [15], as if the test particle being hampered inside horizon.

It is worth noting that this non-monotonic behavior of a 3-velocity is a specific property of the observers, static outside and resting inside horizon [15] and does not appear in the frame connected with Lemaître coordinates. Indeed, it was shown that the 3-velocity with respect to the Lemaître frame at a horizon can take any value from 0 to 1, and this velocity decreases in a monotonic way (if it is not equal to zero identically) in the Schwarzschild black hole reaching 0 at a singularity (if the angular momentum is zero) [16]. When we are interested in the asymptotics near a singularity the behavior of the velocity near horizon does not matter, so we use the static frame.

As for the second possibility, we can consider a frame connected with the first particle, thus, falling non-radially. This frame is convenient when we are interested not only in asymptotic but in the overall particle motion under the horizon. This requires generalization of formulae from [14] for non-radially falling frames. Thus in the present paper we make a next step in which we generalize the results of the previous paper and get the most general formulae for velocity with respect to an arbitrary freely falling frame.

The paper is organized as follows. In Sect. 2 we define the line element and the tetrad for resting observer inside Schwarzschild BH and we apply them (Sect. 3) for description of the kinematics of a freely falling test particle. In Sect. 4 the anisotropy of the this space-time is described. Particle collisions are analyzed in the following three sections: Sect. 5—general setup, Sect. 6—in-plane collisions, Sect. 7—collisions of particles moving in different planes. In Sects. 8 and 9 the effects of action of external force are

described. In Sect. 10 we analyze another type of singularity when the scale factor in the longitudinal direction remains finite. In Sect. 11 we generalize the Lemaître frame to include reference particles with nonzero angular momentum. In Sect. 12 we compare properties of local velocity with respect to a frame with those of non-local velocity, defined as a time derivative of a proper distance between two different points. Discussion and final remarks are presented in Sect. 13.

2 Metric, tetrad

Let us consider the black hole metric

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\omega^2. \quad (1)$$

Here, $f(r_+) = 0$, where r_+ is the radius of the event horizon. Our main concern is the Schwarzschild metric for which $f = 1 - \frac{r_+}{r}$, though we need not to fix f for most of our results. Inside the horizon, the mutual role of temporal, t and spatial, r , coordinates interchanges. We can choose $T = -r$, $y = t$, where $-r_+ \leq T \leq 0$, $-\infty < y < \infty$ [17]. Then

$$ds^2 = -\frac{dT^2}{g} + g dy^2 + T^2 d\omega^2, \quad (2)$$

where $g = -f$.

In what follows, it is convenient to use the tetrad attached to a resting observer with constant spatial, y, θ, ϕ coordinates. Such an observer follows a geodesic that has no analogue in the outer part of space-time [18]. Namely, in the coordinates (T, y, θ, ϕ)

$$\hat{h}_{(0)\mu} = -\frac{1}{\sqrt{g}}(1, 0, 0, 0), \quad (3)$$

$$\hat{h}_{(1)\mu} = (0, \sqrt{g}, 0, 0), \quad (4)$$

$$\hat{h}_{(2)\mu} = (0, 0, |T|, 0)$$

$$\hat{h}_{(3)\mu} = (0, 0, 0, |T| \sin \theta). \quad (5)$$

3 Motion of a free particle

Outside the horizon there exists the time-like Killing vector that corresponds to the time translation and it leads to the conservation of a particle's energy. Inside the horizon it becomes a space-like one and this leads to the conservation of the y -component of momentum. The angular momentum is conserved everywhere.

Then, the four-velocity u^μ of a particle that moves within the plane $\theta = \frac{\pi}{2}$ has the form in the coordinate system (2)

$$u^\mu = \left(P, -\frac{P}{g}, 0, \frac{L}{T^2} \right), \quad (6)$$

where

$$P = \sqrt{p^2 + g \left(\frac{L^2}{T^2} + 1 \right)}, \tag{7}$$

$p = -u_y$ is the specific conserved momentum (the sign minus is chosen to keep the maximum similarity with the region outside the horizon), $L = u_\phi$ is the specific, conserved angular momentum.

The corresponding tetrad components read

$$u^{(a)} = u^\mu h_\mu^{(a)} = \left(\frac{P}{\sqrt{g}}, -\frac{p}{\sqrt{g}}, 0, \frac{L}{|T|} \right). \tag{8}$$

Then, one can obtain (see Sect. 8 in [19]) that the three-velocity has components

$$V^{(1)} = -\frac{p}{P}, \tag{9}$$

$$V^{(3)} = \frac{L\sqrt{g}}{|T|P}. \tag{10}$$

The absolute value of the velocity $V = \sqrt{(V^{(1)})^2 + (V^{(3)})^2}$,

$$V = \sqrt{1 - \frac{g}{P^2}}. \tag{11}$$

The Lorentz gamma factor

$$\gamma = \frac{1}{\sqrt{1 - V^2}} = \frac{P}{\sqrt{g}}. \tag{12}$$

For given p and L , the velocity under discussion obeys the condition

$$\frac{V^2}{1 - V^2} = \frac{p^2}{g} + \frac{L^2}{T^2}. \tag{13}$$

If the singularity is being approached, $T \rightarrow 0, g \rightarrow \infty$.

Then, if $L \neq 0$, we have

$$|V| \approx 1 - \frac{1}{2} \left(\frac{T}{L} \right)^2. \tag{14}$$

In doing so, $V^{(1)} \rightarrow 0, V^{(3)} \rightarrow \pm 1$.

If $L = 0, V^{(3)} = 0$. In the limit under discussion $V^{(1)} \rightarrow 0$.

Thus, in any case $V^{(1)} \rightarrow 0$.

4 Geometry and dynamics

The above result can be given the following geometric interpretation. The space-time described by the line element (2) may be referred to as a T -“sphere” [9]. It has got some particular properties: it is non-static, homogeneous, finite in time extent. It has a hypercylindrical a space-like section $V^3 = R^1 \times S^2$ with no symmetry center, open ($-\infty < y < \infty$) in radial, homogeneity direction R^1 . This may be regarded as an anisotropic cosmological model,

expanding longitudinally and contracting transversely in a two-sphere S^2 of radius $|T|$ (see also [18–20]). Expansion along y axis is finally getting extremely violent: all of the objects are carried away in such a manner that their “own” speeds are getting negligible - they are finally in a relative rest. And that is the meaning of the first of the results of the former section: the resting (in y axis) observer measures the speed of the test object travelling along this axis as diminishing to zero, $V \rightarrow 0$, as is seen from (13) when $L = 0$ and the singularity is approached, so $g \rightarrow \infty$.

If the velocity vector of a test object has got a transverse (to y axis) component, i.e. its angular momentum is non-zero, $L \neq 0$, it is also carried away transversely due to the transverse contraction. This transverse contraction is of critical character: hypercylinder V^3 collapses to the line as the radius of the two-sphere tends to zero. All of the massive and massless particles are carried away in the following manner. The value of the speed of the massive test objects as measured by resting observers goes to that of light, $V \rightarrow 1$, and the light rays (massless test objects) are perceived by the resting observer as indefinitely blueshifted [11].

Then, the following interesting question arises: what is relative speed of the two observers depending on their angular momenta? If particles collide, whether their energy in the center of mass frame remains finite or grows indefinitely? In particular, it concerns particles travelling with (a) parallel, (b) antiparallel angular momenta. These questions are considered below.

5 Particle collisions: general setup

Now, we consider collisions of two particles of masses m_1 and m_2 and briefly analyze the behavior of the energy $E_{c.m.}$ in the center of mass at the point of collision. By definition,

$$E_{c.m.}^2 = -P_\mu P^\mu, \tag{15}$$

where $P^\mu = m_1 u_1^\mu + m_2 u_2^\mu$ is the total four-momentum. Then,

$$E_{c.m.}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma_{12}, \tag{16}$$

where w has the meaning of the relative speed, the Lorentz factor of relative motion

$$\gamma_{12} = -u_{1\mu} u_2^\mu = \frac{1}{\sqrt{1 - w^2}} \tag{17}$$

should not be confused with the individual gamma factor of each particle (12).

Below, we discuss two cases separately.

6 Particles move in the same plane

Then, it follows from (6) and (17) that

$$\gamma_{12} = \frac{P_1 P_2 - p_1 p_2}{g} - \frac{L_1 L_2}{T^2}, \tag{18}$$

It is instructive to describe collisions in terms of kinematic characteristics. One can define the angle ψ between particles 1 and 2 according to

$$\cos \psi = \frac{\vec{V}_1 \vec{V}_2}{V_1 V_2}, \tag{19}$$

where $\vec{V}_1 \vec{V}_2 = V_1^{(1)} V_2^{(1)} + V_1^{(3)} V_2^{(3)}$. Then, it follows from (9), (10) that

$$\cos \psi = \frac{1}{\sqrt{p_1^2 + g \frac{L_1^2}{T^2}} \sqrt{p_2^2 + g \frac{L_2^2}{T^2}}} \left(p_1 p_2 + \frac{L_1 L_2 g}{T^2} \right), \tag{20}$$

$$\gamma_{12} = \gamma_1 \gamma_2 (1 - \cos \psi). \tag{21}$$

Our main concern is the behavior of γ_{12} near the singularity. For fixed L_1, L_2 , the absolute velocity of each particle in the limit when the singularity is approached, can take only two values: either $V = 0$ or $V = 1$ [14, 19]. Below, we enumerate different sub-cases separately depending on the angular momentum of each particle.

A $L_1 = 0 = L_2$

Then,

$$P = \sqrt{p^2 + g}. \tag{22}$$

For $g \rightarrow \infty$ we have

$$\gamma_{12} \approx 1 + \frac{w^2}{2}, \tag{23}$$

where

$$w \approx \frac{|p_1 - p_2|}{\sqrt{g}} \rightarrow 0. \tag{24}$$

Also,

$$V_1 \rightarrow 0, V_2 \rightarrow 0, \tag{25}$$

$$\cos \psi \rightarrow \text{sign}(p_1 p_2), \tag{26}$$

If a particle entered the interior of the horizon from its exterior, $p > 0$. If it entered from the left (mirror) region, $p < 0$. Thus in a physically relevant case when both particles came from infinity, $\psi \rightarrow 0$.

B $L_1 = 0, L_2 = L \neq 0$

$$\gamma_{12} \approx \left| \frac{L}{T} \right| \rightarrow \infty, \tag{27}$$

$$w^2 \approx 1 - \frac{T^2}{L^2} \rightarrow 1, \tag{28}$$

$$V_1 \rightarrow 0, V_2 \rightarrow 1, \tag{29}$$

$$\cos \psi \rightarrow 0. \tag{30}$$

C $L_1 L_2 > 0$

$$\gamma_{12} \rightarrow \frac{L_1^2 + L_2^2}{L_1 L_2}, \tag{31}$$

$$w \rightarrow \frac{|L_1^2 - L_2^2|}{L_1^2 + L_2^2} < 1, \tag{32}$$

$$V_1 \rightarrow 1, V_2 \rightarrow 1, \tag{33}$$

$$\cos \psi \approx 1 - \frac{T^2 (p_1 L_2 - p_2 L_1)^2}{g L_1^2 L_2^2}. \tag{34}$$

Particles move almost parallel to each other near the singularity.

D $L_1 L_2 < 0$

$$\gamma_{12} \approx 2 \frac{|L_1 L_2|}{T^2} \rightarrow \infty, \tag{35}$$

$$w \approx 1 - \frac{T^4}{4L_1^2 L_2^2} \rightarrow 1, \tag{36}$$

$$V_1 \rightarrow 1, V_2 \rightarrow 1, \tag{37}$$

$$\cos \psi \rightarrow -1. \tag{38}$$

This means that head-on collision occurs, $\psi \rightarrow \pi$.

Now, we can summarize the results of the present section in Table 1.

7 Particles move within different planes

As is well-known, in the case of conserved angular momentum a particle moves within a plane. According to above consideration, we can choose this plane to be $\theta = \frac{\pi}{2}$ for, say, particle 1. However, in general, this is not the case for particle 2, the variable θ will be varying in time. Will it significantly affect the results for the relative velocity and Lorentz factor γ_{12} near the singularity? To answer this question, we generalize the results of the previous section. Omitting the details of derivation, we give the corresponding formulas below. Now,

$$u^\mu = \left(P, -\frac{p}{g}, \frac{\sigma Q}{T^2}, \frac{L}{T^2 \sin^2 \theta} \right), \tag{39}$$

where $\sigma = \pm 1$,

$$Q = \sqrt{L_{tot}^2 - \frac{L^2}{\sin^2 \theta}}, \tag{40}$$

$$P = \sqrt{p^2 + g \left(1 + \frac{L_{tot}^2}{T^2} \right)}, \tag{41}$$

it is implied that

$$L_{tot} \geq \frac{|L|}{\sin \theta}. \tag{42}$$

Here, the integral of motion L_{tot} has the meaning of the total angular momentum of a particle, while L is its component corresponding to a variable ϕ . Then, for $V^{(a)} = (V^{(1)}, V^{(2)}, V^{(3)})$ one finds

$$V^{(a)} = \left(-\frac{p}{P}, \frac{\sigma Q \sqrt{g}}{|T| P}, \frac{L \sqrt{g}}{|T| P \sin \theta} \right), \tag{43}$$

Equation (12) is still valid but now with (41). Obviously,

$$V_{\perp} = \sqrt{(V^{(2)})^2 + (V^{(3)})^2} = \frac{\sqrt{g}}{|T| P} L_{tot}. \tag{44}$$

We assume that for particle 1 $\theta = \frac{\pi}{2}$, $Q_1 = 0$, $L_{1tot} = |L_1|$. Then, in the point of collision both particles have the same coordinates, so $\theta = \frac{\pi}{2}$ for particle 2 as well. It is convenient to introduce an angle α for particle 2. so that $L_2 = L_{tot} \cos \alpha$, where $\cos \alpha$ can have any sign. Then, in the point of collision we have for particle 2

$$u^{\mu} = \left(P, -\frac{p}{g}, \frac{L_{2tot} \sin \alpha}{T^2}, \frac{L_{2tot} \cos \alpha}{T^2} \right). \tag{45}$$

Equations (18) and (21) are also valid but in P the quantity L_{tot} appears instead of L .

Now,

$$\cos \psi = \frac{1}{\sqrt{p_1^2 + g \frac{L_1^2}{T^2}}} \frac{1}{\sqrt{p_2^2 + g \frac{L_{2tot}^2}{T^2}}} \left(p_1 p_2 + \frac{L_1 L_2 g}{T^2} \right), \tag{46}$$

$$\gamma_{12} = \frac{P_1 P_2 - p_1 p_2}{g} - \frac{L_1 L_2}{T^2}. \tag{47}$$

When a singularity is approached, $V^{(1)} \rightarrow 0$ as before, while $V_{\perp} \rightarrow 1$, so $V \rightarrow 1$ as well. Let us denote the cases A-D depending on the L_1, L_2 in the manner similar to that in the former section. Then, one can see that cases A and B coincide with those from Table 1. Indeed, if one of angular momenta is zero, one can choose the equatorial plane for another particle to be $\theta = \frac{\pi}{2}$, so nothing new happens. Obviously, case D is similar to that from Table 1. It remains to check what happens in case C. Then,

$$\cos \psi \rightarrow \frac{L_2}{L_{2tot}} = \cos \alpha, \tag{48}$$

$\psi = \alpha$. Taking into account that $V_1 \rightarrow 1$ and $V_2 \rightarrow 1$, we see that according to (21), in case C a new possibility arises:

$$\gamma_{12} \approx \frac{|L_1| (L_{2tot} - L_2)}{T^2}, \tag{49}$$

so $\gamma_{12} \rightarrow \infty$ in spite of $L_1 L_2 > 0$. Such a possibility was absent when both particles had been moving within the same

Table 1 Types of particles collisions near the singularity

	L_1	L_2	w	ψ
A	0	0	0	0
B	0	$\neq 0$	1	$\frac{\pi}{2}$
C	$\neq 0$	$\neq 0, L_2$ parallel to L_1	Separated from 1	0
D	$\neq 0$	$\neq 0, L_2$ antiparallel to L_1	1	π

plane (see Table 1 above). The similar phenomenon for massless particles was discussed in [21].

8 Motion under the action of force

Let now some force act on a particle. Then, the equations of motion formally retain their form but the quantities p and L cease to be integrals of motion and become the functions of time. If there is an acceleration a^{μ} , one finds its tetrad components using (3)–(5) that (assuming $\theta = \frac{\pi}{2}$)

$$a^{(3)} = |T| a^{\phi} = \frac{a_{\phi}}{|T|}, \tag{50}$$

$$a^{(y)} = \sqrt{g} a^y = \frac{a_y}{\sqrt{g}}, \tag{51}$$

$$a^{(\hat{t})} = -\frac{a^T}{\sqrt{g}} = a_T \sqrt{g}. \tag{52}$$

If ξ^{μ} is the Killing vector, it is easy to notice that

$$\frac{d}{d\tau} (\xi^{\mu} u_{\mu}) = \xi^{\mu} a_{\mu}. \tag{53}$$

Then,

$$\frac{dp}{d\tau} = a_y = \sqrt{g} a^{(y)}, \tag{54}$$

$$\frac{dL}{d\tau} = a_{\phi} = |T| a^{(3)}, \tag{55}$$

where we used the same definitions $p = -u_y$ and $L = u_{\phi}$ as for free particles. Now,

$$\frac{p^2}{g} - \frac{(u^T)^2}{g} + \frac{L^2}{T^2} = -1, \tag{56}$$

where

$$u^T = \frac{dT}{d\tau} = \sqrt{p^2 + g(1 + \frac{L^2}{T^2})} = P. \tag{57}$$

It follows from equations of motion that

$$\frac{dp}{dT} = -\frac{dp}{dr} = \frac{\sqrt{g} a^{(y)}}{\sqrt{p^2 + g(1 + \frac{L^2}{T^2})}}, \tag{58}$$

$$\frac{dL}{dT} = -\frac{dL}{dr} = \frac{|T| a^{(3)}}{\sqrt{p^2 + g(1 + \frac{L^2}{T^2})}}. \tag{59}$$

It is clear from (58), (59) that p and L remain finite, if $a^{(y)}$ and $a^{(3)}$ are finite.

This has important consequences for the properties of velocities. In particular, in the tetrad (3)–(5) $V^{(1)} \rightarrow 0$ and $V^{(3)} \rightarrow \pm 1$, if $L \neq 0$ ($V^{(3)} = 0$ for $L = 0$). These conclusions are valid for any finite E, L [14], so they apply to the case under discussion as well. Therefore, we come to an important conclusion: the presence of finite force does not abolish the effect of high energy collision. This statement is the counterpart of similar results for the BSW effect near the horizon [22].

9 When particle velocity can approach the speed of light

It follows from the above consideration that eq. (13) indeed retains its validity, if the constants of motion p and L are replaced by their momentary values $p(T)$ and $L(T)$. In turn, this has an important consequence. For a finite acceleration, the velocity can reach the limiting value $V = 1$ only in two cases: when approaching the horizon and/or singularity. In the first case, the right hand side of (13) diverges due to the first term where $g \rightarrow 0$. In the second one it does so due to the second term where $T \rightarrow 0$.

All these conclusions are obtained with the assumption that $a_{(i)}$ are finite and hence p and L are finite as well. If we relax the requirement of finiteness of $a_{(i)}$, an additional possibility opens that $V \rightarrow 1$ due to unbounded acceleration and, correspondingly, unbounded p and L . Thus there are three possibilities for getting $V \rightarrow 1$: (i) horizon, (ii) singularity, (iii) infinite acceleration.

This result is valid for the velocities with respect to the Lemaître frame as well. Moreover, it is valid with respect to a general radially free falling system formed by particles with the specific energy e_0 . It is known that in such a general case the radial component of velocity of a particle with specific energy e is given by (see [14])

$$V^{(1)} = \frac{P_0 e - P e_0}{e_0 e - P P_0} \tag{60}$$

and the angular component is

$$V^{(3)} = \frac{\mathcal{L} f}{r(e_0 e - P P_0)}, \tag{61}$$

where

$$P_0 = \sqrt{e_0^2 - f}. \tag{62}$$

Formulae of the present paper for the components of velocity of an individual particle can be thought of as a particular case $e_0 = 0$ (corresponding to a resting observer with $y = const$ under the horizon) of these general formulae. Assuming that e is finite, we see that $V^{(3)} < 1$ as it should be and $V^{(3)} \rightarrow 1$ at singularity. As for the radial component,

substituting P and P_0 into the condition $V^{(1)} = 1$ and considering finite e we get, after a simple algebra, that $f = 0$. This means that $V^{(1)}$ can take the value 1 at the horizon only, provided e is finite. It is worth noting that this result is valid for any spherically symmetric static space-times since we do not specify the function f . The coordinate system $e_0 = 0$ considered here becomes singular at the horizon itself. The discussion contained in Sect. 6 of [14] explains, why $V^{(1)} = 1$ for other systems, regular at a horizon, is safe from physical point of view (note, that unlike a singularity we have equality, not a limit here!).

10 Another types of space-like singularity: finite g

Up to now, we considered the singularity of the Schwarzschild type. This implied that in the limit when $T = -r \rightarrow 0$, the scale factor $g \rightarrow \infty$. In this section, we consider what happens if g remains finite there. In particular, this case is encountered in so-called T-models [9]. Note, that we still assume a metric without inner horizons. Now, in (18) the only potential source of divergences is the term with T in the denominator. Analyzing this expression and (9, 10), we arrive at the following conclusions.

Consider motion of two particles within the same plane. If a particle has $L = 0$, $|V^{(1)}| \rightarrow \frac{|p|}{\sqrt{p^2 + g(0)}} < 1$, $V^{(3)} \equiv 0$.

This differs from the singular case where $V^{(1)}$ tends to zero. However, if $L \neq 0$, $|V^{(1)}| \rightarrow 0$, $V^{(3)} \rightarrow \pm 1$ as in the singular case. Correspondingly, if $L_1 = 0 = L_2$, γ_{12} is finite (though does not equal to 1 in the case when particles have different values of p). If $L_1 = 0$, $L_2 \neq 0$, $\gamma_{12} \rightarrow \infty$. If $L_1 L_2 < 0$, $\gamma_{12} \rightarrow \infty$ as well. If $L_1 L_2 > 0$, γ_{12} is finite. Straightforward calculation shows that the formula (32) for a relative velocity changes quantitatively:

$$w = \frac{|L_2^2(1 - p_1/g(0)) - L_1^2(1 + p_2/g(0))|}{L_2^2(1 + p_1/g(0)) + L_1^2(1 + p_2/g(0))}. \tag{63}$$

In the limit $g(0) \rightarrow \infty$ Eq. (63) turns into Eq. (32) as it should be. Comparing with Table 1, we see that there is no qualitative difference between the Schwarzschild case and the one under discussion.

11 Generalizing Lemaître reference frame

In the above consideration, we mentioned the Lemaître frame that is used sometimes for the description of the black hole interior. Its standard use implies that reference particles compose the set of observers free falling from infinity from the state of rest, so their specific energy $e_0 = 1$. Meanwhile, this frame can be generalized if the role of reference particles is played by observers with $e_0 \neq 0$. This is done in [14] where

even cases with $e_0 < 0$ and $e_0 = 0$ were included in a general scheme. This leads to formulae like (60) and (10). Now, we make the next step and build the generalized Lemaître frame based on particles with nonzero specific angular momentum L_0 . For brevity, we call it L -Lemaître frame. It is relevant, in particular, under the horizon where, as we saw, radial particle motion is unstable, so if we want to know how dynamics looks like from the viewpoint of a falling particle, it is natural to use the L -Lemaître frame. Below, we list main formulas for it. In principle, in order to describe mutual motion of two freely falling particle, we can use a standard frame and perform the Lorentz rotation, as we did when we got the singularity asymptotic for mutual velocity. Considering the L -Lemaître frame is an alternative and more direct way to get the results.

As consideration runs along the same lines as above and in [14], we give the main formulas briefly.

We choose the reference particle to move in the plane $\theta = \frac{\pi}{2}$ and restrict ourselves by the region under the horizon. We attach to it a zeroth vector of a tetrad and choose the other three orthogonal to it. In coordinates (T, y, θ, ϕ)

$$u_0^\mu = h_{(0)}^\mu = \left(P_0, -\frac{p_0}{g}, 0, \frac{L_0}{T^2} \right), \tag{64}$$

$$h_{(0)\mu} = \left(-\frac{P_0}{g}, -p_0, 0, L_0 \right). \tag{65}$$

Here, the parameters of a reference particle have the subscript 0,

$$P_0 = \sqrt{p_0^2 + g \left(1 + \frac{L_0^2}{r^2} \right)}. \tag{66}$$

Also,

$$h_{(1)\mu} = \frac{1}{\sqrt{1 + \frac{L_0^2}{r^2}}} \left(\frac{p_0}{g}, P_0, 0, 0 \right), \tag{67}$$

$$h_{(2)\mu} = |T| (0, 0, 1, 0), \tag{68}$$

$$h_{(3)\mu} = \left(-\frac{P_0 L_0}{g |T| \sqrt{1 + \frac{L_0^2}{r^2}}}, -\frac{p_0 L_0}{|T| \sqrt{1 + \frac{L_0^2}{r^2}}}, 0, r \sqrt{1 + \frac{L_0^2}{T^2}} \right). \tag{69}$$

Then, using the general definition [23]

$$V^{(i)} = V_{(i)} = -\frac{h_{(i)\mu} u^\mu}{h_{(0)\mu} u^\mu}, \tag{70}$$

we have for a freely falling particle

$$V^{(1)} = \frac{1}{\gamma g \sqrt{1 + \frac{L_0^2}{T^2}}} (P p_0 - P_0 p), \tag{71}$$

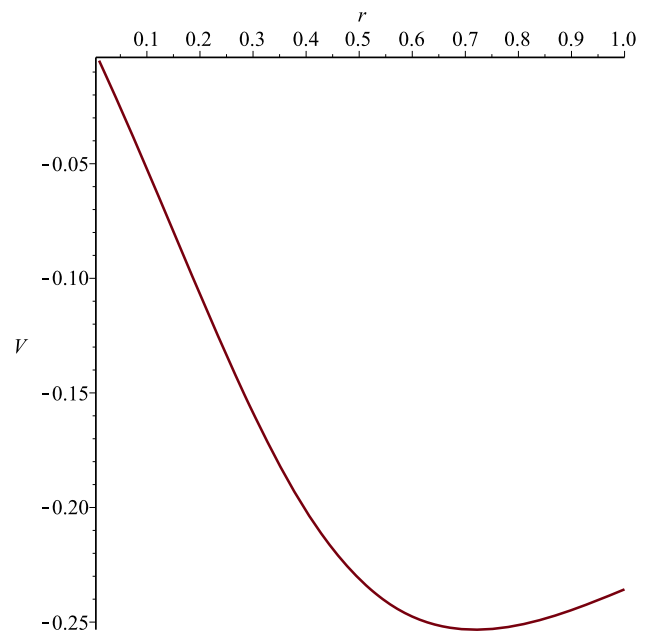


Fig. 1 The angular component $V^{(3)}$ of velocity of a particle with $L = m^2$ and $e = 2$ with respect to the frame with $L_0 = m^2$ and $e_0 = 1$. The unit for $V^{(3)}$ is c

$$V^{(2)} = \frac{\sigma Q}{\gamma |T|} \tag{72}$$

$$V^{(3)} = \frac{1}{\gamma |T| \sqrt{1 + \frac{L_0^2}{T^2}}} \left(\frac{L}{\gamma} - L_0 \right). \tag{73}$$

where γ is the individual Lorentz gamma factor of a particle

$$\gamma = -u_{(0)\mu} u^\mu. \tag{74}$$

It follows from (6), (39), (65) that

$$\gamma = \frac{P_0 P - p p_0}{g} - \frac{L L_0}{T^2}. \tag{75}$$

The quantity Q is given by Eq. (40) and contains L_{tot} .

These formulae give the most general expressions for 3-velocity components with respect to an arbitrary freely falling frame. At a singularity they reduce to the results obtained above.

It is worth to note that in the case of $L_0 = L_1$ and two equatorial planes coinciding, the angular component of velocity $V^{(3)}$ vanishes only in a singularity, being non-zero everywhere else (see Fig. 1), so the Eq. (32) is the limiting equation even if two angular moments are the same.

12 Motion with respect to a frame versus motion of nearby particles

Apart from the behavior of velocity with respect to a fixed frame, another interesting question is mutual movement of nearby points. The properties of such a motion can be very

different from the motion with respect to a fixed frame, even if one of the points considered is at rest with respect to the coordinate system in question. The reason is that the velocity with respect to a frame is a local entity, while a distance between two points is a space-like variable. In the present section we consider both local and non-local velocities. The goal of the section is mostly pedagogical since it contains a large part of known issues, though reinterpreted in our notations.

First, consider the radial motion. The velocity along the leg of a hypercylinder in metric (2) is known to decrease and vanish in a singularity [14, 19]. As for the distance to a nearby point, it increases as it can be clearly seen from the form of the metrics (1), (2). At the singularity the proper distance diverges. This means that a velocity of nearby (but anyway located in a different point!) particle should diverge if we consider “velocity” as a rate of change of a proper distance.

A reasonable way to define a velocity of a distant object is to consider the derivative of a proper distance with respect to a *proper* time of an observer. The metric (2) is not a synchronous one, so we introduce a proper time of an observer being at rest from the viewpoint of the metric (2) as

$$d\tilde{t} = dT/\sqrt{g}. \tag{76}$$

With this definition we have for the velocity in question

$$\frac{dl}{d\tilde{t}} = \frac{d(\sqrt{g}y)}{d\tilde{t}} = \frac{d\sqrt{g}}{d\tilde{t}}y + \sqrt{g}\frac{dy}{d\tilde{t}} = \frac{1}{2}\frac{dg}{dT}y + V^{(1)}, \tag{77}$$

since $\sqrt{g}dy/d\tilde{t} = \sqrt{g}(dy/dT)(dT/d\tilde{t}) = -p/(P) = V^{(1)}$, where Eqs. (6) and (9) were used. So that, we got a relation resembling the known one for cosmology – the overall radial velocity is decomposed into a non-local part, which can be rewritten as $(1/2\sqrt{g})(dg/dT)l$ being proportional to the proper distance $l = \sqrt{g}y$ to the particle (an analog of Hubble velocity in cosmology), and a local “peculiar velocity” $V^{(1)}$. The later term tends to zero in a singularity while the first term diverges (being a non-local entity, it is not bound by the speed of light). The picture is similar to the Big Rip cosmological singularity, apart from the fact that the Big Rip is isotropic.

In popular books, when describing influence of tidal forces to an unhappy observer falling freely into a black hole, authors usually illustrate the text by emotional pictures of an observer “spaghettified” in the direction toward a singularity. This has no sense in coordinates like T, y since (i) they are homogeneous inside a horizon, and (ii) a singularity, being space-like and in absolute future for an observer, is not present in any of an observer’s $T = const$ slices. By itself, “spaghettization” does occur but has another meaning: if we make a series of snapshots of cross-sections $T = const$ for different T , the object extends more and more when T grows approaching $T = 0$.

Another picture arises in the Lemaître coordinates. Singularity is present in the sections of constant Lemaître time, so the direction towards a singularity makes sense. Since for two radially separated particles the singularity occurs at different moments of the proper time τ along the trajectory, the separation between two points reaches its finite maximum when the “inner” particle hits a singularity (in this context the word “hits” is conditional since the singularity is space-like, we use it for brevity only).

It is easy to estimate this maximum for a pure radial motion. For the Schwarzschild black hole this can be a student seminar exercise. Suppose two particles, being at rest with respect to the Lemaître system are separated by some distance. Let a particle move with $E = m$, so it would start its motion from the rest at infinity. Then, it is known (see, e.g. Eq. 2.3.12 of [24]) that if a particle moves from some r to $r_f < r$, the proper time τ is equal to

$$\tau(r, r_f) = (2/3)r_g^{-1/2}(r^{3/2} - r_f^{3/2}). \tag{78}$$

In particular, the proper time between a given position r and the singularity $r = 0$ is obtained from (78) if we put there $r = 0$, so

$$\tau(r, 0) = (2/3)r_g^{-1/2}r^{3/2}. \tag{79}$$

It is also worth mentioning that for such a particle the Lemaître time coincides with the proper one (see, e.g. Eq. 14 of [19]). Also, the proper distance in this case is equal to the difference of the coordinate values of r .

Let us have two such particles initially separated by the coordinate distance l . We want to find the location r_f of the “outer” particle initially located at $r + l$, at the moment τ when the “inner” particle hits the singularity. Equating $\tau(r + l, r_f) = \tau(r, 0)$, assuming small l and expanding the right hand side with respect to l/r we get

$$r_f = [(3/2)l]^{2/3}r^{1/3}, \tag{80}$$

which gives for the ratio

$$r_f/l = (3/2)^{2/3}(r/l)^{1/3}. \tag{81}$$

Thus small absolute displacements remain small. However, relative displacement may be arbitrary large.

As a trilling example we can consider the following situation: suppose that different parts of human body ($l \sim 1m$) start to move geodesically after *tidal* acceleration g_t exceeds the free fall acceleration at the surface of Earth ($g_E \sim 10m/s^2 \sim 10^{-16}m^{-1}$ in natural units $c = 1$). For small l , $g_t \approx r_g l/r^3$ (see, e.g. page B-20 of [10]). Thus free fall begins at $r = (r_g l/g_E)^{1/3}$. Using these data we can estimate

$$r_f/l \sim 60r_g^{1/9}, \tag{82}$$

where r_g is expressed in meters. This indeed indicates “spaghettization” - the size in r -direction enlarges from about

100 times for a stellar mass black hole to about 1000 times for a supermassive (10^9 solar masses) one (note, that the dependence upon black hole mass is rather weak).

We can add also that in the Lemaître coordinates the coordinate velocity itself admit a decomposition

$$dr/d\tilde{\tau} = -v + V_{(r)} \tag{83}$$

where $\tilde{\tau}$ is the Lemaître time, $v = \sqrt{1 - f}$ and V_r is a local velocity with respect to the Lemaître frame (see the equation (54) in [19]). It should not to be confused with $V^{(1)}$ in (77) where the frame with $p = 0$ was used. Nevertheless, the asymptotic of the terms in the right hand side of this equation at a singularity are the same: $v \rightarrow \infty$ and $V_{(r)} \rightarrow 0$ (see corresponding formulae in [19]). On the other hand, in a direct analog of (77) the non-local flow velocity (the first term in the decomposition) is equal to difference of the values of v at the positions of the particle and the observer, and this difference is, in general, not factorizable, which means that it is no longer proportional to the proper distance between the particle and the observer.

For the motion in the angular direction the situation is quite opposite. Suppose we have a particle with a zero angular momentum, so it falls along $\phi = 0$ line, and a nearby particle does so with some small but non-zero L . We know that $V^{(3)}$ of the second particle tends to 1 when the singularity is approached. Does this mean that the proper distance between these two particles increase rapidly? The answer is “no” as the direct dependence $\phi(r)$ in the Schwarzschild metric shows (Fig. 2). In this picture we plot $V^{(3)}$ of a particle with $L = m^2$ inside a horizon. It tends to 1 near a singularity. In the same plot we show the distance from the line $\phi = 0$ to this particle (we assume that this particle crosses the line $\phi = 0$ at the horizon) which is equal to $r\phi = -T\phi$. This distance first increases due to non-zero L (as it would be in a flat space also), then it starts to decrease despite growing velocity $V^{(3)}$. The contraction in the angular direction overcomes, and the distance in the ϕ direction appears to be always smaller than it would be without gravity.

This picture is qualitatively the same in the Lemaître coordinates as well. The only difference is that $V^{(3)}$ in static coordinate always vanishes at a horizon (this is a counterpart of the statement that radial velocity is always 1 at a horizon), while the analog of this value with respect to the Lemaître system can take any value from 0 to 1.

As for the angle ϕ itself, it reaches a finite value at singularity. This value grows with growing L , tending to π for $L \rightarrow \infty$ (see Eq. 21 of [20]).

It is worthwhile to mention that static coordinates admit a “Hubble-like” decomposition for angular motion as well. Indeed, if the first particle has $\phi = 0$, then using Eqs. (76),

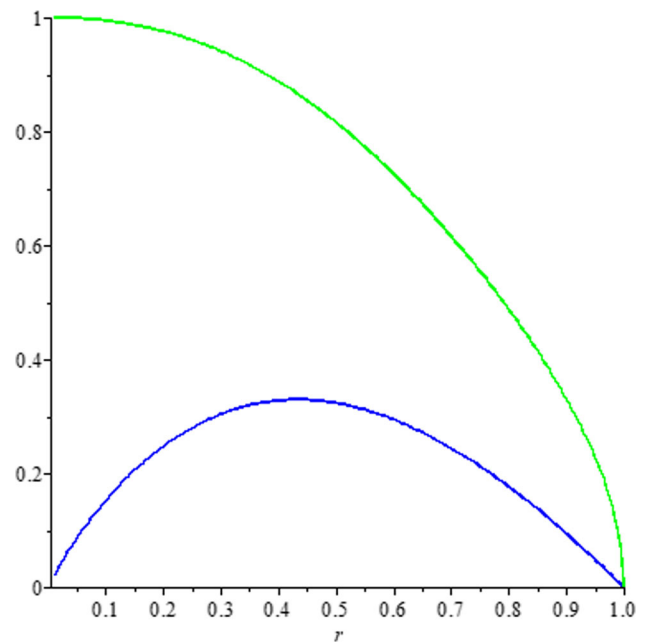


Fig. 2 The angular component $V^{(3)}$ of velocity of a particle with $L = m^2$ inside a horizon (green) and distance to the particle in angular direction from the radius $\phi = 0$ crossed by this particle at a horizon (blue). The unit for $V^{(3)}$ is c , the unit for the distance is r_g

(6) and (10), we have

$$\frac{dl}{d\tilde{t}} = -\frac{d(T\phi)}{d\tilde{t}} = -\sqrt{g}\phi + \frac{L\sqrt{g}}{TP} = -\sqrt{g}\phi + V^{(3)}, \tag{84}$$

so an analog of the “Hubble flow” is directed inwards, its velocity is proportional to ϕ of the second particle, and if $g(0)$ is singular, the corresponding velocity diverges. This means that it dominates near a singularity since $V^{(3)}$, being a physical velocity, is bounded by the speed of light while the first term does not.

This decomposition takes place only for the “static” coordinates (corresponding to the observer with $p = 0$, $y = const$), in the Lemaître coordinates the spatial sections are flat, so the distance in angular direction is $l = r \sin \phi$, and the above property is lost since the angular component of the peculiar velocity in the Lemaître frame does not depend on the value of ϕ (see [19]) while the second term in the decomposition of $dl/d\tilde{\tau}$ does.

13 Discussion and conclusions

In this paper we have considered dynamical phenomena in the vicinity of the singularity of the Schwarzschild-like space-time. This concerns not only the Schwarzschild metric but any singularity of the same type when $r \rightarrow 0$ and $g \rightarrow \infty$. In this case one can regard horizon’s interior as an anisotropic, dynamical V^4 space-time with a hypercylinder $V^3 = R^1 \times S^2$

space-like sections. There is a longitudinal, R^1 -expansion and transversal, S^2 -contraction. Due to extremely violent R^1 -expansion in its final stage one could expect the asymptotic state of mutual rest of all the particles moving along y -direction (see e.g. [20]). This picture has been completed by a Doppler's blueshift, for the case of transverse component of trajectories: a light-like, non-zero angular momentum geodesics have been recorded blueshifted [11]. We have verified here the kinematics of the test particles following time-like, non-zero angular momentum trajectories of both geodesic and non-geodesic character. If a test particle moves along an arbitrary non-zero angular momentum trajectory, then its speed as measured by resting observers, those with constant spatial coordinates, approaches that of light, $w \rightarrow 1$ as $T \rightarrow 0$. Previously, it was found that this is valid for geodesic trajectories [14, 19]. Now, we showed that this is valid for an arbitrary finite force. Moreover, the presence of a finite force is compatible with high energy collisions near the singularity.

If, instead of one particle, we take the two particles following non-zero angular momenta trajectories, their relative velocity $w \rightarrow 1$ with only one exceptional case. It occurs if both particles move in the same plane and have parallel angular momenta; then the value of their relative speed w is smaller than that of light, $w < 1$. This also happens if both particles have zero angular momenta. Otherwise, non-zero angular momentum of a test particle is a necessary and sufficient condition for $w \rightarrow 1$.

It should be pointed out that there exists the reason, common for both the indefinite blueshift for the class of non-zero angular momentum light-like geodesics and indefinite tendency of the relative speed of the particles following their non-zero angular momentum trajectories to the speed of light when approaching the ultimate singularity $T \rightarrow 0$ of Schwarzschild BH's interior. This effect is caused by a contraction in the course of highly anisotropic dynamics of space-time. Indeed, when approaching $T \rightarrow 0$, [9] the hypercylinder is critically contracting, i.e. the radius $|T|$ of the two-sphere, diminishes to the zero value, $T \rightarrow 0$. This critical contraction carries all of the objects, massive and massless, in such a way that the light recorded by a resting or moving along y -axis observer turns out to be indefinitely blueshifted and the speed of a test particle as measured by resting or moving along y axis observer tends indefinitely to the speed of light, $w \rightarrow 1$. When two colliding massive particles follow their non-zero angular momenta trajectories, then in general they experience head-on collision and their relative speed approaches that of light. (For motion within the same plane and the same directions of the angular momenta the effect is moderate: the relative speed of the colliding particles is found to be smaller than the speed of light, $w < 1$. This is an analogy of the finding in [11] where for motion within the

same plane and the same directions of the angular momenta of the observer and the light a finite blueshift is found).

The result $w \rightarrow 1$ may be regarded as a center of mass energy collision tending to infinity. This interpretation provides a particular perspective. All of the variety of the BSW effect, unbounded energy collisions in the vicinity of the black hole horizons, outer or inner, have lead to the conclusion about arbitrary large limit which, however, is not reached in any particular collision, so an infinite limit cannot be realized. This is called a principle of kinematic censorship [25]. Meanwhile, in the case under discussion this principle is violated when $T \rightarrow 0$ ($r \rightarrow 0$). This is probably quite natural since in the singularity itself all known laws of physics can be violated and geometry as such ceases to exist.

Since exact vanishing of angular momentum and exact coinciding of planes of motion for two particles represent zero-measure set of initial conditions and cannot be exactly satisfied in any realistic physical situation, we can conclude that tending w to 1 at a singularity is unavoidable. Correspondingly, indefinite growth of $E_{c.m.}$ is general feature for particle collisions near the singularity.

It is also shown for spherically symmetric space-times of a quite general form that a particle velocity can approach the speed of light only in three cases: (i) on the horizon, (ii) in the singularity, (iii) when a proper acceleration diverges.

We also considered another type of singularity when $g(0)$ is finite. This applies, for example to T-models of sphere that represent a special type of the Kantowski-Sachs metric. This includes the solution of for the collapse of dust complementary to the LTB models [9]. The comparison of particle dynamics near both types of singularity is carried out, it showed the possibility of high energy collisions in both cases.

We also generalized the Lemaître type of frame which is built from particles with nonzero angular momenta. It can be useful for description of particle flows under the horizon.

The most important and intriguing issue that arises from our finding is the question about stability or instability of the vacuum singularity if backreaction of matter is taken into account. This remains an open question.

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