



Purely virtual extension of quantum field theory for gauge invariant fields: Yang–Mills theory

Damiano Anselmi^{1,2,a}

¹ Dipartimento di Fisica “E.Fermi”, Università di Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy

² INFN, Sezione di Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy

Received: 8 March 2023 / Accepted: 15 June 2023 / Published online: 28 June 2023
© The Author(s) 2023

Abstract We extend quantum field theory by including purely virtual “cloud” sectors, to define physical off-shell correlation functions of gauge invariant quark and gluon fields, without affecting the S matrix amplitudes. The extension is made of certain cloud bosons, plus their anticommuting partners. Both are quantized as purely virtual, to ensure that they do not propagate ghosts. The extended theory is renormalizable and unitary. In particular, the off-shell, diagrammatic version of the optical theorem holds. We calculate the one-loop two-point functions of dressed quarks and gluons, and show that their absorptive parts are gauge independent, cloud independent and positive (while they are generically unphysical if the cloud sectors are not purely virtual). A gauge/cloud duality simplifies the computations and shows that the gauge choice is just a particular cloud. It is possible to dress every field insertion with a different cloud. We compare the purely virtual extension to previous approaches to similar problems.

1 Introduction

The greatest success of perturbative quantum field theory relies on the theory of scattering. However, quantum field theory is not just scattering processes. It is also off-shell correlation functions, including correlation functions of composite fields. Gauge invariant composite fields can be divided in two classes: those that are at least quadratic in the elementary fields, and those that contain linear terms. It is straightforward to build representatives of the first class, not equally easy to build composite fields of the second class. The latter are particularly important, because they provide a complete basis of observables and can eventually be used to replace the elementary fields altogether. In this paper we extend quan-

tum field theory in a way that overcomes this difficulty and preserves the fundamental physics.

Specifically, we add purely virtual “cloud” sectors to the Yang–Mills action, built by means of certain cloud fields and their anticommuting partners. The sectors are arranged so as to satisfy certain “cloud symmetries”, which ensure that the scattering amplitudes coincide with the usual ones, and the correlation functions of the ordinary fields are also unaffected. Each field insertion in a correlation function can be rendered gauge invariant by “dressing” it with an independent cloud. Each cloud is specified by a cloud function and a cloud Feddeev–Popov determinant.

To ensure that the extended theory is unitary and propagates no additional degrees of freedom, we quantize the clouds as purely virtual [1]. This way, the correlation functions of the dressed fields satisfy the off-shell, diagrammatic version of the optical theorem. Moreover, we show that the extended theory is renormalizable and polynomial in all the fields except for the basic cloud fields (which are dimensionless) and their anticommuting partners.

The extension is perturbative, and the expansion in powers of the gauge coupling g coincides with the expansion in the number of loops. Renormalizability is proved to all orders by means of an extended Batalin–Vilkovisky formalism and its Zinn–Justin master equations.

Note that the dressed fields we build are invariant under infinitesimal gauge transformations, but are not required to be invariant under global gauge transformations. This is indeed the way out to have physical non singlet states without violating unitarity.

If we wish, we can use the formalism developed here to downgrade the elementary fields (which are not gauge invariant) to mere integration and diagrammatic tools, and use the dressed fields (which are manifestly gauge invariant) everywhere. This way, we know from the start that everything we compute is manifestly gauge independent.

^a e-mail: damiano.anselmi@unipi.it (corresponding author)

As said, the purely virtual nature of the clouds ensures that no unwanted degrees of freedom are propagated. This opens the way to extract physical information from the off-shell correlation functions of the elementary fields in a systematic way. We illustrate the basic properties of the formalism by calculating the one-loop two-point functions of the dressed quarks and gluons, and showing that their absorptive parts are physical. When the clouds are not purely virtual (which occurs, for example, if we quantize the cloud sectors by means of the Feynman $i\epsilon$ prescription), the absorptive parts are generically unphysical.

A certain gauge/cloud duality, which is sometimes helpful to simplify the computations, shows that the usual gauge choice is ultimately nothing but a particular cloud, as long as the gauge trivial modes are rendered purely virtual. This suggests to use a “purely virtual gauge” as a valid alternative to the so-called physical gauges [2]. Among those, we mention the Coulomb gauge, the temporal gauge, the light-cone gauge and, more generally, the axial gauges. Normally, such gauges lead to mathematical complications. What they miss is the concept of pure virtuality, although in some cases (like the Coulomb gauge), they incorporate it by accident, so to speak. In our approach, we do not change the gauge fixing to make it physical. Rather, we make a gauge-fixing physical by changing the prescription we use to define it.

We compare our formalism with other approaches to similar issues available in the literature, putting particular emphasis on the “Coulomb” approaches, that is to say, the clouds defined by Dirac in QED [3] and those studied by Lavelle and McMullan in non-Abelian gauge theories [4–6]. Earlier definitions of gauge invariant variables in Yang–Mills theory are due to Chang [7, 8]. Different lines of thinking exist as well, such as the ’t Hooft approach, based on composite fields and a symmetry breaking mechanism [9], and the approach based on Wilson lines.

In a parallel paper [10], we explore similar issues in gravity.

We point out some physical applications of our results. The main one is the possibility of studying new types of scattering processes. As said, the usual S matrix amplitudes do not change, after the extension. Those amplitudes concern asymptotic states, which become free in the infinite past and in the infinite future. The correlation functions of dressed fields overcome this restriction, and allow us to define “short-distance scattering processes” among colored states of quarks and gluons, which are the processes where the incoming and outgoing states are not allowed (or not not have enough time) to become free. In the same spirit, we can study transition amplitudes in Yang–Mills theories on compact manifolds, merging the formalism of this paper with the one of [11], for situations where the experimental apparatus surrounding the physical process actively influences the process itself. See also [12] about this.

Experimental situations of this type are, among others, the interactions inside a quark gluon plasma, or the interactions between quarks and gluons at distances comparable to the proton radius, where we cannot use the notion of asymptotic state. The increasing precision of present colliders and the colliders of new generations make us hope that in a non distant future we can be less dependent on the paradigms of quantum field theory that have dominated the scene since its birth. The formalism of this paper breaks the main technical barriers for the undertaking of such studies, and is a first step towards devising feasible experiments.

The processes we have just mentioned are not, or do not need to be, on the mass shell. If treated with the usual approaches, they are gauge dependent, and unphysical. Our results imply that we can actually define them by means of dressed fields, as long as the dressings are purely virtual. The results we obtain are physical (i.e., gauge invariant and gauge independent – in addition, they obey the optical theorem), but depend on the dressing parameters, which we denote by $\tilde{\lambda}$. These parameters do not belong to the fundamental theory, but describe features of the experimental setup, such as experimental resolutions, finite volume effects, finite temperature effects, dependences on a background, or an external field, etc.

The $\tilde{\lambda}$ dependence of the results is not unexpected. Think, for example, of the correlation functions built by means of Wilson lines: they depend on the Wilson lines themselves. Another situation where the physical predictions depend on the details of the instrumentation is when the amplitudes are affected by infrared divergences, which are compensated by soft and collinear photons, gluons, or gravitons [13–16]. In those cases, the predictions depend on the energy resolution and the angular resolution. Something similar occurs, to some extent, when we observe unstable particles, like the muon [17], which do not admit asymptotic states in a strict sense.

The $\tilde{\lambda}$ dependencies mean that it is impossible to eliminate the influence of the observer on the observed phenomenon. Yet, this does not prevent us from making testable predictions. We can eliminate the $\tilde{\lambda}$ dependences by calibrating the instrumentation, i.e., by sacrificing a few initial measurements to determine the values of the parameters $\tilde{\lambda}$, after which everything is predicted uniquely, and can be confirmed or falsified experimentally.

It is also useful to point out the differences between the goals of our approach and the goals of other approaches to gauge theories that are available in the literature, such as the compensator field approach [18] and the Stueckelberg approach [19]. The first one is a rephrasing of the theory and its gauge symmetries, but does not change the cohomology of physical observables. The second one is used to describe massive vectors. Our purpose, instead, is to define “gauge-invariant gauge fields”, so to speak, that is to say,

colored physical states of quarks and gluons. This is possible by means of dressed fields.

The first difference between our approach and the compensator field approach is that, after introducing the extra sectors, we still define the physical observables as being gauge invariant: they are not required to be invariant under the extra (cloud) transformations. The dressed fields, which are indeed cloud dependent, are built on this premise. Once we have done that, we can consider new correlation functions (those that contain insertions of dressed fields) and study new scattering processes (the short distance processes mentioned above). These goals cannot be achieved in the compensator field approach. The correlation functions of ordinary gauge invariant composite fields (built without using the cloud sector), such as $\bar{\psi}\psi$, $F_{\mu\nu}F^{\mu\nu}$, etc., and well as the S matrix amplitudes, instead, do not change.

Since the dressed fields are just required to be gauge invariant, but not cloud invariant, the extra fields become propagating. Generically, this can be dangerous: if those fields are not treated properly, they may affect the observable spectra in undesirable ways. We show, by means of explicit calculations, that if they are quantized by means of the usual Feynman prescription, they inject ghosts into the theory. Since our definition of physical fields prevents us from getting rid of them cohomologically, we must achieve the goal in a different, non cohomological way: we make them purely virtual.

To make the whole construction work, we need to keep the usual sector and the cloud sectors to some extent separated. In particular, the functions that define the clouds should be gauge invariant, while the usual gauge-fixing functions should be cloud invariant. We show that these restrictions are consistent, because they are preserved by renormalization. Restrictions on the gauge-fixing choices are not unusual. A familiar one is adopted in the context of the background field method, where the gauge-fixing must be invariant under the background transformations.

We recall that purely virtual particles, also called fake particles, or “fakeons”, are defined by a new diagrammatics [1], which takes advantage of the possibility of splitting the usual optical theorem [20–25] into independent, algebraic spectral optical identities. Each identity is associated with a different (multi)threshold. By removing subsets of such identities, and projecting the whole theory to the physical subspace, certain degrees of freedom can be removed at all energies, while preserving unitarity and the optical theorem in a manifest way. The main application of this idea is the formulation of a consistent theory of quantum gravity [26], which is observationally testable due to its predictions in inflationary cosmology [27]. At the phenomenological level, fakeons evade common constraints that limit the employment of normal particles [28, 29].

Throughout the paper we work with the dimensional regularization [30–32], $\varepsilon = 4 - D$ denoting the difference

between the physical dimension and the continued one.

The paper is organized as follows. In Sect. 2 we give the basic definitions that are necessary to build the cloud sectors. In Sect. 3 we recall the standard Batalin–Vilkovisky formalism for gauge theories, and the Zinn–Justin master equation. In Sect. 4 we extend the formalism and the master equation to define the cloud sector. In Sect. 5 we show that the ordinary correlation functions of elementary and composite fields are unaffected by the cloud sector. In Sect. 6 we prove the same for the S matrix amplitudes. In Sect. 7 we build the correlation functions of the dressed fields. In Sect. 8 we prove that the cloud sector and the gauge-trivial sector are related by a certain duality relation. In Sect. 9 we add several copies of the cloud sector and show that each insertion in a correlation function can be dressed with its own, independent cloud. In Sect. 10 we define the absorptive parts and study their properties. In Sect. 11 we compute the two-point function of the dressed fermions at one loop with a covariant cloud and show that its absorptive part is unphysical. In Sect. 12 we overcome this difficulty by introducing purely virtual clouds. In Sect. 13 we repeat the analysis for the two-point function of the dressed gauge fields, and show that the absorptive part is physical, if purely virtual clouds are used. In Sect. 14 we prove that the extended theory is renormalizable, and show how the renormalization works in detail. In Sect. 15 we compare our approach with other approaches available in the literature. Section 16 contains the conclusions, while Appendix A contains the notation and some useful formulas. In Appendix B we prove that the dressed fields are unique. In Appendix C we study how the cloud independence goes through renormalization.

2 The cloud field, its anticommuting partner, and the dressed fields

In this section we lay out the basic notions that are needed to build the cloud sectors. For definiteness, we consider Yang–Mills theory with gauge group $G = SU(N_c)$ and quarks ψ in the fundamental representation.

The dressings can be easily worked out, once the theory contains a field $\hat{U}(x)$, with values in G , that transforms as

$$\delta_\Lambda \hat{U} = -ig\hat{U}\Lambda \quad (2.1)$$

under a gauge transformation. Here, $\Lambda(x) = \Lambda^a(x)T^a$ are the parameters of the transformation and T^a are the Hermitian matrices of the fundamental representation.

For example, if ψ is a fermion, the product $\hat{U}\psi$ is obviously gauge invariant, because (2.1) and the ψ transformation law $\delta_\Lambda \psi = ig\Lambda\psi$ imply $\delta_\Lambda(\hat{U}\psi) = 0$. Nevertheless, it is not convenient to use \hat{U} as an elementary field for the perturbative expansion, since we also need the inverse matrix \hat{U}^{-1} .

It is better to write $\hat{U} = e^{-ig\phi}$, and define $\phi = \phi^a T^a$ as the fundamental ‘‘cloud field’’.

The gauge transformation $\delta_\Lambda \phi$ of ϕ can be derived from (2.1). A version of the Campbell–Baker–Hausdorff formula reads

$$de^X = e^X \sum_{n=0}^\infty \frac{(-\text{ad}_X)^n}{(n+1)!} dX = e^X \frac{1 - e^{-\text{ad}_X}}{\text{ad}_X} dX \tag{2.2}$$

where $\text{ad}_X Y \equiv [X, Y]$, X and Y being matrices or operators. After rearranging the formula into the form

$$dX = \frac{\text{ad}_X}{1 - e^{-\text{ad}_X}} e^{-X} de^X,$$

we apply it with $\hat{U} = e^{-ig\phi}$, $X = -ig\phi$, $de^X \rightarrow \delta_\Lambda \hat{U}$, $d\phi \rightarrow \delta_\Lambda \phi$. This way, the desired ϕ gauge transformation is easily found. It reads

$$\begin{aligned} \delta_\Lambda \phi &= \frac{ig \text{ad}_\phi}{e^{ig \text{ad}_\phi} - 1} \Lambda \equiv R(\phi, \Lambda) \equiv T^a R^a(\phi, \Lambda) \\ &= \Lambda - \frac{ig}{2} [\phi, \Lambda] - \frac{g^2}{12} [\phi, [\phi, \Lambda]] + \mathcal{O}(g^3). \end{aligned} \tag{2.3}$$

For the moment, we define the cloud field ϕ as a field that transforms according to this law. In the next sections we introduce it at the level of the action, derive its Feynman rules and study its diagrammatic properties.

It is possible to check the closure of the transformation, i.e.,

$$[\delta_\Lambda, \delta_\Sigma] \phi = \delta_{-ig[\Lambda, \Sigma]} \phi, \tag{2.4}$$

where $-ig[\Lambda, \Sigma] = gT^a f^{abc} \Lambda^b \Sigma^c$.

The gauge-invariant dressed fields are

$$\begin{aligned} A_{\mu d} &= \hat{U} A_\mu \hat{U}^{-1} + \frac{i}{g} \hat{U} (\partial_\mu \hat{U}^{-1}) \\ &= e^{-ig \text{ad}_\phi} A_\mu - \frac{1 - e^{-ig \text{ad}_\phi}}{ig \text{ad}_\phi} (\partial_\mu \phi), \\ \psi_d &= e^{-ig\phi} \psi, \quad \bar{\psi}_d = \bar{\psi} e^{ig\phi}. \end{aligned} \tag{2.5}$$

The explicit expression of $A_{\mu d}$ is obtained by means of (2.2) and $e^X Y e^{-X} = e^{\text{ad}_X} Y$.

It is easy to prove that the fields (2.5) are indeed gauge invariant: $\delta_\Lambda A_{\mu d} = \delta_\Lambda \psi_d = \delta_\Lambda \bar{\psi}_d = 0$. It is also possible to prove (see appendix B) that they are unique, given the ϕ transformation law.

It is also crucial to introduce an anticommuting partner $H = H^a T^a$ of ϕ , transforming as

$$\delta_\Lambda H^a = H^b \frac{\delta R^a(\phi, \Lambda)}{\delta \phi^b}. \tag{2.6}$$

The consistency of this transformation law can be readily proved from its closure:

$$[\delta_\Lambda, \delta_\Sigma] H^a = H^b \frac{\delta R^c(\phi, \Lambda)}{\delta \phi^b} \frac{\delta R^a(\phi, \Sigma)}{\delta \phi^c}$$

$$\begin{aligned} &+ H^b R^c(\phi, \Lambda) \frac{\delta^2 R^a(\phi, \Sigma)}{\delta \phi^b \delta \phi^c} - (\Lambda \leftrightarrow \Sigma) \\ &= H^b \frac{\delta}{\delta \phi^b} \left(R^c(\phi, \Lambda) \frac{\delta R^a(\phi, \Sigma)}{\delta \phi^c} \right. \\ &\quad \left. - R^c(\phi, \Sigma) \frac{\delta R^a(\phi, \Lambda)}{\delta \phi^c} \right) \\ &= H^b \frac{\delta R^a(\phi, -ig[\Lambda, \Sigma])}{\delta \phi^b} = \delta_{-ig[\Lambda, \Sigma]} H^a. \end{aligned} \tag{2.7}$$

We have used (2.6) in the first two lines, (2.4) in the third and fourth lines and (2.6) again in the last step.

We have achieved what we wanted, that is to say, define gauge-invariant dressings for quarks and gluons. However, we have done it at the cost of introducing new fields: the cloud field ϕ and its anticommuting partner H . The next problem is to include the extra fields into the action, and ensure that:

- (a) the extension does not change the fundamental physics;
- (b) in particular, no unphysical degrees of freedom propagate.

The field H plays a crucial role to achieve objective (a). Specifically, we use it to endow the cloud sector with a certain symmetry, which ensures that the correlation functions of the undressed fields are unmodified (despite the presence of nontrivial interactions between them and the extra fields), and so are the S matrix amplitudes. Moreover, we render the extra fields purely virtual, to ensure that requirement (b) is manifestly fulfilled as well.

We also want to preserve locality, renormalizability and unitarity, and do everything without affecting the usual structure of the perturbative expansion. In particular, the expansion in powers of the gauge coupling g should coincide with the expansion in the number of loops.

3 Batalin–Vilkovisky formalism and Zinn–Justin master equation

In this section we recall the standard formalism to treat gauge theories. In the next sections we generalize it to build the cloud sector.

We start from the classical action

$$S_{\text{cl}}(A, \bar{\psi}, \psi) = -\frac{1}{4} \int F_{\mu\nu}^a F^{\mu\nu a} + \int \bar{\psi} (i\gamma^\mu D_\mu - m) \psi \tag{3.1}$$

of a non-Abelian gauge theory with gauge group $SU(N_c)$, coupled to matter. For concreteness, we assume that the matter sector is made of fermions ψ in the fundamental representation, $D_\mu \psi = \partial_\mu \psi - ig A_\mu \psi$ being their covariant derivative. The specific form of S_{cl} is not important for the

formalism we are going to develop. However, (3.1) will be used in the explicit computations of this paper. We do not write the measure d^Dx of the spacetime integrals explicitly, when no confusion can arise.

We introduce the set of fields $\Phi^\alpha = (A_\mu, C, \bar{C}, B, \psi, \bar{\psi})$, where $A_\mu = T^a A_\mu^a$ are the gauge fields, $C = C^a T^a$ are the Faddeev–Popov ghosts [33], $B = B^a T^a$ are the Nakanishi–Lautrup Lagrange multipliers [34,35] and $\bar{C} = \bar{C}^a T^a$ are the antighosts. The superscript α collects all the indices. To have control on the Ward–Takahashi–Slavnov–Taylor identities [36–39] to all orders in a compact form, we use the Batalin–Vilkovisky formalism [40,41].

We couple sources $K^\alpha = (K_A^\mu, K_C, K_{\bar{C}}, K_B, K_\psi, K_{\bar{\psi}})$ to the field transformations by means of the functional

$$S_K(\Phi, K) = - \int (D_\mu C)^a K_A^{\mu a} + \frac{g}{2} \int f^{abc} C^b C^c K_C^a - i g \int \bar{\psi} C K_{\bar{\psi}} - i g \int K_\psi C \psi - \int B^a K_{\bar{C}}^a, \tag{3.2}$$

where $D_\mu C^a = \partial_\mu C^a + g f^{abc} A_\mu^b C^c$ is the covariant derivative of C^a . Precisely, the infinitesimal field transformations are

$$\delta_\Lambda \Phi^\alpha = \theta(S_K, \Phi^\alpha) = -\theta \frac{\delta_r S_K}{\delta K^\alpha}, \tag{3.3}$$

where $\Lambda = \theta C$, θ is a constant anticommuting (Grassmann) variable and

$$(X, Y) = \int \left(\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K^\alpha} - \frac{\delta_r X}{\delta K^\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} \right) \tag{3.4}$$

are the Batalin–Vilkovisky antiparentheses [40,41], the subscripts r and l denoting the right and left derivatives, respectively.

The closure of the algebra of transformations is encoded into the identities

$$(S_K, S_K) = 0, \quad (S_K, (S_K, X)) = 0 \tag{3.5}$$

for every X . The Jacobi identity satisfied by the antiparentheses [40,41] implies that the two properties just stated are equivalent. The second one is called nilpotence relation.

The gauge-fixed action reads

$$S_{\text{gf}}(\Phi) = S_{\text{cl}}(A, \bar{\psi}, \psi) + (S_K, \Psi(\Phi)), \tag{3.6}$$

where $\Psi(\Phi)$ is a certain functional that fixes the gauge, commonly known as “gauge fermion”. A typical form of $\Psi(\Phi)$ is

$$\Psi(\Phi) = \int \bar{C}^a \left(G^a(A) + \frac{\lambda}{2} B^a \right), \tag{3.7}$$

where $G^a(A)$ is the gauge-fixing function. For example, the covariant gauge is the one with $G^a(A) = \partial^\mu A_\mu^a$, which gives

$$(S_K, \Psi) = \int B^a \left(\partial^\mu A_\mu^a + \frac{\lambda}{2} B^a \right) - \int \bar{C}^a \partial^\mu D_\mu C^a \rightarrow -\frac{1}{2\lambda} \int (\partial^\mu A_\mu^a)^2 - \int \bar{C}^a \partial^\mu D_\mu C^a, \tag{3.8}$$

where the arrow denotes the integration over B^a . The gauge-fixed action then reads

$$S_{\text{gf}}(\Phi) = -\frac{1}{4} \int F_{\mu\nu}^a F^{\mu\nu a} + \int \bar{\psi} (i\gamma^\mu D_\mu - m)\psi - \frac{1}{2\lambda} \int (\partial^\mu A_\mu^a)^2 - \int \bar{C}^a \partial^\mu D_\mu C^a. \tag{3.9}$$

Other gauge choices will be considered in the paper.

To have control on the renormalization of the gauge transformations, it is useful to include them as composite fields. This is achieved by adding S_K to the gauge-fixed action and working with the new action

$$S(\Phi, K) = S_{\text{gf}}(\Phi) + S_K(\Phi, K). \tag{3.10}$$

The identities (3.5) imply that $S(\Phi, K)$ satisfies the Zinn–Justin equation [42]

$$(S, S) = 0, \tag{3.11}$$

also known as master equation. Formula (3.11) collects the gauge invariance of the classical action, the triviality of the gauge-fixing sector, and the closure of the gauge algebra. The Jacobi identity implies the nilpotence relation $(S, (S, X)) = 0$ for every X .

4 Cloud sector

In this section we define the cloud sector. The idea is to add the cloud field ϕ to the action, but trivialize its presence, in some sense, by means of a new symmetry (which we call cloud symmetry), built with the anticommuting partner H , so as to keep the correlation functions built without involving the cloud sector and the S matrix elements unchanged.

The goal is achieved as follows. First, we introduce a new set of fields $\tilde{\Phi}^\alpha = (\phi^a, H^a, \bar{H}^a, E^a)$ and the sources $\tilde{K}^\alpha = (\tilde{K}_\phi^a, \tilde{K}_H^a, \tilde{K}_{\bar{H}}^a, \tilde{K}_E^a)$ coupled to their transformations, where H can be understood as “cloud Faddeev–Popov ghosts”, E are new Lagrange multipliers and \bar{H} are the cloud antighosts. Second, we extend the definition (3.4) of antiparentheses to include the new sector:

$$(X, Y) = \int \left(\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K^\alpha} - \frac{\delta_r X}{\delta K^\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} + \frac{\delta_r X}{\delta \tilde{\Phi}^\alpha} \frac{\delta_l Y}{\delta \tilde{K}^\alpha} - \frac{\delta_r X}{\delta \tilde{K}^\alpha} \frac{\delta_l Y}{\delta \tilde{\Phi}^\alpha} \right).$$

Third, we collect the transformations of the old and new fields into the functionals

$$\begin{aligned}
 S_K^{\text{gauge}} &= S_K - \int R^a(\phi, C) \tilde{K}_\phi^a + \int H^b \frac{\delta R^a(\phi, C)}{\delta \phi^b} \tilde{K}_H^a, \\
 S_K^{\text{cloud}} &= \int H^a \tilde{K}_\phi^a - \int E^a \tilde{K}_H^a, \quad S_K^{\text{tot}} = S_K^{\text{gauge}} + S_K^{\text{cloud}},
 \end{aligned}
 \tag{4.1}$$

where $R^a(\phi, C)$, defined in (2.3), is just another way to write $\delta_C \phi^a$. The first functional collects the gauge transformations (2.3) and (2.6) of ϕ and H , while the second functional encodes the cloud transformations, which are the most general shifts of ϕ and \tilde{H} . For example, the total transformation of ϕ is

$$\delta_{\Lambda, \mathcal{H}} \phi^a = \theta (S_K^{\text{tot}}, \phi^a) = -\theta H^a + R^a(\phi, \Lambda) = -\mathcal{H}^a + \delta_\Lambda \phi^a,$$

where $\mathcal{H} = \theta H$ can be viewed as an arbitrary function that translates the cloud field ϕ .

It is easy to check the identities

$$(S_K^{\text{gauge}}, S_K^{\text{gauge}}) = (S_K^{\text{cloud}}, S_K^{\text{cloud}}) = 0,
 \tag{4.2}$$

which express the closures of the algebras of the gauge and cloud transformations. The first identity follows from (3.5), (2.4) and (2.7).

We can also check the closure of the combined transformations, i.e.,

$$(S_K^{\text{tot}}, S_K^{\text{tot}}) = (S_K^{\text{cloud}}, S_K^{\text{gauge}}) = 0.
 \tag{4.3}$$

The proof follows from

$$\begin{aligned}
 &(S_K^{\text{cloud}}, S_K^{\text{gauge}}) \\
 &= \int \left(\frac{\delta_r S_K^{\text{cloud}}}{\delta H^a} \frac{\delta_l S_K^{\text{gauge}}}{\delta \tilde{K}_H^a} - \frac{\delta_r S_K^{\text{cloud}}}{\delta \tilde{K}_\phi^a} \frac{\delta_l S_K^{\text{gauge}}}{\delta \phi^a} \right) = 0.
 \end{aligned}$$

The following identities also hold:

$$\begin{aligned}
 S_K^{\text{gauge}} &= S_K - \left(S_K^{\text{cloud}}, \int R^a(\phi, C) \tilde{K}_H^a \right), \\
 S_K^{\text{cloud}} &= - \left(S_K^{\text{cloud}}, \int \phi^a \tilde{K}_\phi^a + \int \tilde{H}^a \tilde{K}_H^a \right).
 \end{aligned}
 \tag{4.4}$$

They show that the two functionals $S_K^{\text{gauge}} - S_K$ and S_K^{cloud} are ‘‘cohomologically exact’’ under the cloud symmetry, i.e., they have the form $(S_K^{\text{cloud}}, \text{local functional})$.

4.1 The cloud and the total action

To specify the cloud we want to use, we add $(S_K^{\text{tot}}, \tilde{\Psi})$ to the action, where $\tilde{\Psi}(\Phi, \tilde{\Phi})$ is the ‘‘cloud fermion’’. A typical form of it is

$$\tilde{\Psi}(\Phi, \tilde{\Phi}) = \int \tilde{H}^a \left(V^a(\phi, A) + \frac{\tilde{\lambda}}{2} E^a \right),
 \tag{4.5}$$

where $V^a(\phi, A)$ denotes the ‘‘cloud-fixing function’’, i.e., the function that defines the cloud. We assume that V^a is gauge invariant,

$$(S_K^{\text{gauge}}, V^a) = 0, \quad (S_K^{\text{gauge}}, \tilde{\Psi}) = 0.
 \tag{4.6}$$

In practice, $V^a(\phi, A)$ depends on ϕ and A_μ only through the dressed gauge field $A_{\mu d}$. Sometimes, with an abuse of notation, we just write $V^a(A_{\mu d})$. More generally, V^a may depend on the other dressed fields as well.

The gauge fermion (3.7) was implicitly assumed to be cloud invariant, because it was built before adding the extra sectors. This is an assumption we have to maintain after the extension, and will be crucial for the construction of the correlation functions of the dressed fields. In the end, the cloud-fermion must be gauge invariant and the gauge-fermion must be cloud invariant.

We find

$$\begin{aligned}
 (S_K^{\text{tot}}, \tilde{\Psi}) &= (S_K^{\text{cloud}}, \tilde{\Psi}) = \int E^a \left(V^a(\phi, A) + \frac{\tilde{\lambda}}{2} E^a \right) \\
 &\quad + \int \tilde{H}^a \frac{\delta V^a(\phi, A)}{\delta \phi^b} H^b.
 \end{aligned}
 \tag{4.7}$$

Note that the last term provides a sort of Faddeev-Popov determinant for the cloud, which is crucial for the properties that we want to prove.

The total action of the extended theory is then

$$S_{\text{tot}}(\Phi, K, \tilde{\Phi}, \tilde{K}) = S_{\text{cl}} + (S_K^{\text{tot}}, \Psi + \tilde{\Psi}) + S_K^{\text{tot}}
 \tag{4.8}$$

and satisfies its own master equation

$$(S_{\text{tot}}, S_{\text{tot}}) = 0.
 \tag{4.9}$$

Note that S_{tot} is separately gauge invariant and cloud invariant, since Eqs. (4.2), (4.3) and (4.6) imply

$$(S_K^{\text{gauge}}, S_{\text{tot}}) = (S_K^{\text{cloud}}, S_{\text{tot}}) = 0.
 \tag{4.10}$$

Finally, using (4.4) we can write

$$\begin{aligned}
 S_{\text{tot}}(\Phi, K, \tilde{\Phi}, \tilde{K}) &= S(\Phi, K) + (S_K^{\text{cloud}}, \Theta), \\
 \Theta &= \tilde{\Psi} - \int R^a(\phi, C) \tilde{K}_H^a - \int \phi^a \tilde{K}_\phi^a \\
 &\quad - \int \tilde{H}^a \tilde{K}_H^a,
 \end{aligned}$$

which shows that the difference between the total action and the ordinary action $S(\Phi, K)$ of formula (3.10) is exact under the cloud symmetry.

4.2 Covariant cloud and propagators

To make explicit calculations, we need to choose the cloud function. There is a large arbitrariness in this choice. A convenient starting point is the covariant cloud

$$V^a(\phi, A) = \partial^\mu A_{\mu d}^a.
 \tag{4.11}$$

Other choices will be considered later on.

Formula (4.7) then gives

$$(S_K^{\text{tot}}, \tilde{\Psi}) = (S_K^{\text{cloud}}, \tilde{\Psi}) = -\frac{1}{2\tilde{\lambda}} \int (\partial^\mu A_{\mu d}^a)^2 - \int (\partial^\mu \bar{H}^b) \frac{\delta A_{\mu d}^b}{\delta \phi^a} H_d^a, \tag{4.12}$$

after integrating E out. To the lowest order, (4.12) reads

$$(S_K^{\text{cloud}}, \tilde{\Psi}) = -\frac{1}{2\tilde{\lambda}} \int (\partial^\mu A_\mu^a - \square \phi^a)^2 + \int (\partial^\mu \bar{H}^a)(\partial_\mu H^a) + \mathcal{O}(g). \tag{4.13}$$

Together with (3.9), this expression allows us to derive the propagators in the covariant framework. We find

$$\begin{aligned} \langle A_\mu^a(p) A_\nu^b(-p) \rangle_0 &= -\frac{i\delta^{ab}}{p^2 + i\epsilon} \left(\eta_{\mu\nu} - \frac{(1-\lambda)p_\mu p_\nu}{p^2 + i\epsilon} \right), \\ \langle \psi(p) \bar{\psi}(-p) \rangle_0 &= \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon}, \\ \langle A_\mu^a(p) \phi^b(-p) \rangle_0 &= -\frac{\lambda\delta^{ab} p_\mu}{(p^2 + i\epsilon)^2}, \\ \langle \phi^a(p) \phi^b(-p) \rangle_0 &= -\frac{i(\lambda + \tilde{\lambda})\delta^{ab}}{(p^2 + i\epsilon)^2}, \end{aligned} \tag{4.14}$$

plus the ghost propagators. With an abuse of notation, we use the same symbols for the fields and their Fourier transforms, since the meaning is clear from the context. For the moment, we use the Feynman $i\epsilon$ prescription for every pole. Later on we switch to the purely virtual (fakeon) prescription for the cloud poles.

The correlation functions of the dressed fields do not suffer from infrared divergences (even without advocating the properties of the dimensional regularization), although the denominators of some propagators contain the square of p^2 . A quick way to prove this statement is by noting that there exists a gauge choice ($\lambda = -\tilde{\lambda}$) where such problems are manifestly absent. Indeed, the concerning denominators are only those originated by $\langle \phi^a(p) \phi^b(-p) \rangle_0$, which vanishes for $\lambda = -\tilde{\lambda}$. Since the dressed correlation functions are gauge independent, they are also infrared finite. With more general gauge choices (such as $\lambda \neq -\tilde{\lambda}$), the infrared divergences cancel out among different diagrams contributing to the same order.

5 Cloud independence of the ordinary correlation functions

In this section and the next one we prove the cloud independence of the non-cloud sector. We start by showing that the ordinary correlation functions are unaffected by the clouds.

This also ensures that the renormalization of the fundamental (i.e., non-cloud) sector of the theory is the same as usual.

The generating functional of the correlation functions is

$$\begin{aligned} Z_{\text{tot}}(J, K, \tilde{J}, \tilde{K}) &= \int [d\Phi d\tilde{\Phi}] \exp \left(iS_{\text{tot}}(\Phi, K, \tilde{\Phi}, \tilde{K}) \right. \\ &\quad \left. + i \int \Phi^\alpha J^\alpha + i \int \tilde{\Phi}^\alpha \tilde{J}^\alpha \right) \end{aligned} \tag{5.1}$$

and $W_{\text{tot}}(J, K, \tilde{J}, \tilde{K}) = -i \ln Z_{\text{tot}}(J, K, \tilde{J}, \tilde{K})$ is the generating functional of the connected ones. The functional derivatives of Z_{tot} or W_{tot} with respect to the sources J^α , calculated at $\tilde{J} = \tilde{K} = 0$, are the ordinary correlation functions of the elementary fields (and their transformations). They are collected in

$$\begin{aligned} Z_{\text{tot}}(J, K, 0, 0) &= \int [d\Phi d\tilde{\Phi}] \exp \left(iS(\Phi, K) \right. \\ &\quad \left. + i(S_K^{\text{cloud}}, \tilde{\Psi}) + i \int \Phi^\alpha J^\alpha \right) \\ &= \int [d\Phi] \exp \left(iS(\Phi, K) \right. \\ &\quad \left. + i \int \Phi^\alpha J_\alpha \right) \int [d\tilde{\Phi}] e^{i(S_K^{\text{cloud}}, \tilde{\Psi})}. \end{aligned} \tag{5.2}$$

We want to prove that this expression coincides with the ordinary generating functional, thanks to the identity

$$\int [d\tilde{\Phi}] e^{i(S_K^{\text{cloud}}, \tilde{\Psi})} = 1. \tag{5.3}$$

Using (4.7) and integrating over the cloud ghosts and antighosts, the left-hand side of (5.3) becomes

$$\int [d\phi dE] \exp \left(i \int E^a \left(V^a + \frac{\tilde{\lambda}}{2} E^a \right) \right) \det \left[\frac{\delta V^a}{\delta \phi^b} \right].$$

Inserting

$$1 = \int [dQ] \exp \left(-\frac{i\tilde{\lambda}}{2} \int (Q^a - E^a)^2 \right),$$

it also becomes

$$\int [d\phi dQ dE] e^{-\frac{i\tilde{\lambda}}{2} \int (Q^a)^2} \exp \left(i \int E^a (V^a + \tilde{\lambda} Q^a) \right) \det \left[\frac{\delta V^a}{\delta \phi^b} \right].$$

Integrating on the Lagrange multipliers E , we obtain a functional δ function, and conclude

$$\begin{aligned} &\int [dQ] e^{-\frac{i\tilde{\lambda}}{2} \int (Q^a)^2} \int [d\phi] \delta(V^a + \tilde{\lambda} Q^a) \det \left[\frac{\delta V^a}{\delta \phi^b} \right] \\ &= \int [dQ] e^{-\frac{i\tilde{\lambda}}{2} \int (Q^a)^2} = 1, \end{aligned}$$

as desired.

Note that the cloud Faddeev–Popov determinant is crucial to trivialize the ϕ integral. Without it, the cloud sector would affect the non-cloud one and change the fundamental theory.

6 Cloud independence of the S matrix amplitudes

Now we prove that the scattering amplitudes of the dressed fields coincide with the usual scattering amplitudes (of undressed fields). Specifically, the clouds have no effect on shell, when the polarizations are attached to the amputated external legs.

First, consider a generic theory of scalar fields φ , described by some classical action $S(\varphi)$. If $\mathcal{O}(\varphi)$ denotes a composite field that is at least quadratic in φ , the connected two-point function of $\varphi' \equiv \varphi + \mathcal{O}(\varphi)$ can be decomposed, in momentum space, as

$$\begin{aligned} \langle \varphi' | \varphi' \rangle &= \langle \varphi | \varphi \rangle + \langle \mathcal{O} | \varphi \rangle \langle \varphi | \varphi \rangle + \langle \varphi | \varphi \rangle \langle \mathcal{O} | \mathcal{O} \rangle + \langle \mathcal{O} | \mathcal{O} \rangle \\ &\quad + \langle \mathcal{O} | \varphi \rangle \langle \varphi | \varphi \rangle \langle \mathcal{O} | \mathcal{O} \rangle \\ &= [1 + \langle \mathcal{O} | \varphi \rangle]^2 \langle \varphi | \varphi \rangle + \langle \mathcal{O} | \mathcal{O} \rangle. \end{aligned} \tag{6.1}$$

Here and below, a vertical bar is used to separate the (elementary or composite) field of momentum p (to the left) from the one of momentum $-p$ (to the right). The symbol $\langle \dots \rangle$ collects the “nonamputable” diagrams, which are those that do not contain propagators of momentum p . The first equality of identity (6.1) can be easily proved diagrammatically.

Formula (6.1) shows that the location of the pole is the same in $\langle \varphi | \varphi \rangle$ and $\langle \varphi' | \varphi' \rangle$. We write it as $p^2 = m_{\text{ph}}^2$, where m_{ph} denotes the physical mass (possibly equipped with an imaginary part, if the particle is unstable). On the other hand, the residue at the pole may change. Precisely, we have

$$\langle \varphi | \varphi \rangle \simeq \frac{iZ}{p^2 - m_{\text{ph}}^2 + i\epsilon}, \quad \langle \varphi' | \varphi' \rangle \simeq \frac{iZ'}{p^2 - m_{\text{ph}}^2 + i\epsilon},$$

where Z is the usual normalization factor and

$$Z' = Z [1 + \langle \mathcal{O} | \varphi \rangle]_{p^2=m_{\text{ph}}^2}^2$$

is the new normalization factor.

Now, consider the correlation functions that contain more than two φ' insertions. Singling out one insertion at a time, the diagrammatics easily gives

$$\langle \varphi' \text{ rest} \rangle = [1 + \langle \mathcal{O} | \varphi \rangle] \langle \varphi | \varphi \rangle \langle \varphi \text{ rest} \rangle + \langle \mathcal{O} \text{ rest} \rangle,$$

where the nonamputation $\langle \dots \rangle$ only refers to the leg under consideration. Thus, the identity

$$\begin{aligned} &\left\langle \prod_{a=1}^j \frac{p_a^2 - m_{\text{ph}}^2}{\sqrt{Z'}} \varphi'(p_a) \right\rangle_{\text{on-shell}} \\ &= \left\langle \prod_{a=1}^j \frac{p_a^2 - m_{\text{ph}}^2}{\sqrt{Z}} \varphi(p_a) \right\rangle_{\text{on-shell}} \end{aligned} \tag{6.2}$$

holds, which proves that the S matrix amplitudes do not change when we make a (perturbative) change of field variables from φ to φ' .

Applying this result to the cloud extension of Yang–Mills theory, we obtain, in momentum space

$$\begin{aligned} &\left\langle \prod_{i=1}^n k_i^2 \varepsilon_i^{\mu_i}(k_i) A_{\mu_i d}(k_i) \prod_{a=1}^j \bar{u}_{s_a d}(p_a) (\not{p}_a - \tilde{m}) \psi_d(p_a) \right. \\ &\quad \left. \prod_{b=1}^j \bar{\psi}_d(q_b) (\not{q}_b - \tilde{m}) u_{s_b d}(q_b) \right\rangle_{\text{on-shell}} \\ &= \left\langle \prod_{i=1}^n k_i^2 \varepsilon_i^{\mu_i}(k_i) A_{\mu_i}(k_i) \prod_{a=1}^j \bar{u}_{s_a}(p_a) (\not{p}_a - \tilde{m}) \psi(p_a) \right. \\ &\quad \left. \prod_{b=1}^j \bar{\psi}(q_b) (\not{q}_b - \tilde{m}) u_{s_b}(q_b) \right\rangle_{\text{on-shell}}, \end{aligned} \tag{6.3}$$

where $\not{p} = \gamma^\mu p_\mu$ and \tilde{m} denotes the physical mass. The polarizations $\varepsilon^\mu(k)$ and $u_s(p)$ satisfy $k_\mu \varepsilon^\mu(k) = 0$ and $(\not{p} - \tilde{m})u_s(p) = 0$ and include the normalization factors $1/\sqrt{Z}$. The “dressed” polarizations $\varepsilon_d^\mu(k)$ and $u_{s_d}(p)$ are the same, apart from having normalization factors $1/\sqrt{Z'}$. By the theorem proved in the previous section, the right-hand side of (6.3) is cloud independent and coincides with the usual S matrix amplitude.

Two differences between (6.2) and (6.3) deserve to be singled out. Formulas (2.5) show that the expansion of $A_{\mu d}$ contains a linear contribution $-\partial_\mu \phi$, besides A_μ itself, plus nonlinear terms. Thus, $A_{\mu d}$ is not of the form $\varphi' = \varphi + \mathcal{O}(\varphi)$. Nevertheless, the linear term $-\partial_\mu \phi$ becomes $ik_\mu \phi(k)$, after the Fourier transform, and is killed by the polarization $\varepsilon^\mu(k)$. This means that $\varepsilon_d^\mu(k) A_{\mu d}(k)$ is of the required form, apart from an unimportant normalization factor.

Second, we are not comparing correlation functions of the same theory, as in (6.2). We are jumping from one theory (the extended one, to which the left-hand side of (6.3) refers) to another theory (the non extended one, to which the right-hand side of (6.3) refers). This is possible, thanks to the result of the previous section.

In the end, the product $k^2 \varepsilon_d^\mu(k) A_{\mu d}(k)$ is gauge invariant (and gauge independent, for the arguments we give below) and its dressing is trivial:

$$\lim_{k^2 \rightarrow 0} k^2 \varepsilon_d^\mu(k) \langle A_{\mu d}(k) \dots \rangle = \lim_{k^2 \rightarrow 0} k^2 \varepsilon^\mu(k) \langle A_\mu(k) \dots \rangle.$$

The same result holds for the fermions and any other elementary fields, if present.

The identity (6.3) proves that the ordinary theory of scattering can be rephrased as a theory of scattering of dressed fields. We could even forget about the ordinary fields altogether, and always work with dressed fields, which have the advantage of being manifestly gauge invariant. In so doing,

both gauge invariance and gauge independence become manifest.

A straightforward consequence is that the usual S -matrix amplitudes are gauge independent. With the usual methods, the proof of this result is relatively simple in the Abelian case, but more demanding in the non-Abelian one [43].

7 Dressed correlation functions

In this section we study the correlation functions of the dressed fields. A way to deal with their insertions systematically is by coupling them to new sources and extending the generating functionals again. We replace the action S_{tot} inside (5.1) by

$$S_{\text{tot}}^{\text{ext}} = S_{\text{tot}} + \int (J_d^\mu A_{\mu d} + \bar{J}_{\psi d} \psi_d + \bar{\psi}_d J_{\psi d}), \tag{7.1}$$

and denote the extended generating functionals by $Z_{\text{tot}}^{\text{ext}}(J, K, \tilde{J}, \tilde{K}, J_d) = \exp(iW_{\text{tot}}^{\text{ext}}(J, K, \tilde{J}, \tilde{K}, J_d))$. Note that the extended action is gauge invariant, since (4.10) implies

$$(S_K^{\text{gauge}}, S_{\text{tot}}^{\text{ext}}) = 0. \tag{7.2}$$

The correlation functions of the dressed fields are the functional derivatives with respect to the dressed sources $J_d = (J_d^\mu, \tilde{J}_{\psi d}, J_{\psi d})$.

7.1 Gauge independence

It is straightforward to prove that the dressed correlation functions, collected in the functional $Z_{\text{tot}}^{\text{ext}}(J_d) = \exp(iW_{\text{tot}}^{\text{ext}}(J_d)) \equiv Z_{\text{tot}}^{\text{ext}}(0, 0, 0, 0, J_d)$, are gauge independent.

Assume that the gauge fermion Ψ depends on some gauge-fixing parameter λ . A derivative with respect to λ amounts to an insertion of an S_K^{gauge} -exact functional:

$$\frac{\partial}{\partial \lambda} W_{\text{tot}}^{\text{ext}}(J_d) = \langle (S_K^{\text{gauge}}, \Psi_\lambda) \rangle_{J=K=\tilde{J}=\tilde{K}=0}, \tag{7.3}$$

where $\Psi_\lambda = \partial\Psi/\partial\lambda$. We want to show that the right-hand side of this identity vanishes.

Consider $\langle \Psi_\lambda \rangle_{J=K=\tilde{J}=\tilde{K}=0}$ and perform a change of field variables

$$\delta\Phi^\alpha = \theta(S_K^{\text{gauge}}, \Phi^\alpha), \quad \delta\tilde{\Phi}^\alpha = \theta(S_K^{\text{gauge}}, \tilde{\Phi}^\alpha),$$

in the functional integral that defines the numerator of the average. Because of (7.2), which also implies $(S_K^{\text{gauge}}, S_{\text{tot}}^{\text{ext}}|_{K=\tilde{K}=0}) = 0$, everything is invariant, but Ψ_λ . Thus,

$$0 = \langle \delta\Psi_\lambda \rangle_{J=K=\tilde{J}=\tilde{K}=0} = \theta \langle (S_K^{\text{gauge}}, \Psi_\lambda) \rangle_{J=K=\tilde{J}=\tilde{K}=0}, \tag{7.4}$$

as we wished to prove. Gauge independence will be verified explicitly in the computations of the next sections. In Sect. 14 we prove that it survives the renormalization.

8 Gauge/cloud duality

In this section we prove a *gauge/cloud duality*, which relates the gauge-trivial sector of the theory to the cloud sector.

We start by deriving the could transformation of the dressed gauge field from (2.1) and (2.5). The result is

$$(S_K^{\text{tot}}, A_{\mu d}) = (S_K^{\text{cloud}}, A_{\mu d}) = D_\mu(A_d)H_d, \tag{8.1}$$

where $D_\mu(A_d)$ denotes the covariant derivative, evaluated on the dressed field A_d . Instead,

$$\begin{aligned} H_d &\equiv -\frac{i}{g}(S_K^{\text{cloud}}, \hat{U})\hat{U}^{-1} = \frac{i}{g}\hat{U}(S_K^{\text{cloud}}, \hat{U}^{-1}) \\ &= \frac{e^{-igad_\phi} - 1}{-igad_\phi} H \end{aligned} \tag{8.2}$$

denotes the dressed cloud ghost H . In the last step of (8.2) we have used (2.2) with $X = ig\phi$. It is easy to check that H_d is indeed gauge invariant, $(S_K^{\text{gauge}}, H_d) = 0$. We see that the could transformation of $A_{\mu d}$ is analogous to the gauge transformation of A_μ , provided the dressed fields replace the undressed ones. By inverting (8.2), we obtain

$$H = R(-\phi, H_d). \tag{8.3}$$

Similarly, when we work out the cloud transformation of H_d , we find that it mimics the gauge transformation of C :

$$(S_K^{\text{cloud}}, H_d) = igH_dH_d. \tag{8.4}$$

We can also introduce the dressed Faddeev-Popov ghosts

$$C_d \equiv R(\phi, C) = -\frac{igad_\phi}{1 - e^{igad_\phi}} C = (S_K^{\text{gauge}}, \phi), \tag{8.5}$$

which are clearly gauge invariant. Their cloud transformations read

$$\begin{aligned} (S_K^{\text{cloud}}, C_d^a) &= (S_K^{\text{gauge}}, H^a) = (S_K^{\text{gauge}}, R^a(-\phi, H_d)) \\ &= -C_d^b \frac{\delta R^a(\phi_d, H_d)}{\delta \phi_d^b}, \end{aligned} \tag{8.6}$$

having defined $\phi_d = -\phi$ and used (8.3).

Next, we consider the change of field variables

$$\Phi, \tilde{\Phi} \rightarrow \Phi_d, \tilde{\Phi}_d \tag{8.7}$$

from undressed fields to dressed fields, by means of the definitions (2.5), (8.2), (8.5) and $\phi_d = -\phi$, leaving all the other fields unchanged: $\bar{C}_d = \bar{C}$, $B_d = B$, $\bar{H}_d = \bar{H}$ and $E_d = E$. The transformations (8.7) are perturbatively local, which means that when we use them as changes of field variables in the functional integral, the Jacobian determinant is equal to one (using the dimensional regularization).

To ensure that all the properties derived so far continue to hold, we need to preserve the antiparentheses. We can achieve this goal by embedding (8.7) into a canonical transformation $\Phi, \tilde{\Phi}, K, \tilde{K} \rightarrow \Phi_d, \tilde{\Phi}_d, K_d, \tilde{K}_d$, (8.8)

of the Batalin-Vilkovisky type. Its generating functional is $F(\Phi, \tilde{\Phi}, K_d, \tilde{K}_d) = \int \Phi_d(\Phi, \tilde{\Phi})K_d + \int \tilde{\Phi}_d(\Phi, \tilde{\Phi})\tilde{K}_d$.

At the practical level, the whole operation amounts to work out the transformations of the dressed fields, which we have already done, and couple them to the dressed sources. Using (8.1), (8.4) and (8.6), we find

$$S_K^{\text{gauge}} = \int C_d^a \tilde{K}_{\phi_d}^a - \int B_d^a K_{C_d}^a,$$

$$S_K^{\text{cloud}} = - \int (D_\mu(A_d)H_d)^a K_{A_d}^{\mu a} + \frac{g}{2} \int f^{abc} H_d^b H_d^c \tilde{K}_{H_d}^a$$

$$- i g \int \tilde{\psi}_d H_d K_{\tilde{\psi}_d} - i g \int K_{\psi_d} H_d \psi_d$$

$$- \int E_d^a \tilde{K}_{H_d}^a - \int R^a(\phi_d, H_d) \tilde{K}_{\phi_d}^a$$

$$+ \int C_d^b \frac{\delta R^a(\phi_d, H_d)}{\delta \phi_d^b} K_{C_d}^a.$$

We see that (8.8) switches the gauge transformations and the cloud transformations.

Similarly, it exchanges the roles of the gauge-fixing function G^a and the cloud function V^a : $G^a(A(\phi, A_d)) \leftrightarrow V^a(A_d)$. An important caveat of such an exchange is that it understands that the prescription adopted for the gauge-trivial sector is exchanged with the prescription adopted for the cloud sector.

For example, choosing the covariant gauge in (3.7) and the covariant cloud in (4.5), we have

$$\Psi = \int \tilde{C}_d^a \left(\partial^\mu (\hat{U}_d A_{\mu d} \hat{U}_d^{-1} + \frac{i}{g} \hat{U}_d (\partial_\mu \hat{U}_d^{-1}))^a + \frac{\lambda}{2} B_d^a \right),$$

$$\tilde{\Psi} = \int \tilde{H}_d^a \left(\partial^\mu A_{\mu d}^a + \frac{\tilde{\lambda}}{2} E_d^a \right), \tag{8.9}$$

where $\hat{U}_d = \exp(-i g \phi_d) = \hat{U}^{-1}$.

Collecting the various pieces together, the dual action reads

$$S_{\text{tot}} = -\frac{1}{4} \int F_{\mu\nu}^a(A_d) F^{\mu\nu a}(A_d)$$

$$+ \int \tilde{\psi}_d (i \gamma^\mu D_\mu(A_d) - m) \psi_d$$

$$+ (S_K^{\text{tot}}, \Psi + \tilde{\Psi}) + S_K^{\text{tot}}. \tag{8.10}$$

Using the duality just proved, it is possible to simplify the calculations of the correlation functions of the dressed fields. Actually, if we choose a unique cloud for every insertion (see next section for the generalization to multiclouds),

the correlation functions of the dressed fields coincide with the correlation functions of the undressed fields in a specific gauge.

This property can be proved by applying the canonical transformation (8.8) to the dressed correlation functions. The result is an identical correlation function where the dressed fields are replaced by the undressed ones, the gauge-fixing is replaced by the cloud and the cloud is replaced by the gauge-fixing. For example, if we use the covariant gauge (3.9) and the covariant cloud function (4.11), we obtain

$$\left\langle A_{\mu_1 d}(x_1) \cdots A_{\mu_n d}(x_n) \psi_d(y_1) \cdots \psi_d(y_j) \tilde{\psi}_d(z_1) \cdots \tilde{\psi}_d(z_j) \right\rangle$$

$$= \left\langle A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n) \psi(y_1) \cdots \psi(y_j) \tilde{\psi}(z_1) \cdots \tilde{\psi}(z_j) \right\rangle_{\lambda \rightarrow \tilde{\lambda}}. \tag{8.11}$$

This property will be verified in the computations of the next sections. It ensures that the left-hand side (which does not depend on λ by gauge independence), can be worked out by replacing λ with $\tilde{\lambda}$ in the undressed correlation function appearing on the right-hand side (which does not depend on $\tilde{\lambda}$ by cloud independence).

Typically, the left-hand side of (8.11) receives contributions from a huge number of diagrams. However, the identity (8.11) implies that most contributions cancel out in the end. For example, the two-point function of the dressed gauge field amounts to just one diagram, if it is computed as the right-hand side of (8.11), but tenths of diagrams if it is computed as the left-hand side of (8.11).

9 Multiclouds

In this section we extend the formalism of the previous ones by adding several copies of the cloud sector. This allows us to dress each insertion, in a correlation function, with its own cloud, independently of the clouds of the other insertions.

We introduce many cloud fields ϕ^i , where i labels the copies, together with their anticommuting partners H^i (the cloud ghosts), the antighosts \tilde{H}^i and the Lagrange multipliers E^i , collected in the list $\tilde{\Phi}^{\alpha i} = (\phi^i, H^i, \tilde{H}^i, E^i)$. Then we couple sources $\tilde{K}^{\alpha i}$ to their transformations, which include the gauge transformations and the cloud transformations of each copy. We collect them in the functionals

$$S_K^{\text{gauge}} = S_K - \sum_i \int R^a(\phi^i, C) \tilde{K}_\phi^{ai}$$

$$+ \sum_i \int H^{bi} \frac{\delta R^a(\phi^i, C)}{\delta \phi^{bi}} \tilde{K}_H^{ai},$$

$$S_K^{\text{cloud}i} = \int (H^{ai} \tilde{K}_\phi^{ai} - E^{ai} \tilde{K}_{\tilde{H}}^{ai}),$$

$$S_K^{\text{cloud}} = \sum_i S_K^{\text{cloud}i},$$

$$S_K^{\text{tot}} = S_K^{\text{gauge}} + S_K^{\text{cloud}}. \tag{9.1}$$

Finally, we extend the definition (3.4) of antiparentheses to include all the copies:

$$(X, Y) = \int \left[\frac{\delta_r X}{\delta \Phi^\alpha} \frac{\delta_l Y}{\delta K^\alpha} - \frac{\delta_r X}{\delta K^\alpha} \frac{\delta_l Y}{\delta \Phi^\alpha} + \sum_i \left(\frac{\delta_r X}{\delta \tilde{\Phi}^{ai}} \frac{\delta_l Y}{\delta \tilde{K}^{ai}} - \frac{\delta_r X}{\delta \tilde{K}^{ai}} \frac{\delta_l Y}{\delta \tilde{\Phi}^{ai}} \right) \right]. \tag{9.2}$$

It is easy to check that the identities (4.2) and (4.3) continue to hold. The total cloud fermion can be just the sum of the cloud fermions of each copy. We take

$$\tilde{\Psi}(\Phi, \tilde{\Phi}) = \sum_i \tilde{\Psi}_i, \quad \tilde{\Psi}_i = \int \bar{H}^{ai} \left(V^{ai} + \frac{\tilde{\lambda}_i}{2} E^{ai} \right), \tag{9.3}$$

where V^i are the gauge invariant cloud functions: $(S_K^{\text{gauge}}, V^i) = 0$. For simplicity, we also assume that each V^i depends on the i th cloud field ϕ^i only (besides A_μ), i.e., different cloud sectors are not mixed by the cloud functions. It can be proved that renormalization preserves the unmixing (see Sect. 14).

The total action of the extended theory is still (4.8), and satisfies (4.9) and (4.10). Moreover,

$$(S_K^{\text{cloud}i}, S_{\text{tot}}) = 0 \tag{9.4}$$

for every i .

We can always build gauge invariant functions with two cloud fields, since the product $\hat{U}(\phi_j)\hat{U}^{-1}(\phi_k) \equiv \hat{U}_{jk}$ is gauge invariant for every j and k . We have no powerful control on how such functions propagate through the operations we make, once they are turned on. The cloud unmixing just mentioned is an important simplification, as long as we can prove that it is not ruined by renormalization and our own manipulations.

The correlation functions that do not contain insertions of some cloud sector are independent of that sector. Indeed, the proof of (5.3) can be repeated for every sector separately. The multicloud propagators can be easily derived using this property. Consider, for example, the case of two clouds. Denote the cloud fields by ϕ_1 and ϕ_2 and choose the cloud fermions (4.5) with parameters $\tilde{\lambda}_1, \tilde{\lambda}_2$, and the covariant cloud functions (4.11). Finally, choose the covariant gauge (3.8). Then, the propagators (4.14) hold in each sector. In addition, we have

$$\langle \phi^1 | \phi^2 \rangle = -\frac{1}{\tilde{\lambda}_1 - \tilde{\lambda}_2} \left(\tilde{\lambda}_2 \langle \phi^1 | \phi^1 \rangle - \tilde{\lambda}_1 \langle \phi^2 | \phi^2 \rangle \right)$$

$$= -\frac{i\lambda\delta^{ab}}{(p^2 + i\epsilon)^2}. \tag{9.5}$$

The identity (9.5) is easily proved from the sum

$$-\frac{1}{2\tilde{\lambda}_1} \int \left(\partial^\mu A_{d\mu}^{a1} \right)^2 - \frac{1}{2\tilde{\lambda}_2} \int \left(\partial^\mu A_{d\mu}^{a2} \right)^2,$$

which shows that, at the quadratic level, the combination $\phi^1 - \phi^2$ decouples from A_μ and from the combination $\tilde{\lambda}_2\phi^1 + \tilde{\lambda}_1\phi^2$. This implies $\langle \phi^1 - \phi^2 | \tilde{\lambda}_2\phi^1 + \tilde{\lambda}_1\phi^2 \rangle_0 = 0$. Note that (9.5) may suggest that the cloud sectors mix. Nevertheless, renormalization does not mix them, as shown in section 14.

The correlation functions of dressed fields can be studied by means of the extension

$$S_{\text{tot}}^{\text{ext}} = S_{\text{tot}} + \sum_i \int \left(J_d^{\mu i} A_{\mu d}^i + \bar{J}_{\psi d}^i \psi_d^i + \bar{\psi}_d^i J_{\psi d}^i \right), \tag{9.6}$$

where $A_{\mu d}^i, \psi_d^i$ and $\bar{\psi}_d^i$ denote the dressed fields of the i th cloud sector.

The gauge/cloud duality is less powerful in the presence of many clouds. It can be used to eliminate one cloud, or a combination of clouds, but not all of them. For example, a correlation function

$$\langle A_{\mu_1 d}^{(1)}(x_1) \cdots A_{\mu_n d}^{(n)}(x_n) \psi_d^{(n+1)}(y_1) \cdots \psi_d^{(n+j)}(y_j) \bar{\psi}_d^{(n+j+1)}(z_1) \cdots \bar{\psi}_d^{(n+2j)}(z_j) \rangle, \tag{9.7}$$

with different clouds for every field, can be converted into

$$\langle A_{\mu_1}^{(1)}(x_1) A_{\mu_2 d}^{(2)'}(x_1) \cdots A_{\mu_n d}^{(n)'}(x_n) \psi_d^{(n+1)'}(y_1) \cdots \psi_d^{(n+j)'}(y_j) \bar{\psi}_d^{(n+j+1)'}(z_1) \cdots \bar{\psi}_d^{(n+2j)'}(z_j) \rangle, \tag{9.8}$$

by means of a canonical transformation of the form (8.8), which turns the first dressed field into its undressed version. The clouds of the other insertions are redefined as a consequence. We have emphasized this by means of primes in (9.8).

These operations preserve the unmixing, after further redefinitions of the cloud fields themselves. Indeed, the transformation (8.8) leads to

$$A_{\mu_i d}^{(i)'} = \hat{U}_{i1} A_{\mu_i d}^{(1)} \hat{U}_{i1}^{-1} + \frac{i}{g} \hat{U}_{i1} (\partial_{\mu_i} \hat{U}_{i1}^{-1}), \quad i > 1.$$

To restore the unmixing, after relabeling $A_{\mu d}^{(1)}$ as A_μ , it is sufficient to define the new i th cloud field $\phi_i'(\phi_i, \phi_1), i > 1$, so as to have $\hat{U}_{i1} = \exp(-ig\phi_i')$.

The proof that the usual correlation functions are cloud independent, given in Sect. 5, can be straightforwardly generalized to the multicloud case. Similarly, the proof of formula (6.3), which states that the S matrix amplitudes coincide with the usual ones, can be generalized to the case where each insertion is dressed by means of its own, independent cloud. Note that each insertion may require a different normalization factor $1/\sqrt{Z'}$, depending on the cloud.

10 Absorptive parts

In this section we define the absorptive parts of the off-shell correlation functions that contain insertions of dressed fields, and study their properties.

If $S = 1 + iT$ denotes the S matrix, the (amputated, connected) diagrams give iT and the amplitudes are T . If the unitarity equation $SS^\dagger = 1$ holds, it implies $-2\text{Re}[iT] = TT^\dagger \geq 0$. A virtue of the identity $-2\text{Re}[iT] = TT^\dagger$, known as optical theorem, is that it holds diagram by diagram (which means: if we replace iT by any diagram we want, and TT^\dagger by a suitable sum of “cut diagrams”, built with “cut propagators”, the usual vertices and their complex conjugates [20–25]). It also holds without putting the external legs on shell. Moreover, as shown in ref. [1], it splits into many independent, purely algebraic spectral optical identities, because different thresholds do not talk to one another.

It also holds with non amputated diagrams. To see this, it is sufficient to attach fictitious vertices to the legs that we do not want to amputate. At the practical level, this amounts to multiplying each of them by a factor $-i$. The identity $-2\text{Re}[iT] = TT^\dagger$ also holds with insertions of local composite fields (which can be attached to other fictitious vertices – for this reason, each of them brings a further factor $-i$). Combining elementary and composite fields, the identity also holds with insertions of dressed fields (each of which must be multiplied by $-i$).

We define the absorptive part of an off-shell correlation function, with or without insertions of dressed fields, as minus twice the real part of its amputated version, multiplied by the polarizations and the normalization factors \sqrt{Z} of the external states. It is expected to be nonnegative if the unitarity equation $SS^\dagger = 1$ holds, by the arguments given above. In our calculations, the factors \sqrt{Z} can be set to one, since the absorptive parts we are going to compute vanish at the tree level. Note that more polarizations may be allowed off shell than on shell.

The extended action $S_{\text{tot}}^{\text{ext}}$ is local and Hermitian. However, formula (4.13) shows that the cloud fields ϕ^i (which are dimensionless) do not have ordinary kinetic terms, but higher-derivative ones. For this reason, unitarity and the diagrammatic optical theorem are guaranteed to hold only if we use the fakeon prescription and projection for the cloud fields (see below). If not, we expect to find unphysical absorptive parts. The results of our computations confirm these claims.

11 Dressed fermion self-energy

In this section and the next two we illustrate the properties proved so far in explicit calculations. We concentrate on the two-point functions of the gauge fields and the fermions to order g^2 , which means one loop. For simplicity, we use the

same cloud for all the insertions. We compare several types of clouds, gauge-fixings and prescriptions.

The four-leg vertices, which are multiplied by g^2 , contribute only to tadpoles, which vanish using the dimensional regularization. Ignoring them, it is sufficient to expand the dressed fields (2.5) and the cloud action to order g . We find

$$A_{\mu d} = A_\mu - \partial_\mu \phi - \frac{ig}{2}[\phi, 2A_\mu - \partial_\mu \phi] + \mathcal{O}(g^2), \quad \psi_d = \psi - ig\phi\psi + \mathcal{O}(g^2).$$

For the moment, we concentrate on the covariant clouds and the covariant gauge-fixings, and use the Feynman $i\epsilon$ prescription everywhere. The cloud action (4.12) reads

$$\begin{aligned} (S_K^{\text{cloud}}, \tilde{\Psi}) = & -\frac{1}{2\lambda} \int (\partial^\mu A_\mu^a - \square\phi^a)^2 \\ & + \frac{g}{2\lambda} f^{abc} \int (\partial^\mu \partial^\nu A_\mu^a - \square\partial^\nu \phi^a) \times \\ & \phi^b (2A_\nu^c - \partial_\nu \phi^c) + \int (\partial^\mu \bar{H}^a)(\partial_\mu H^a) \\ & - \frac{g}{2} f^{abc} \int [(\square\bar{H}^a)\phi^b - 2(\partial^\mu \bar{H}^a)(A_\mu^b \\ & - \partial_\mu \phi^b)] H^c + \mathcal{O}(g^2). \end{aligned} \tag{11.1}$$

The two-point function of the dressed gauge fields is calculated in section 13. Here we concentrate on the two-point function of the dressed fermion, which reads

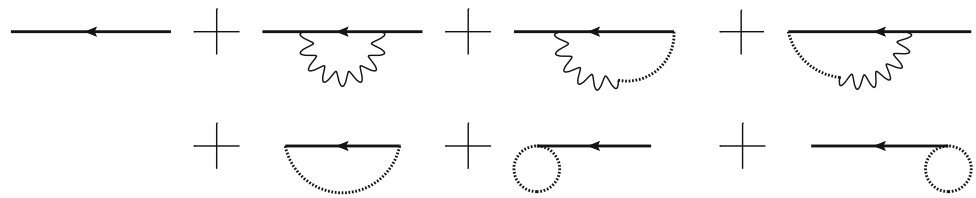
$$\begin{aligned} \langle \psi_d | \bar{\psi}_d \rangle = & \langle \psi | \bar{\psi} \rangle - ig\langle \phi\psi | \bar{\psi} \rangle + ig\langle \psi | \bar{\psi}\phi \rangle \\ & + g^2\langle \phi\psi | \bar{\psi}\phi \rangle - \frac{g^2}{2}\langle \phi^2\psi | \bar{\psi} \rangle \\ & - \frac{g^2}{2}\langle \psi | \bar{\psi}\phi^2 \rangle + \mathcal{O}(g^3). \end{aligned} \tag{11.2}$$

The diagrams contributing to the right-hand side of (11.2) are shown in Fig. 1.

We restrict to the massless limit, which makes the formulas more explicit.

The last two diagrams of Fig. 1 are tadpoles, which vanish using the dimensional regularization. We have included them just to show that $\langle \psi_d | \bar{\psi}_d \rangle$ is not plagued by infrared problems, no matter what regularization we use. The reason is that the denominators $\sim 1/(p^2)^2$ (which give potential infrared divergences with a generic regularization technique) cancel out among the last three diagrams. Their cancellation can be verified by taking the loop momentum to zero. In that limit, a fermion propagator $i\gamma^\mu p_\mu/(p^2 + i\epsilon)$ factors out and all the diagrams become identical tadpoles. Since last two are multiplied by $-g^2/2$, while the third to last one is multiplied by g^2 , the total vanishes. We have not included other tadpole diagrams, because they vanish identically (they factorize the trace of T^a or a contraction like f^{abb}).

Fig. 1 Two-point function of the dressed fermion to order g^2 . The dashed line denotes the cloud field ϕ



The second diagram is the ordinary fermion self-energy. Added to the tree propagator, it gives

$$\langle \psi | \bar{\psi} \rangle = \frac{i\gamma^\mu p_\mu}{p^2 + i\epsilon} \left[1 - \frac{g^2 \tilde{\lambda}}{8\pi^2 \epsilon} \frac{N_c^2 - 1}{2N_c} (-p^2 - i\epsilon)^{-\epsilon/2} \right] + \mathcal{O}(g^4), \tag{11.3}$$

where $\epsilon = 4 - D$ and D is the continued spacetime dimension. The remaining diagrams of Fig. 1 give

$$\frac{i\gamma^\mu p_\mu}{p^2 + i\epsilon} \frac{g^2(\lambda - \tilde{\lambda})}{8\pi^2 \epsilon} \frac{N_c^2 - 1}{2N_c} (-p^2 - i\epsilon)^{-\epsilon/2} + \mathcal{O}(g^4), \tag{11.4}$$

so in total we get

$$\langle \psi_d | \bar{\psi}_d \rangle = \frac{i\gamma^\mu p_\mu}{p^2 + i\epsilon} \left[1 - \frac{g^2 \tilde{\lambda}}{8\pi^2 \epsilon} \frac{N_c^2 - 1}{2N_c} (-p^2 - i\epsilon)^{-\epsilon/2} \right] + \mathcal{O}(g^4). \tag{11.5}$$

The dependence on the gauge-fixing parameter λ has disappeared, as expected. Nevertheless, the result depends on the choice of the cloud, through the parameter $\tilde{\lambda}$.

We see that

$$\langle \psi_d | \bar{\psi}_d \rangle = \langle \psi | \bar{\psi} \rangle_{\lambda \rightarrow \tilde{\lambda}},$$

in agreement with (8.11).

The renormalization of $\langle \psi_d | \bar{\psi}_d \rangle$ requires the counterterm

$$\frac{g^2 \tilde{\lambda}}{8\pi^2 \epsilon} \frac{N_c^2 - 1}{2N_c} \int \bar{J}_{\psi_d} \frac{i\gamma^\mu \partial_\mu}{\square - i\epsilon} J_{\psi_d}. \tag{11.6}$$

However, there is no need to insert it explicitly into the action $S_{\text{tot}}^{\text{ext}}$ of (7.1). As the nonlocal nature of (11.6) suggests, (11.6) is generated automatically by the Legendre transform that relates the generating functional $\Gamma_{\text{tot}}^{\text{ext}}$ of the one-particle irreducible diagrams to the generating functional $W_{\text{tot}}^{\text{ext}}$ of the connected Green functions.

After renormalization, we take ϵ to zero and find

$$\begin{aligned} \langle \psi_d | \bar{\psi}_d \rangle_R &= \frac{i\gamma^\mu p_\mu}{p^2 + i\epsilon} \left[1 + \frac{g^2 \tilde{\lambda}}{(4\pi)^2} \frac{N_c^2 - 1}{2N_c} \ln(-p^2 - i\epsilon) \right] \\ &+ \mathcal{O}(g^4). \end{aligned} \tag{11.7}$$

Now we extract the absorptive part of this expression. Since we have prescribed every field à la Feynman, the arguments of the previous section alert us that the result might be unphysical.

Recall that we are working in the massless limit. We study the sign of the absorptive part by summing on the external polarization states u_s . We use the identity $\sum_s u_s \bar{u}_s = \gamma^\mu p_\mu$ without assuming $p^2 = 0$, because we stay off shell. This amounts to choosing the states $u_s = \sqrt{p^0}(0, \dots, 1, \dots, 0)^t$ (with 1 in position s and zeros elsewhere) and the basis $\gamma^\mu = ((0, \sigma^\mu), (\bar{\sigma}^\mu, 0))^t$, $\sigma^\mu = (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, where t denotes the transpose, $p^0 > 0$ and $s = 1, 2, 3, 4$. Then, the absorptive part of $\langle \psi_d | \bar{\psi}_d \rangle_R$ is

$$\begin{aligned} \text{Abso}[\langle \psi_d | \bar{\psi}_d \rangle_R] &= -2\text{Re} \sum_s \bar{u}_s (-i\gamma^\mu p_\mu) \langle \psi_d | \bar{\psi}_d \rangle_R (-i\gamma^\nu p_\nu) u_s \\ &= \frac{g^2 \tilde{\lambda}}{2\pi} \frac{N_c^2 - 1}{2N_c} p^2 \theta(p^2) + \mathcal{O}(g^4). \end{aligned} \tag{11.8}$$

A factor 4 comes from the spinor trace.

We see that the sign of (11.8) is positive or negative, depending on the sign of the cloud parameter $\tilde{\lambda}$. As already remarked, the cloud field ϕ has higher-derivative kinetic terms. Thus, if we quantize it by means of the Feynman prescription, as we have done so far, it can propagate ghosts and violate the optical theorem.

We need to pay more attention to the clouds we choose, otherwise they can inject unphysical degrees of freedom into the theory and make the computations of the dressed correlation functions uninteresting.

12 Purely virtual clouds

The clouds we have been using so far are physically unacceptable, because they add degrees of freedom that do not belong to the fundamental theory. A way to preserve the content of the fundamental theory is to switch to purely virtual clouds. This way, we can use the correlation functions of the dressed fields as tools to extract the physical content of the off-shell correlation functions of the fundamental fields.

For a clearer understanding of what is going on, it may be helpful to introduce the special gauge of Ref. [44], which is defined by the gauge function $G^a(A) = \lambda \partial_0 A^{0a} + \nabla \cdot \mathbf{A}^a$ in (3.7), where $A^{\mu a} = (A^{0a}, \mathbf{A}^a)$. The gauge fermion reads

$$\Psi(\Phi) = \int \bar{C}^a \left(\lambda \partial_0 A^{0a} + \nabla \cdot \mathbf{A}^a + \frac{\lambda}{2} B^a \right). \tag{12.1}$$

For a more direct gauge/cloud duality and a convenient switch back and forth between the gauge-trivial sector and

the cloud sector, we mimic the special gauge into a “special cloud”, by choosing the cloud function $V^a(A, \phi) = \tilde{\lambda} \partial_0 A_d^{0a} + \nabla \cdot \mathbf{A}_d^a$, instead of (4.11). Then the cloud fermion (4.5) reads

$$\tilde{\Psi}(\Phi, \tilde{\Phi}) = \int \bar{H}^a \left(\tilde{\lambda} \partial_0 A_d^{0a} + \nabla \cdot \mathbf{A}_d^a + \frac{\tilde{\lambda}}{2} E^a \right). \tag{12.2}$$

With these choices the propagators of the gauge fields and the cloud field become, after integrating the Lagrange multipliers B and E out,

$$\begin{aligned} \langle A^{ia}(p) A^{jb}(-p) \rangle_0 &= \frac{i \delta^{ab} \Pi^{ij}}{p^2 + i\epsilon} + \frac{i \lambda \delta^{ab} p^i p^j}{\mathbf{p}^2 (\hat{p} \cdot p)}, \\ \langle A^{0a}(p) A^{0b}(-p) \rangle_0 &= -\frac{i \delta^{ab}}{\hat{p} \cdot p}, \\ \langle A^{\mu a}(p) \phi^b(-p) \rangle_0 &= -\frac{\delta^{ab} (\tilde{\lambda} E, \lambda \mathbf{p})}{(\tilde{p} \cdot p) (\hat{p} \cdot p)}, \\ \langle \phi^a(p) \phi^b(-p) \rangle_0 &= -\frac{i \delta^{ab} (\lambda + \tilde{\lambda})}{(\tilde{p} \cdot p) (\hat{p} \cdot p)}, \end{aligned} \tag{12.3}$$

together with $\langle A^{ia}(p) A^{0b}(-p) \rangle_0 = 0$, where $\Pi^{ij} = \delta^{ij} - (p^i p^j / \mathbf{p}^2)$, $p^\mu = (p^0, \mathbf{p}) = (E, \mathbf{p})$, $\hat{p}^\mu = (\lambda E, \mathbf{p})$ and $\tilde{p}^\mu = (\tilde{\lambda} E, \mathbf{p})$. The ghost propagators are

$$\langle C^a(p) \bar{C}^b(-p) \rangle_0 = \frac{i \delta^{ab}}{\hat{p} \cdot p}, \quad \langle H^a(p) \bar{H}^b(-p) \rangle_0 = \frac{i \delta^{ab}}{\tilde{p} \cdot p}.$$

We have left the denominators $1/(\hat{p} \cdot p)$ and $1/(\tilde{p} \cdot p)$ unprescribed. The former belong to the gauge-trivial sector, while the latter belong to the cloud sector. Obviously, the poles $1/p^2$ belong to the physical sector. The virtue of the special gauge, combined with the special cloud, is that it keeps the three sectors distinct throughout the calculations (at generic λ and $\tilde{\lambda}$). The distinction also holds through the threshold decomposition of [1], which is crucial to define the diagrammatics of purely virtual particles. Specifically, the physical thresholds, which are those originated solely by the physical poles $1/p^2$, are kept distinct from the unphysical thresholds, which are those that receive any contributions from the poles $1/(\hat{p} \cdot p)$ and $1/(\tilde{p} \cdot p)$. Also note that at generic λ and $\tilde{\lambda}$ there are no double poles (which is what makes the special gauge “special” [44]).

Since the physical quantities are gauge independent, it does not matter which prescription (e.g., Feynman $i\epsilon$, or purely virtual) we use for the poles $1/(\hat{p} \cdot p)$ of the gauge-trivial sector, as long as they are all prescribed the same way. The poles $1/(\tilde{p} \cdot p)$ belonging to the cloud sector, instead, should be quantized as purely virtual, according to the rules of [1].

Practically, this means that we start with the Feynman prescription everywhere, as we would normally do, then make the threshold decomposition of Ref. [1], and finally drop

every $\tilde{\lambda}$ -dependent threshold. These operations render the whole cloud sector purely virtual.

For a variety of applications (and to have a more direct gauge/cloud match), it may be convenient to work with a purely virtual gauge-trivial sector as well. To do so, it is sufficient to adopt the fakeon prescription for the poles $1/(\hat{p} \cdot p)$ as well. This option was introduced in [45] to provide a more direct proof of unitarity in gauge theories.

At the end, we just drop all the $\lambda, \tilde{\lambda}$ -dependent thresholds that we find in the decomposition of Ref. [1], and keep only the physical ones. What we obtain is a powerful physical gauge.

12.1 Dressed fermion self-energy, again

We use the framework just defined to calculate the dressed fermion self-energy anew. For simplicity, we calculate it at rest, and assume $\lambda > 0, \tilde{\lambda} > 0$. First, we use the Feynman prescription for every pole. Then, we describe what changes when we use purely virtual clouds.

The undressed two-point function turns out to be

$$\langle \psi | \bar{\psi} \rangle = \frac{i \gamma^\mu p_\mu}{p^2 + i\epsilon} \left[1 - \frac{g^2 (1 - \sqrt{\lambda} + 2\lambda)}{8\pi^2 \epsilon (1 + \sqrt{\lambda})} \times \frac{N_c^2 - 1}{2N_c} (-p^2 - i\epsilon)^{-\epsilon/2} \right] + \mathcal{O}(g^4), \tag{12.4}$$

instead of (11.3). The other diagrams of Fig. 1 give

$$\begin{aligned} & -\frac{i \gamma^\mu p_\mu}{p^2 + i\epsilon} \frac{g^2 (\sqrt{\lambda} - \sqrt{\tilde{\lambda}}) (1 - \sqrt{\lambda} - \sqrt{\tilde{\lambda}} - \sqrt{\lambda \tilde{\lambda}})}{4\pi^2 \epsilon (1 + \sqrt{\lambda}) (1 + \sqrt{\tilde{\lambda}})} \times \\ & \frac{N_c^2 - 1}{2N_c} (-p^2 - i\epsilon)^{-\epsilon/2} + \mathcal{O}(g^4), \end{aligned}$$

instead of (11.4). In total we get

$$\langle \psi_d | \bar{\psi}_d \rangle = \frac{i \gamma^\mu p_\mu}{p^2 + i\epsilon} \left[1 - \frac{g^2 (1 - \sqrt{\tilde{\lambda}} + 2\tilde{\lambda})}{8\pi^2 \epsilon (1 + \sqrt{\tilde{\lambda}})} \frac{N_c^2 - 1}{2N_c} \times (-p^2 - i\epsilon)^{-\epsilon/2} \right] + \mathcal{O}(g^4) = \langle \psi | \bar{\psi} \rangle_{\lambda \rightarrow \tilde{\lambda}},$$

in agreement with formula (8.11). Note that although formula (8.11) was derived in the covariant gauge and with a covariant cloud, it also applies to the present calculation, because the exchange $G^a(A) \leftrightarrow V^a(\phi, A)$ still amounts to $\lambda \leftrightarrow \tilde{\lambda}$, for the choices of gauge-fixing and cloud that we have made.

Now we switch to purely virtual clouds, using the fakeon prescription for the poles $1/(\tilde{p} \cdot p)$ of (12.3). The imaginary part of $\langle \psi_d | \bar{\psi}_d \rangle$ remains the same, so we can focus on the absorptive part

$$\begin{aligned} & \text{Abso}[\langle \psi_d | \bar{\psi}_d \rangle_R] \\ & = -2\text{Re} \sum_s \bar{u}_s (-i \gamma^\mu p_\mu) \langle \psi_d | \bar{\psi}_d \rangle_R (-i \gamma^\nu p_\nu) u_s \end{aligned}$$

$$= 2p^2 \text{ReTr}[\gamma^\mu p_\mu \langle \psi_d | \bar{\psi}_d \rangle_R], \tag{12.5}$$

which is the one affected by the prescription. By definition, all the contributions to (12.5) coming from the $\tilde{\lambda}$ -dependent thresholds drop out, due to the fakeon prescription for the poles $1/(\tilde{p} \cdot p)$. The contributions of the λ -dependent thresholds compensate one another, so we can use the prescription we want for the poles $1/(\hat{p} \cdot p)$. Choosing the fakeon prescription for them as well, we see that the absorptive part of $\langle \psi_d | \bar{\psi}_d \rangle$ comes from the sole second diagram of Fig. 1, which is the usual self-energy diagram, provided we restrict the gauge-field propagator (12.3) to its physical part

$$\langle A^{\mu a}(p) A^{vb}(-p) \rangle_{0\text{phys}} = i \frac{\delta^{ab} \delta_i^\mu \delta_j^v \Pi^{ij}}{p^2 + i\epsilon}, \tag{12.6}$$

which is the one obtained by dropping the poles $1/(\hat{p} \cdot p)$ and $1/(\tilde{p} \cdot p)$ in (12.3). Finally, the absorptive part does not need renormalization. The final result is

$$\text{Abso}[\langle \psi_d | \bar{\psi}_d \rangle_R] = \frac{g^2}{2\pi} p^2 \theta(p^2). \tag{12.7}$$

As desired, it is gauge independent, cloud independent and positive. Ultimately, this is the physical content of the fermion two-point function at one loop.

The calculation has been done for fermions at rest. The general, off-shell result is not Lorentz invariant. The reason is that, in order to compensate for the gauge dependence of the undressed fermion ψ , the cloud must be built with the longitudinal and temporal components of the gauge fields. The very definition of such components requires to specify a Lorentz frame. If we want, we can even choose different Lorentz frames for each cloud and for the gauge-fixing.

The cloud Faddeev–Popov determinant did not contribute so far. It contributes to $\langle \psi_d | \bar{\psi}_d \rangle$ starting from two loops. It also contributes to the one-loop two-point function $\langle A_d | A_d \rangle$ of the dressed gauge fields (see below). Clearly, it is crucial for the gauge/cloud duality.

Sometimes, it may be convenient to simplify the calculations by choosing $\lambda = \tilde{\lambda} = 1$. In that case, the propagators (12.3) become

$$\begin{aligned} \langle A^{ia}(p) A^{jb}(-p) \rangle_0 &= \frac{i\delta^{ab} \Pi^{ij}}{p^2 + i\epsilon} + \frac{i\delta^{ab} p^i p^j}{\mathbf{p}^2 p^2} \Big|_f, \\ \langle A^{0a}(p) A^{0b}(-p) \rangle_0 &= - \frac{i\delta^{ab}}{p^2} \Big|_f, \\ \langle A^{\mu a}(p) \phi^b(-p) \rangle_0 &= - \frac{\delta^{ab} p^\mu}{(p^2)^2} \Big|_f, \\ \langle \phi^a(p) \phi^b(-p) \rangle_0 &= - \frac{2i\delta^{ab}}{(p^2)^2} \Big|_f, \end{aligned} \tag{12.8}$$

where the subscript “f” denotes the fakeon prescription. The various types of thresholds (physical, gauge or cloud) are not

manifestly distinct at $\lambda = \tilde{\lambda} = 1$, so we have to keep track of their origins in different ways.

Note that (12.8) involves square denominators like $1/(p^2)^2$. Their fakeon prescription in a diagram G is defined as follows:

$$G \left[\frac{1}{(p^2)^2} \Big|_f \right] \rightarrow - \lim_{\mu^2 \rightarrow 0} \frac{d}{d\mu^2} G \left[\frac{1}{p^2 - \mu^2} \Big|_f \right].$$

Power counting is straightforward with both the covariant and special gauge and clouds. It may not work equally well with other choices of gauges and clouds. An example is the Coulomb gauge, which can be obtained from the special gauge by letting λ tend to zero. Similarly, the “Coulomb cloud” can be obtained by letting $\tilde{\lambda}$ tend to zero in the special cloud. In those limits, the integrals on the loop energies and the integrals on the space components of the loop momenta obey different power counting rules.

No particular prescription is needed to treat the Coulomb poles $1/\mathbf{p}^2$. They are purely virtual by accident, in some sense. Furthermore, formula (8.11) shows that the correlation functions of the dressed fields at $\tilde{\lambda} = 0$ coincide with those of the undressed fields in the Coulomb gauge. The absorptive part at rest clearly coincides with (12.7), because it is cloud independent.

13 Dressed gauge-field two-point function

In this section we study the dressed gauge-field two-point function at one loop, sticking to pure Yang–Mills theory for simplicity. As before, we start from the covariant gauge and the covariant cloud, with the Feynman prescription everywhere. At a second stage we switch to purely virtual clouds.

First, we verify that the ordinary two-point function $\langle A_\mu^a | A_\nu^b \rangle$ is cloud independent, to check the results of Sect. 5. The diagrams contributing to $\langle A_\mu^a | A_\nu^b \rangle$ are too many to be listed here and include loops of cloud ghosts $H\text{-}\bar{H}$. Collecting everything together and subtracting the divergent part, the result is the same as usual, i.e.,

$$\begin{aligned} \langle A^{\mu a} | A^{\nu b} \rangle_{\text{one-loop}} &= \frac{ig^2 N_c (13 - 3\lambda)}{6(4\pi)^2 (p^2)^2} \delta^{ab} (p^2 \eta^{\mu\nu} - p^\mu p^\nu) \ln(-p^2 - i\epsilon). \end{aligned} \tag{13.1}$$

As expected, the dependence on the cloud parameter $\tilde{\lambda}$ disappears and the gauge dependence remains.

Next, we compute the two-point function $\langle A_{d\mu}^a | A_{d\nu}^b \rangle$ of the dressed gauge fields, still in the covariant gauge. The number of diagrams is even larger, but the final result is extremely simple and coincides with (13.1), apart from the replacement $\lambda \rightarrow \tilde{\lambda}$, in agreement with the general property

(8.11):

$$\begin{aligned} &\langle A_d^{\mu a} | A_d^{v b} \rangle_{\text{one-loop}} \\ &= \frac{i g^2 N_c (13 - 3\tilde{\lambda})}{6(4\pi)^2 (p^2)^2} \delta^{ab} (p^2 \eta^{\mu\nu} - p^\mu p^\nu) \ln(-p^2 - i\epsilon). \end{aligned}$$

As in the case of the fermion self-energy, the absorptive part,

$$\begin{aligned} \text{Abso} \left[\langle A_d^{\mu a} | A_d^{v b} \rangle \right] &= -2\text{Re} \left[(ip^2) \langle A_d^{\mu a} | A_d^{v b} \rangle (ip^2) \right] \\ &= \frac{g^2 N_c (13 - 3\tilde{\lambda})}{48\pi} \delta^{ab} (p^2 \eta^{\mu\nu} \\ &\quad - p^\mu p^\nu) \theta(p^2), \end{aligned}$$

is not physical, because the cloud is not physical.

Switching to purely virtual clouds, the imaginary part does not change. To work out the real part, we just need to compute one diagram, i.e., the self-energy diagram where physical gauge fields circulate with the propagator (12.6). Note that we do not need to use the vertex $AA\phi$, because it involves at least one divergence $\partial^\mu A_\mu$, by formula (11.1). We obtain, for $p^\mu = (p^0, \mathbf{0})$,

$$\begin{aligned} \text{Abso}[\langle A_d^{ia}(p) A_d^{jb}(-p) \rangle] &= \frac{g^2 N_c \delta^{ab} \delta^{ij}}{24\pi} p^2 \theta(p^2), \\ \text{Abso}[\langle A_d^{0a}(p) A_d^{0b}(-p) \rangle] &= \text{Abso}[\langle A_d^{0a}(p) A_d^{ib}(-p) \rangle] = 0. \end{aligned}$$

The results are again gauge independent, cloud independent and nonnegative.

14 Renormalization

In this section we study the renormalization of the extended theory. We show that everything goes through in the usual way (by means of renormalization constants for the couplings, the masses, the fields and the sources) apart from non-polynomial, nonderivative redefinitions of the cloud fields and their anticommuting partners into functions of themselves, with no mixings among different cloud sectors.

14.1 Master equations

First, the master equation (4.9) implies an analogous master equation

$$(\Gamma_{\text{tot}}, \Gamma_{\text{tot}}) = 0 \tag{14.1}$$

for the generating functional $\Gamma_{\text{tot}} = W_{\text{tot}}(J, \tilde{J}, K, \tilde{K}) - \int \Phi^\alpha J_\alpha - \sum_i \int \tilde{\Phi}^{\alpha i} \tilde{J}_\alpha^i$ of the connected, one-particle irreducible (1PI) Green functions, where $\Phi^\alpha = \delta_r W_{\text{tot}} / \delta J^\alpha$, $\tilde{\Phi}^{\alpha i} = \delta_r W_{\text{tot}} / \delta \tilde{J}^{\alpha i}$. The proof follows from a change of field variables

$$\Phi^\alpha \rightarrow \Phi^\alpha + \theta(S_{\text{tot}}, \Phi^\alpha), \quad \tilde{\Phi}^{\alpha i} \rightarrow \tilde{\Phi}^{\alpha i} + \theta(S_{\text{tot}}, \tilde{\Phi}^{\alpha i}),$$

in the functional integral (5.1) that defines $Z_{\text{tot}} = \exp(i W_{\text{tot}})$. Only the source terms $\int \Phi J + \int \tilde{\Phi} \tilde{J}$ contribute, giving

$$\int \langle (S_{\text{tot}}, \Phi^\alpha) \rangle J^\alpha + \sum_i \int \langle (S_{\text{tot}}, \tilde{\Phi}^{\alpha i}) \rangle \tilde{J}^{\alpha i} = 0,$$

which can easily be rewritten as (14.1). When (4.9) does not hold, the same argument gives $(\Gamma_{\text{tot}}, \Gamma_{\text{tot}}) = \langle (S_{\text{tot}}, S_{\text{tot}}) \rangle$.

Second, the i th cloud invariance of the total action S_{tot} , i.e., the identity $(S_K^{\text{cloud}i}, S_{\text{tot}}) = 0$ of (9.4), implies the i th cloud invariance

$$(S_K^{\text{cloud}i}, \Gamma_{\text{tot}}) = 0 \tag{14.2}$$

of the Γ functional. The proof follows from the change of field variables

$$\begin{aligned} \Phi^\alpha &\rightarrow \Phi^\alpha, \quad \tilde{\Phi}^{\alpha i} \rightarrow \tilde{\Phi}^{\alpha i} + \theta(S_K^{\text{cloud}i}, \tilde{\Phi}^{\alpha i}), \\ \tilde{\Phi}^{\alpha j} &\rightarrow \tilde{\Phi}^{\alpha j} \text{ for } i \neq j, \end{aligned} \tag{14.3}$$

in Z_{tot} . Both the source terms and the action S_{tot} contribute now, giving

$$\left\langle \int \frac{\delta_r S_K^{\text{cloud}i}}{\delta \tilde{K}^{\alpha i}} \frac{\delta_l S_{\text{tot}}}{\delta \tilde{\Phi}^{\alpha i}} \right\rangle = \int (S_K^{\text{cloud}i}, \tilde{\Phi}^{\alpha i}) \tilde{J}^{\alpha i},$$

which can be rewritten as (14.2), after using $(S_K^{\text{cloud}i}, S_{\text{tot}}) = 0$ in the left-hand side.

14.2 Renormalization algorithm

Proceeding inductively, we denote the order of the loop expansion by the power of \hbar (although \hbar is set to one everywhere else in this paper). We assume that we have renormalized the theory up to n loops. We denote the so-renormalized action by $S_{n \text{ tot}}$ and the Γ functional associated with it by $\Gamma_{n \text{ tot}}$. We also assume that $S_{n \text{ tot}}$ has the form $S_{n \text{ tot}} = S_{\text{tot}} +$ divergent counterterms (in some subtraction scheme) and satisfies

$$(S_{n \text{ tot}}, S_{n \text{ tot}}) = \mathcal{O}(\hbar^{n+1}), \quad (S_K^{\text{cloud}i}, S_{n \text{ tot}}) = 0.$$

The inductive assumptions are clearly satisfied at order zero.

The locality of counterterms ensures, as usual, that the order $(n + 1)$ divergent part $\Gamma_{n \text{ tot div}}^{(n+1)}$ of $\Gamma_{n \text{ tot}}$ is local. By the argument above, the effective action $\Gamma_{n \text{ tot}}$ satisfies $(\Gamma_{n \text{ tot}}, \Gamma_{n \text{ tot}}) = \langle (S_{n \text{ tot}}, S_{n \text{ tot}}) \rangle = (S_{n \text{ tot}}, S_{n \text{ tot}}) + \mathcal{O}(\hbar^{n+2})$. The $(n + 1)$ -th order divergent part of this equation gives

$$2(S_{\text{tot}}, \Gamma_{n \text{ tot div}}^{(n+1)}) = (S_{n \text{ tot}}, S_{n \text{ tot}}) + \mathcal{O}(\hbar^{n+2}),$$

having noted that $(S_{n \text{ tot}}, S_{n \text{ tot}})$ is divergent. Defining

$$\begin{aligned} S_{n+1 \text{ tot}} &= S_{n \text{ tot}} - \Gamma_{n \text{ tot div}}^{(n+1)} \\ &= S_{\text{tot}} + \text{divergent counterterms}, \end{aligned} \tag{14.4}$$

we have

$$(S_{n+1 \text{ tot}}, S_{n+1 \text{ tot}}) = (S_{n \text{ tot}}, S_{n \text{ tot}}) - 2(S_{n \text{ tot}}, \Gamma_{n \text{ tot div}}^{(n+1)}) + \mathcal{O}(\hbar^{n+2}) = \mathcal{O}(\hbar^{n+2}).$$

Moreover, $(S_K^{\text{cloud}i}, S_{n \text{ tot}}) = 0$ implies $(S_K^{\text{cloud}i}, \Gamma_{n \text{ tot}}) = 0$ and its $(n+1)$ -th order divergent part gives $(S_K^{\text{cloud}i}, \Gamma_{n \text{ tot div}}^{(n+1)}) = 0$, which in turn implies $(S_K^{\text{cloud}i}, S_{n+1 \text{ tot}}) = 0$. Finally, by (14.4) $\Gamma_{n+1 \text{ tot}}$ is convergent up to the order $n + 1$ included. Thus, the inductive assumptions are fully replicated to that order. This allows us to take the argument to $n \rightarrow \infty$, where we obtain the renormalized action $S_{R \text{ tot}} \equiv S_{\infty \text{ tot}}$, and conclude that it satisfies the renormalized master equations

$$(S_{R \text{ tot}}, S_{R \text{ tot}}) = 0, \quad (S_K^{\text{cloud}i}, S_{R \text{ tot}}) = 0. \tag{14.5}$$

14.3 Renormalized action

Now we characterize $S_{R \text{ tot}}$ more precisely. Besides the usual ghost number, we introduce ‘‘cloud numbers’’ for each cloud. The usual ghost number is equal to 1 for C , minus 1 for \bar{C} , K_B , K_A^μ , K_ψ , $K_{\bar{\psi}}$, \tilde{K}_ϕ^i and \tilde{K}_H^i , minus 2 for K_C , and 0 for every other field and source. The i th cloud number is equal to one for H^i , minus one for \bar{H}^i , \tilde{K}_H^i and \tilde{K}_E^i , and zero in all the other cases.

Every term of the action S_{tot} is neutral with respect to the ghost and cloud numbers just defined, with the exception of the source terms $\int H^i \tilde{K}_\phi^i$. Since, however, such terms cannot be used in nontrivial 1PI diagrams, all the counterterms are neutral. Thus, each cloud number is separately conserved by the 1PI diagrams beyond the tree level.

By power counting, the counterterms can be at most linear in the sources. Indeed, the dimensions of \tilde{K}_ϕ^i , \tilde{K}_H^i , $\tilde{K}_{\bar{H}}^i$, \tilde{K}_E^i are 3, 2, 2 and 1, respectively, but \tilde{K}_E^i never appears, $\tilde{K}_{\bar{H}}^i$ appears trivially and does not participate in the counterterms, while a bilinear in \tilde{K}_H^i is prohibited by the conservation of the i th cloud number.

We recall that the cloud symmetry generated by S_K^{cloud} collects the most general shifts of the cloud fields ϕ^i , combined with analogous shifts of \bar{H}^i and the sources \tilde{K}_H^i and \tilde{K}_E^i . A general theorem (which is easily proved by switching to the language of differential forms – see, for example, the appendix of [46]) ensures that a local functional X that is closed with respect to a symmetry of this type (i.e., such that $(S_K^{\text{cloud}}, X) = 0$) is the sum of an exact local functional (i.e., a functional of the form (S_K^{cloud}, Y) , for some other local functional Y) plus a local functional that is independent on the shifted fields, as well as their shifts.

Since $S_{R \text{ tot}}$ satisfies the second Eq. (14.5) for every i , hence $(S_K^{\text{cloud}}, S_{R \text{ tot}}) = 0$, it can be written as the sum

$$S_{R \text{ tot}} = S_R + (S_K^{\text{cloud}}, \Upsilon_R) + S_K^{\text{cloud}} \tag{14.6}$$

of a local functional S_R that does not depend on the cloud fields and the cloud sources, plus a cloud exact functional, where Υ_R is local. We have separated S_K^{cloud} from the rest, because S_K^{cloud} is nonrenormalized, due to its triviality.

14.4 Cloud independence through renormalization

Now we prove that the functional S_R coincides with the usual renormalized action. To achieve this goal, we need to show that the cloud independence theorem of Sect. 5 safely goes through the renormalization algorithm. The proof given in Sect. 5, which relies on the specific forms of the action and the cloud fermions used there, needs to be upgraded in a nontrivial way.

Since every renormalized action $S_{n \text{ tot}}$, as well as the functionals $\Gamma_{n \text{ tot div}}^{(n+1)}$, satisfy $(S_K^{\text{cloud}}, S_{n \text{ tot}}) = (S_K^{\text{cloud}}, \Gamma_{n \text{ tot div}}^{(n+1)}) = 0$, the argument used for (14.6) allows us to write them as

$$S_{n \text{ tot}} = S_n + (S_K^{\text{cloud}}, \Upsilon_n) + S_K^{\text{cloud}}, \tag{14.7}$$

$$\Gamma_{n \text{ tot div}}^{(n+1)} = \Gamma_{n \text{ div}}^{(n+1)} + (S_K^{\text{cloud}}, R_n^{(n+1)}),$$

where S_n and $\Gamma_{n \text{ div}}^{(n+1)}$ are independent of the cloud fields $\tilde{\Phi}^{ai}$ and the cloud sources \tilde{K}^{ai} .

Assume, by induction, that S_n is cloud independent (that is to say, independent of the cloud parameters $\tilde{\lambda}$). Let $Z_{n \text{ tot}}(J, K, \tilde{J}, \tilde{K})$ denote the generating functional associated with the action $S_{n \text{ tot}}$. At $\tilde{J} = \tilde{K} = 0$ it reads

$$Z_{n \text{ tot}}(J, K, 0, 0) = \int [d\Phi d\tilde{\Phi}] \exp \left(i S_n(\Phi, K) + i (S_K^{\text{cloud}}, \tilde{\Upsilon}_n) + i \int \Phi^\alpha J_\alpha \right) = \int [d\Phi] \exp \left(i S_n(\Phi, K) + i \int \Phi^\alpha J_\alpha \right) \int [d\tilde{\Phi}] e^{i (S_K^{\text{cloud}}, \tilde{\Upsilon}_n)}, \tag{14.8}$$

where $\tilde{\Upsilon}_n$ denotes Υ_n at $\tilde{K} = 0$. The cloud sector does not contribute, because

$$\int [d\tilde{\Phi}] e^{i (S_K^{\text{cloud}}, \tilde{\Upsilon}_n)} = 1. \tag{14.9}$$

This identity can be proved as follows. The left-hand side is in principle a functional of the fields Φ and the sources K , since $\tilde{\Upsilon}_n$ may depend on them. To show that it is actually a constant, we consider arbitrary infinitesimal variations of Φ and K . If $\delta \tilde{\Upsilon}_n$ denotes the variation of $\tilde{\Upsilon}_n$ due to them, the variation of the integral is

$$\delta \int [d\tilde{\Phi}] e^{i (S_K^{\text{cloud}}, \tilde{\Upsilon}_n)} = i \int [d\tilde{\Phi}] (S_K^{\text{cloud}}, \delta \tilde{\Upsilon}_n) e^{i (S_K^{\text{cloud}}, \tilde{\Upsilon}_n)}, \tag{14.10}$$

Performing the change of field variables $\tilde{\Phi}^\alpha \rightarrow \tilde{\Phi}^\alpha + \theta (S_K^{\text{cloud}}, \tilde{\Phi}^\alpha)$ in the integral

$$\int [d\tilde{\Phi}] \delta \tilde{\Upsilon}_R e^{i (S_K^{\text{cloud}}, \tilde{\Upsilon}_n)},$$

we obtain

$$\int [d\tilde{\Phi}] \delta \tilde{\Upsilon}_n e^{i(S_K^{\text{cloud}}, \tilde{\Upsilon}_n)} = \int [d\tilde{\Phi}] \left[\delta \tilde{\Upsilon}_n + \theta(S_K^{\text{cloud}}, \delta \tilde{\Upsilon}_n) \right] e^{i(S_K^{\text{cloud}}, \tilde{\Upsilon}_n)}. \tag{14.11}$$

We have used the fact that $(S_K^{\text{cloud}}, \tilde{\Upsilon}_n)$ is independent of the sources \tilde{K} , so $(S_K^{\text{cloud}}, \tilde{\Upsilon}_n) \rightarrow (S_K^{\text{cloud}}, \tilde{\Upsilon}_n) + \theta(S_K^{\text{cloud}}, (S_K^{\text{cloud}}, \tilde{\Upsilon}_n)) = (S_K^{\text{cloud}}, \tilde{\Upsilon}_n)$. The equality (14.11) shows that the right-hand side of (14.10) vanishes, as we wished to prove.

Thus, all the connected correlation functions of the undressed fields, which are collected in $W_{n\text{tot}}(J, K, 0, 0)$, coincide with the usual ones, even at the renormalized level. Not only, we can also show that the IPI correlation functions of the undressed fields, collected in $\Gamma_{n\text{tot}}(\Phi, K, 0, 0)$, coincide with the usual ones. Indeed, it is easy to see, using the second equation of (14.7), that setting $\tilde{\Phi}^i = 0$ is equivalent to setting $\tilde{J}^i = 0$ in all cases apart from \tilde{J}_E^i . The proof given above also works if we keep the sources \tilde{J}_E^i arbitrary, since $\sum_i \int E^i \tilde{J}_E^i = (S_K^{\text{cloud}}, \sum_i \int \tilde{H}^i \tilde{J}_E^i)$: the derivation can be repeated with $\tilde{\Upsilon}_n \rightarrow \tilde{\Upsilon}_n + \sum_i \int \tilde{H}^i \tilde{J}_E^i$. Thus, even $W_{n\text{tot}}(J, K, \hat{0}, 0)$ coincides with the usual one, where $\hat{0}$ means that all the sources \tilde{J}^i are set to zero but \tilde{J}_E^i . Actually, $W_{n\text{tot}}(J, K, \hat{0}, 0)$ does not even depend on \tilde{J}_E^i . These facts imply that $\Gamma_{n\text{tot}}(\Phi, K, 0, 0)$ coincides with the usual non-cloud one, and so does $\Gamma_{n\text{div}}^{(n+1)}$. In particular, $\Gamma_{n\text{div}}^{(n+1)}$ is cloud independent. Then, (14.4) shows that S_{n+1} , inside $S_{n+1\text{tot}}$, is cloud independent. Finally, we can take n to infinity, and infer that S_R is cloud independent and coincides with the usual renormalized action.

We have thus achieved a neat separation between the fundamental theory and the cloud sectors, and ensured that the separation is compatible with renormalization. We recall that S_R is determined by gauge-independent renormalization constants Z_g and Z_m for the coupling g and the fermion mass m , respectively, plus a generically gauge-dependent canonical transformation that incorporates the wave-function renormalization constants of the fields Φ and the sources K .

The arguments of this subsection can be specialized to every cloud, to prove that the i th cloud parameters $\tilde{\lambda}_i$ do not propagate to the other cloud sectors.

14.5 Renormalized clouds

Now we analyze the functional Υ_R . Separating the source-independent part $\tilde{\Psi}_R$ of Υ_R from the source-dependent part, we can write

$$(S_K^{\text{cloud}}, \Upsilon_R) = (S_K^{\text{cloud}}, \tilde{\Psi}_R) + \sum_i \int (C^a S^{abi}(\phi) \tilde{K}_\phi^{bi} + C^a H^{bi} S^{abc}(\phi) \tilde{K}_H^{ci}), \tag{14.12}$$

for some functions $S^{abi}(\phi)$ and $S^{abc}(\phi)$, which encode the gauge transformations of the cloud fields and the cloud ghosts. The structures of the last two terms are determined by the conservations of the ghost and cloud numbers, as well as power counting and cloud exactness. These same properties exclude any other source-dependent terms.

The functions $S^{abi}(\phi)$ and $S^{abc}(\phi)$ are not independent, since by cloud exactness it must be possible to collect the last two terms of (14.12) into

$$\left(S_K^{\text{cloud}}, \sum_i \int C^a S^{abi}(\phi) \tilde{K}_H^{bi} \right). \tag{14.13}$$

Moreover, each function $S^{abi}(\phi)$ can depend only on the i th cloud field ϕ^i , because otherwise (14.13) is not neutral with respect to each cloud number separately. Thus, from now on we write $S^{abi}(\phi^i)$.

Consider $(S_K^{\text{cloud}}, \tilde{\Psi}_R)$. The gauge and cloud conditions we have used in this paper, which are (3.8), (4.11), (12.1) and (12.2), ensure that the counterterms can depend on \tilde{C} , \tilde{H}^i and E^i only through the derivatives $\partial_\mu \tilde{C}$, $\partial_\mu \tilde{H}^i$, $\partial_\mu E^i$. Moreover, they cannot depend on B . We want to show that the renormalized gauge fermion $\tilde{\Psi}_R$ has the form

$$\tilde{\Psi}_R = \sum_i \int \tilde{H}^{ai} \left(V_R^{ai}(A, \phi^i) + \frac{\tilde{\lambda}_i}{2} E^{ai} \right) \tag{14.14}$$

for some local functions V_R^{ai} that depend only on the i th cloud fields ϕ^i . First, note that a term proportional to \tilde{C} cannot appear in $\tilde{\Psi}_R$, because its antiparenthesis with S_K^{cloud} would have dimension greater than four, or not be neutral with respect to the ghost and cloud numbers.

Second, the coefficient of \tilde{H}^{ai} in $\tilde{\Psi}_R$ has dimension 2. It cannot contain B , because B appears trivially in the action. It cannot depend on \tilde{C} and C either, because $(S_K^{\text{cloud}}, \tilde{\Psi}_R)$ would contain a term $\sim E\tilde{C}C$, which cannot be generated, since E and \tilde{C} can appear only through their derivatives. For the same reason, the terms $\sim \tilde{H}E$ of $\tilde{\Psi}_R$ are nonrenormalized, because any corrections would bring counterterms $\sim E^2$ in $(S_K^{\text{cloud}}, \tilde{\Psi}_R)$.

Thus, $\tilde{\Psi}_R$ can only contain the gauge fields and the cloud fields in the way we have shown in (14.14). Moreover, the functions V_R^{ai} can only depend on the i th cloud field ϕ^i , otherwise $(S_K^{\text{cloud}}, \tilde{\Psi}_R)$ would violate the i th cloud number conservation. This proves that there are no mixings among different cloud sectors.

Finally, the renormalized action reads

$$S_{R\text{tot}} = S_R + \sum_i \left(S_K^{\text{cloud } i}, \int \tilde{H}^{ai} \left(V_R^{ai}(A, \phi^i) + \frac{\tilde{\lambda}_i}{2} E^{ai} \right) + \int C^a S^{abi}(\phi^i) \tilde{K}_H^{bi} \right) + S_K^{\text{cloud}}, \tag{14.15}$$

It may also be convenient to organize it as

$$S_{R\text{tot}} = S_{R\text{tot}0} + S_{K,R}^{\text{gauge}} + S_K^{\text{cloud}}, \tag{14.16}$$

where $S_{R\text{tot}0} = S_{R\text{tot}}|_{K=\tilde{K}=0}$. The renormalized gauge transformations are encoded in $S_{K,R}^{\text{gauge}}$. Separating the various contributions according to their dependences on the sources K and \tilde{K} , the master equations (14.5) imply

$$\begin{aligned} (S_{K,R}^{\text{gauge}} + S_K^{\text{cloud}}, S_{R\text{tot}0}) &= 0, \\ (S_{K,R}^{\text{gauge}} + S_K^{\text{cloud}}, S_{K,R}^{\text{gauge}} + S_K^{\text{cloud}}) &= 0, \\ (S_K^{\text{cloud}i}, S_{R\text{tot}0}) = (S_K^{\text{cloud}i}, S_{K,R}^{\text{gauge}} + S_K^{\text{cloud}}) &= 0, \end{aligned}$$

which immediately give

$$\begin{aligned} (S_{K,R}^{\text{gauge}}, S_{R\text{tot}0}) = (S_K^{\text{cloud}i}, S_{R\text{tot}0}) &= 0, \\ (S_{K,R}^{\text{gauge}}, S_{K,R}^{\text{gauge}}) = (S_K^{\text{cloud}i}, S_{K,R}^{\text{gauge}}) &= 0. \end{aligned} \tag{14.17}$$

Combined, the two sets of equations imply $(S_{K,R}^{\text{gauge}}, S_{R\text{tot}}) = 0$, which was not obvious from (14.5). The left equations ensure that $S_{R\text{tot}0}$ is gauge and cloud invariant. In particular, all the functions V_R^{ai} of (14.15) must be gauge invariant, by the gauge invariance of the terms $E^{ai}V_R^{ai}$ contained in $(S_K^{\text{cloud}}, \tilde{\Psi}_R)$.

The functions $S^{abi}(\phi^i)$ are further constrained by the closure of the renormalized gauge transformations. Apart from that, they are arbitrary. Indeed, it is always possible to make nontrivial redefinitions that send each ϕ^i into a function of itself. Then, for consistency, each H^i must be sent into H^i times a suitable function of ϕ^i . Since the fields ϕ^i are dimensionless, renormalization can activate nonpolynomial, nonderivative redefinitions of this type. By the second equation of (C.8), these redefinitions can depend on the gauge-fixing parameters and the i th cloud parameters, but not on the parameters of the other clouds.

14.6 Renormalized dressed fields

Using the renormalized gauge transformations, which are encoded in the functional $S_{K,R}^{\text{gauge}}$, we can build the renormalized, gauge invariant dressed fields $A_{\mu dR}$, ψ_{dR} and $\bar{\psi}_{dR}$. By the result of appendix B, their expressions are unique up to constant (matrix) factors. We want to prove that we can fix those factors so that the correlation functions of $A_{\mu dR}$, ψ_{dR} and $\bar{\psi}_{dR}$ are gauge independent.

Including appropriate sources, the extended renormalized action is

$$S_{R\text{tot}}^{\text{ext}} = S_{R\text{tot}} + \int (J_{dR}^\mu A_{\mu dR} + \bar{J}_{\psi dR} \psi_{dR} + \bar{\psi}_{dR} J_{\psi dR}).$$

Clearly, $(S_{K,R}^{\text{gauge}}, S_{R\text{tot}0}^{\text{ext}}) = 0$, where $S_{R\text{tot}0}^{\text{ext}} = S_{R\text{tot}}^{\text{ext}}|_{K=\tilde{K}=0}$.

The equations of gauge dependence (C.9), derived in appendix C, evaluated at $K = \tilde{K} = 0$, give

$$\begin{aligned} \frac{\partial S_{R\text{tot}0}}{\partial \lambda} &= (S_{R\text{tot}0}, \Psi_{\lambda R1}) + (S_{K,R}^{\text{gauge}}, \Psi_{\lambda R0}), \\ \frac{\partial S_{K,R}^{\text{gauge}}}{\partial \lambda} &= (S_{K,R}^{\text{gauge}}, \Psi_{\lambda R1}), \\ (S_K^{\text{cloud}i}, \Psi_{\lambda R0}) = (S_K^{\text{cloud}i}, \Psi_{\lambda R1}) &= 0, \end{aligned} \tag{14.18}$$

where $\Psi_{\lambda R0} = \Psi_{\lambda R}|_{K=\tilde{K}=0}$ and $\Psi_{\lambda R1} = \Psi_{\lambda R} - \Psi_{\lambda R0}$. In particular, the first formula gives

$$\begin{aligned} \frac{\partial S_{R\text{tot}0}}{\partial \lambda} &= \int \frac{\delta_r S_{R\text{tot}0}}{\delta \hat{\Phi}^\alpha} \Delta \hat{\Phi}^\alpha + (S_{K,R}^{\text{gauge}}, \Psi_{\lambda R0}), \\ \Delta \hat{\Phi}^\alpha &= \frac{\delta_l \Psi_{\lambda R1}}{\delta \hat{K}^\alpha}, \end{aligned}$$

where $\hat{\Phi}^\alpha = (\Phi^\alpha, \tilde{\Phi}^\alpha)$ and $\hat{K}^\alpha = (K^\alpha, \tilde{K}^\alpha)$. This result tells us the the whole gauge dependence of $S_{R\text{tot}0}$ is encoded into a field redefinition, plus a gauge-exact term. The field redefinition $\hat{\Phi}^\alpha = \hat{\Phi}^\alpha(\hat{\Phi}', \lambda)$ is the solution of

$$\frac{\partial \hat{\Phi}^\alpha(\hat{\Phi}', \lambda)}{\partial \lambda} = -\Delta \hat{\Phi}^\alpha(\hat{\Phi}(\hat{\Phi}', \lambda), \lambda),$$

with arbitrary initial conditions. It can be worked out perturbatively in g , starting from $\hat{\Phi}^\alpha = \hat{\Phi}^{\alpha'}$.

It is convenient to switch to the new variables $\hat{\Phi}^{\alpha'}$, $\hat{K}^{\alpha'}$ by means of the canonical transformation generated by $F(\hat{\Phi}, \hat{K}') = \int \hat{\Phi}'(\hat{\Phi}, \lambda) \hat{K}'$. In so doing, we obtain

$$\frac{\partial S'_{R\text{tot}0}}{\partial \lambda} = (S_{K,R}^{\text{gauge}'}, \Psi'_{\lambda R0}), \quad \frac{\partial S_{K,R}^{\text{gauge}'}}{\partial \lambda} = 0, \tag{14.19}$$

where the prime on the antiparentheses refers to the new variables. The last equation follows from

$$\frac{\partial \hat{K}^\alpha(\hat{\Phi}', \hat{K}', \lambda)}{\partial \lambda} = \frac{\delta_l \Psi_{\lambda R1}}{\delta \hat{\Phi}^\alpha}(\hat{\Phi}(\hat{\Phi}', \hat{K}(\hat{\Phi}', \hat{K}')),$$

which is easy to prove from the transformation.

Now we consider the correlations functions that contain insertions of $A_{\mu dR}$, ψ_{dR} and $\bar{\psi}_{dR}$, and apply arguments that are analogous to those of Sect. 7.1. First, we switch to the variables with primes everywhere. We denote the transformed action $S_{R\text{tot}}(\hat{\Phi}(\hat{\Phi}'), \hat{K}(\hat{\Phi}', \hat{K}'))$ by $S'_{R\text{tot}}$ and the transformed renormalized dressed fields $A_{\mu dR}(\hat{\Phi}(\hat{\Phi}'))$, $\psi_{dR}(\hat{\Phi}(\hat{\Phi}'))$ and $\bar{\psi}_{dR}(\hat{\Phi}(\hat{\Phi}'))$ by $A'_{\mu dR}$, ψ'_{dR} and $\bar{\psi}'_{dR}$. Gauge invariance, which reads $(S_{K,R}^{\text{gauge}}, S_{R\text{tot}0}^{\text{ext}}) = 0$, is obviously preserved by the transformation: $(S_{K,R}^{\text{gauge}'}, S_{R\text{tot}0}^{\text{ext}'}) = 0$. The transformed fields $A'_{\mu dR}$, ψ'_{dR} and $\bar{\psi}'_{dR}$ are $S_{K,R}^{\text{gauge}'}$ -closed, i.e., solutions X' of $(S_{K,R}^{\text{gauge}'}, X') = 0$, using variables with primes.

This is where we fix the arbitrary constant factors in front of such solutions: it is sufficient to require that $A'_{\mu dR}$, ψ'_{dR} and $\bar{\psi}'_{dR}$ be gauge independent. Such a requirement does make sense, because the second formula of (14.19) ensures that the

gauge transformations themselves are gauge independent in the new variables. We just have to pay attention that the overall factors of the solutions do not introduce spurious gauge dependencies. Once this is done, we are ready to repeat the arguments of Sect. 7.1, with the replacements

$$S_{\text{tot}}^{\text{ext}} \rightarrow S'_{R\text{tot}} + \int \left(J_{\text{d}R}^{\mu} A'_{\mu\text{d}R} + \bar{J}_{\psi\text{d}R} \psi'_{\text{d}R} + \bar{\psi}'_{\text{d}R} J_{\psi\text{d}R} \right),$$

$$\Psi_{\lambda} \rightarrow \Psi'_{\lambda R0}, \quad S_K^{\text{gauge}} \rightarrow S_{K,R}^{\text{gauge}'},$$

besides of course $\hat{\Phi}, \hat{K} \rightarrow \hat{\Phi}', \hat{K}'$. The result is that the correlation functions of the renormalized dressed fields $A_{\mu\text{d}R}$, $\psi_{\text{d}R}$ and $\bar{\psi}_{\text{d}R}$ are gauge independent.

The renormalized sources $J_{\text{d}R}^{\mu}$, $J_{\psi\text{d}R}$ and $\bar{J}_{\psi\text{d}R}$ are equal to J_{d}^{μ} , $J_{\psi\text{d}}$ and $\bar{J}_{\psi\text{d}}$ times suitable renormalization constants. Apart from that, the correlation functions of $A_{\mu\text{d}R}$, $\psi_{\text{d}R}$ and $\bar{\psi}_{\text{d}R}$ do not need further renormalization. Indeed, the sources $J_{\text{d}R}^{\mu}$, $J_{\psi\text{d}R}$ and $\bar{J}_{\psi\text{d}R}$ have dimensions 3, 5/2 and 5/2, respectively, so no local counterterms with two or more of them are allowed.

Finally, the arguments that lead to the identity (6.3) continue to hold after renormalization. We obtain an identity analogous to (6.3), where the dressed and undressed fields are replaced by their renormalized versions. In particular, the S -matrix amplitudes of the renormalized dressed fields are cloud independent, and coincide with the usual amplitudes of the renormalized undressed fields. Since the former are manifestly gauge independent, the latter are gauge independent as well.

14.7 Renormalization recap

Summarizing, the renormalized action has the structure of the starting action, with standard multiplicative renormalization constants for the coupling and the masses, combined with a canonical transformation that encodes multiplicative renormalizations of the sources and the fields, except for the cloud fields ϕ^i and their anticommuting partners H^i , which are renormalized in nonpolynomial, nonderivative ways. Different cloud sectors do not mix with one another.

From a specific gauge, every other gauge can be reached by means of a canonical transformation. Thus, the renormalization in every other gauge is the same as above, up to a renormalized canonical transformation. Finally, every cloud choice can be reached from specific cloud choices by means of canonical transformations. Again, the renormalization is the same as above up to renormalized canonical transformations.

Equipped with the renormalized action and the renormalized gauge transformations, we can build dressed fields that are gauge-invariant with respect to the latter. They are unique up to constant factors, by the theorem proved in appendix B.

The constant factors can be fixed so that their correlation functions are gauge independent.

The proof of the gauge/cloud duality can be repeated for the renormalized theory.

15 Comparison with other approaches

In this section we compare the approach of this paper with other approaches that are available in the literature.

In the Dirac approach [3] the gauge invariant dressings of electrons in QED are defined by means of nonlocal operator insertions, such as

$$\exp\left(-ie\frac{\nabla\cdot\mathbf{A}}{\Delta}\right)\psi, \quad (15.1)$$

where Δ denotes the Laplacian. We may call the exponential prefactor ‘‘Coulomb–Dirac cloud’’. Photons do not need a particular dressing, since we can work directly with the field strength, which is linear in the gauge field.

The extension of the Dirac approach to quarks and non-Abelian gauge fields has been done by Lavelle and McMullan in refs. [4–6]. Static quarks are dressed similarly to (15.1), while moving quarks are described by means of boosted Coulomb–Dirac clouds. The renormalization is studied in [47].

In the static case, the Lavelle–McMullan expressions of the dressed gauge fields and fermions, and their correlation functions, are related to the ones defined here as follows. First, choose a cloud function $V^a(A, \phi)$ of the Coulomb–Dirac type and take $\tilde{\lambda} = 0$:

$$\tilde{\Psi}(\Phi, \tilde{\Phi}) = \int \bar{H}^a(\nabla\cdot\mathbf{A}_d^a),$$

$$(S_K^{\text{cloud}}, \tilde{\Psi}) = \int E^a(\nabla\cdot\mathbf{A}_d^a) - \int \bar{H}^a \nabla\cdot\frac{\partial\mathbf{A}_d^a(A, \phi)}{\partial\phi^b} H^b. \quad (15.2)$$

Then integrate E^a out. This gives the cloud condition

$$0 = \nabla\cdot\mathbf{A}_d^a(A, \phi) = \nabla\cdot\mathbf{A}^a + \Delta\phi^a + \mathcal{O}(g), \quad (15.3)$$

which can be solved perturbatively for ϕ . The solution $\phi(\mathbf{A})$ is nonlocal in space, but unambiguous (and does not need a particular prescription, since it is of the Coulomb type). If we insert it in the expressions (2.5) of $A_{\mu\text{d}}$ and ψ_{d} , the Lavelle–McMullan correlation functions are the correlation functions of such dressed fields. Note that after these operations the cloud sector of the action can be dropped, since it integrates to one, as in (5.3).

The comparison in the nonstatic case is less straightforward. Without a general notion of pure virtuality, like the one provided by the fakeon prescription, clouds of the Coulomb–Dirac and Lavelle–McMullan types seem to be the meaningful choices.

Another approach to build gauge invariant correlation functions is the one suggested by 't Hooft in Ref. [9]. Consider fermions ψ in the fundamental representation and introduce scalar fields ϕ , also in the fundamental representation. The bilinear $\phi^\dagger \psi$ is obviously a color singlet. If a spontaneous symmetry breaking mechanism gives ϕ an expectation value v , then the product $\phi^\dagger \psi$ can be expanded, and its expansion begins linearly in the fields. Writing $\phi = v + \eta$, we have

$$\langle \phi^\dagger \psi | \bar{\psi} \phi \rangle = \langle v^\dagger \psi | \bar{\psi} v \rangle + \langle \eta^\dagger \psi | \bar{\psi} v \rangle + \langle v^\dagger \psi | \bar{\psi} \eta \rangle + \langle \eta^\dagger \psi | \bar{\psi} \eta \rangle.$$

The last three terms show that one-loop diagrams contribute to the lowest order. In other words, the expansion in powers of the coupling does not match the loop expansion. It might be interesting to study the 't Hooft approach with purely virtual scalar fields ϕ .

A well-known way to build manifestly gauge invariant correlation functions of gauge fields and quarks is by means of Wilson lines. We show that, in general, this method introduces unwanted degrees of freedom, so it cannot be used naively to define physical absorptive parts.

We define the Wilson line $W(x, y)$ by the formula

$$W(x, y) = \mathbb{P} \exp \left(ig \int_0^1 ds \mathcal{A}(x, y; s) \right), \tag{15.4}$$

where \mathbb{P} denotes the path ordering and

$$\mathcal{A}(x, y; s) \equiv (x - y)^\mu A_\mu(y(1 - s) + xs).$$

We could consider arbitrary paths connecting x to y , but for simplicity we restrict to the straight segment.

We concentrate on the fermion two-point function

$$G_\psi(x, y) \equiv \text{Tr}(W(y, x)\psi(x)\bar{\psi}(y)),$$

which is clearly gauge invariant. At one loop, we can truncate the expansion of the Wilson line to the order g^2 :

$$W(x, y) = 1 + ig \int_0^1 ds \mathcal{A}(x, y; s) - g^2 \int_0^1 ds \mathcal{A}(x, y; s) \int_0^s d\tau \mathcal{A}(x, y; \tau) + \mathcal{O}(g^3).$$

We have

$$G_\psi(x, y) \equiv \text{Tr}(\psi(x)\bar{\psi}(y)) + ig \int_0^1 ds \text{Tr}(\mathcal{A}(y, x; s)\psi(x)\bar{\psi}(y)) - g^2 \int_0^1 ds \int_0^s d\tau \text{Tr}(\mathcal{A}(y, x; s)\mathcal{A}(y, x; \tau)\psi(x)\bar{\psi}(y)) + \mathcal{O}(g^3).$$

Working in momentum space, the first contribution can be read from (11.3). The second contribution is

$$\frac{i\gamma^\mu p_\mu}{p^2 + i\epsilon} \frac{g^2 \lambda}{8\pi^2 \epsilon} (N_c^2 - 1)(-p^2 - i\epsilon)^{-\epsilon/2},$$

while the third contribution reads

$$\frac{i\gamma^\mu p_\mu}{p^2 + i\epsilon} \frac{g^2(3 - \lambda)}{16\pi^2 \epsilon} (N_c^2 - 1)(-p^2 - i\epsilon)^{-\epsilon/2}$$

In total, we find

$$G_\psi(p) = N_c \frac{i\gamma^\mu p_\mu}{p^2 + i\epsilon} \left[1 + \frac{3g^2}{8\pi^2 \epsilon} \frac{N_c^2 - 1}{2N_c} (-p^2 - i\epsilon)^{-\epsilon/2} \right] + \mathcal{O}(g^4),$$

the factor N_c being due to the trace. The λ -dependence disappears, as expected, but the absorptive part is negative,

$$\text{Abso}[G_\psi(p)] = -\frac{3g^2}{2\pi} \frac{N_c^2 - 1}{2} p^2 \theta(p^2) + \mathcal{O}(g^4). \tag{15.5}$$

therefore, unphysical.

16 Conclusions

We have extended quantum field theory to include purely virtual ‘‘cloud’’ sectors, which allow us to define gauge invariant dressed fields and study their correlation functions. The cloud diagrammatics and its Feynman rules are derived from a local action, built by means of cloud fields ϕ^i and their anticommuting partners H^i . It includes the cloud functions, the cloud Faddeev-Popov determinants and the cloud symmetries. The usual gauge-fixing must be cloud invariant, while the cloud-fixings must be gauge invariant. The dressed fields are gauge invariant, but not necessarily cloud invariant.

The extended theory is unitary, renormalizable and polynomial in all the fields except for ϕ^i , which are dimensionless. No extra degrees of freedom are propagated. The extension is perturbative and the expansion in powers of the gauge coupling g coincides with the expansion in the number of loops. Each insertion in a correlation function can be equipped with its own, independent cloud. The correlation functions of the undressed fields are unaffected by the extra sectors. The S matrix amplitudes of the dressed fields coincide with the usual scattering amplitudes. This ensures, among the other things, that the latter are gauge independent.

The results allow us to define short-distance scattering processes, where the products do not have enough time to become noninteracting, asymptotic states. Then the predictions depend on the clouds, because the observer necessarily disturbs the observed phenomenon. A few initial measurements must be sacrificed to calibrate the instrumentation. After that, everything else is testable, and possibly falsifiable.

The dressed fields are invariant under infinitesimal gauge transformations, but not necessarily under global gauge transformations. This is what allows us to build physical non singlet states without violating unitarity. The dressed fields can also be used to replace the elementary fields everywhere,

reducing the latter to mere integration and diagrammatic tools. This way, all the calculations are manifestly gauge independent.

An extended Batalin–Vilkovisky formalism and its Zinn–Justin master equations allow us to prove that the symmetries are preserved by renormalization to all orders. Renormalizability by power counting is manifest with a variety of gauge and cloud choices. With more general choices it can be proved by means of canonical transformations.

The extra sectors propagate ghosts, and give unphysical results, if the extra fields are quantized by means of the usual Feynman $i\epsilon$ prescription. To avoid this, those sectors are rendered purely virtual, which means that the extra fields are quantized as fake fields. The purely virtual nature of the cloud sectors ensures that, in the end, no unwanted degrees of freedom propagate. This allows us to extract the physical absorptive parts of ordinary correlation functions from the correlation functions of the dressed fields. More generally, it opens the way to extract physical information from off-shell correlation functions in a systematic way.

A certain gauge/cloud duality can be used to simplify the computations. At the conceptual level, it shows that a gauge choice is ultimately nothing but a particular cloud, provided the gauge trivial modes are rendered purely virtual.

We have illustrated the basic properties of the formalism by calculating the one-loop two-point functions of the dressed quarks and gluons. Their absorptive parts are gauge independent, cloud independent and positive. Instead, they are unphysical if the clouds are not purely virtual (such as those defined by the Feynman prescription). They are also unphysical, generically speaking, if Wilson lines are used.

Among the other things, the purely virtual cloud formalism can be used as an alternative to the popular physical gauges. Here, instead of changing the gauge fixing to make it physical, we make a gauge-fixing physical by changing the prescription we use for it.

The cloud fields and their partners are massless. Massless purely virtual particles can in principle violate causality (not just microcausality) [48, 49]. This aspect deserves further study. Here we just note that the absorptive parts we have calculated are not concerned by this fact, because they are cloud independent, so they are properties of the fundamental theory.

Acknowledgements We are grateful to U. Aglietti, M. Bochicchio, D. Comelli, E. Gabrielli, C. Marzo and L. Marzola for helpful discussions. This work was supported in part by the European Regional Development Fund through the CoE program grant TK133 and the Estonian Research Council grant PRG803. We thank the CERN theory group for hospitality during the final stage of the project.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study].

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Funded by SCOAP³. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

Appendices

Appendix A: Notation

In this appendix we collect the notation and some useful formulas, starting from

$$\begin{aligned} A_\mu &= A_\mu^a T^a, & [T^a, T^b] &= i f^{abc} T^c, \\ \text{Tr} [T^a T^b] &= \frac{1}{2} \delta^{ab}, & D_\mu \psi &= \partial_\mu \psi - i g A_\mu \psi, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] = F_{\mu\nu}^a T^a, \\ F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \end{aligned}$$

where A_μ^a are the gauge fields, $F_{\mu\nu}^a$ is the field strength, T^a are the generators of the Lie group (Hermitian matrices of the fundamental representation), f^{abc} are the structure constants of the Lie algebra, D_μ is the covariant derivative and ψ is a field that belongs to the fundamental representation. The gauge transformations read

$$\begin{aligned} A_\mu^U &= U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1}, & F_{\mu\nu}^U &= U F_{\mu\nu} U^{-1}, \\ \psi^U &= U \psi, \\ D_\mu^U &= U D_\mu U^{-1}, & W^U(x, y) &= U(x) W(x, y) U^{-1}(y), \end{aligned} \quad (\text{A.1})$$

where $U(x) = \exp(i g \alpha^a(x) T^a)$ is a point-dependent matrix of $SU(N_c)$, $\alpha^a(x)$ are arbitrary functions, and $W(x, y)$ is the Wilson line defined in formula (15.4).

Appendix B: Uniqueness of the dressed fields

We prove that for every transformation law $\delta_C \phi^a = E^{ab}(\phi) C^b$ of the cloud field ϕ^a , the dressed gauge fields are unique, up to linear combinations. Let us write

$$A_{\mu d}^a = F^{ab}(\phi) A_\mu^b - G^{ab}(\phi) \partial_\mu \phi^b. \quad (\text{B.1})$$

If we require $(S_K, A_{\mu d}^a) = 0$, we obtain three equations, from the vanishing of the terms proportional to $A_\mu^c, \partial_\mu C^c$ and $C^c \partial_\mu \phi^d$, which are

$$\begin{aligned} F^{ab} - G^{ac} E^{cb} &= 0, & F^{ab,c} E^{cd} + g F^{ac} f^{cbd} &= 0, \\ G^{ab,c} E^{cd} + G^{ac} E^{cd,b} &= 0, \end{aligned} \tag{B.2}$$

where the comma denotes the derivative with respect to ϕ .

We expand in powers of g , writing $F^{ab} = \delta^{ab} + \sum_{n=1}^\infty g^n F_n^{ab}$, $G^{ab} = \delta^{ab} + \sum_{n=1}^\infty g^n G_n^{ab}$ and $E^{ab} = \delta^{ab} + \sum_{n=1}^\infty g^n E_n^{ab}$. The expansions of (B.2) to order g^n give equations of the form

$$F_n^{ab} - G_n^{ab} = X_n^{ab}, \quad F_n^{ab,c} = Y_n^{abc}, \quad G_n^{ab,c} = Z_n^{abc},$$

where X_n^{ab}, Y_n^{abc} and Z_n^{abc} are known from the previous orders. The ambiguity of the solution is $\Delta F_n^{ab} = \Delta G_n^{ab} = \text{constant}$, which can be absorbed into a redefinition $A_{\mu d}^a \rightarrow (\delta^{ab} + g^n \Delta F_n^{ab}) A_{\mu d}^b$. Thus, the most general solution $A_{\mu d}^a$ is a linear combination $M^{ab} \hat{A}_{\mu d}^b$ of a particular solution $\hat{A}_{\mu d}^b$, where M^{ab} is a constant matrix. If $E^{ab}(\phi)$ is an expansion in powers of $g\phi$, a unique solution has the same property, since $M^{ab} = \delta^{ab}$ in that case.

Similarly, the dressed fermions are unique, up to constant factors.

Appendix C: Cloud and gauge dependence through renormalization

In this appendix, we prove further results about the dependence of the renormalized action on the cloud parameters, and derive some already proved results in a more general way. See [50] for details and examples of derivations like the ones that follow.

We add the inductive assumption that the renormalized action $S_{n \text{ tot}}$ to order n satisfies the equation

$$\frac{\partial S_{n \text{ tot}}}{\partial \tilde{\lambda}} = (S_K^{\text{cloud}}, \Upsilon_{n, \tilde{\lambda}}), \tag{C.1}$$

where $\Upsilon_{n, \tilde{\lambda}} = \partial \Upsilon_n / \partial \tilde{\lambda}$. Referring to formulas (14.7), this means that S_n does not depend on the cloud parameters $\tilde{\lambda}$ introduced through the cloud fermions (a result proved in a different way in subsection 14.4). We also assume that $\langle \Upsilon_{n, \tilde{\lambda}} \rangle$ is convergent to the order \hbar^n included. Here and below averages such as $\langle \Upsilon_{n, \tilde{\lambda}} \rangle$ are regarded as functionals of $\Phi, K, \tilde{\Phi}$ and \tilde{K} .

Equation (C.1) implies

$$\begin{aligned} \frac{\partial \Gamma_{n \text{ tot}}}{\partial \tilde{\lambda}} &= \frac{\partial W_{n \text{ tot}}}{\partial \tilde{\lambda}} = \left\langle \frac{\partial S_{n \text{ tot}}}{\partial \tilde{\lambda}} \right\rangle = \langle (S_K^{\text{cloud}}, \Upsilon_{n, \tilde{\lambda}}) \rangle \\ &= (S_K^{\text{cloud}}, \langle \Upsilon_{n, \tilde{\lambda}} \rangle). \end{aligned} \tag{C.2}$$

The first equality follows from the definition of $\Gamma_{n \text{ tot}}$ as the Legendre transform of $W_{n \text{ tot}}$. The fourth equality can be proved as follows (see also [50]). Consider an arbitrary action S'_{tot} that does not satisfy $(S_K^{\text{cloud}}, S'_{\text{tot}}) = 0$. Repeat the argument that leads to the proof of (14.2) with the change of field variables

$$\Phi^\alpha \rightarrow \Phi^\alpha, \quad \tilde{\Phi}^{\alpha i} \rightarrow \tilde{\Phi}^{\alpha i} + \theta (S_K^{\text{cloud}}, \tilde{\Phi}^{\alpha i}),$$

in the generating functional Z'_{tot} associated with S'_{tot} . The result is

$$(S_K^{\text{cloud}}, \Gamma'_{\text{tot}}) = \langle (S_K^{\text{cloud}}, S'_{\text{tot}}) \rangle, \tag{C.3}$$

instead of (14.2). The average with prime is calculated with the action S'_{tot} . Applying formula (C.3) to $S'_{\text{tot}} = S_{n \text{ tot}} + \theta \Upsilon_{n, \tilde{\lambda}}$, where θ is a constant anticommuting parameter, and noting that $\Gamma'_{\text{tot}} = \Gamma_{n \text{ tot}} + \theta \langle \Upsilon_{n, \tilde{\lambda}} \rangle$ (which easily follows from the definition of the Γ functional as a Legendre transform), we immediately get the fourth equality of (C.2). We need to use the identities $(S_K^{\text{cloud}}, S_{n \text{ tot}}) = (S_K^{\text{cloud}}, \Gamma_{n \text{ tot}}) = 0$ for this. They also ensure that the average on the right-hand side can be calculated with the unperturbed action $S_{n \text{ tot}}$.

Now, taking the $(n + 1)$ -th order divergent part of (C.2), we obtain

$$\frac{\partial \Gamma_{n \text{ tot div}}^{(n+1)}}{\partial \tilde{\lambda}} = (S_K^{\text{cloud}}, \Upsilon_{n, \tilde{\lambda} \text{ div}}^{(n+1)}), \tag{C.4}$$

where $\Upsilon_{n, \tilde{\lambda} \text{ div}}^{(n+1)}$ is the $(n + 1)$ -th order divergent part of $\langle \Upsilon_{n, \tilde{\lambda}} \rangle$. It is local, since $\langle \Upsilon_{n, \tilde{\lambda}} \rangle$ is convergent to order n by assumption. Integrating (C.4), we find

$$\Gamma_{n \text{ tot div}}^{(n+1)} = \Gamma_{n \text{ tot div}}^{(n+1)}(\tilde{\lambda}_0) + \left(S_K^{\text{cloud}}, \int_{\tilde{\lambda}_0}^{\tilde{\lambda}} \Upsilon_{n, \tilde{\lambda} \text{ div}}^{(n+1)}(\tilde{\lambda}') d\tilde{\lambda}' \right),$$

where the $\tilde{\lambda}$ -dependence has been emphasized in parentheses. The second equation of (14.7) then implies

$$\begin{aligned} \Gamma_{n \text{ tot div}}^{(n+1)} &= \Gamma_{n \text{ div}}^{(n+1)} + (S_K^{\text{cloud}}, R_{n \text{ div } 0}^{(n+1)}) \\ &\quad + \left(S_K^{\text{cloud}}, \int_{\tilde{\lambda}_0}^{\tilde{\lambda}} \Upsilon_{n, \tilde{\lambda} \text{ div}}^{(n+1)}(\tilde{\lambda}') d\tilde{\lambda}' \right), \end{aligned}$$

where $\Gamma_{n \text{ div}}^{(n+1)}$ and $R_{n \text{ div } 0}^{(n+1)}$ are $\tilde{\lambda}$ independent. In particular, the $\tilde{\lambda}$ -independence of $\Gamma_{n \text{ div}}^{(n+1)}$ can be immediately proved by taking $\tilde{\Phi}^{\alpha i} = \tilde{K}^{\alpha i} = 0$. Finally, (14.4) gives

$$\begin{aligned} S_{n+1 \text{ tot}} &= S_{n \text{ tot}} - \Gamma_{n \text{ tot div}}^{(n+1)} \\ &= S_{n+1} + (S_K^{\text{cloud}}, \Upsilon_{n+1}) + S_K^{\text{cloud}}, \\ S_{n+1} &= S_n - \Gamma_{n \text{ div}}^{(n+1)}, \\ \Upsilon_{n+1} &= \Upsilon_n - R_{n \text{ div } 0}^{(n+1)} - \int_{\tilde{\lambda}_0}^{\tilde{\lambda}} \Upsilon_{n, \tilde{\lambda} \text{ div}}^{(n+1)}(\tilde{\lambda}') d\tilde{\lambda}'. \end{aligned}$$

We have

$$\frac{\partial S_{n+1 \text{ tot}}}{\partial \tilde{\lambda}} = (S_K^{\text{cloud}}, \Upsilon_{n+1, \tilde{\lambda}}), \quad \Upsilon_{n+1, \tilde{\lambda}} = \Upsilon_{n, \tilde{\lambda}} - \Upsilon_{n, \tilde{\lambda} \text{ div}}^{(n+1)}.$$

Clearly, $\langle \Upsilon_{n+1, \tilde{\lambda}} \rangle$ is convergent to the order \hbar^{n+1} included, so the inductive assumptions are replicated to order $n + 1$. Taking n to infinity, we obtain $\Upsilon_R = \Upsilon_\infty$ and

$$\frac{\partial S_{R\text{tot}}}{\partial \tilde{\lambda}} = (S_K^{\text{cloud}}, \Upsilon_{R, \tilde{\lambda}}). \quad (\text{C.5})$$

Using (14.6) at $\tilde{\Phi}^{\alpha i} = \tilde{K}^{\alpha i} = 0$, we see that S_R is cloud independent.

The arguments can be specialized to every cloud separately. This allows us to prove the further identities

$$\frac{\partial S_{R\text{tot}}}{\partial \tilde{\lambda}_i} = (S_K^{\text{cloud } i}, \Upsilon_{R, \tilde{\lambda}_i}), \quad (\text{C.6})$$

which ensure that the i th cloud parameters $\tilde{\lambda}_i$ do not propagate to the other cloud sectors. Moreover, formula (9.3) tells us that $\Upsilon_{0, \tilde{\lambda}_i} = \tilde{\Psi}_{i, \tilde{\lambda}_i} \equiv \partial \tilde{\Psi}_i / \partial \tilde{\lambda}_i$. The derivation just given ensures that $\Upsilon_{R, \tilde{\lambda}_i}$ is the renormalized version of the functional $\Upsilon_{0, \tilde{\lambda}_i} = \tilde{\Psi}_{i, \tilde{\lambda}_i}$, which we denote by $\tilde{\Psi}_{i, \tilde{\lambda}_i R}$.

Finally, if X is a functional such that $(S_K^{\text{cloud } i}, X) = 0$, the same derivation shows that the renormalized functional X_R satisfies $(S_K^{\text{cloud } i}, X_R) = 0$. In particular, $(S_K^{\text{cloud } j}, \Upsilon_{R, \tilde{\lambda}_i}) = 0$ for every $j \neq i$.

Similar procedures (see [50]) allow us to prove

$$\begin{aligned} \frac{\partial S_{R\text{tot}}}{\partial \tilde{\lambda}_i} &= (S_{R\text{tot}}, \tilde{\Psi}_{i, \tilde{\lambda}_i R}), \\ \frac{\partial S_{R\text{tot}}}{\partial \lambda} &= (S_{R\text{tot}}, \Psi_{\lambda R}), \quad (S_K^{\text{cloud } i}, \Psi_{\lambda R}) = 0, \end{aligned} \quad (\text{C.7})$$

where $\Psi_{\lambda R}$ is the renormalized version of the functional $\Psi_\lambda = \partial \Psi / \partial \lambda$.

Summarizing the cloud dependence is encoded in the equations

$$\begin{aligned} \frac{\partial S_{R\text{tot}}}{\partial \tilde{\lambda}_i} &= (S_{R\text{tot}}, \tilde{\Psi}_{i, \tilde{\lambda}_i R}), \quad \frac{\partial S_{R\text{tot}}}{\partial \tilde{\lambda}_i} = (S_K^{\text{cloud } i}, \tilde{\Psi}_{i, \tilde{\lambda}_i R}), \\ (S_K^{\text{cloud } j}, \tilde{\Psi}_{i, \tilde{\lambda}_i R}) &= 0 \text{ for } j \neq i, \end{aligned} \quad (\text{C.8})$$

while the gauge dependence is encoded in

$$\frac{\partial S_{R\text{tot}}}{\partial \lambda} = (S_{R\text{tot}}, \Psi_{\lambda R}), \quad (S_K^{\text{cloud } i}, \Psi_{\lambda R}) = 0. \quad (\text{C.9})$$

Integrating the first equations of (C.8) and (C.9) as shown in [50], it is straightforward to prove that the dependence on the cloud parameters is a canonical transformation and the dependence on the gauge fixing parameters is also a canonical transformation.

References

1. D. Anselmi, Diagrammar of physical and fake particles and spectral optical theorem. *J. High Energy Phys.* **11**, 030 (2021). [arXiv:2109.06889](#) [hep-th]
2. P. Gaigg, W. Kummer, M. Schweda (eds.), *Physical and Nonstandard Gauges, Lecture Notes in Physics*, vol. 361 (Springer, Heidelberg, 1990)
3. P.A.M. Dirac, Gauge invariant formulation of quantum electrodynamics. *Can. J. Phys.* **33**, 650 (1955)
4. M. Lavelle, D. McMullan, Observables and gauge fixing in spontaneously broken gauge theories. *Phys. Lett. B* **347**, 89 (1995). [arXiv:hep-ph/9412145](#)
5. M. Lavelle, D. McMullan, The color of quarks. *Phys. Lett. B* **371**, 83 (1996). [arXiv:hep-ph/9509343](#)
6. M. Lavelle, D. McMullan, Constituent quarks from QCD. *Phys. Rep.* **279**, 1 (1997). [arXiv:hep-ph/9509344](#)
7. S.-S. Chang, Quantization of Yang-Mills fields by separation of gauge variables. *Phys. Rev. D* **19**, 2958 (1979)
8. S.-S. Chang, Manifestly gauge invariant gluon equations. *Phys. Rev. D* **21**, 3466 (1980)
9. G. 't Hooft, Which topological features of a gauge theory can be responsible for permanent confinement?. *NATO Sci. Ser. B* **59**, 117 (1980). In: G. 't Hooft, et al., *Recent Developments in Gauge Theories*. NATO Advanced Study Institutes Series, vol 59. Springer, Boston
10. D. Anselmi, Purely virtual extension of quantum field theory for gauge invariant fields: quantum gravity. [arXiv:2207.12401](#) [hep-th]
11. D. Anselmi, Quantum field theory of physical and purely virtual particles in a finite interval of time on a compact space manifold: diagrams, amplitudes and unitarity. [arXiv:2304.07642](#) [hep-th]
12. D. Anselmi, Gauge theories and quantum gravity in a finite interval of time, on a compact space manifold. [arXiv:2306.07333](#) [hep-th]
13. F. Bloch, A. Nordsieck, Note on the radiation field of the electron. *Phys. Rev.* **52**, 54 (1937)
14. T. Kinoshita, Mass singularities of Feynman amplitudes. *J. Math. Phys.* **3**, 650 (1962)
15. T.D. Lee, M. Nauenberg, Degenerate systems and mass singularities. *Phys. Rev.* **133**, B1549 (1964)
16. S. Weinberg, Infrared photons and gravitons. *Phys. Rev.* **140**, B516 (1965)
17. D. Anselmi, Dressed propagators, fakeon self-energy and peak uncertainty. *J. High Energy Phys.* **06**, 058 (2022). [arXiv:2201.00832](#) [hep-ph]
18. B. de Wit, M.T. Grisaru, Compensating fields and anomalies, in *Quantum Field Theory and Quantum Statistics*, vol. 2, ed. by I.A. Batalin, C.J. Isham, G.A. Vilkovisky (Adam Hilger, London, 1987)
19. E.C.G. Stueckelberg, Die Wechselwirkungen Kraefte in der Elektrodynamik und in der Feldtheorie der Kernkraefte (I), [The interaction forces in electrodynamics and in the field theory of nuclear forces (I)]. *Helv. Phys. Acta* **11**, 225 (1938)
20. R.E. Cutkosky, Singularities and discontinuities of Feynman amplitudes. *J. Math. Phys.* **1**, 429 (1960)
21. M. Veltman, Unitarity and causality in a renormalizable field theory with unstable particles. *Physica* **29**, 186 (1963)
22. G. 't Hooft, Renormalization of massless Yang-Mills fields. *Nucl. Phys. B* **33**, 173 (1971)
23. G. 't Hooft, Renormalizable Lagrangians for massive Yang-Mills fields. *Nucl. Phys. B* **35** 167 (1971)
24. G. 't Hooft, M. Veltman, Diagrammar, CERN report CERN-73-09
25. M. Veltman, *Diagrammatica. The path to Feynman rules* (Cambridge University Press, New York, 1994)
26. D. Anselmi, On the quantum field theory of the gravitational interactions. *J. High Energy Phys.* **06**, 086 (2017). [arXiv:1704.07728](#) [hep-th]
27. D. Anselmi, E. Bianchi, M. Piva, Predictions of quantum gravity in inflationary cosmology: effects of the Weyl-squared term. *J. High Energy Phys.* **07**, 211 (2020). [arXiv:2005.10293](#) [hep-th]
28. D. Anselmi, K. Kannike, C. Marzo, L. Marzola, A. Melis, K. Müürsepp, M. Piva, M. Raidal, Phenomenology of a fake inert dou-

- blet model. *J. High Energy Phys.* **10**, 132 (2021). [arXiv:2104.02071](https://arxiv.org/abs/2104.02071) [hep-ph]
29. D. Anselmi, K. Kannike, C. Marzo, L. Marzola, A. Melis, K. Mürsepp, M. Piva, M. Raidal, A fake doublet solution to the muon anomalous magnetic moment. *Phys. Rev. D* **104**, 035009 (2021). [arXiv:2104.03249](https://arxiv.org/abs/2104.03249) [hep-ph]
 30. C.G. Bollini, J.J. Giambiagi, Lowest order divergent graphs in ν -dimensional space. *Phys. Lett. B* **40**, 566 (1972)
 31. G.t. 't Hooft, M. Veltman, Regularization and renormalization of gauge fields. *Nucl. Phys. B* **44**, 189 (1972)
 32. G.M. Cicuta, E. Montaldi, Analytic renormalization via continuous space dimension. *Lett. Nuovo Cim.* **4**, 329 (1972)
 33. L.D. Faddeev, V. Popov, Feynman diagrams for the Yang–Mills field. *Phys. Lett. B* **25**, 29 (1967)
 34. N. Nakanishi, Covariant quantization of the electromagnetic field in the Landau gauge. *Prog. Teor. Phys.* **35**, 1111 (1966)
 35. B. Lautrup, Canonical quantum electrodynamics in covariant gauges. *Kgl. Dan. Vid. Se. Mat. Fys. Medd.* **35**(11), 1 (1967)
 36. J.C. Ward, An identity in quantum electrodynamics. *Phys. Rev.* **78**, 182 (1950)
 37. Y. Takahashi, On the generalized Ward identity. *Nuovo Cimento* **6**, 371 (1957)
 38. A.A. Slavnov, Ward identities in gauge theories. *Theor. Math. Phys.* **10**, 99 (1972)
 39. J.C. Taylor, Ward identities and charge renormalization of Yang–Mills field. *Nucl. Phys. B* **33**, 436 (1971)
 40. I.A. Batalin, G.A. Vilkovisky, Gauge algebra and quantization. *Phys. Lett. B* **102**, 27 (1981)
 41. I.A. Batalin, G.A. Vilkovisky, Quantization of gauge theories with linearly dependent generator. *Phys. Rev. D* **28**, 2567 (1983). Erratum-ibid. *D* **30**, 508 (1984)
 42. J. Zinn-Justin, Renormalization of gauge theories, in Rollnik, H., Dietz, K. (eds.) *Bonn Lectures 1974, Trends in Elementary Particle Physics, Lecture Notes in Physics*, vol. 37 (Springer, Berlin, 1975), p. 1
 43. R.E. Kallosh, I.V. Tyutin, The equivalence theorem and gauge invariance in renormalisable theories. *Sov. J. Nucl. Phys.* **17**, 98 (1973); *Yad. Fiz.* **17**, 190 (1973)
 44. D. Anselmi, Aspects of perturbative unitarity. *Phys. Rev. D* **94**, 025028 (2016). [arXiv:1606.06348](https://arxiv.org/abs/1606.06348) [hep-th]
 45. D. Anselmi, Fakeons, unitarity, massive gravitons and the cosmological constant. *J. High Energy Phys.* **12**, 027 (2019). [arXiv:1909.04955](https://arxiv.org/abs/1909.04955) [hep-th]
 46. D. Anselmi, Master functional and proper formalism for quantum gauge field theory. *Eur. Phys. J. C* **73**, 2363 (2013). [arXiv:1205.3862](https://arxiv.org/abs/1205.3862) [hep-th]
 47. E. Bagan, M. Lavelle, D. McMullan, Charges from dressed matter: physics and renormalisation. *Ann. Phys.* **282**, 503 (2000). [arXiv:hep-ph/9909262](https://arxiv.org/abs/hep-ph/9909262)
 48. D. Anselmi, M. Piva, Quantum gravity, fakeons and microcausality. *J. High Energy Phys.* **11**, 21 (2018). [arXiv:1806.03605](https://arxiv.org/abs/1806.03605) [hep-th]
 49. D. Anselmi, Fakeons, microcausality and the classical limit of quantum gravity. *Class. Quantum Gravity* **36**, 065010 (2019). [arXiv:1809.05037](https://arxiv.org/abs/1809.05037) [hep-th]
 50. D. Anselmi, Ward identities and gauge independence in general chiral gauge theories. *Phys. Rev. D* **92**, 025027 (2015). [arXiv:1501.06692](https://arxiv.org/abs/1501.06692) [hep-th]