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Can $1/N_c$ corrections destroy the saturation of dipole densities?

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Abstract In this paper we discuss a well known QCD result: the steep increase of Green's function for exchange of n BFKL Pomerons $G_{n\mathbb{P}}\left(Y\right) \propto \exp\left(\frac{n^2}{N_c^2}\Delta_{\text{BFKL}}Y\right)$ where N_c is the number of colours ,Y is the rapidity and Δ_{BFKL} is the intercept of the BFKL Pomeron. We consider this problem in the framework of the simple Pomeon models in zero transverse dimensions, which have two advantages :(i) they allow to take into account all shadowing corrections, including the summation of the Pomeron loops and (ii) they have the same as in QCD striking increase of $G_{n\mathbb{P}}\left(Y\right)$. We found that the strength of shadowing corrections is not enough to stop the increase of the scattering amplitude with energy in contradiction to the unitarity constraints. Hence, our answer to the question in the title is positive. We believe that we need to search an approach beyond of the BFKL Pomeron calculus to treat $1/N_c$ corrections in Colour Glass Condensate effective theory.

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1 Introduction

The only candidate for the effective theory of high energy QCD is Colour Glass Condensate (CGC) approach (see Ref. [1] for a review). Two main ideas of CGC: the saturation of the dipole density and the new dimensional scale (Q_s), which increases with energy, have become widely accepted language for discussing the high energy scattering in QCD. However the CGC approach suffers several problems. The most known of them is the power-like behaviour of the scattering amplitude at large impact parameters [2–5] that violated the Froissart theorem [6,7]. We have to introduce non-perturbative corrections at large impact parameters and the embryonic stage of our understanding of confinement of quarks and gluons does not allow us to come up with a reasonable theoretical approach to the problem. The second well known problem is summing of the BFKL Pomeron loops.[1] This problem is a technical one, but in spite of intensive work [8–68] it is still far away from being solved.This situation makes the problem one of the principle problems, without solving which we cannot consider the dilute–dilute and dense–dense parton densities collisions. As has been recently shown [66,67], even the Balitsky–Kovchegov (BK) equation, that governs the dilute-dense parton density scattering (deep inelastic scattering (DIS) of electron with proton), has to be modified due to contributions of Pomeron loops.

In this paper we wish to draw an attention of the reader to a different problem of the CGC approach. In the CGC approach the scattering amplitude does not exceed the unitarity limit due to shadowing corrections. The Balitsky–Kovchegov [48] non-linear equation, which sums the 'fan' diagrams of interacting BFKL Pomerons (see Fig. 1a), generates the amplitude, which tends to unity (the unitary limit) at high energies. However, it has been shown in Refs. [69–73] that Green's

[1] The abbreviation BFKL Pomeron stands for Balitsky, Fadin, Kuraev and Lipatov Pomeron.

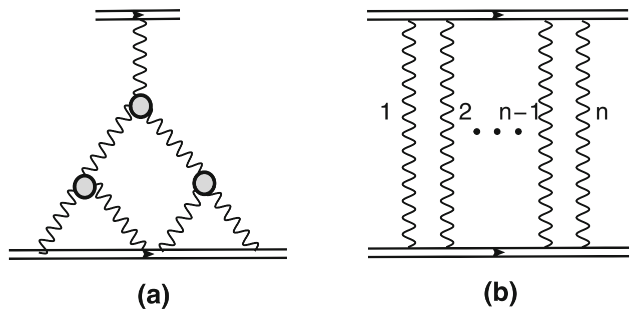

Fig. 1 **a** The 'fan' diagrams of the Balitsky–Kovchegov non-linear equation. The wavy lines denote the BFKL Pomerons. The blob shows the triple Pomeron vertex. **b** Is the Green's function of the exchange of n BFKL Pomerons

function of the exchange of n BFKL Pomerons does not increase as $\exp\left(n\,\Delta_{\rm BFKL} Y\right)$ where $\Delta_{\rm BFKL}$ is the intercept of the BFKL Pomeron and Y is the rapidity of two colliding dipoles. It turns out that Green's function grows as $\exp\left(\frac{n^2}{N_c^2}\Delta_{\rm BFKL} Y\right)$ (N_c is the number of colours) and this increase cannot be suppressed by the shadowing corrections.[2] The N_c suppression gives rise for the hope that such an increase can manifest itself only at very high energies, but recently A. Kovner and M. Li showed that for nucleus-nucleus scattering in the CGC approach there exist large $1/N_c^2$ corrections that are larger than the shadowing suppression.[3] In spite of these dangerous results for the CGC approach, the references [69–73] show that these large corrections can be treated in the BFKL Pomeron calculus if we introduce the vertices of interaction of four Pomerons ($I\!P + I\!P \to I\!P + I\!P$).

Bearing this in mind we wish to return to discussion of these corrections in simple, but exactly solvable, zero dimensional models [38,39,78–92]. In these models we can sum Pomeron loops and we have the same kind of corrections, which go under slang name of many particle Regge poles [74–77]. In the next section we will discuss these corrections in details. Here we wish to tell that the main goal of this paper is to find the scattering amplitude taking into account the $I\!P + I\!P \to I\!P + I\!P$ vertices in the simple zero dimensional models to get experience what we can expect in QCD in the CGC approach for these disastrous contributions.

2 Setting the problem

Green's function for one Pomeron in the zero dimensional model can be viewed as a sum of the contributions:

$$G_{I\!P}\left(Y\right) \;=\; \sum_{k=0}^{\infty}\frac{\left(\Delta\,Y\right)^n}{n!} \;=\; e^{\Delta\,Y} \tag{1}$$

Equation 1 sums the ladder diagrams in leading log(1/x) approximation in which all produced dipoles have a strong ordering in the fractions of total momentum x_i:

$$1 \gg x_1 \gg x_2 \gg \cdots x_i \gg x_{i+1} \gg \cdots \gg x_{n-1} \gg x_n \tag{2}$$

this leads to $1/n!$ in Eq. (1). The two Pomeron exchange is shown in Fig. 2a and corresponds to the Feynman diagram of Fig. 2b in which all dipoles emitted by the ladder 1 are absorbed by ladder 1', and all dipoles produced by ladder 2 are absorbed by ladder 2'. These diagrams lead to the Green's function of the exchange of two Pomerons: $G_{2I\!P} = G^2_{I\!P}\left(Y\right)$. However, the diagrams in which the dipole, emitted from ladder 1, will be absorbed by ladder 2' are not small in the zero dimensional models. We are going to call these diagrams "switch diagrams" , using the terminology suggested in Refs. [69–73]. Indeed, after first exchange ladder 1 and 2' will give the Pomeron exchange, leading to the diagram of Fig. 2d. Since at given rapidity we have two 'switch' diagrams: dipole emitted between ladders 1 and 2' and the dipole emitted between ladders 2 and 1' we can obtain the vertex $I\!P + I\!P \to I\!P + I\!P$, which is equal to 2Δ.
The diagrams of Fig. 2d can be summed in ω -representation:

$$G_2\left(Y\right) \;=\; \int\limits_{\epsilon - i\infty}^{\epsilon + i\infty}\frac{d\omega}{2\,\pi\,i} e^{\omega\,\tilde{Y}}\,G_2\left(\omega\right) \tag{3}$$

In this representation

$$G_2\left(\omega\right) \;=\; \sum_{k=0}^{\infty}\frac{1}{\omega\,-\,2\,\Delta}\left(\frac{2\,\Delta}{\omega\,-\,2\,\Delta}\right)^k \;=\; \frac{1}{\omega\,-\,4\,\Delta} \tag{4}$$

where $1/\left(\omega - 2\Delta\right)$ is the contribution of the exchange of two Pomerons (see Fig. 2b). Coming back to Y representation one can see that $G_2\left(Y\right) \;=\; \exp\left(4\,\Delta\,Y\right)$ instead of $G_2\left(Y\right) \;=\; \exp\left(2\,\Delta\,Y\right)$ which is expected for two Pomeron exchange.

Summing the diagrams of Fig. 2e we obtain the Green's function for the exchange of n Pomerons: $G_n\left(Y\right) = \exp\left(n^2\Delta Y\right)$. This Green's function was derived in Refs. [74–77] for the parton approach to high energy scattering. In QCD the structure of the corrections remain to be the same and the only difference is that the 'switch' diagrams as well the intercept of n Pomeron state have the smallness of the order of $1/N_c^2$. Therefore, the $I\!P + I\!P \to I\!P + I\!P$

[2] Actually, in Refs. [69–73] the exchange of n-BFKL Pomerons were considered in the double log approximation, which leads to the anomalous dimension $\frac{\bar{\alpha}_S}{(N^2-1)^2\omega}n^4$ instead of $\frac{\bar{\alpha}_S}{\omega}n^2$ for the exchange of n BFKL Pomerons .

[3] We thank A, Kovner and M. Li for sharing with us their finding.

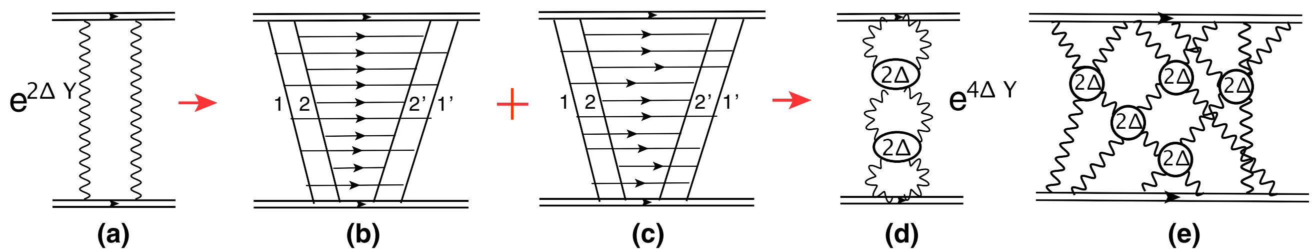

Fig. 2 Summing the switch diagrams for the exchange of two BFKL Pomerons. **a** The exchange of two BFKL Pomerons. **b** The exchange of two BFKL Pomeron in the zero dimensional model. **c** The switch diagram for two ladder exchange in which the produced dipoles from the Pomeron 1 are absorbed by the Pomeron 2' and vise versa. **d** The sum of all diagrams in the zero dimensional model that contribute in the Green's function of the exchange of two Pomerons in t-channel. **e** The Green's function of the exchange of n Pomerons. The wavy lines denote the exchange of one Pomeron with the Green's function $\exp(\Delta Y)$. The $I\!P + I\!P \to I\!P + I\!P$ vertex is denoted by blob and equals to 2Δ

vertex in QCD is equal to $\frac{2}{N_c^2}\,\Delta_{\text{BFKL}}$. leading to $G_n(Y) = \exp\left(n\Delta\,Y + \frac{n(n-1)}{N_c^2}\Delta Y\right)$.

3 The structure of the parton cascade

The useful tool for discussing the structure of the parton cascade is the generating function, which has the following form [23,93,94]:

$$Z(Y,u) = \sum_n P_n(Y)\,u^n \tag{5}$$

where $P_n(Y)$ is the probability to find n dipoles with rapidity Y. For the scattering of one dipole with the target we have the following initial and boundary conditions:

$$\begin{aligned}&\text{initial condition: } Z(Y=0,u) = u;\\ &\text{boundary condition: } Z(Y,u=1) = 1;\end{aligned} \tag{6}$$

The boundary condition follows from $P_n(Y)$ being probabilities.

We need to write the evolution equation for $Z(Y,u)$ taking into account two Pomeron vertices: $\Gamma_{3I\!P} = \Delta$ for $I\!P \to I\!P + I\!P$ and $\Gamma^{2I\!P}_{2I\!P} = 2\,\Delta$ for $I\!P + I\!P \to I\!P + I\!P$. Fortunately the equation for Z for such a cascade has been written in Ref. [93,94] and it takes the form:

$$\frac{\partial Z(Y,u)}{\partial Y} = -\Delta u(1-u)\frac{\partial Z(Y,u)}{\partial u} + \Delta\,u(u-1)\frac{\partial^2 Z(Y,u)}{\partial u^2} \tag{7}$$

As has been mentioned that Eq. (7) has been suggested in Refs. [93,94] , but Eq. (7) has a remarkable difference in comparison with equations in this reference: the sign in front of $u^2\partial^2/\partial\,u^2$ is plus, not minus as in our previous attempts. In Refs. [93,94] the equations for the generating functional in QCD are considered and general equations for it are proposed, which take into account two vertices: the decay of one dipole into two $\Gamma(1\to 2)$ and the merging of two dipoles in

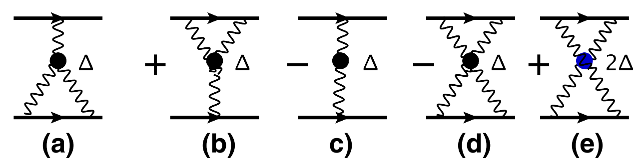

Fig. 3 The first Pomeron diagrams in $Z(Y,u)$, which are generated by Eq. (7). The black circles denote the triple Pomeron vertices, while the blue one describes the four Pomeron interactions induced by 'switch' diagrams

one $\Gamma(2\to 1)$. In Eq. (6.50) of this paper the general QCD equations are simplified for the zero dimensional model and they have the form:

$$\begin{aligned}\frac{\partial Z(Y,u)}{\partial Y} &= -\Gamma(1\to 2)\,u(1-u)\frac{\partial Z(Y,u)}{\partial u}\\ &\quad+\Gamma(2\to 1)\,u(1-u)\frac{\partial^2 Z(Y,u)}{\partial u^2}\\ &\to -\Delta u(1-u)\frac{\partial Z(Y,u)}{\partial u}\\ &\quad+\Delta u(1-u)\frac{\partial^2 Z(Y,u)}{\partial u^2}\end{aligned} \tag{8}$$

The last equation is written for our case without $\Gamma^{2I\!P}_{2I\!P} = 2\Delta$. Two terms $u^2\frac{\partial Z(Y,u)}{\partial u}$ and $u\frac{\partial^2 Z(Y,u)}{\partial u^2}$ generate the diagrams of Fig. 3a, b, respectively. The relative sign in Eq. (8) of these two terms is correct since they obtain the equal contributions. The diagrams of Fig. 3c, d stem from the the terms $u\frac{\partial Z(Y,u)}{\partial u}$ and $u^2\frac{\partial^2 Z(Y,u)}{\partial u^2}$. Their contributions are needed to provide that $Z(Y,u=1)=1$. In our approach we added the direct four Pomeron interaction with the strength of 2Δ (see Fig. 3e), Writing this term in the way to keep $Z(Y,u=1)=1$ one can see that we obtain Eq. (7), which differs by the sign of the second term from Eq. (8).

Below we discuss a bit different equation:

$$\frac{\partial Z\left(\tilde{Y},u\right)}{\partial \tilde{Y}} = -u(1-u)\frac{\partial Z\left(\tilde{Y},u\right)}{\partial u} + \kappa\, u(u-1)\frac{\partial^2 Z\left(\tilde{Y},u\right)}{\partial u^2} \tag{9}$$

where $\tilde{Y} = \Delta Y$ and factor $\kappa = 1/N_c^2$ takes into account a suppression for $\Gamma(2\to 1) = 1/N_c^2$ in QCD (see Ref. [93,94]). Comparing Eq. (10) with (7) one can see that $\kappa = 1$ for zero dimensional model , which we consider. Introducing κ in Eq. (10) we wish to study how the correct colour structure of QCD could influence the behaviour of the scattering amplitude.

3.1 BFKL cascade

Neglecting $\Gamma_{2I\!P}^{2I\!P}$ we obtain the well known equation for the BFKL cascade.

$$\frac{\partial Z\left(\tilde{Y},u\right)}{\partial \tilde{Y}} = -u(1-u)\frac{\partial Z\left(\tilde{Y},u\right)}{\partial u} \tag{10}$$

It is instructive to observe that Eq. (10) leads to a non-linear equation for $Z(Y,u)$ [23,93,94]. Indeed, the general solution to Eq. (10) is of the form $Z(Y,u) = Z(u(Y))$; if we substitute this function into Eq. (10), the derivatives $\partial Z/\partial Y$ on the l.h.s. and r.h.s. of Eq. (10) cancel, and we obtain a differential equation for the function $u(Y)$. Using the initial condition of Eq. (17) we can re-write Eq. (10) in the form:

$$\frac{\partial Z\left(\tilde{Y},u\right)}{\partial \tilde{Y}} = -Z(Y,u) + Z^2(Y,u) \tag{11}$$

Note, that the scattering amplitude in our models is equal to $N(Y) = 1 - Z(Y, 1-\gamma)$ where γ is the amplitude of the interaction of the dipole with the target at low energy. For $N(Y)$ we have the nonlinear equation:

$$\frac{\partial N\left(\tilde{Y}\right)}{\partial \tilde{Y}} = -N(Y) + N^2(Y) \tag{12}$$

which is the Balitsky–Kovchegov equation [48] for our simple models.

The solution to Eq. (11), which satisfies the initial and boundary conditions of Eq. (17) has the following form:

$$Z(Y,u) = \frac{u\, e^{-\Delta Y}}{1 + u\left(e^{-\Delta Y} - 1\right)} \tag{13}$$

One can see that at large values of Y $Z \to 0$ ($N \to 1$). In other words, the nonlinear corrections, which stem from triple Pomeron interactions , suppress the increase of the scattering amplitude ($N \propto e^{\Delta Y}$) and lead to the scattering amplitude which reaches the unitarity bound. We call this phenomenon the saturation of parton densities.

3.2 Asymptotic solution

The general solution to Eq. (10) has been found in Ref. [88]. However, before applying the developed technique to this particular equation we wish to see a qualitative changes in the behaviour of Z at large values of Y that stems from a different sign of $\Gamma_{2I\!P}^{2I\!P}$ than in previous attempts to develop a similar cascade.

For this purpose we are going to find the asymptotic solution at large Y from the following equation:

$$\frac{\partial Z^{\text{asymp}}\left(\tilde{Y},u\right)}{\partial u} + \kappa\frac{\partial^2 Z^{\text{asymp}}\left(\tilde{Y},u\right)}{\partial u^2} = 0 \tag{14}$$

It has an obvious solution

$$Z^{\text{asymp}}(u) = \frac{1 - e^{-\frac{u}{\kappa}}}{1 - e^{-\frac{1}{\kappa}}}, \tag{15}$$

which satisfies the boundary condition: $Z^{\text{asymp}}(u=1) = 1$ and $Z^{\text{asymp}}(u) \propto u$ at $u \ll 1$. To find, how the solution approaches the asymptotic one, we are looking for the solution in the form: $Z(Y,u) = \left(1 - e^{-\frac{u}{\kappa} - \phi(Y,u)}\right)/\left(1 - e^{-\frac{1}{\kappa}}\right)$, assuming that ϕ''_{uu} and ϕ'^2_u are small. The equation for ϕ takes the form:

$$\phi'_{\tilde{Y}}\left(\tilde{Y},u\right) = u(1-u)\,\phi'_u\left(\tilde{Y},u\right) \tag{16}$$

with the initial condition at Y=0:

$$\phi\left(\tilde{Y}=0,u\right) = \left(\frac{1}{C} - \frac{1}{\kappa}\right)u(1-u) \quad \text{with} \quad C = \frac{1}{1-\exp(1/\kappa)} \tag{17}$$

which corresponds to $Z(Y=0,u) = u$ and $Z(Y=0, u=1) = 1$. A general solution to Eq. (16) has the form:

$$\phi(Y,u) = \Phi\left(\tilde{Y} + \ln\frac{u}{1-u}\right) \tag{18}$$

where the arbitrary function Φ has to be found from the initial conditions of Eq. (17).

Finally, the solution takes the form:

$$Z(Y,u) = \frac{1}{1-e^{-1/\kappa}} \times\left(1 - \exp\left(-\frac{u}{\kappa} - \left((1-\exp(-1/\kappa)) - \frac{1}{\kappa}\right) \times \frac{u(1-u)\,e^{\tilde{Y}}}{(1-u+ue^{\tilde{Y}})^2}\right)\right) \tag{19}$$

Figure 4 shows that this solution approaches the unitarity limit, giving a hope that the shadowing corrections could suppress the increase of the Green's function of the n-Pomerons in t-channel. However, this figure shows that $Z\left(\tilde{Y},u\right) < 0$

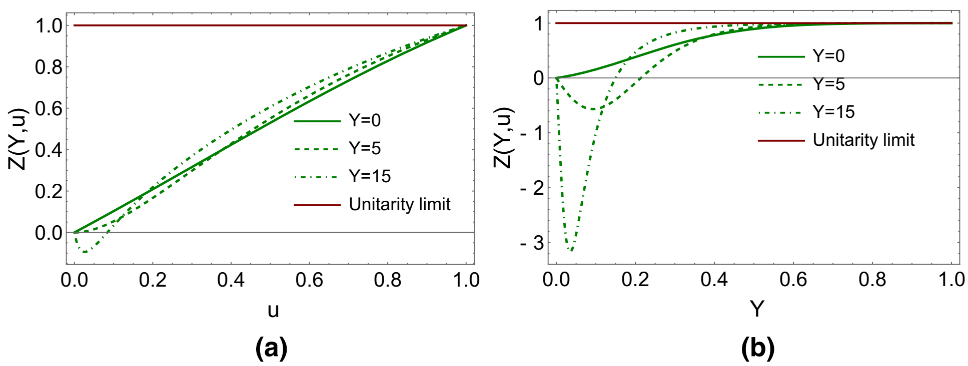

Fig. 4 Graphic form of the solution of Eq. (19). **a** $\kappa = 1$. **b** $\kappa = 1/8$. $\Delta = 0.2$

in the limited range of u, where this solution violates the unitarity constraints. Hence, we need to find exact solution for the final conclusions.

3.3 Exact solution

Fortunately, the general solution to Eq. (10) has been found in Ref. [88]. We discuss this solution here, repeating all steps of Ref. [88] and paying special attention to the sign in the r.h.s. of Eq. (10). First, we consider $Z(Y, u)$ in ω representation:

$$Z(Y, u) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\omega}{2\pi i} e^{\omega \tilde{Y}} z(\omega, u) \tag{20}$$

For $z(\omega, u)$ Eq. (10) takes the form:

$$\omega z(\omega, u) = -u(1-u)\Big(\frac{\partial z(\omega, u)}{\partial u} + \kappa \frac{\partial^2 z(\omega, u)}{\partial u^2}\Big) \tag{21}$$

Plugging $z(\omega, u) = \exp\left(-\frac{u}{2\kappa}\right) \tilde{z}(\omega, u)$ in Eq. (21) we obtain:

$$4\kappa\,\omega\,\tilde{z}(\omega, u) = -u(1-u)\Big(-\tilde{z}(\omega, u) + 4\kappa^2\, \tilde{z}''_{uu}(\omega, u)\Big) \tag{22}$$

Introducing $\tilde{z}(\omega, u) = u(1-u)\mathcal{G}(\omega, u) = \frac{1-v^2}{4}\mathcal{G}(\omega, v)$ with $1 - 2u = v$, we can rewrite Eq. (22) in the following form:

$$\begin{aligned} 4\kappa\,\omega\mathcal{G}(\omega, u) &= u(1-u)\mathcal{G}(\omega, u) - 4\kappa^2\left(u(1-u)\,\mathcal{G}(\omega, u)\right)''_{uu} \\ &= \left(u(1-u) + 8\kappa^2\right)\mathcal{G}(\omega, u) \\ &\quad + 8\kappa^2(2u-1)\mathcal{G}'_u(\omega, u) - 4\kappa^2 u(1-u)\mathcal{G}''_{uu}(\omega, u) \end{aligned} \tag{23}$$

Using v we have

$$(1-v^2)\,\mathcal{G}''_{vv}(\omega, v) - 4\,v\,\mathcal{G}'_v(\omega, v) + \left\{-2 + \frac{1}{\kappa}\omega - \frac{1-v^2}{4\kappa^2}\right\}\mathcal{G}(\omega, v) = 0 \tag{24}$$

Table 1 First five eigenvalues ω_n of $\omega_n = \kappa\,\lambda^1_n$

ω	κ	1	2	3	4	5
ω_n	$8\ (N_c = 3)$	1.636	1.766	2.616	3.579	4.8
ω_n	$1\ (N_c = 1)$	2.2	6.14	12.133	20.113	30.128

In Ref. [88] it is noted, that functions $\mathcal{G}(\omega, u)$ are intimately related to prolate spheroidal wave functions $S_{n,m}(c, v) = (1-v^2)^{\frac{m}{2}}\,\mathcal{G}(c, v)$ [95–97], which satisfy the following equation:

$$\frac{d}{dv}\left((1-v^2)\frac{d\,S_{n,m}(c, v)}{d\,v}\right) + \left(\lambda^m_n - c^2 v^2 - \frac{m}{1-v^2}\right) S_{n,m}(c, v) = 0 \tag{25}$$

with n and m being integer numbers.

For functions $\mathcal{G}(c, v)$ Eq. (25) takes the form:

$$(1-v^2)\,\mathcal{G}''_{vv}(c, v) - 2(m+1)v\,\mathcal{G}'_v(c, v) + \left\{\lambda^m_n - c^2 v^2 - m(m+1)\right\}\mathcal{G}(c, v) = 0 \tag{26}$$

Comparing Eqs. (24) and (26) we obtain that

$$m = 1;\quad c^2 = -\frac{1}{4\kappa^2};\quad \omega_n = \kappa\,\lambda^1_n + \frac{1}{4\kappa}; \tag{27}$$

Hence, the set of the eigenfunctions for the generating function $Z(\omega, u)$ has the form:

$$Z_n(v) = \sqrt{1-v^2}\,e^{-\frac{1-v}{4\kappa}}\,S_{n,1}\left(\frac{i}{2\kappa}, v\right) \tag{28}$$

Going back to rapidity representation, one can see that

$$\begin{aligned} Z_n(Y, v) &= e^{\omega_n \tilde{Y}}\sqrt{1-v^2}\,e^{-\frac{1-v}{4\kappa}}\,S_{n,1}\left(\frac{i}{2\kappa}, v\right) \\ &= e^{\kappa\lambda^1_n \tilde{Y}}\sqrt{1-v^2}\,e^{-\frac{1-v}{4\kappa}}\,S_{n,1}\left(\frac{i}{2\kappa}, v\right) \end{aligned} \tag{29}$$

Therefore, one can see that each eigenfunction increases as function of $\tilde{Y}$ (see Table 1 for the values of ω_n).

Generally speaking the generating function is equal to

$$Z(Y, v) = Z^{\text{asymp}}(u) + \sum_{n=1}^{\infty} \mathrm{C_n} e^{\omega_n \tilde{Y}} \sqrt{1-v^2}\, e^{-\frac{1-v}{4\kappa}}\, S_{n,1}\left(\frac{i}{2\kappa}, v\right) \tag{30}$$

where

$$\mathrm{C_n} = \int_{-1}^{1} dv \Bigg(Z(Y=0, u) - Z^{\text{asymp}}(u) \Bigg) \frac{e^{\frac{1-v}{4\kappa}}}{\sqrt{1-v^2}} \frac{S_{n,1}\left(\frac{i}{2\kappa}, v\right)}{||S_{n,1}\left(\frac{i}{2\kappa}, v\right)||} \tag{31}$$

with

$$||S_{n,1}\left(\frac{i}{2\kappa}, v\right)|| = \int_{-1}^{1} dv\, |S_{n,1}\left(\frac{i}{2\kappa}, v\right)|^2 = \frac{2\,n\,(n+1)}{2\,n\,+\,1} \tag{32}$$

$Z^{\text{asymp}}(u)$ is the solution to Eq. (14) which correponds to the minimal value of $\omega = 0$. It is convenient to choose it in the following form:

$$Z^{\text{asymp}}(u) = \frac{1 - e^{-\frac{u}{\kappa}}}{1 - e^{-\frac{1}{\kappa}}} \tag{33}$$

This solution gives $Z^{\text{asymp}}(u=1) = 1$ and $Z^{\text{asymp}}(u=0) = 0$. Hence the difference $Z(Y=0, u) - Z^{\text{asymp}}(u)$ satisfies the following boundary condition:

$$\begin{aligned} &Z(Y=0, u=0) - Z^{\text{asymp}}(u=0) = 0;\\ &Z(Y=0, su=1) - Z^{\text{asymp}}(u=1) = 0; \end{aligned} \tag{34}$$

This difference can be expanded in the series of Eq. (30) since functions $S_{n,1}\left(\frac{i}{2\kappa}, v\right)$ satisfy the boundary conditions of Eq. (34) being equal to zero at $v = \pm 1$[4]

From Eq. (30) we conclude that $Z(Y, u)$ increases with energy (rapidity Y). In other words even in the simple case of the initial condition of Eq. (17), which corresponds to the deep inelastic scattering the shadowing corrections failed to stop the increase of the scattering amplitude.

3.4 Energy growth from the general structure of equation.

In this section we demonstrate that the energy increase actually stems from the general structure of Eq. (10). Indeed, Eq. (21) can be rewritten in the form of the Sturm–Liouville equation:

$$\begin{aligned} &s(u)\,\omega\, Z(\omega, u) + \frac{d}{d\,u}\Big(p(u) Z'_u(\omega, u) \Big) = 0 \quad \text{with}\\ &s(u) = \frac{\kappa}{u(1-u)} e^{\frac{u}{\kappa}} \quad \text{and} \quad p(u) = e^{\frac{u}{\kappa}} \end{aligned} \tag{35}$$

[4] We wish to refer to Refs. [88,98] for more detailed analysis of solutions since Eq. (10) belong to this class.

The Sturm–Liouville equation has the following general features [88,98]:

1. Equation 35 has infinite set of eigenvalues $\omega_m = \lambda_n$. λ_n monotonically increases with n with $\lambda_n \to \infty$ at large n. In our case of Eq. (35) all $\lambda_n > 0$. The least value of λ_n is $\lambda_0 = 0$ which corresponds the asymptotic solution of Eq. (15).
2. The multiplicity of each eigenvalue is equal to 1.
3. The eigenfunctions $Z_n(u)$ are orthogonal

$$\int_0^1 du\, s(u)\, Z_n(u)\, Z_m(u) = 0 \quad \text{for} \quad n \neq n \tag{36}$$

4. For large n

$$\lambda_n = \frac{\pi^2\, n^2}{\delta^2} \quad \text{with} \quad \delta = \int_0^1 du \sqrt{\frac{s(u)}{p(u)}} \tag{37}$$

5. For our equation

$$\delta = \sqrt{\kappa} \int_0^1 \frac{du}{\sqrt{u(1-u)}} = \sqrt{\kappa}\,\pi \tag{38}$$

giving

$$\lambda_n = n^2/\kappa \tag{39}$$

Having Eq. (39) we can conclude, that the generating function $Z_n\left(\tilde{Y}, u\right)$ increases with Y. This feature is based on the general features of the Sturm–Liouville equation and can be stated without finding the exact solution. Since the class of Sturm–Lioville equations is much wider than our particular equation (see Eq. (10)), we believe that more complicate equations in the case of QCD will still have these property,

4 Conclusions

The main question, that we have answered in this paper, whether the shadowing correction can stop the steep increase of Green's function for the exchange of the n BFKL Pomerons: $G_{n\,I\!P}(Y) \propto \exp\left(\frac{n^2}{N_c^2}\Delta_{\text{BFKL}} Y\right)$. In this paper we considering the simple Pomeron calculus in zero transverse dimension. This approach has two great advantages: (i) it takes into account all shadowing corrections including the summation of the Pomeron loops and (ii) it has the same as in QCD striking increase of $G_{n\,I\!P}(Y)$. Solving exactly the evolution equation we demonstrate that the shadowing corrections cannot stop the increase of scattering amplitude which violates the unitarity constraints. Hence, our answer to the question in the title is positive.

However, another phenomenon could considerably increase the shadowing corrections: the saturation effects inside the parton cascade. In the zero dimensional Pomeron calculus such corrections have been included in UTM model [38,39,84,91,92].[5] On one hand, we have demonstrated in Ref. [92] the scattering amplitude for this model in spite of saturation in the parton cascade coincides with the BFKL cascades at high energies. On the other hand, we need to include $\Gamma^{2I\!P}_{2I\!P}$ in this model. It has not been done and we consider this as a next problem to be solved. If we think about the theoretical realization of the parton model (see for example Refs. [99,100]), we do not expect that the unitarity would be violated. Therefore, we need to understand why this theoretical approach cannot be reduced to Eq. (7) at high energies.

The violation of unitarity stems from the vertex $\Gamma^{2I\!P}_{2I\!P} > 0$, which also appears in QCD (see Refs. [69–73]) with the same sign. In Ref. [101] has been demonstrated that generalization of Balitsky–Kovchegov non-linear equation in the CGC approach, which takes into account the $1/N_c$ corrections, but without treating the 'switch' diagrams of Refs. [69–73] , do not change drastically the scattering amplitude. In my opinion CGC approach could describe the 'switch' diagrams. However, including them in CGC approach we will loose the connection with the Pomeron calculus. We do not believe that the technical complications, coming with the QCD analysis, could provide the stronger shadowing than in the simple zero dimensional models. However, we are aware that we have no idea how to sum Pomeron loops in QCD and how they influence the strength of the shadowing corrections. On the other hand, we need to consider a possibility to go out of the Pomeron calculus to treat the high energy amplitude in CGC and consider CGC approach or/and the effective Lagrangian of Refs. [102,103] without reducing them to the Pomeron calculus.

Concluding, we believe that CGC approach correctly describes the high energy interaction at $N_c \to \infty$ and we do not have a clue how the shadowing correction could suppress the growth of the scattering amplitude in $1/N_c$ order.

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[5] Following Refs. [91,92] we call UTM (Unitarity Toy Model) the zero dimensional model that takes into account both t and s channel unitarity, In this model the scattering amplitudes of two dipoles have been introduced, which generate the saturation effects in the parton cascade. Such saturation effects are not included in Eq. (7).

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