



Quantum gravity constraints on fine structure constant from GUP in braneworlds

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Abstract The Generalized Uncertainty Principle (GUP) has been discussed in the thick braneworld scenario. By considering Rydberg atoms in this background, we show that the spacetime geometry affects Maxwell equations inducing an effective dielectric constant on the space. In its turn, the corrected Coulomb potential by the gravitational interaction yields a deviation on the 3-dimensional Bohr radius. Then, we compute the corrections on the fine structure constant owing to the GUP in higher-dimensional spacetime. We also found constraints for the deformation parameter β and D -dimensional Planck length l_D by comparing the predicted deviations with the recent empirical data of the fine structure constant. We compute the intermediate length scale, which in principle may be larger than the Planck length scale. It is conjectured that below such scale Quantum Gravity effects should take place.

1 Introduction

There are theoretical motivations for the assumption of the high dimensionality of spacetime. Originally this idea was presented by the schemes of the well-known unification theories. Currently, its interest has been renewed, mainly due to braneworld models. In this case, our observable 4-dimensional universe would be a thin hypersurface embedding in a higher-dimensional space [1–4]. In this scenario, the particles and fields of the Standard Model (SM) would be trapped on the brane, while gravity could have free access to the supplementary space. The leakage of gravity to the extra dimensions would justify the weakness of the gravitational interaction compared to the other fundamental forces of nature and thus would solve the well-known hierarchy

problem [1,3]. From the field theory point of view, the braneworld deals as a topological defect e.g., a domain wall [5], where the localization of fermions, for instance, can be ensured through an interaction of the Yukawa-type between the Dirac fields and the defect [5].

The well-known thick braneworld, which is an extension of the thin brane models, has received increasing attention in the last years [6]. According to these models, the wave function of the SM particles would have a finite penetration depth in the transverse directions of the brane. Thus, the SM fields are confined on the submanifold whose thickness in the extra dimensions should be of the order of $(\text{TeV}/\hbar c)^{-1}$, which would allow the emission of SM particles into the bulk through collisions at TeV energies [1,2].

Although the gravitational interaction is negligible in the atomic domain according to 3-dimensional physics, it will be amplified in the extra-dimensional scenario. In this way, the atomic energy spectrum is modified by the gravitational interaction. Thus, precise measurements of atomic transitions can be used to test and constrain extra-dimensional models. Indeed, the search for empirical traces of extra dimensions in the thick brane scenario has motivated several studies in atomic systems [7–9]. In this context, we can highlight the recent work [9], in which one computes the gravitational correction on the energy levels of Hydrogen-like ions that lies in Rydberg states. Though the gravitational interaction is amplified in the short-distances regime, one has found that the gravitational influence of the electrovacuum is the main contribution to gravitational potential energy [9].

While atomic spectroscopy has proven to be an alternative way to seek traces of hidden dimensions, here, on the other hand, we intend to discuss the possibility of measuring the effects of spatial extra-dimensionality through the variations of the fine structure constant in a thick brane scenario. Indeed, modern theories such as brane models or string/M-theory offer a natural and self-consistent framework for the

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variation of fundamental couplings [10, 11]. Assuming that the multidimensional constants are genuinely fundamental, they determine the couplings in three dimensions along with the size R of the space of extra dimensions. On its turn, in the thick brane model, we expect that the brane structure leads to a deviation of the 3-dimensional value of the fine structure constant.

According to the quantum gravity theories – such as string theory, for instance – it is expected that the strength of the gravitational interaction will be comparable to other fundamental interactions on the fundamental length scale that is assumably to be on the Planck scale [12] or electroweak scale in extra-dimensional physics [1]. Since the absence of knowledge of how the physical laws work on the Planck scale, also known as Planck length or minimal length, it seems reasonable to seek to understand the theoretical implications of the combined gravitational and quantum effects on the properties of the physical systems in this fundamental scale.

Several theoretical models, such as the unification schemes of string theory, black hole physics, and even doubly special relativity (DSR) conjecture that, due to the existence of gravity, the Heisenberg Uncertainty Principle (HUP) must be modified, which in turn, leads to the prediction of the existence of a minimal observable length [13–21]. The extension of the HUP that encompasses these modifications is called the Generalized Uncertainty Principle (GUP) [22]. Furthermore, it is known that there are some derivations, and even proposals of Gedanken experiments, generally associated with the gravitational phenomena, which lead to the prediction of the GUP [23]. Thus, the GUP formulation incorporates gravitational corrections into a principle of quantum mechanics.

Many works have employed the GUP formalism to the study of properties and possible corrections to several physical systems such as BTZ black hole [24–26] and acoustic black hole [27], approach to entropic nature of the gravitational interaction [28], study of the inflationary era [29], description of brane cosmology dynamic [30], and also corrections for fine structure constant [31–33]. From a phenomenological point of view, experiments of gravitational and non-gravitational sources have been investigated, aiming to obtain constraints on the GUP deformation parameter [34–47]. In this perspective, it becomes important to study the phenomenology of quantum gravity because we could infer measurable effects of physics beyond the SM [41].

Given the recent developments in the field of spectroscopy of Rydberg states [48], some works have investigated the possibility of testing for theories that lie beyond the Standard Model from the study of the atomic transition between Rydberg states [9, 49, 50]. Thus, this work aims to discuss the constraints obtained from GUP in a thick brane scenario by considering hydrogen-like atoms in Rydberg states. As we will see, the GUP introduces the dependence on the higher-dimensional Planck length l_D and the β -parameter, called

the deformation parameter, into the corrected fine structure constant. The β -parameter estimates the deformation on the canonical commutation relation from quantum mechanics, and is directly related to the intermediate length scale $\ell \equiv l_D\beta/2$ below which it is expected new physics could come to play a fundamental role [36]. We must also worth highlight that, motivated by the extra-dimensional scenarios, it seems reasonable to consider the Planck length as a parameter to be constrained by the experiment. Then, we intend to constrain the minimal length by comparing our theoretical prediction with the experimental data from fine structure constant measurement. This analysis will allow us, simultaneously, to constrain the deformation parameter β from GUP.

The paper is organized as follows. In Sect. 2, by considering hydrogen-type atoms in Rydberg states, we show the potential produced from the electromagnetic field generated by the atomic nucleus in a thick braneworld scenario. For this case, the metric mimics the Reissner–Nordström geometry. Then we compute the gravitational correction to the Coulomb electrostatic potential. In Sect. 3, we address the GUP in the thick brane model and thus, obtain the corrected fine structure constant. From this result, one find constraints on higher-dimensional Planck length and deformation parameter by using recent measurements of the fine structure constant. In its turn, in Sect. 4, we compute the constraints on the deformation parameter and the intermediate length scale, which must be a Quantum Gravity effects regulator. Finally, in Sect. 5, we discuss some results and present the final remarks.

2 Rydberg atom in the thick brane

We start by considering the proposed Arkani–Dimopoulos–Dvali (ADD) braneworld model [1, 2]. In this case, the space-time has δ large extra dimensions compactified on a torus T^δ with a radius R . The SM fields must be localized in the 4-dimensional hypersurface by means of some confinement mechanism, while only the gravitational field have access to extra-dimensional space [1]. The classical action describing the gravitational field produced by an atom in higher-dimensional space can be written as

$$S_G = \frac{c^3}{16\pi G_D} \int d^4x d^\delta z \sqrt{-\hat{g}} \hat{R}, \quad (1)$$

where G_D is the higher-dimensional gravitational constant, $D = \delta + 3$ labels the number of spatial dimensions, \hat{g} is the determinant of the metric with a signature $(-, +, \dots, +)$ and \hat{R} is the scalar curvature of the bulk. We should also mention that the x -coordinates label the parallel directions to the brane while z -coordinates label the transverse directions.

In the atomic domain, we can consider gravity in the weak field regime and thus write the metric as being composed of

a four-dimensional part and a perturbation term due to the existence of the supplementary space. In this case, the metric assumes the form $g_{AB} = \eta_{AB} + h_{AB}$, where η_{AB} is the metric of Minkowski spacetime, and h_{AB} is the perturbed metric that describes tensor fluctuations on flat spacetime. Here, the atomic nucleus is responsible for generating the geometry which resembles the Reissner–Nordstrom spacetime. Although the charges are confined to 3-brane and can not escape into the bulk – unless excited through highly energetic processes as in the LHC experiment – the electric field produced by the atomic nucleus can penetrate a finite region ε into the transverse directions.

By calculating the solution to the linearized Einstein equations in a coordinate system that satisfies the harmonic gauge condition and considering the appropriate energy-momentum tensor, it was shown that the metric takes the form [9]:

$$\begin{aligned}
 ds^2 = & - \left(1 + \frac{2}{c^2} \varphi_s + \frac{2(2+\delta)}{c^2(1+\delta)} \chi_s \right) (dx^0)^2 \\
 & + \left(1 - \frac{2}{c^2(1+\delta)} \varphi_s \right) \left[\left(1 + \frac{2(2+\delta)}{c^2(1+\delta)} \lambda_{1,s} \right) dr^2 \right. \\
 & + \left. \left(1 + \frac{2(2+\delta)}{c^2(1+\delta)} \lambda_{2,s} \right) r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \\
 & + \left(1 - \frac{2}{c^2(1+\delta)} \varphi_s \right) dz^2, \tag{2}
 \end{aligned}$$

where φ_s is the Newtonian potential generated by an atom while χ_s is the gravitational potential associated with the energy of the electromagnetic field. The functions $\lambda_{1,s} \cong \chi_s$, and $\lambda_{2,s} \cong -\chi_s$ are found from spatial components of the electromagnetic stress-energy tensor. For further details, see [9].

Recently, the potential φ_s has been obtained for hydrogen-like atoms in S-states by considering a Gaussian profile to nuclear wave-function in the extra-dimensional directions [7, 8]. For this case, one shows that the interior contribution of the gravitational potential energy to atomic energy levels overcomes the outside contribution. On the other hand, here we are interested in analyzing atoms with a high principal quantum number known as Rydberg atoms. Therefore, for the atoms which lie in Rydberg states the internal gravitational energy contribution is suppressed by the external [9]. In this case, the leading term of the gravitational potential can be written as:

$$\varphi_s = - \frac{\hat{G}_D M}{r^{1+\delta}}, \tag{3}$$

where $\hat{G}_D = 4G_D \Gamma(\frac{3+\delta}{2}) / [(2+\delta)\pi^{(1+\delta)/2}]$. This result is found by assuming that the Dirac delta approach to describe the proton wave-function profile is valid. From (3) it can be shown that for Rydberg atoms on braneworld scenario, the

gravitational potential energy is amplified if $R \gg a_0$, where a_0 is the Bohr radius.

Contrary to the gravitational potential φ_s , the gravitational potential produced by the electromagnetic field, χ_s , can not be obtained in the thin brane limit by considering the delta distribution to the atomic nucleus wave-function. Here the χ_s -potential diverges on all space due to the spread of the electromagnetic field on 3-brane. Therefore, for Rydberg atoms, we must admit that the electric field flux lines have a penetration depth $\varepsilon \lesssim \mathcal{O}(10^{-19} \text{ m})$ into the transversal directions to braneworld. In light of this fact, one may calculate the gravitational potential inside the thick brane due to the distribution of the electric field on spacetime and get [9],

$$\chi_s = - \frac{\xi(\delta) \hat{G}_D Q^2}{16\pi\epsilon_0 c^2 \varepsilon^{\delta-2} r^4}, \tag{4}$$

where ε is the depth penetration of electric field lines on the transverse direction to the brane, and $\xi(\delta) = \delta\sqrt{\pi}\Gamma(\frac{\delta-2}{2})/2(\delta-1)\Gamma(\frac{\delta-1}{2})$. Note that,

$$\varphi_s/\chi_s = \frac{16\pi\epsilon_0 c^2 M}{\xi(\delta) Q^2} \varepsilon \left(\frac{\varepsilon}{r}\right)^{\delta-3} \ll 1, \text{ if } \delta \geq 3 \text{ and } r \geq r_N, \tag{5}$$

where r_N is the nuclear radius. Its shown that χ_s surpass the gravitational potential φ_s for external regions to the atomic nucleus by considering realistic values for the brane thickness $\varepsilon < 10^{-19} \text{ m}$. In the next section, we intend to study the corrections induced by this spacetime geometry to 3-dimensional electrostatic potential. Thereby we expect to find corrections in the electric potential due to extra-dimensional scenario.

2.1 Gravitational modified Maxwell’s equations

Theoretical proposals for modifications of the electrostatic potential have been presented in several scenarios. In string theory, for instance, corrections to electrostatic potential were obtained and explained due to the existence of the dilaton and modulus fields on effective 4-dimensional theory [51]. On the other hand, from a field theory point of view, modifications to Coulomb electric field have been studied and associated with an exotic dielectric function arising from the tachyon matter, which would correct Maxwell’s equations. It is shown that this approach mimics what occurs with the field of gluons accountable for quarks binding in the hadronic matter [52].

In its turn, in our study, we expect that geometric effects associated with the extra-dimensionality of spacetime might induce modifications to electrostatic potential. Although the SM fields are localized on the brane, the electric flux lines have a penetration depth ε into transversal direction. How-

ever, at low energy scales, is expected that the SM fields do not couple to the bulk geometry. Therefore, without loss of generality, we can assume that the brane is located at $z = 0$, and so we write the metric induced into isotropic coordinates as:

$$ds^2 = -w^2 (dx^0)^2 + v^2 (d\vec{x} \cdot d\vec{x}), \tag{6}$$

where

$$w^2 = 1 + \frac{2}{c^2} \varphi_s + \frac{2(2 + \delta)}{c^2(1 + \delta)} \chi_s, \tag{7}$$

$$v^2 = 1 - \frac{2}{c^2(1 + \delta)} \varphi_s - \frac{(2 + \delta)}{c^2(1 + \delta)} \chi_s. \tag{8}$$

In curved spacetime, i.e., in the presence of gravity, the well-known Maxwell’s equation with source takes the form:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} F^{\mu\nu}) = \mu_0 J^\nu, \tag{9}$$

where g is the determinant of the metric tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor, A_μ is the four-potential, J_ν is the four-current and μ_0 is the vacuum magnetic permeability.

We should note that since this problem is spherically symmetric due to the presence of the static gravitational field produced by the atomic nucleus, the equation of motion for the electric field reduces to:

$$\nabla \cdot (\epsilon \vec{E}) = \rho_0, \tag{10}$$

where $\epsilon = (v/w)\epsilon_0$ is an effective dielectric constant, ϵ_0 is the vacuum permittivity, and ρ_0 is the charge density. Therefore, we observe that the effective dielectric function ϵ modifies the electric field due to geometrical effects. We must emphasize that, for conformally flat metrics whose condition between the components of the metric $v = w$ is satisfied, the 3-dimensional Maxwell equation is recovered.

At low energy scales, we can write the electric field as being composed of a 3-dimensional part corresponding to the Coulomb electric field \vec{E}_0 and a correction owing to the gravitational contribution \vec{E}_g . In this case, we admit the weak field limit, where the gravitational interaction contributes with a small correction to the electrostatic potential,

$$\phi = \phi_0 + \phi_g, \tag{11}$$

where $\phi_0 = Q/4\pi\epsilon_0 r$ is the 3-dimensional electrostatic potential and $\phi_g \sim \mathcal{O}(\hat{G}_D)$ is the gravitational correction to the electrostatic potential, so that the electric field assumes the form $\vec{E} = -\nabla(\phi_0 + \phi_g)$. In this case, the Eq. (10) reduces to

$$\nabla^2 \phi_g = -\frac{1}{\epsilon_0} \nabla \cdot [(1 - \epsilon/\epsilon_0) \vec{E}_0]. \tag{12}$$

From the Eqs. (7) and (8) we get the solution to Eq. (12), that will take the form:

$$\phi_g = \frac{\kappa(\delta)}{16\pi^2 \epsilon_0^2 c^4} \frac{\hat{G}_D Q^3}{r_N^4 r \epsilon^{\delta-2}} \left(1 + \mathcal{O}(r_N/r)^4\right), \tag{13}$$

where $\kappa(\delta) = 3\sqrt{\pi}\delta(\delta + 2)\Gamma(\frac{\delta-2}{2})/64\Gamma(\frac{\delta+3}{2})$, $r_N = r_0 A^{1/3}$ is the nuclear radius, where A is the nucleon number and $r_0 = 1.2$ fm [53]. Note that to exterior region $r \gg r_N$ the contributions of higher orders will be negligible since $\mathcal{O}(r_N/r)^4 \ll 1$.

3 Fine structure constant from GUP in higher-dimensional space

In this section, we obtain the corrections to the fine structure constant owing to the GUP in the thick braneworld scenario. According to the thick brane model, although the SM interactions are confined to 3-dimensional hypersurface, ensured by the existence of a higher-dimensional fundamental Planck scale $M_D \sim (\text{TeV}/c^2)$ [1,2], the wave-function of the SM particles have access to a limited region into transverse directions to brane. As we have seen, by considering a Rydberg atom in this background, a correction to the electrical force experienced by charged particles orbiting the atomic nucleus must arise and can be written as

$$F_e = \frac{Q^2}{4\pi\epsilon_0 r^2} + \frac{\kappa(\delta)}{16\pi^2 \epsilon_0^2 c^4} \frac{\hat{G}_D Q^4}{r_N^4 r^2 \epsilon^{\delta-2}}. \tag{14}$$

Note that if we turn off the gravitational effects present in the second term of the Eq. (14), we should recover the expected classical result. From this relation, it is possible to calculate the Bohr radius in the thick braneworld scenario. In this case, the maximum uncertainty in the position of the lepton orbiting the atomic nucleus of a hydrogen-like atom is related to the well-known Bohr radius. So, for this higher-dimensional spacetime, we can easily obtain the radius of the orbit of the n th state for a Rydberg atom:

$$(\Delta x_i)_{\max} \equiv \tilde{a}_0 = a_0 n^2 \left(1 - \frac{a_0 \hat{G}_D m Q^4 \kappa(\delta)}{16\pi^2 \epsilon_0^2 c^4 r_N^4 \hbar^2 \epsilon^{\delta-2}}\right), \tag{15}$$

where $a_0 = 4\pi\epsilon_0 \hbar^2 / m Q^2$ is the 3-dimensional Bohr radius, m is the lepton mass orbiting the atomic nucleus, and n is the principal quantum number. As seen, the extra dimensions induce a correction to the 3-dimensional Bohr radius.

3.1 Fine structure constant corrected by GUP

It is known that the limit to simultaneous measurement accuracy between the position and momentum, for instance, is predicted by the HUP, although there is no restriction for measuring the particle position. On the other hand, the existence of a minimal measurable length has been proposed in several contexts, including extra-dimensional scenarios, and this leads to GUP [22]. Therefore, it is expected that on the Planck scale – that within the higher-dimensional framework is $\sim (\text{TeV}/\hbar c)^{-1}$ – the effects of quantum gravity would manifest leading to correction to the Heisenberg uncertainty relation. Following these ideas, the most general GUP, including linear and quadratic contributions in momentum, can be expressed as [25],

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2} \left(1 - \beta \frac{l_D}{\hbar} \Delta p_i + \beta^2 \frac{l_D^2}{\hbar^2} \Delta p_i^2 \right), \tag{16}$$

where $l_D = (\hbar G_D/c^3)^{1/(\delta+2)}$ is the Planck length of the higher-dimensional spacetime, which from braneworld point of view acts as a fundamental scale, β is a dimensionless positive deformation parameter. The commutation relation that leads to the GUP (16) and which is consistent with string theory, DSR theories, and black holes physics has been originally proposed in Refs. [20,42,54–56]. Let us note that it is possible to construct an intermediate length scale related to the minimal uncertainty on the position. Thereby, we may rewrite the Eq. (16) as follows:

$$\Delta x_i \geq \frac{\hbar}{2} \left(\frac{1}{\Delta p_i} - \beta \frac{l_D}{\hbar} + \beta^2 \frac{l_D^2}{\hbar^2} \Delta p_i \right). \tag{17}$$

The minimum value to Δx may be easily found by

$$\frac{d(\Delta x_i)}{d(\Delta p_i)} = \frac{\hbar}{2} \left(-\frac{1}{\Delta p_i^2} + \beta^2 \frac{l_D^2}{\hbar^2} \right) = 0 \tag{18}$$

Thus, the minimization of the Eq. (17) is guaranteed for $(\Delta p_i)_{\max} = \hbar/\beta l_D$, such that now we can define an intermediate length scale bellow which we expect that signals of new physics appear [36],

$$(\Delta x_i)_{\min} = \ell \equiv \beta l_D/2. \tag{19}$$

As we will see later, the existence of this new intermediate length scale ensures that $\beta \approx 1$ and, in addition, effects of Quantum Gravity can be measurable.

Here we must also state that, without loss of generality, one has assumed that the additional terms to uncertainty present on the Eq. (16) are due to gravitational interaction

since the coupling constant is the higher-dimensional gravitational constant G_D . Finally, note that by a direct inspection from HUP, $\Delta x \Delta p \geq \hbar/2$, and the GUP Eq. (16), it is possible to define an effective Planck constant

$$\hbar_{\text{eff}} = \hbar \left(1 - \beta \frac{l_D}{\hbar} \Delta p_i + \beta^2 \frac{l_D^2}{\hbar^2} \Delta p_i^2 \right). \tag{20}$$

Now, let us analyze the solution for the momentum uncertainty from Eq. (16), aiming to determine the effective Planck constant corrected due to the GUP in the braneworld framework. So, from Eq. (16), we find the uncertainty for the momentum as follows:

$$\Delta p_i = \frac{\hbar \left(\beta l_D + 2\Delta x_i - \sqrt{4\Delta x_i^2 + 4\beta \Delta x_i l_D - 3\beta^2 l_D^2} \right)}{2\beta^2 l_D^2}. \tag{21}$$

Since the Bohr radius can be related to the maximum uncertainty in the position of lepton within the atom, we can substitute the results (15) and (21) into Eq. (20), to obtain

$$\hbar_{\text{eff}} \simeq \hbar \left[1 - \beta \left(\frac{l_D}{2a_0 n^2} + \frac{\hat{G}_D l_D m Q^4 \kappa(\delta)}{32\pi^2 \epsilon_0^2 c^4 n^2 \hbar^2 r_N^4 \epsilon^{\delta-2}} \right) + \beta^2 \left(\frac{l_D^2}{2a_0^2 n^4} + \frac{\hat{G}_D l_D^2 m Q^4 \kappa(\delta)}{16\pi^2 \epsilon_0^2 c^4 n^4 \hbar^2 a_0 r_N^4 \epsilon^{\delta-2}} \right) \right]. \tag{22}$$

This result has been found by considering a power series expansion for $\hat{G}_D \ll 1$ and $l_D \ll 1$.

In the following, since we aim to find deviations to the 3-dimensional fine structure constant α , we will analyze the induced gravitational corrections on the fine structure constant owing to the GUP. Thus, we define an effective fine structure constant α_{eff} from the effective Planck constant (22) so that

$$\alpha_{\text{eff}} = \frac{Q^2}{4\pi \epsilon_0 \hbar_{\text{eff}} c}. \tag{23}$$

Finally, substituting the result (22) into Eq. (23), we get

$$\alpha_{\text{eff}} \simeq \frac{Q^2}{4\pi \epsilon_0 \hbar c} \left[1 + \beta \left(\frac{l_D}{2a_0 n^2} + \frac{\hat{G}_D l_D m Q^4 \kappa(\delta)}{32\pi^2 \epsilon_0^2 c^4 n^2 \hbar^2 r_N^4 \epsilon^{\delta-2}} \right) - \beta^2 \left(\frac{l_D^2}{4a_0^2 n^4} + \frac{\hat{G}_D l_D^2 m Q^4 \kappa(\delta)}{32\pi^2 \epsilon_0^2 c^4 n^4 \hbar^2 a_0 r_N^4 \epsilon^{\delta-2}} \right) \right]. \tag{24}$$

Here, again, we have performed a power series expansion by assuming that $\hat{G}_D \ll 1$ and $l_D \ll 1$. As seen, due to geometrical effects, we find that the effective fine structure constant in higher-dimensional spacetime presents deviations compared to its expected classical 3-dimensional value. Although

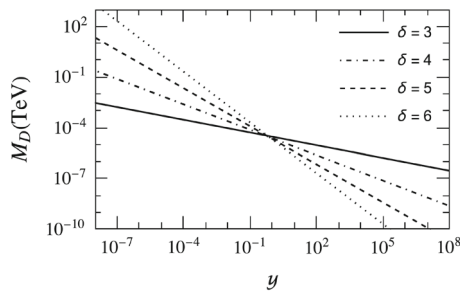


Fig. 1 Requirements on higher-dimensional Planck mass so that the correction from the electrovacuum exceeds the contribution from $\mathcal{O}(\hat{G}_D)^0$, by considering a hydrogen-like Rydberg atom

α can be assumed as a “coupling constant” of the electromagnetic interaction, the gravitational field induced corrections on their value extracted from Rydberg atoms in the thick brane scenario. On its turn, in the absence of extra-dimensional effects, i.e., for $\delta = 0$ the l_D reduces to 3-dimensional Planck length l_{Pl} , and so we obtain the fine structure constant corrected by 3-dimensional GUP, which for an hydrogen-like atom with $Z = 1$ in the fundamental state can be written as follows:

$$\alpha_{eff} \simeq \alpha_0 \left(1 + 1.53 \times 10^{-25} \beta - 2.33 \times 10^{-50} \beta^2 \right), \quad (25)$$

where $\alpha_0 = Q^2/4\pi\epsilon_0\hbar c$ is the 3-dimensional fine structure constant. This small correction found shows us that the fine structure constant has a value very close to the 3-dimensional expected value.

On the other hand, if we assume the condition predicted by the string theory such that $\beta \ll 1$, and since as $l_D/a_0 \ll 1$, one can see from Eq. (24) that the first-order correction in β will be stronger than the second-order $\sim \beta^2$. In this case, we can determine the regime for which the extra-dimensional contribution owing of the brane structure for atoms in Rydberg states will surpass the term arising from the generalization of the HUP, i.e., the term of $\mathcal{O}(l_D)$. Thus, we get the condition on the higher-dimensional Planck mass

$$M_D < \left[\frac{16\Gamma\left(\frac{\delta+3}{2}\right)\kappa(\delta)}{(\delta+2)\pi^{(\delta+1)/2}} \frac{m_e a_0}{\epsilon^{\delta-2} r_N^2} \frac{\hbar^{\delta-1}}{c^{\delta+3}} \left(\frac{Q^2}{8\pi\epsilon_0 r_N} \right)^2 \right]^{1/(\delta+2)}. \quad (26)$$

The condition (26) is shown in Fig. 1 and presents bounds on M_D so that the contribution due to the brane structure overcomes that one owing the term $l_D/2a_0n^2$. For this purpose, let us admit that l_D is related to ϵ by a factor y , such that $\epsilon = yl_D$.

From Fig. 1, we should note that the curves represent the upper limit for which the condition Eq. (26) is valid. For values of M_D in the area, below the curves, the gravitational

correction from electrovacuum overcomes the contribution due to the first term $l_D/2a_0n^2$, i.e., the effects from thickness braneworld should amplify the correction on the fine structure constant. In this case, for small values of the y -parameter, the brane structure deviation will dominate fine structure correction, while higher values imply the weakening of such correction. As we will see in the following subsection, strict corrections coming from 3-dimensional GUP are smaller than uncertainty in the measure of α . In its turn, we expect that the extra-dimensional scenario amplifies such corrections.

3.2 Constraints for the GUP deformation parameter β in braneworld scenario

In this subsection, we compute the bounds on the fine structure constant owing to the corrections from the GUP in the thick brane scenario. As proposed for theoretical models of quantum gravity, especially in some string theory inspired models, the deformation parameter β present on GUP must be of order unity [43]. On the other hand, the empirical constraints owing to experiments of gravitational and non-gravitational sources have been investigated, aiming to obtain upper bounds on the deformation parameter. In this case, the deformation parameter β can assume high values ($\beta \gg 1$) without such results implying conflicts with the empirical data [44]. Here, in order to present simultaneously constraints on the parameter β and l_D , let us analyze recent measurements of the fine structure constant.

A recent measurement of the fine structure constant by using matter-wave interferometry technique with rubidium atoms has been obtained with an unprecedented precision [57],

$$\alpha^{-1} = 137.035999206 (11), \quad (27)$$

whose experimental uncertainty $\delta\alpha_{exp}$ is expressed in parenthesis. As we have seen, the theoretical GUP model at the presence of a supplementary space predicts deviations on the fine structure constant. So, from the measure (27), we intend to obtain bounds to proposed corrections. Firstly, from Eq. (24), one can rewrite the effective fine structure constant as

$$\alpha_{eff} = \alpha_0 + \Delta\alpha, \quad (28)$$

where $\Delta\alpha$ is a correction due to GUP into thick brane framework. Aiming to infer empirical bounds, we must require that $|\Delta\alpha| < \delta\alpha_{exp}$, which demands that the correction obtained does not exceed the uncertainty in the measurement of the fine structure constant. In this way, we ensure that, so far, no indications that confirm the existence of a fundamental length

of the large scale proposed by GUP or even extra dimensions have been detected.

By specifying the principal quantum number n of a hydrogen-like atom, we have found upper limits to deformation parameter β and Planck length l_D in higher-dimensional space. Moreover, the constraints obtained from experiments of gravitational and non-gravitational sources also are shown in Figs. 2 and 3, by considering electronic and muonic hydrogen-like atoms, respectively. In these figures, the shaded areas represent excluded regions by our analysis. The “VEP” point corresponds to constraint found from violation of equivalence principle on Earth, while the marker labeled “mechanical oscillator” describes bound obtained from a non-gravitational experiment studying optomechanical interaction between micro- and nano-oscillators [44].

As we can see, the constraints found by our analysis are more stringent than those obtained by other physical systems from gravitational and non-gravitational sources [44]. For instance, for the gravitational experiment, strong constraints have been extracted from the analysis of violation of the equivalence principle, where it was shown that $\beta < 10^{21}$ [40]. In another way, considering constraints derived from micro- and nano-harmonic-oscillators, the constraint $\beta < 10^{18}$ was imposed to the deformation parameter [38]. If we assume that the fundamental scale is $l_D = 1.6 \times 10^{-35}$ m – for instance, keeping fixed the values $n = 10$, $y = 10^2$, and $\delta = 5$ –, our analysis gives us the restriction to the deformation parameter $\beta < 5.3 \times 10^{16}$. While for $l_D = 10^{-20}$ m ($\sim \mathcal{O}(\text{TeV}/\hbar c)^{-1}$) we get that $\beta < 85$.

As expected, to small values of y , the bounds present strong dependence with δ . Besides, to higher δ -values, with $l_D > 10^{-22}$ m, we found the most stringent constraints [see Fig. 2a]. On the other hand, higher y -values weaken the correction coming from the brane thickness and therefore make the bounds independent of spacetime codimensions [see Fig. 2b]. In its turn, for the electronic atom the intermediate length scale is $\ell = 4.2 \times 10^{-19}$ m. Below this scale, we expect the effects of Quantum Gravity to become measurable. We should emphasize that these constraints on β ensure that the effects from GUP have not manifested themselves until currently in the fine structure constant measure.

Recent developments in the field of spectroscopy have provided motivation for exploring the potential of precise measurements of optical transitions between Rydberg states as an effective means of testing QED theory and obtaining a more accurate value for the Rydberg constant [48]. In this context, some works address the possibility of measuring deviations to the inverse square law, being able to detect traces of extra dimensions, or even discussing new Physics using the spectroscopy of Rydberg states [9, 49, 50]. Somewhat, these studies aiming circumvent the proton radius puzzle [58]. Here, the high principal quantum number n ensure that we con-

sidered the Rydberg states. Notice that, for high values of the principal quantum number, one observed the weakening of the constraints obtained in our analysis. This result may be associated with the fact that the lepton orbiting a Rydberg atom will be far away from the atomic nucleus. In this case, the lepton would feel the field produced by the atomic nucleus more weakly, inducing a smaller correction on the fine structure constant.

Although we have obtained bounds from electronic hydrogen-like atoms data, we must emphasize that experiments involving muonic atoms have received growing attention recently [58–61]. This is partly due to the possibility of looking at extremely small regions of the order of the proton radius, allowing the study of the atomic structure [58]. In addition, such physical systems would act as tests for Quantum Electrodynamics (QED) since atomic physics can be applied to measure the fundamental properties of elementary particles. At last, the muonic systems can still allow us to seek physics beyond the SM [58]. Briefly stated, a muonic atom can be described as an atom whose lepton, usually an electron, that orbits the atomic nucleus has been replaced by a muon, which possesses an electric charge identical to the electron [59].

In this case, if we admit that measurements of the fine structure constant might be extracted from experiments using muonic atoms, we expect more stringent bounds than early obtained since $m_\mu/m_e \simeq 206$ and $a_0/a_{0\mu} \simeq 185$, where $a_{0\mu}$ is the Bohr radius of the muonic hydrogen-like atom. Thus, as previously discussed, we might find prospective constraints by assuming that the fine structure constant has been extracted from measurements into a muonic atom with the same uncertainty that to electronic atom [57]. This analysis yields Fig. 3.

As shown in Fig. 3, the bounds from the muonic hydrogen allows us to find stronger constraints than those obtained from the gravitational and non-gravitational experiments [44]. To obtain these results we have considered $y = 10^{-6}$ (Fig. 3.a) and $y = 10^6$ (Fig. 3.b). Although the lepton is away from the atomic nucleus in the Rydberg atoms, the gravitational effects will be amplified due to the smaller Bohr radius of the muonic atom. As we can see by tightest the allowed region for the deformation parameter in Fig. 3, the muonic system provides us with the strongest prospective constraints than found in the electronic atom.

4 Intermediate length scale as a regulator effect of new physics

Here we aim to present the constraints on the intermediate length of large scale ℓ and the deformation parameter β from the empirical data. Thus, we consider empirical bounds data on β obtained from several physical systems of gravitational

Fig. 2 Constraints are shown to corrections of the fine structure constant induced from the β deforming parameter owing to GUP into higher-dimensional spacetime by considering electronic hydrogen-like atoms in Rydberg states. The symbols “♦” and “▼” labels respectively bounds from gravitational and non-gravitational empirical data [44]. In **a**, we have assumed $y = 10^{-6}$, while in **b $y = 10^6$ was considered**

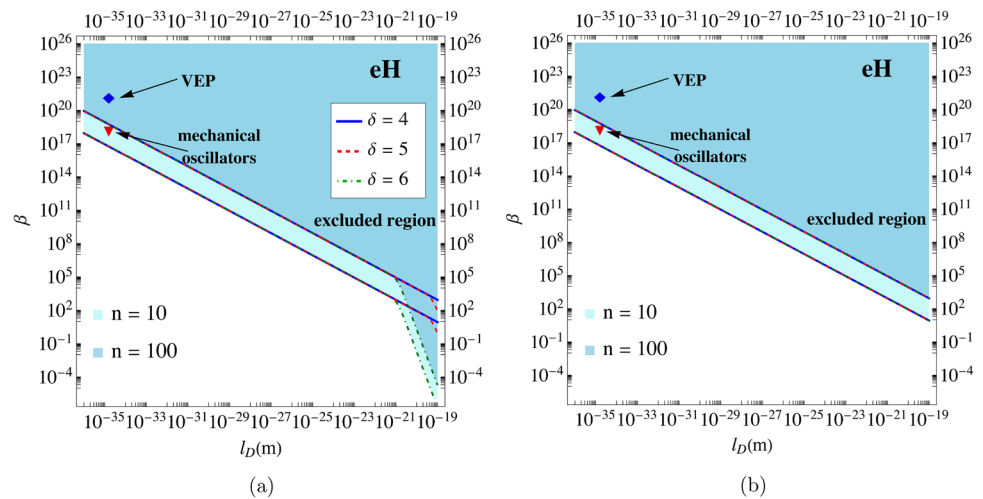
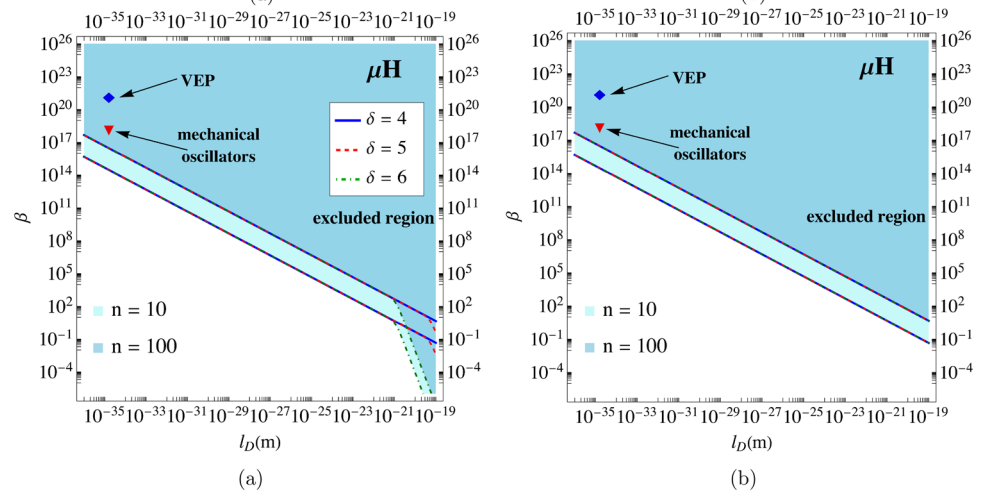


Fig. 3 Prospective bounds for β -parameter and the D -dimensional Planck length are obtained from fine structure constant measurements by considering the muonic hydrogen-like atom assuming **a** $y = 10^{-6}$ and **b** $y = 10^6$



and non-gravitational sources [34–39]. Since the limits on the deformation parameter have been found in the 3-dimensional scenario ($l_{Pl} = 1.6 \times 10^{-35}$ m), one can compute the value to intermediate length scale ℓ from these data as shown in Table 1, knowing the predicted deformation on uncertainty relation. Besides, the constraints have been obtained by considering quadratic-like GUP corrections (QGUP). In this framework, one expect that $\beta \sim l_{Pl}^{-2} \ell^2$. Indeed, the empirical data obtained from Table 1 are shown in Fig. 4 by the red dots and fitted by the solid curve described by the function $\beta = 1.3 \times 10^{69} \ell^{1.98}$. Thus, for instance, $\beta = 1$ leads to $\ell = 1.4 \times 10^{-35}$ m. The dash-dotted curve is an prediction found by considering $l_D = 10^{-20}$ m. We expect that any measurement of Quantum Gravity effects must be ruled out in the area above the solid curve.

On the other hand, the constraints on ℓ found by our analysis (see Eq. (19)), which considers empirical data from the fine structure constant measure, have been labeled by the marks “■” (electronic atom constraints) and “♦” (muonic atom constraints) in the plot. The fitted data by $\beta(\ell)$ that considers the linear and quadratic GUP (LQGUP) will be represented in Fig. 4 by the linear dashed and dotted curves

Table 1 Upper limits to constraints on the deformation parameter and intermediate length scale from gravitational and non-gravitational experimental sources

$\beta <$	$\ell(\text{m}) <$	References
5.3×10^{78}	7.4×10^4	[34]
3×10^{72}	5.5×10^1	
2×10^{71}	1.4×10^1	
2×10^{69}	1.4	
2.3×10^{60}	2.4×10^{-5}	[35]
10^{50}	2.0×10^{-10}	[36]
10^{36}	2.0×10^{-17}	
10^{21}	6.2×10^{-25}	
3×10^{33}	8.8×10^{-19}	[37]
2×10^{19}	7.2×10^{-26}	[38]
5×10^{13}	1.1×10^{-28}	
6×10^{12}	3.9×10^{-29}	
3×10^7	8.8×10^{-32}	
5.2×10^6	3.6×10^{-32}	[39]
4×10^4	3.2×10^{-33}	
10^{-4}	1.6×10^{-37}	

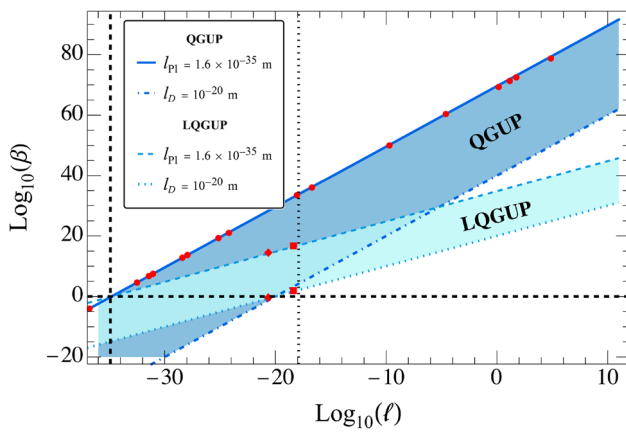


Fig. 4 The horizontal black dashed line represents the curve $\beta = 1$, while the vertical black dashed is found for fixed value $\ell = 1.6 \times 10^{-35}$ m. The vertical dotted line defines the region below which $\ell \lesssim \mathcal{O}(\text{TeV}/\hbar c)^{-1}$

found keeping the fixed values for $l_D = l_{Pl} = 1.6 \times 10^{-35}$ m and $l_D = 10^{-20}$ m, respectively. The dashed curve is plotted from the best fit function $\beta = 1.2 \times 10^{35} \ell$. In this case, $\beta < 1$ implies that $\ell < 8.0 \times 10^{-36}$ m. In its turn, for predictive dotted curve the function which best fits the data is $\beta = 2 \times 10^{20} \ell$, so that for $\beta < 1$ we get $\ell < 5.0 \times 10^{-21}$ m.

It is worthwhile to note that the constraints are weakened in 3-dimensional scenario ($l_D = l_{Pl} = 1.6 \times 10^{-35}$ m). Furthermore, if $l_D = l_{Pl} = 1.6 \times 10^{-35}$ m, for any values of ℓ there will not exist $\beta < 1$ such that $\ell > l_{Pl} = 1.6 \times 10^{-35}$ m. Indeed, the experiments with physical pendula have provided the most stringent limit on the deformation parameter β [39]. However, as one can see, such bounds require that $\ell < l_{Pl}$. Nevertheless, if the spacetime has hidden dimensions, there is a wide range to ℓ -scale so that $\beta < 1$ and also $\ell > l_{Pl}$, although we must have the condition on the intermediate scale $\ell < l_D$. In this context, the existence of a supplementary space leads to the prospective predictions of bounds $\beta \sim 1$. Thus, by considering the extra-dimensional scenario, the LQGUP strongly constrains the region such that $\beta \gg 1$, while QGUP presents more stringent constraints for $\beta \ll 1$. Traces of new physics are expected to reveal in the shadowed areas.

5 Concluding remarks

In this paper, we discuss the generalization of the Heisenberg Uncertainty Principle (HUP) in thick braneworld background. As is known, the Generalized Uncertainty Principle (GUP) predicts the existence of a minimal measurable length. In its turn, in braneworld scenario, the minimal length is related to the true fundamental scale, called higher-dimensional Planck length. By considering the hydrogen-

like atoms in Rydberg states within the thick brane framework, we show that electrostatic potential must be corrected due to gravitational contribution computable for the electromagnetic interaction. This correction induces a modification on Coulomb law responsible for describing the electrostatic interaction between charged particles.

For the sake of simplicity, we address the origin of the correction for the HUP as being owing to the gravitational interaction effects at a fundamental length scale. Thus, assuming a combination of linear and quadratic GUP approaches, we have found the effective fine structure constant, which depends on the deformation parameter β and $(3 + \delta)$ -dimensional Planck length l_D . By comparing a recent measure of the fine structure constant with the predicted deviation by our study, we obtain stringent constraints for the deformation parameter and the higher-dimensional Planck length. From our analysis – for instance, by fixing values $n = 10$, $y = 10^2$, and $\delta = 5 -$, if we consider that the fundamental length scale $l_D = l_{Pl} = 1.6 \times 10^{-35}$ m, the upper bound on $\beta < 5.3 \times 10^{16}$, which avoids any conflicts with the empirical results. On the other hand, if the true fundamental scale $l_D = 10^{-20}$ m ($\sim \mathcal{O}(\text{TeV}/\hbar c)^{-1}$), this implies in a strengthening of bounds, so that $\beta < 85$. From these results, we compute the intermediate length scale that must regulate the quantum effects of gravity $\ell = 4.2 \times 10^{-19}$ m for the electronic atom. We must still emphasize that, in this analysis, the precision of α -measurement is related to the tightening of bounds.

For atoms that lie in higher states with $n \gg 1$, it becomes difficult to probe regions where the supposed effects of gravitational interaction are amplified. However, the smallest Bohr radius for the muonic system leads to the strongest constraints than those found for electronic hydrogen. From this analysis, we present prospective bounds by considering Rydberg muonic atoms in the Fig. 3, which have approximately intensified the constraints by a factor of 10^2 times. Therefore, muonic atoms have shown to be as possible physical systems capable of testing, simultaneously, GUP and extra-dimensional theories.

Finally, we have discussed the definition of an intermediate length scale that would act as a regulator of the effects of Quantum Gravity. We showed that the extra-dimensional scenario leads to constraints such that $\beta \sim 1$ and, notwithstanding the GUP effects may be measurable. If GUP deviations are detected, in principle, we may infer the bounds for the higher-dimensional Planck length, which would indicate the existence of extra dimensions. Although we have obtained strong constraints on the deformation parameter and Planck length of the higher-dimensional space, we must highlight that, in our analysis, such bounds on β and l_D were obtained by keeping the other free parameters fixed. In this case, in a sense, this approach presents prospective constraints get from the investigation of the space of parameters.

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References

- N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B **429**, 263 (1998)
- I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali, Phys. Lett. B **436**, 257 (1998)
- L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999)
- L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999)
- V. Rubakov, M. Shaposhnikov, Phys. Lett. B **125**, 136 (1983)
- C. Csaki, J. Erlich, T.J. Hollowood, Y. Shirman, Nucl. Phys. B **581**, 309–338 (2000)
- F. Dahia, A.S. Lemos, Phys. Rev. D **94**, 084033 (2016)
- F. Dahia, A.S. Lemos, Eur. Phys. J. C **76**, 435 (2016)
- F. Dahia, E. Maciel, A.S. Lemos, Eur. Phys. J. C **78**(6), 526 (2018)
- G.A. Palma, P. Brax, A.C. Davis, C. van de Bruck, Phys. Rev. D **68**, 123519 (2003)
- P. Loren-Aguilar, E. Garcia-Berro, J. Isern, Y.A. Kubyshev, Class. Quantum Gravity **20**, 3885–3896 (2003)
- C.M. Will, Living Rev. Relativ. **17**, 4 (2014)
- D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B **216**, 41–47 (1989)
- F. Scardigli, Phys. Lett. B **452**, 39–44 (1999)
- R.J. Adler, D.I. Santiago, Mod. Phys. Lett. A **14**, 1371 (1999)
- S. Capozziello, G. Lambiase, G. Scarpetta, Int. J. Theor. Phys. **39**, 15–22 (2000)
- R. Casadio, W. Feng, I. Kuntz, F. Scardigli, Phys. Lett. B **838**, 137722 (2023)
- L.J. Garay, Int. J. Mod. Phys. A **10**, 145 (1995)
- G. Amelino-Camelia, Phys. Lett. B **510**, 255 (2001)
- A.F. Ali, S. Das, E.C. Vagenas, Phys. Lett. B **678**, 497 (2009)
- S. Hossenfelder, Living Rev. Relativ. **16**, 2 (2013)
- A.N. Tawfik, A.M. Diab, Int. J. Mod. Phys. D **23**(12), 1430025 (2014)
- M. Maggiore, Phys. Lett. B **304**, 65–69 (1993)
- M.A. Anacleto, F.A. Brito, A.G. Cavalcanti, E. Passos, J. Spinelly, Gen. Relativ. Gravit. **50**(2), 23 (2018)
- M.A. Anacleto, F.A. Brito, B.R. Carvalho, E. Passos, Adv. High Energy Phys. **2021**, 6633684 (2021)
- A. Iorio, G. Lambiase, P. Pais, F. Scardigli, Phys. Rev. D **101**(10), 105002 (2020)
- M.A. Anacleto, F.A. Brito, E. Passos, W.P. Santos, Phys. Lett. B **737**, 6–11 (2014)
- P. Nicolini, Phys. Rev. D **82**, 044030 (2010)
- A. Tawfik, H. Magdy, A. Farag Ali, Gen. Relativ. Gravit. **45**, 1227–1246 (2013)
- G. Barca, E. Giovannetti, G. Montani, Int. J. Geom. Meth. Mod. Phys. **19**(07), 2250097 (2022)
- F. Nasserri, Phys. Lett. B **618**, 229–232 (2005)
- F. Nasserri, Phys. Lett. B **632**, 151–154 (2006)
- G. de A. Marques, V. B. Bezerra, Int. J. Mod. Phys. Conf. Ser. **18**, 25–30 (2012)
- F. Scardigli, R. Casadio, Eur. Phys. J. C **75**, 425 (2015)
- Z.W. Feng, S.Z. Yang, H.L. Li, X.T. Zu, Phys. Lett. B **768**, 81 (2017)
- S. Das, E.C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008)
- F. Marin, F. Marino, M. Bonaldi, M. Cerdonio, L. Conti, P. Falferi, R. Mezzena, A. Ortolan, G.A. Prodi, L. Taffarello et al., Nat. Phys. **9**, 71–73 (2013)
- M. Bawaj, C. Biancofiore, F. Marin, Nat. Commun. **6**, 7503 (2015)
- P.A. Bushev, J. Bourhill, M. Goryachev, N. Kukharchyk, E. Ivanov, S. Gallioui, M.E. Tobar, S. Danilishin, Phys. Rev. D **100**(6), 066020 (2019)
- S. Ghosh, Class. Quantum Gravity **31**, 025025 (2014)
- S. Das, E.C. Vagenas, Can. J. Phys. **87**, 233 (2009)
- A.F. Ali, S. Das, E.C. Vagenas, Phys. Rev. D **84**, 044013 (2011)
- L. Buoninfante, G.G. Luciano, L. Petruzzello, Eur. Phys. J. C **79**(8), 663 (2019)
- T. Kanazawa, G. Lambiase, G. Vilasi, A. Yoshioka, Eur. Phys. J. C **79**(2), 95 (2019)
- F. Scardigli, Symmetry **12**(9), 1519 (2020)
- F. Scardigli, G. Lambiase, E. Vagenas, Phys. Lett. B **767**, 242–246 (2017)
- F. Scardigli, J. Phys. Conf. Ser. **1275**(1), 012004 (2019)
- U.D. Jentschura, P.J. Mohr, J.N. Tan, B.J. Wundt, Phys. Rev. Lett. **100**, 160404 (2008)
- A.S. Lemos, EPL **135**(1), 11001 (2021)
- M.P.A. Jones, R.M. Potvliege, M. Spannowsky, Phys. Rev. Res. **2**(1), 013244 (2020)
- M. Cvetič, A.A. Tseytlin, Nucl. Phys. B **416**, 137–172 (1994)
- F.A. Brito, M.L.F. Freire, W. Serafim, Eur. Phys. J. C **74**(12), 3202 (2014)
- S.S.M. Wong, *Introductory Nuclear Physics* (Wiley, New York, 2008)
- A.N. Tawfik, E.A. El Dahab, Int. J. Mod. Phys. A **30**(09), 1550030 (2015)
- S. Gangopadhyay, A. Dutta, M. Faizal, EPL **112**(2), 20006 (2015)
- E.C. Vagenas, A.F. Ali, M. Hameda, H. Alshal, Eur. Phys. J. C **79**(5), 398 (2019)
- L. Morel, Z. Yao, P. Cladé, S. Guellati-Khélifa, Nature **588**(7836), 61–65 (2020)
- R. Pohl, R. Gilman, G.A. Miller, K. Pachucki, Annu. Rev. Nucl. Part. Sci. **63**, 175–204 (2013)
- M.I. Eides, H. Grotch, V.A. Shelyuto, Phys. Rep. **342**, 63–261 (2001)
- J.J. Krauth, M. Diepold, B. Franke, A. Antognini, F. Kottmann, R. Pohl, Ann. Phys. **366**, 168–196 (2016)
- A. Antognini, F. Kottmann, R. Pohl, SciPost Phys. Proc. **5**, 021 (2021)