Eur. Phys. J. C (2023) 83:339
https://doi.org/10.1140/epjc/s10052-023-11450-w

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

The existence of null circular geodesics outside extremal spherically symmetric asymptotically flat hairy black holes

Yan Peng[a]

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China

Received: 22 February 2023 / Accepted: 30 March 2023 / Published online: 27 April 2023

Abstract The existence of null circular geodesics in the background of non-extremal spherically symmetric asymptotically flat black holes has been proved in previous works. An interesting question that remains, however, is whether extremal black holes possess null circular geodesics outside horizons. In the present paper, we focus on the extremal spherically symmetric asymptotically flat hairy black holes. We show the existence of the fastest circular trajectory around an extremal black hole. As the fastest trajectory corresponds to the position of null circular geodesics, we prove that null circular geodesics exist outside extremal spherically symmetric asymptotically flat hairy black holes.

1 Introduction

It is generally acknowledged that highly curved black hole spacetimes are characterized by the existence of null circular geodesics outside horizons [1,2]. Such geodesics are how massless particles can orbit the central black holes. They provide valuable information on the structure of the spacetime geometry, which is closely related to various remarkable properties of black hole spacetimes and has attracted considerable attention [3–5].

The presence of circular null geodesics has many important implications for astronomical observations, such as gravitational lensing and black hole shadow, as well as gravitational waves [4,6–15]. Circular null geodesics also provide a lower bound for the effective radius of scalar field hairs outside black holes [16–21]. The characteristic resonances of perturbed black holes can be determined by the instability properties of circular null geodesics [22–29]. Massless fields can accumulate on stable circular null geodesics, leading to nonlinear instabilities in highly curved spacetimes [4,30–36]. It has also been shown that the null circular geodesic corresponds to the fastest orbital trajectory around a central black hole [37].

The existence of null circular geodesics can be predicted by analyzing the non-linearly coupled Einstein-matter field equations. In non-extremal hairy black hole spacetimes, the existence of null circular geodesics has been proved in the asymptotically flat background [38]. Null circular geodesics also exist in the stationary axisymmetric non-extremal black hole spacetimes [39]. Hod raised an interesting question of whether null circular geodesics can also exist outside extremal black holes [40]. Motivated by the discussion of [40], our work and [42] independently provide alternative elegant proofs of the same result indicating that null geodesics do exist outside spherically symmetric extremal black holes. Specifically, [42] established remarkably compact proofs based on the dominant energy condition, and we will establish the proof that null circular geodesics provide the fastest orbital trajectory around a black hole [37].

In this work, we first introduce the extremal hairy black hole spacetimes. We prove the existence of null circular geodesics by analyzing the existence of the circular trajectory around central extremal hairy black holes. Finally, we summarize our main results and draw conclusions.

2 The existence of null circular geodesics in extremal black holes

We study the physical and mathematical properties of null circular geodesics in the background of extremal hairy black hole spacetimes. The spherically symmetric curved line element of a four-dimensional asymptotically flat hairy black hole is described by [18,41]

$$ds^2 = -e^{-2\delta}\mu dt^2 + \mu^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

[a] e-mail: yanpengphy@163.com (corresponding author)

Here, δ and μ are metric solutions depending only on the Schwarzschild areal coordinate r. Angular coordinates are $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The black hole horizon r_H is obtained from $\mu(r_H) = 0$. At the regular horizon, δ is finite. At the infinity, there is $\mu(\infty) = 1$ and $\delta(\infty) = 0$ in the asymptotically flat spacetimes. We study null circular geodesics in the equatorial plane with $\theta = \frac{\pi}{2}$.

In the following, we follow the analysis of [17] to obtain the null circular geodesic equation. The Lagrangian describing the geodesics is given by

$$2\mathcal{L} = -\mathrm{e}^{-2\delta}\mu\dot{t}^2 + \mu^{-1}\dot{r}^2 + r^2\dot{\phi}^2, \tag{2}$$

where the dot represents differentiation with respect to proper time.

The Lagrangian is independent of coordinates t and ϕ, leading to two constants of motion referred to as E and L. From the Lagrangian (2), we obtain the generalized momenta expressed as [2]

$$p_t = g_{tt}\dot{t} = -\mathrm{e}^{-2\delta}\mu\dot{t} = -E = \text{const}, \tag{3}$$

$$p_\phi = g_{\phi\phi}\dot{\phi} = r^2\dot{\phi} = L = \text{const}, \tag{4}$$

$$p_r = g_{rr}\dot{r} = \mu^{-1}\dot{r}. \tag{5}$$

The Hamiltonian of the system is $\mathcal{H} = p_t\dot{t} + p_r\dot{r} + p_\phi\dot{\phi} - \mathcal{L}$, which implies

$$2\mathcal{H} = -E\dot{t} + L\dot{\phi} + g_{rr}\dot{r}^2 = \varepsilon = \text{const}. \tag{6}$$

The case of null geodesics corresponds to the value $\varepsilon = 0$.

From relations (3), (4) and (6), we find

$$\dot{r}^2 = \frac{1}{g_{rr}}[E\dot{t} - L\dot{\phi}] \tag{7}$$

for null geodesics.

Equations (3) and (4) yield relations

$$\dot{t} = \frac{E}{\mathrm{e}^{2\delta}\mu}, \quad \dot{\phi} = \frac{L}{r^2}. \tag{8}$$

Substituting Eq. (8) into Eq. (7), one finds

$$\dot{r}^2 = \mu\left[\frac{E^2}{\mathrm{e}^{2\delta}\mu} - \frac{L^2}{r^2}\right]. \tag{9}$$

The null circular geodesic equation $\dot{r}^2 = (\dot{r}^2)' = 0$ can be expressed as

$$2\mathrm{e}^{-2\delta}\mu - r(\mathrm{e}^{2\delta}\mu)' = 0. \tag{10}$$

The null circular geodesics usually correspond to the circular trajectory with the shortest orbital period [37]. In the following, we would like to search for the circular trajectory with the shortest orbital period as measured by asymptotic observers. In order to minimize the orbital period for a given radius r, one should move as close as possible to the speed of light. In this case, the orbital period can be obtained from Eq. (1) with $\mathrm{d}s = \mathrm{d}r = \mathrm{d}\theta = 0$ and $\Delta\phi = 2\pi$ [37,38]:

$$T(r) = -\frac{2\pi\sqrt{-g_{tt}g_{\phi\phi}}}{g_{tt}} = \frac{2\pi r}{\mathrm{e}^{-\delta}\sqrt{\mu}}. \tag{11}$$

The circular trajectory with the shortest orbital period is characterized by

$$T'(r) = 0, \tag{12}$$

where the solution is the radius of the fast circular trajectory.

The condition (12) yields the fast circular trajectory equation

$$2\mathrm{e}^{-2\delta}\mu - r(\mathrm{e}^{2\delta}\mu)' = 0. \tag{13}$$

We find that the null circular geodesic equation (10) is identical to the fastest circular trajectory equation (13). Therefore, the extreme period circle radius equation and the null circular geodesics equation share the same roots. If we can prove the existence of the fastest circular trajectory, the null circular geodesic exists outside black holes. In fact, there are relations $T(r_H) = T(\infty) = \infty$ according to the formula (11). Thus, the function $T(r)$ must possess a minimum at some finite radius $r = r_{extrem}$. At this radius, Eqs. (10) and (13) hold, which means that $r = r_{extrem}$ corresponds to the position of the null circular geodesics. In other words, we prove the existence of null circular geodesics outside extremal black holes.

3 Conclusions

We have investigated the existence of null circular geodesics outside extremal spherically symmetric asymptotically flat hairy black holes. We first obtained the null circular geodesic characteristic equations. We then obtained the equation governing the fastest circular trajectory that particles can follow around the central extremal spherically symmetric asymptotically flat hairy black hole. We showed that the null circular geodesic equation is identical to the fastest circular trajectory equation. Thus, the extreme period circular radius equation and the null circular geodesic equation share the same roots. We found that solutions exist for the extreme period circular radius equation; therefore, we proved that the null circular geodesic exists outside extremal spherically symmetric asymptotically flat hairy black holes.

Acknowledgements We would like to thank the anonymous referee and editor for the constructive suggestions to improve the manuscript. This work was supported by the Shandong Provincial Natural Science Foundation of China under Grant no. ZR2022MA074. This work was also supported by a grant from Qufu Normal University of China under Grant no. xkjjc201906, the Youth Innovations and Talents Project of Shandong Provincial Colleges and Universities (Grant no. 201909118), Taishan Scholar Project of Shandong Province (Grant no. tsqn202103062) and the Higher Educational Youth Innovation Science and Technology Program Shandong Province (Grant no. 2020KJJ004).

It should be emphasized that an analytical proof for the existence of null circular geodesics in extremal black hole spacetimes has also recently been given in Ref. [42] (published in the arXiv in the same week as the current manuscript).

Data availability statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This work is based on analytical studies. So there is no numerical data and all analysis has been presented in the paper].

Funded by SCOAP^3. SCOAP^3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

1. J.M. Bardeen, W.H. Press, S.A. Teukolsky, Rotating black holes: locally nonrotating frames, energy extraction, and scalar synchrotron radiation. Astrophys. J. **178**, 347 (1972)
2. S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983)
3. S.L. Shapiro, S.A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* (Wiley, New York, 1983), p.645p
4. V. Cardoso, A.S. Miranda, E. Berti, H. Witek, V.T. Zanchin, Geodesic stability, Lyapunov exponents and quasinormal modes. Phys. Rev. D **79**, 064016 (2009). arXiv:0812.1806 [hep-th]
5. S. Hod, Spherical null geodesics of rotating Kerr black holes. Phys. Lett. B **718**, 1552 (2013)
6. Z.I. Stefanov, S.S. Yazadjiev, G.G. Gyulchev, Connection between black-hole quasinormal modes and lensing in the strong deflection limit. Phys. Rev. Lett. **104**, 251103 (2010)
7. I.Z. Stefanov, S.S. Yazadjiev, G.G. Gyulchev, Connection between black-hole quasinormal modes and lensing in the strong deflection limit. Phys. Rev. Lett. **104**, 251103 (2010)
8. S.-W. Wei, Y.-X. Liu, H. Guo, Relationship between high-energy absorption cross section and strong gravitational lensing for black hole. Phys. Rev. D **84**, 041501(R) (2011)
9. S.-W. Wei, Y.-X. Liu, Establishing a universal relation between gravitational waves and black hole lensing. Phys. Rev. D **89**, 047502 (2014)
10. B. Raffaelli, Strong gravitational lensing and black hole quasinormal modes: towards a semiclassical unified description. Gen. Relativ. Gravit. **48**, 16 (2016)
11. V. Cardoso, E. Franzin, P. Pani, Is the gravitational-wave ringdown a probe of the event horizon?. Phys. Rev. Lett. **116**, 171101 (2016) [Erratum: Ibid. 117, 089902 (2016)]
12. R.A. Konoplya, Z. Stuchlik, Are eikonal quasinormal modes linked to the unstable circular null geodesics? Phys. Lett. B **771**, 597 (2017)
13. E. Gallo, J.R. Villanueva, Photon spheres in Einstein and Einstein–Gauss–Bonnet theories and circular null geodesics in axially-symmetric spacetimes. Phys. Rev. D **92**(6), 064048 (2015)
14. Z. Stuchlik, J. Schee, B. Toshmatov, J. Hladik, J. Novotny, Gravitational instability of polytropic spheres containing region of trapped null geodesics: a possible explanation of central supermassive black holes in galactic halos. JCAP **1706**(06), 056 (2017)
15. Z. Stuchlik, S. Hledik, Jan Novotny General relativistic polytropes with a repulsive cosmological constant. Phys. Rev. D **94**(10), 103513 (2016)
16. D.N*úñ*ez, H. Quevedo, D. Sudarsky, Black holes have no short hair. Phys. Rev. Lett. **76**, 571 (1996)
17. S. Hod, A no-short scalar hair theorem for rotating Kerr black holes. Class. Quantum Gravity **33**, 114001 (2016)
18. S. Hod, Hairy black holes and null circular geodesics. Phys. Rev. D **84**, 124030 (2011)
19. Y.S. Myung, T. Moon, Hairy mass bound in the Einstein–Born–Infeld black hole. Phys. Rev. D **86**, 084047 (2012)
20. Y. Peng, Hair mass bound in the black hole with nonzero cosmological constants. Phys. Rev. D **98**, 104041 (2018)
21. Y. Peng, Hair distributions in noncommutative Einstein–Born–Infeld black holes. Nucl. Phys. B **941**, 1–10 (2019)
22. B. Mashhoon, Stability of charged rotating black holes in the eikonal approximation. Phys. Rev. D **31**(2), 290–293 (1985)
23. S. Hod, Universal bound on dynamical relaxation times and black-hole quasinormal ringing. Phys. Rev. D **75**, 064013 (2007)
24. S. Hod, Black-hole quasinormal resonances: wave analysis versus a geometric-optics approximation. Phys. Rev. D **80**, 064004 (2009)
25. R.D. Dolan, The quasinormal mode spectrum of a kerr black hole in the eikonal limit. Phys. Rev. D **82**, 104003 (2010)
26. Y. Decanini, A. Folacci, B. Raffaelli, Unstable circular null geodesics of static spherically symmetric black holes, Regge poles and quasinormal frequencies. Phys. Rev. D **81**, 104039 (2010)
27. Y. Decanini, A. Folacci, B. Raffaelli, Resonance and absorption spectra of the Schwarzschild black hole for massive scalar perturbations: a complex angular momentum analysis. Phys. Rev. D **84**, 084035 (2011)
28. S. Hod, Resonance spectrum of near-extremal Kerr black holes in the eikonal limit. Phys. Lett. B **715**, 348–351 (2012)
29. H. Yang, D.A. Nichols, F. Zhang, A. Zimmerman, Z. Zhang, Y. Chen, Quasinormal-mode spectrum of Kerr black holes and its geometric interpretation. Phys. Rev. D **86**, 104006 (2012)
30. J. Keir, Slowly decaying waves on spherically symmetric spacetimes and ultracompact neutron stars. Class. Quantum Gravity **33**, 135009 (2016)
31. S. Hod, Upper bound on the radii of black-hole photonspheres. Phys. Lett. B **727**, 345 (2013)
32. P.V.P. Cunha, E. Berti, C.A.R. Herdeiro, Light-ring stability for ultracompact objects. Phys. Rev. Lett. **119**, 251102 (2017)
33. S. Hod, On the number of light rings in curved spacetimes of ultra-compact objects. Phys. Lett. B **776**, 1 (2018)
34. S. Hod, Upper bound on the gravitational masses of stable spatially regular charged compact objects. Phys. Rev. D **98**, 064014 (2018)
35. Y. Peng, Upper bound on the radii of regular ultra-compact star photonspheres. PLB **790**, 396–399 (2019)
36. Y. Peng, On instabilities of scalar hairy regular compact reflecting stars. JHEP **1810**, 185 (2018)
37. S. Hod, The fastest way to circle a black hole. Phys. Rev. D **84**, 104024 (2011)
38. S. Hod, Phys. Lett. B **727**, 345 (2013). arXiv:1701.06587
39. P.V.P. Cunha, C.A.R. Herdeiro, Phys. Rev. Lett. **124**, 181101 (2020)
40. S. Hod, Eur. Phys. J. C **82**, 663 (2022)
41. D. N*úñ*ez, H. Quevedo, D. Sudarsky, Black holes have no short hair. Phys. Rev. Lett. **76**, 571 (1996)
42. S. Hod, Extremal black holes have external light rings. Phys. Rev. D **107**, 024028 (2023). arXiv:2211.15983