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A semi-tetrad decomposition of the Kerr spacetime

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Abstract In this paper we perform a semi-tetrad decomposition of the Kerr spacetime. We apply the 1+1+2 covariant method to the Kerr spacetime in order to describe its geometry outside the horizon. Comparisons are drawn between an observer belonging to the Killing frame and a ZAMO (zero angular momentum observer), a locally non-rotating observer in a zero angular momentum frame, and their resulting geometrical quantities that generate the evolution and propagation equations. This enhances the study of the Kerr geometry as the results are valid in the ergoregion from where energy can be extracted. Using this formalism allows us to present the kinematic and dynamic quantities in a transparent geometrical manner not present in alternate approaches. We find significant relationships between the properties of shear, vorticity and acceleration. Additionally we observe that in the Killing frame, the gravitational wave is a direct consequence of vorticity and in the ZAMO frame, the gravitational wave is a direct consequence of shear. To our knowledge, using the 1+1+2 formalism to investigate the Kerr spacetime is a novel approach, and this provides new insights into the spacetime geometry in an easier manner than alternate approaches. Furthermore we make corrections to earlier equations in the 1+1+2 formalism applied to the Kerr spacetime.

1 Introduction

The Kerr spacetime, discovered in 1963 by Roy Kerr, is extremely relevant to the understanding of black hole physics and modern astrophysics. It is known that outside a rotating star the geometry asymptotically approaches the Kerr geometry. Hence, to a very good approximation, the spacetime near each rotating black hole in the observable universe is given by the Kerr solution. The Kerr metric [1] is a unique exact solution of Einstein's vacuum field equations which extends the Schwarzschild metric to include angular momentum. It is crucially important to study the effects of gravity in the Kerr geometry. Only then will it be possible to build a complete model of a rotating isolated body in general relativity which is an unsolved problem in astrophysics.

Examples of well known tetrad or semi-tetrad methods are the complex null tetrads of Newman and Penrose [2], the 1+3 covariant approach developed by Ehlers and Ellis [3,4] and the 1+1+2 covariant approach developed by Clarkson and Barrett [5]. The 1+3 formalism has generated new results in areas like gauge-invariant study [6,7], the cosmic microwave background [8] and specific spacetimes [9–12]. An extension of the 1+3 covariant approach is the 1+1+2 covariant approach which has generated new results in locally rotationally symmetric spacetimes in general relativity [13] and $f(R)$ gravity [14], and spacetimes with conformal symmetry [15]. In a breakthrough manner the 1+1+2 covariant approach was utilized to prove that tidal forces are gravitational waves [16]. It is clear that the 1+1+2 formalism is capable of generating new results when applied to current and previously analyzed astrophysical problems.

Through the years particular interest in the Kerr geometry has been generated through various studies [17–19]. In [17] the Newman-Penrose formalism involving null tetrads was used to explore the solution of Maxwell's equations in the Kerr geometry. However, our approach in this paper is different, here we use both the 1+3 and 1+1+2 semi-tetrad covariant approaches to describe the Kerr spacetime geometry. The advantage of using these approaches is that the physics and geometry of the spacetime are described by tensor quantities and relations which remain valid in all coor-

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dinate systems. The purpose of using these methods is to extract the geometrical features of the spacetime in an easier manner as the geometric variables have well defined physical interpretations. These geometric variables have been defined explicitly in this paper. A partial study of the Kerr metric in the 1+3 formalism was done in [20]; we complete this analysis and write down the full set of 1+3 equations for the Kerr metric. Noteworthily, we apply the 1+1+2 formalism to the Kerr spacetime and explicitly write down the 1+1+2 Kerr geometrical quantities and the evolution and propagation equations they satisfy. This application is done in two frame choices, the Killing frame and a zero angular momentum frame, to provide a well rounded description of the Kerr geometry. To our knowledge, using the 1+1+2 formalism to investigate the Kerr spacetime is a novel approach and it provides new insights into the spacetime geometry. All quantities and equations have been validated with the mathematical software Maple and the software package GRTensorIII [21].

The paper is structured as follows: in Sect. 2 we define the Kerr metric, consider its key features, and explain our frame choices. In Sect. 3 and Appendix A, we briefly review the 1+3 formalism. Then, in Sect. 4 we explicitly write down the 1+3 Kerr quantities and the equations they satisfy. Section 5 and Appendix B contain a brief review of the 1+1+2 formalism. In Sect. 6 and Appendix C, we apply the 1+1+2 formalism to the Kerr spacetime, a novel approach, and explicitly write down the 1+1+2 Kerr geometrical quantities and the evolution and propagation equations. Some errors in the 1+1+2 formalism equations found earlier are identified and corrected for the Kerr spacetime. Note that these corrections do not alter the linear perturbation results presented in [5,22]. In Sect. 7, we comment on interesting results arising from Sects. 4 and 6. Concluding remarks are made in Sect. 8.

2 The Kerr metric and frame choice

The Kerr metric describes the vacuum, stationary axisymmetric solution corresponding to stationary rotating black holes and depends on angular momentum (relating to a) and the physical mass (m) of the central object as parameters. One of the key features of the Kerr spacetime geometry is that it is Ricci flat ($R_{ab} = 0$). Additionally, there are three off-diagonal terms in the first version of the line element in [1]. Considering the Kerr spacetime in Boyer–Lindquist [23] coordinates which involves a coordinate substitution, results in only one off-diagonal term. Starting from the Boyer–Lindquist coordinates (t, r, θ, φ), a new coordinate $\chi = \cos\theta$ so that $\chi \in [-1; 1]$ is introduced, and we can write the Kerr spacetime in terms of "rational polynomial" coordinates (t, r, χ, φ) as

$$ds^2 = \left(\frac{2mr}{r^2 + a^2\chi^2} - 1\right) dt^2 - \left[\frac{4mar\left(1 - \chi^2\right)}{r^2 + a^2\chi^2}\right] dt d\varphi + \left(\frac{r^2 + a^2\chi^2}{r^2 - 2mr + a^2}\right) dr^2 + \left(\frac{r^2 + a^2\chi^2}{1 - \chi^2}\right) d\chi^2 + \left(1 - \chi^2\right)\left[r^2 + a^2 + \frac{2ma^2r\left(1 - \chi^2\right)}{r^2 + a^2\chi^2}\right] d\varphi^2. \quad (1)$$

For the purpose of this paper we will consider the Kerr spacetime in rational polynomial coordinates. The advantage of using these coordinates is that it eliminates trigonometric functions so that computational calculations can be performed more efficiently. Recent studies on the Kerr spacetime have been done in the context of unit-lapse forms [24], vortex forms [25], its topology [26] and coding simulations [27]. For a comprehensive review of the Kerr spacetime the reader is referred to [28].

In order to further highlight the geometrical properties of the Kerr spacetime, we consider two different types of observers in two different frames. Firstly, we consider a Killing observer belonging to the Killing reference frame (referred to as the chronometric reference frame in [20]). It is named accordingly because the worldlines of the observers in the Killing reference frame are along the timelike Killing vector field of the Kerr spacetime. Secondly we consider what is called a 'ZAMO' (zero angular momentum observer) which is a locally non-rotating observer belonging to a zero angular momentum frame. ZAMOs have zero angular momentum in their proper frame but as they approach any compact object with nonzero angular momentum, frame dragging pulls it along with the geometry so that they acquire the frame dragging angular velocity. At the spatial infinity both frames are the same. For more information about the Kerr spacetime in the context of the ZAMO frame we refer the reader to [29].

In the Kerr model two surfaces have coordinate singularities and the main singularity is a strong curvature ring singularity. The Kerr metric has two physically relevant surfaces on which it appears to be singular: the horizon and the ergosphere. The definition of a Killing frame is that the time axis is along the Killing direction. The frame is moving relative to the black hole so angular momentum is not zero but the observer is staying still in the frame. However this is not possible within the ergoregion which mandates that the observer must co-rotate with the inner mass. Hence the Killing frame only holds outside the ergoregion which is a limitation. This restriction will arise later in our description of a timelike vector which entirely depends on $r^2 - 2mr + a^2\chi^2$. However the ZAMO frame extends till the outer event horizon and consequently our results are valid till the outer event horizon. This means that our results explore the ergoregion

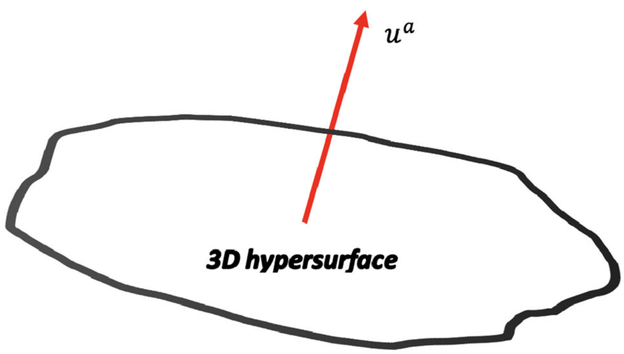

Fig. 1 Diagram illustrating the 1+3 formalism. The $4D$ spacetime is split according to the timelike vector u^a into 'time' and a $3D$ hypersurface

which has generated specific research interest in the context of rotating black holes as this is where energy can be extracted from.

3 1+3 formalism

In the 1+3 formalism, the timelike unit vector u^a ($u^a u_a = -1$) is split in the form $\mathscr{R} \otimes \mathscr{V}$, where $\mathscr{R}$ is the timeline along u^a and $\mathscr{V}$ is the 3-space perpendicular to u^a. The 1+3 covariantly decomposed spacetime is represented by

$$g_{ab} = h_{ab} - u_a u_b, \tag{2}$$

where h_{ab} is a tensor that projects onto the rest space of an observer moving with 4-velocity u^a. A visual representation of the 1+3 spacetime is given in Fig. 1.

The covariant time derivative along the observers' worldlines, denoted by '$\dot{\ }$', is defined using the vector u^a, as

$$\dot{Z}^{a...b}{}_{c...d} = u^e \nabla_e Z^{a...b}{}_{c...d}, \tag{3}$$

for any tensor $Z^{a...b}{}_{c...d}$. The fully orthogonally projected covariant spatial derivative, denoted by 'D', is defined using the spatial projection tensor h_{ab}, as

$$D_e Z^{a...b}{}_{c...d} = h^r{}_e h^p{}_c ... h^q{}_d h^a{}_f ... h^b{}_g \nabla_r Z^{f...g}{}_{p...q}, \tag{4}$$

with total projection on all the free indices. The covariant derivative of the 4-velocity vector u^a is decomposed irreducibly as follows

$$\nabla_a u_b = -u_a \dot{u}_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab} + \varepsilon_{abc} \omega^c, \tag{5}$$

where $\dot{u}_b$ is the acceleration, Θ is the expansion of u_a, σ_{ab} is the shear tensor, ω^a is the vorticity vector representing rotation and ε_{abc} is the effective volume element in the rest space of the comoving observer. The Weyl tensor may be split relative to u^a as

$$E_{ab} = C_{acbd} u^c u^d, \tag{6}$$

$$H_{ab} = \frac{1}{2} \varepsilon_{ade} C^{de}{}_{bc} u^c, \tag{7}$$

where E_{ab} represents the electric part and H_{ab} represents the magnetic part of Weyl curvature and more information can be found in [30]. We refer the reader to [31] for a more detailed review of the 1+3 formalism.

4 The 1+3 Kerr quantities and equations

We apply the 1+3 formalism to the Kerr metric and find the following quantities which have been confirmed by the mathematical software Maple and GRTensorIII [21]. The timelike unit vector for the Killing frame is defined as

$$\begin{aligned} u^a &= \sqrt{\frac{r^2 + a^2\chi^2}{r^2 - 2mr + a^2\chi^2}}\,\delta^a{}_0 \\ &= \left[\sqrt{\tfrac{r^2+a^2\chi^2}{r^2-2mr+a^2\chi^2}},\ 0,\ 0,\ 0\right] \text{ where } u^a u_a = -1, \end{aligned} \tag{8}$$

and for the ZAMO frame it is defined as

$$\begin{aligned} u^a &= \sqrt{\frac{\mathscr{Z}}{\mathscr{F}\left(r^2 - 2mr + a^2\right)}}\left(1,\ \frac{2mar}{\mathscr{Z}}\delta^i{}_3\right) \\ &= \left[\sqrt{\tfrac{\mathscr{Z}}{(r^2-2mr+a^2)\mathscr{F}}},\ 0,\ 0,\ \tfrac{2mar}{\sqrt{\mathscr{Z}(r^2-2mr+a^2)\mathscr{F}}}\right], \end{aligned} \tag{9}$$

where $u^a u_a = -1$, $\mathscr{F} = r^2 + a^2\chi^2$ and $\mathscr{Z} = \left(r^2 + a^2\right)^2 - a^2\left(r^2 - 2mr + a^2\right)\left(1 - \chi^2\right)$, according to the definitions in [20] (where we have taken the charge quantity to be zero). We apply the 1+3 formalism to the Kerr metric and find the following quantities for the Killing frame and the ZAMO frame which have been confirmed by the mathematical software Maple and GRTensorIII [21].

The set of the Kerr 1+3 geometric variables is given by $\{\Theta,\ \omega_{ab},\ \omega^a,\ \sigma_{ab},\ \dot{u}_a,\ E_{ab},\ H_{ab}\}$, and they have the following values given in Table 1

where

$$\begin{aligned} \mathscr{B} &= -r^2 + a^2\chi^2,\ \mathscr{C} = r^2 - 2mr + a^2\chi^2, \\ \mathscr{G} &= r^2 - 2mr + a^2,\ \ \mathscr{J} = \chi^2 - 1,\ \ \mathscr{K} = 3a^2\chi^2 - r^2, \\ \mathscr{M} &= a^2\chi^2 - 3r^2,\ \ \mathscr{N} = a^2\chi^2 - 3a^2 + 4mr - 2r^2, \\ \mathscr{P} &= 2a^2\chi^2 - 3a^2 - r^2 + 2mr, \\ \mathscr{Q} &= a^4\chi^2 - r^2a^2\chi^2 - a^2r^2 - 3r^4, \\ \mathscr{S} &= a^6\chi^2 + 2a^4\chi^2r^2 - 4ma^2\chi^2r^3 + a^2\chi^2r^4 - a^4r^2 \\ &\quad +4ma^2r^3 - 2a^2r^4 - r^6, \\ \mathscr{T} &= 2r^4 - a^4\left(\chi^2 - 3\right) + a^2r\left(-r\left(\chi^2 - 5\right) + 2m\,\mathscr{J}\right), \\ \mathscr{U} &= r^4 - 2a^2r\left(2m - 2r + (r - 2m)\chi^2\right) \\ &\quad +a^4\left(3 - 2\chi^2\right), \\ \mathscr{V} &= -2r^4 + a^2r\left(2m - 5r + (r - 2m)\chi^2\right) \end{aligned}$$

Table 1 The 1+3 Kerr quantities in the Killing frame and the ZAMO frame

	Killing frame	ZAMO frame
Θ	0	0
ω_{ab}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{ma\mathscr{J}\mathscr{B}}{\sqrt{\mathscr{C}^3\mathscr{F}}} \\ 0 & 0 & 0 & \frac{2mar\chi\mathscr{G}}{\sqrt{\mathscr{C}^3\mathscr{F}}} \\ 0 & -\frac{ma\mathscr{J}\mathscr{B}}{\sqrt{\mathscr{C}^3\mathscr{F}}} & -\frac{2mar\chi\mathscr{G}}{\sqrt{\mathscr{C}^3\mathscr{F}}} & 0 \end{bmatrix}$	$\mathbf{0}$
ω^a	$\left[0,\ -\frac{2mar\chi\mathscr{G}}{\mathscr{C}\mathscr{F}^2},\ \frac{ma\mathscr{J}\mathscr{B}}{\mathscr{C}\mathscr{F}^2},\ 0\right]$	$\mathbf{0}$
σ_{ab}	$\mathbf{0}$	$\begin{bmatrix} 0 & \frac{2m^2a^2r\mathscr{J}\mathscr{Q}}{\sqrt{\mathscr{F}^3\mathscr{G}\mathscr{Z}^3}} & -\frac{\varrho\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{F}^3\mathscr{Z}^3}} & 0 \\ \frac{2m^2a^2r\mathscr{J}\mathscr{Q}}{\sqrt{\mathscr{F}^3\mathscr{G}\mathscr{Z}^3}} & 0 & 0 & -\frac{ma\mathscr{J}\mathscr{Q}}{\sqrt{\mathscr{F}^3\mathscr{G}\mathscr{Z}}} \\ -\frac{\varrho\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{F}^3\mathscr{Z}^3}} & 0 & 0 & \frac{2ma^3r\chi\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{F}^3\mathscr{Z}}} \\ 0 & -\frac{ma\mathscr{J}\mathscr{Q}}{\sqrt{\mathscr{F}^3\mathscr{G}\mathscr{Z}}} & \frac{2ma^3r\chi\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{F}^3\mathscr{Z}}} & 0 \end{bmatrix}$
$\dot{u}^a$	$\left[0,\ -\frac{m\mathscr{B}\mathscr{G}}{\mathscr{C}\mathscr{F}^2},\ -\frac{2ma^2r\chi\mathscr{J}}{\mathscr{C}\mathscr{F}^2},\ 0\right]$	$\left[0,\ -\frac{m\mathscr{S}}{\mathscr{Z}\mathscr{F}^2},\ -\frac{2mra^2\chi\left(r^2+a^2\right)\mathscr{J}}{\mathscr{Z}\mathscr{F}^2},\ 0\right]$
E_{ab}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{mr\mathscr{K}\mathscr{N}}{\mathscr{C}\mathscr{F}^2\mathscr{G}} & \frac{3ma^2\chi\mathscr{M}}{\mathscr{C}\mathscr{F}^2} & 0 \\ 0 & \frac{3ma^2\chi\mathscr{M}}{\mathscr{C}\mathscr{F}^2} & -\frac{mr\mathscr{K}\mathscr{P}}{\mathscr{C}\mathscr{F}^2\mathscr{J}} & 0 \\ 0 & 0 & 0 & \frac{mr\mathscr{G}\mathscr{K}\mathscr{J}}{\mathscr{C}\mathscr{F}^2} \end{bmatrix}$	$\begin{bmatrix} \frac{4a^2r^3m^3\mathscr{K}\mathscr{J}}{\mathscr{Z}\mathscr{F}^4} & 0 & 0 & -\frac{2am^2r^2\mathscr{K}\mathscr{J}}{\mathscr{F}^4} \\ 0 & \frac{mr\mathscr{K}\mathscr{T}}{\mathscr{Z}\mathscr{G}\mathscr{F}^2} & \frac{3ma^2\chi\mathscr{Q}}{\mathscr{Z}\mathscr{F}^2} & 0 \\ 0 & \frac{3ma^2\chi\mathscr{Q}}{\mathscr{Z}\mathscr{F}^2} & \frac{mr\mathscr{K}\mathscr{U}}{\mathscr{J}\mathscr{Z}\mathscr{F}^2} & 0 \\ -\frac{2am^2r^2\mathscr{K}\mathscr{J}}{\mathscr{F}^4} & 0 & 0 & \frac{mr\mathscr{K}\mathscr{J}\mathscr{Z}}{\mathscr{F}^4} \end{bmatrix}$
H_{ab}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{ma\chi\mathscr{M}\mathscr{N}}{\mathscr{C}\mathscr{F}^2\mathscr{G}} & -\frac{3mar\mathscr{K}}{\mathscr{C}\mathscr{F}^2} & 0 \\ 0 & -\frac{3mar\mathscr{K}}{\mathscr{C}\mathscr{F}^2} & -\frac{ma\chi\mathscr{M}\mathscr{P}}{\mathscr{C}\mathscr{F}^2\mathscr{J}} & 0 \\ 0 & 0 & 0 & \frac{ma\chi\mathscr{G}\mathscr{M}\mathscr{J}}{\mathscr{C}\mathscr{F}^2} \end{bmatrix}$	$\begin{bmatrix} \frac{4m^3a^3r^2\chi\mathscr{J}\mathscr{M}}{\mathscr{F}^4\mathscr{Z}} & 0 & 0 & -\frac{2ra^2m^2\chi\mathscr{J}\mathscr{M}}{\mathscr{F}^4} \\ 0 & -\frac{ma\chi\mathscr{M}\mathscr{V}}{\mathscr{G}\mathscr{Z}\mathscr{F}^2} & -\frac{3mar\mathscr{W}}{\mathscr{Z}\mathscr{F}^2} & 0 \\ 0 & -\frac{3mar\mathscr{W}}{\mathscr{Z}\mathscr{F}^2} & \frac{ma\chi\mathscr{M}\mathscr{U}}{\mathscr{J}\mathscr{Z}\mathscr{F}^2} & 0 \\ -\frac{2ra^2m^2\chi\mathscr{J}\mathscr{M}}{\mathscr{F}^4} & 0 & 0 & \frac{ma\chi\mathscr{J}\mathscr{M}\mathscr{Z}}{\mathscr{F}^4} \end{bmatrix}$

$$+a^4\left(\chi^2-3\right),$$
$$\mathscr{W}=3a^4\chi^2+3a^2\chi^2r^2-a^2r^2-r^4,\ \ \varrho=4m^2a^4r^2\chi. \tag{10}$$

A 1+3 decomposition on the Kerr spacetime was partially investigated in [20]. We highlight that the quantities of acceleration ($\dot{u}^a$) and vorticity (ω_{ab}) are consistent with the findings of Frolov and Novikov [20]. Now we present the full set of 1+3 Kerr equations, generated by the above tabled quantities, named according to [32]. These equations as well as the relevant identities in Appendix A have been checked in Maple. The 1+3 equations below appear in a similar form to [32] except that the Kerr features come into effect. For example, the expansion scalar Θ is zero in both frames and will not appear in the resulting 1+3 equations. In the Killing frame the shear component is zero and in the ZAMO frame the vorticity components are zero so they will be absent accordingly.

The equation is written first for the Killing frame and thereafter for the ZAMO frame. The first set of equations is derived from the Ricci identities for u^a given by

$$2\nabla_{[a}\nabla_{b]}u^c = R_{ab}{}^c{}_d u^d, \tag{11}$$

where $R_{ab}{}^c{}_d$ is the Riemann curvature tensor and its 1+3 splitting is given in Eq. (A.1). Then using (5) and (A.1) and separating out the orthogonally projected part into trace, symmetric trace-free and skew symmetric parts, and the parallel part, three propagation equations and three constraint equations are obtained as seen in [32]. They are given in the following two subsections.

4.1 Propagation equations I

The Raychaudhuri equation [33] describes gravitational attraction and is a fundamental result to singularity theorems and finding exact solutions to general relativity. It provides an evolution equation for the expansion scalar.

Raychaudhuri equation:

$$\text{Killing frame:}\quad D_a\dot{u}^a = -\dot{u}_a\dot{u}^a - 2\omega_a\omega^a, \tag{12}$$

$$\text{ZAMO frame:}\quad D_a\dot{u}^a = -\dot{u}_a\dot{u}^a + \sigma_{ab}\sigma^{ab}. \tag{13}$$

Vorticity propagation equation:

$$\text{Killing frame:}\quad \dot{\omega}^{<a>} - \frac{1}{2}\varepsilon^{abc}D_b\dot{u}_c = 0, \tag{14}$$

$$\text{ZAMO frame:}\quad \frac{1}{2}\varepsilon^{abc}D_b\dot{u}_c = 0, \tag{15}$$

where angle brackets denote orthogonal projections of covariant time derivatives along u^a as well as represent the projected, symmetric and trace-free part of tensors henceforth as follows

$$Z_{<a>} = h^b{}_a Z_b, \quad Z_{<ab>} = \left(h^c{}_{(a}h^d{}_{b)} - \frac{1}{3}h_{ab}h^{cd}\right) Z_{cd}.$$

Shear propagation equation:

Killing frame:

$$D^{<a}\dot{u}^{b>} = -\dot{u}^{<a}\dot{u}^{b>} + \omega^{<a}\omega^{b>} + E^{ab}, \tag{16}$$

ZAMO frame:

$$\dot{\sigma}^{<ab>} - D^{<a}\dot{u}^{b>} = \dot{u}^{<a}\dot{u}^{b>} - \sigma^{<a}{}_c\sigma^{b>c} - E^{ab}. \tag{17}$$

4.2 Constraint equations I

Shear divergence equation:

$$\text{Killing frame:}\quad 0 = \varepsilon^{abc}\left[D_b\omega_c + 2\dot{u}_b\omega_c\right], \tag{18}$$

$$\text{ZAMO frame:}\quad 0 = D_b\sigma^{ab}. \tag{19}$$

Vorticity divergence equation:

$$\text{Killing frame:}\quad 0 = D_a\omega^a - \dot{u}_a\omega^a. \tag{20}$$

The vorticity divergence equation is zero in the ZAMO frame.
Magnetic constraint equation:

$$\text{Killing frame:}\quad 0 = H^{ab} + 2\dot{u}^{<a}\omega^{b>} + D^{<a}\omega^{b>}, \tag{21}$$

$$\text{ZAMO frame:}\quad 0 = H^{ab} - \varepsilon^{cd<a}D_c\sigma^{b>}{}_d. \tag{22}$$

The next set of equations is derived from the Bianchi identities given by

$$\nabla_{[a}R_{bc]de} = 0. \tag{23}$$

Using the 1+3 splitting of the Riemann curvature tensor (A.1), the once-contracted Bianchi identities give two further propagation equations and two further constraint equations. As seen in [32], these equations are similar to Maxwell's field equations in an expanding universe. They are given in the following two subsections.

4.3 Propagation equations II

These equations describe gravitational radiation.

$\dot{E}$-equation:

Killing frame:

$$\dot{E}^{<ab>} - \varepsilon^{cd<a}D_cH^{b>}{}_d = \varepsilon^{cd<a}\left[2\dot{u}_cH^{b>}{}_d + \omega_cE^{b>}{}_d\right], \tag{24}$$

ZAMO frame:

$$\dot{E}^{<ab>} - \varepsilon^{cd<a}D_cH^{b>}{}_d = 3\sigma^{<a}{}_cE^{b>c} + \varepsilon^{cd<a}\dot{u}_cH^{b>}{}_d. \tag{25}$$

$\dot{H}$-equation:

Killing frame:

$$\dot{H}^{<ab>} + \varepsilon^{cd<a}D_cE^{b>}{}_d = -\varepsilon^{cd<a}\left[2\dot{u}_cE^{b>}{}_d - \omega_cH^{b>}{}_d\right], \tag{26}$$

ZAMO frame:

$$\dot{H}^{<ab>} + \varepsilon^{cd<a}D_cE^{b>}{}_d = 3\sigma^{<a}{}_cH^{b>c} - 2\varepsilon^{cd<a}\dot{u}_cE^{b>}{}_d. \tag{27}$$

4.4 Constraint equations II

The divergence (denoted by 'div') of the electric Weyl tensor and also the magnetic Weyl tensor is given below.

(div E)-equation:

$$\text{Killing frame:}\quad 0 = (C_4)^a = D_bE^{ab} - 3\omega_bH^{ab}, \tag{28}$$

$$\text{ZAMO frame:}\quad 0 = (C_4)^a = D_bE^{ab} - \varepsilon^{abc}\sigma_{bd}H_c{}^d. \tag{29}$$

(div H)-equation:

$$\text{Killing frame:}\quad 0 = (C_5)^a = D_bH^{ab} + 3\omega_bE^{ab}, \tag{30}$$

$$\text{ZAMO frame:}\quad 0 = (C_5)^a = D_bH^{ab} + \varepsilon^{abc}\sigma_{bd}E_c{}^d. \tag{31}$$

5 1+1+2 formalism

In the 1+1+2 formalism, the 3-space $\mathscr{V}$ is now further split by introducing the unit vector e^a orthogonal to u^a ($e^ae_a = 1$, $u^ae_a = 0$). The 1+1+2 covariantly decomposed spacetime is given by

$$g_{ab} = -u_au_b + e_ae_b + N_{ab}, \tag{32}$$

where N_{ab} ($e^aN_{ab} = 0 = u^aN_{ab}$, $N^a{}_a = 2$) projects vectors orthogonal to u^a and e^a onto 2-spaces called 'sheets.' The further splitting is shown in Fig. 2.

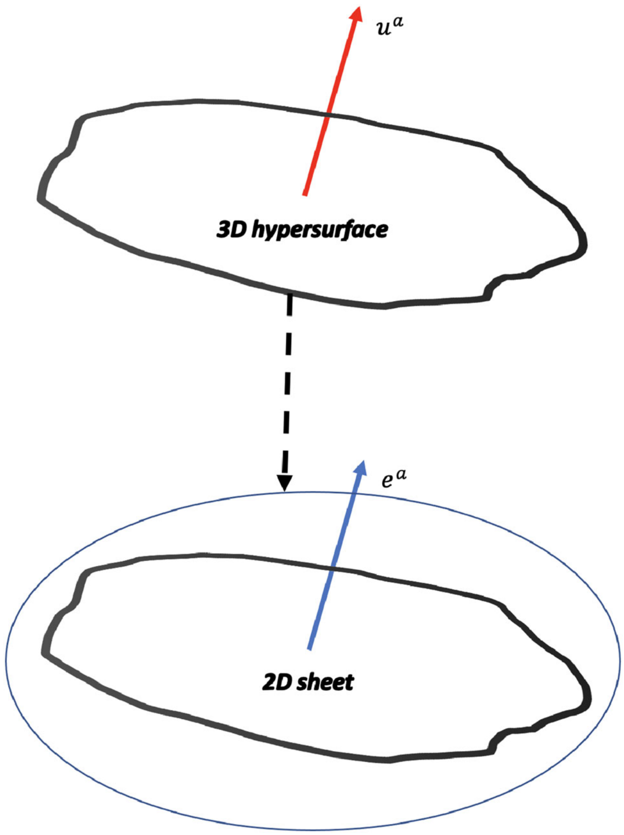

Fig. 2 Diagram illustrating the 1+1+2 formalism. The $3D$ hypersurface is split further according to the spacelike vector e^a resulting in a $2D$ sheet

We introduce two new derivatives for any tensor $\Psi_{a...b}{}^{c...d}$:

$$\hat{\Psi}_{a...b}{}^{c...d} \equiv e^f D_f \Psi_{a...b}{}^{c...d}, \tag{33}$$

$$\delta_f \Psi_{a...b}{}^{c...d} \equiv N_f{}^j N_a{}^l ... N_b{}^g N_h{}^c ... N_i{}^d D_j \Psi_{l...g}{}^{h...i}, \tag{34}$$

defined by the congruence e^a. The hat-derivative (33) is the spatial derivative along the e^a vector field in the surfaces orthogonal to u^a and the delta-derivative (34) is the projected spatial derivative onto the 2-sheet, with projection on every free index.

Taking e^a to be arbitrary, the 1+3 kinematical quantities and Weyl electromagnetic variables are split irreducibly as

$$\dot{u}^a = \mathscr{A} e^a + \mathscr{A}^a, \tag{35}$$

$$\omega^a = \Omega e^a + \Omega^a, \tag{36}$$

$$\sigma_{ab} = \Sigma \left(e_a e_b - \frac{1}{2} N_{ab} \right) + 2\Sigma_{(a} e_{b)} + \Sigma_{ab}, \tag{37}$$

$$E_{ab} = \mathscr{E} \left(e_a e_b - \frac{1}{2} N_{ab} \right) + 2\mathscr{E}_{(a} e_{b)} + \mathscr{E}_{ab}, \tag{38}$$

$$H_{ab} = \mathscr{H} \left(e_a e_b - \frac{1}{2} N_{ab} \right) + 2\mathscr{H}_{(a} e_{b)} + \mathscr{H}_{ab}, \tag{39}$$

respectively, using (B.1) and (B.2).

The covariant derivative of e^a is given by

$$D_a e_b = e_a a_b + \frac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab} + \zeta_{ab}, \tag{40}$$

where traveling along e^a, a_a is the sheet acceleration, ϕ is the sheet expansion, ξ is the vorticity of e^a (the twisting of the sheet) and ζ_{ab} is the shear of e^a (the distortion of the sheet). The 1+1+2 split of the full covariant derivatives of u^a and e^a are as follows

$$\begin{aligned} \nabla_a u_b = & -u_a \left(\mathscr{A} e_b + \mathscr{A}_b\right) + e_a e_b \left(\frac{1}{3}\Theta + \Sigma\right) \\ & + e_a \left(\Sigma_b + \varepsilon_{bc}\Omega^c\right) + \left(\Sigma_a - \varepsilon_{ac}\Omega^c\right) e_b \\ & + N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right) + \Omega \varepsilon_{ab} + \Sigma_{ab}, \end{aligned} \tag{41}$$

$$\begin{aligned} \nabla_a e_b = & -\mathscr{A} u_a u_b - u_a \alpha_b + \left(\frac{1}{3}\Theta + \Sigma\right) e_a u_b + \zeta_{ab}, \\ & + \left(\Sigma_a - \varepsilon_{ac}\Omega^c\right) u_b + e_a a_b + \frac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab}, \end{aligned} \tag{42}$$

where $\mathscr{A}_a \equiv \dot{u}_{\bar{a}}$, $\alpha_a \equiv \dot{e}_{\bar{a}}$ and ε_{ab} is the natural 2-volume element carried by the sheet. The bar on indices denotes projections on the sheet (see (B.1)). The scalars $\{\mathscr{A}, \Omega, \Sigma, \mathscr{E}, \mathscr{H}, \phi, \xi\}$ and vectors $\{\mathscr{A}_a, \Omega_a, \Sigma_a, a_a, \mathscr{E}_a, \mathscr{H}_a, \Sigma_{ab}, \zeta_{ab}, \mathscr{E}_{ab}, \mathscr{H}_{ab}\}$ are the geometric variables that govern the 1+1+2 formalism. The geometric quantities are defined relative to the timelike and spacelike congruences and consequently we can show how the spacetime evolves and behaves in terms of those quantities. The concept of the 1+1+2 splitting was introduced by [34,35] and further expanded upon in [5,36]. A more detailed review of the formalism can be found in Clarkson [22].

6 The 1+1+2 Kerr quantities and equations

The 1+1+2 quantities are defined relative to u^a and e^a and hence the quantities of u^a for the Killing frame (8) and the ZAMO frame (9) still apply. Coincidentally the quantity for e^a, the preferred spatial direction, is the same for both frames and is given by

$$e^a = \left[0, \sqrt{\frac{r^2-2mr+a^2}{r^2+a^2\chi^2}}, 0, 0\right] \quad \text{where} \quad e^a e_a = 1. \tag{43}$$

We took the radial direction as the preferred spatial direction because far away from the rotating black hole, it coincides exactly with the radial direction of the Schwarzschild model which is the preferred spatial direction.

We now apply the 1+1+2 formalism to the Kerr metric. The set of 1+1+2 Kerr geometric variables is given by

$$\begin{aligned} &\{\mathscr{A}, \Omega, \Sigma, \mathscr{E}, \mathscr{H}, \phi, \xi, \mathscr{A}_a, \Omega_a, \Sigma_a, \alpha_a, a_a, \mathscr{E}_a, \mathscr{H}_a, \\ &\Sigma_{ab}, \zeta_{ab}, \mathscr{E}_{ab}, \mathscr{H}_{ab}\}, \end{aligned} \tag{44}$$

Table 2 The 1+1+2 Kerr quantities in the killing frame and the ZAMO frame

	Killing frame	ZAMO frame
Θ	0	0
Ω	$\frac{-2mar\chi\sqrt{\mathscr{G}}}{\mathscr{C}\sqrt{\mathscr{F}^3}}$	0
Ω_a	$\left[0,\ 0,\ \frac{-ma\mathscr{B}}{\mathscr{C}\mathscr{F}},\ 0\right]$	$\mathbf{0}$
Σ	0	0
Σ_a	$\mathbf{0}$	$\left[\frac{2rm^2a^2\mathscr{Q}\mathscr{J}}{\sqrt{\mathscr{Z}^3}\mathscr{F}^2},\ 0,\ 0,\ -\frac{am\mathscr{Q}\mathscr{J}}{\sqrt{\mathscr{Z}}\mathscr{F}^2}\right]$
Σ_{ab}	$\mathbf{0}$	$\begin{bmatrix} 0 & 0 & -\frac{\varrho\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{Z}^3\mathscr{F}^3}} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\varrho\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{Z}^3\mathscr{F}^3}} & 0 & 0 & \frac{2mra^3\chi\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{Z}\mathscr{F}^3}} \\ 0 & 0 & \frac{2mra^3\chi\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{Z}\mathscr{F}^3}} & 0 \end{bmatrix}$
$\mathscr{A}$	$\frac{-m\mathscr{B}\sqrt{\mathscr{G}}}{\mathscr{C}\sqrt{\mathscr{F}^3}}$	$-\frac{m\mathscr{Y}}{\mathscr{Z}\sqrt{\mathscr{G}\mathscr{F}^3}}$
$\mathscr{A}_a$	$\left[0,\ 0,\ \frac{2mra^2\chi}{\mathscr{C}\mathscr{F}},\ 0\right]$	$\left[0,\ 0,\ \frac{2mra^2\chi\left(r^2+a^2\right)}{\mathscr{Z}\mathscr{F}},\ 0\right]$
$\mathscr{E}$	$\frac{-mr\mathscr{K}\mathscr{N}}{\mathscr{C}\mathscr{F}^3}$	$\frac{mr\mathscr{K}\mathscr{T}}{\mathscr{F}^3\mathscr{Z}}$
$\mathscr{E}_a$	$\left[0,\ 0,\ \frac{3ma^2\chi\mathscr{M}\sqrt{\mathscr{G}}}{\mathscr{C}\sqrt{\mathscr{F}^5}},\ 0\right]$	$\left[0,\ 0,\ \frac{3ma^2\chi\mathscr{Q}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{F}^3}\mathscr{Z}},\ 0\right]$
$\mathscr{E}_{ab}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2}\frac{ma^2r\mathscr{K}}{\mathscr{C}\mathscr{F}^2} & 0 \\ 0 & 0 & 0 & \frac{3}{2}\frac{ma^2r\mathscr{G}\mathscr{J}^2\mathscr{K}}{\mathscr{C}^2\mathscr{F}^2} \end{bmatrix}$	$\begin{bmatrix} \frac{6a^4r^3m^3\mathscr{J}^2\mathscr{G}\mathscr{K}}{\mathscr{F}^4\mathscr{Z}^2} & 0 & 0 & \frac{-3a^3r^2m^2\mathscr{J}^2\mathscr{G}\mathscr{K}}{\mathscr{F}^4\mathscr{Z}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3mra^2\mathscr{G}\mathscr{K}}{2\mathscr{F}^2\mathscr{Z}} & 0 \\ \frac{-3a^3r^2m^2\mathscr{J}^2\mathscr{G}\mathscr{K}}{\mathscr{F}^4\mathscr{Z}} & 0 & 0 & \frac{3mra^2\mathscr{G}\mathscr{J}^2\mathscr{K}}{2\mathscr{F}^4} \end{bmatrix}$
$\mathscr{H}$	$\frac{-ma\chi\mathscr{M}\mathscr{N}}{\mathscr{C}\mathscr{F}^3}$	$\frac{ma\chi\mathscr{K}\mathscr{V}}{\mathscr{F}^3\mathscr{Z}}$
$\mathscr{H}_a$	$\left[0,\ 0,\ -\frac{3mar\mathscr{K}\sqrt{\mathscr{G}}}{\mathscr{C}\sqrt{\mathscr{F}^5}},\ 0\right]$	$\left[0,\ 0,\ -\frac{3mar\sqrt{\mathscr{G}}\mathscr{W}}{\sqrt{\mathscr{F}^5}\mathscr{Z}},\ 0\right]$
$\mathscr{H}_{ab}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2}\frac{ma^3\chi\mathscr{M}}{\mathscr{G}\mathscr{F}^2} & 0 \\ 0 & 0 & 0 & \frac{3}{2}\frac{ma^3\chi\mathscr{G}\mathscr{J}^2\mathscr{M}}{\mathscr{C}^2\mathscr{F}^2} \end{bmatrix}$	$\begin{bmatrix} \frac{6m^3a^5r^2\chi\mathscr{J}^2\mathscr{G}\mathscr{M}}{\mathscr{F}^4\mathscr{Z}^2} & 0 & 0 & -\frac{\mathbb{M}\mathscr{J}^2\mathscr{G}\mathscr{M}}{\mathscr{F}^4\mathscr{Z}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3ma^3\chi\mathscr{M}\mathscr{G}}{2\mathscr{F}^2\mathscr{Z}} & 0 \\ -\frac{\mathbb{M}\mathscr{J}^2\mathscr{M}\mathscr{G}}{\mathscr{F}^4\mathscr{Z}} & 0 & 0 & \frac{3ma^3\chi\mathscr{J}^2\mathscr{G}\mathscr{M}}{2\mathscr{F}^4} \end{bmatrix}$
a_a	$\left[0,\ 0,\ \frac{-a^2\chi}{\mathscr{F}},\ 0\right]$	$\left[0,\ 0,\ \frac{-a^2\chi}{\mathscr{F}},\ 0\right]$
ϕ	$-\frac{\mathbb{K}}{\mathscr{C}\sqrt{\mathscr{F}^3\mathscr{G}}}$	$-\frac{\left(ma^2\chi^2-ra^2\chi^2-ma^2-ra^2-2r^3\right)\sqrt{\mathscr{G}}}{\sqrt{\mathscr{F}}\mathscr{Z}}$
ξ	0	0
ζ_{ab}	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\frac{a^2(m-r)\sqrt{\mathscr{F}}}{\mathscr{C}\sqrt{\mathscr{G}}} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\frac{a^2(m-r)\mathscr{J}^2\sqrt{\mathscr{F}\mathscr{G}}}{\mathscr{C}^2} \end{bmatrix}$	$\begin{bmatrix} -\frac{2m^2a^4r^2\mathscr{J}^2\sqrt{\mathscr{G}}\mathbb{J}}{\sqrt{\mathscr{F}^5}\mathscr{Z}^2} & 0 & 0 & \frac{mra^3\mathscr{J}^2\sqrt{\mathscr{G}}\mathbb{J}}{\sqrt{\mathscr{F}^5}\mathscr{Z}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a^2\sqrt{\mathscr{G}}\mathbb{J}}{2\mathscr{Z}\sqrt{\mathscr{F}}} & 0 \\ \frac{mra^3\mathscr{J}^2\sqrt{\mathscr{G}}\mathbb{J}}{\sqrt{\mathscr{F}^5}\mathscr{Z}} & 0 & 0 & -\frac{a^2\mathscr{J}^2\sqrt{\mathscr{G}}\mathbb{J}}{2\sqrt{\mathscr{F}^5}} \end{bmatrix}$
α_a	$\left[0,\ 0,\ 0,\ \frac{ma\mathscr{B}\mathscr{J}\sqrt{\mathscr{G}}}{\sqrt{\mathscr{C}^3}\mathscr{F}}\right]$	$\left[-\frac{2rm^2a^2\mathscr{J}\mathscr{Q}}{\sqrt{\mathscr{Z}^3}\mathscr{F}^2},\ 0,\ 0,\ \frac{ma\mathscr{J}\mathscr{Q}}{\sqrt{\mathscr{Z}}\mathscr{F}^2}\right]$

and they have the following values given in Table 2.

where

$$\mathscr{Y} = a^6\chi^2 + 2a^4\chi^2r^2 - 4mr^3a^2\chi^2 + a^2\chi^2r^4 - a^4r^2 +4ma^2r^3 - 2a^2r^4 - r^6,$$
$$\mathbb{J} = r^2(3m+r) + a^2\chi^2(r-m),$$
$$\mathbb{K} = -2r^3(r-2m)^2 - a^4\chi^2(r+m) + a^4\chi^4(m-r) -a^2r^2\left(r-3m+\chi^2(3r-5m)\right),$$
$$\mathbb{M} = 3rm^2a^4\chi. \qquad (45)$$

All the identities that are listed in Appendix B corresponding to the variables have been confirmed in Maple. The full set of the 1+1+2 Kerr equations for the above covariant variables are obtained by applying the 1+1+2 decomposition procedure to the 1+3 equations, and also by covariantly splitting the Ricci identities for e^a as follows

$$R_{abc} \equiv 2\nabla_{[a}\nabla_{b]}e_c - R_{abcd}e^d = 0. \qquad (46)$$

Splitting (46) using u^a and e^a the evolution (along u^a) and propagation (along e^a) equations below are obtained. We present the full set of the 1+1+2 Kerr equations according to Clarkson [22]. The equation is written first for the Killing frame and thereafter for the ZAMO frame. We note that we corrected some of the equations in [22] and these equations are denoted by the diamond symbol ⋄. A comparison between Clarkson's result and the corrected result for the particular frames can be found in Appendix C. We emphasise that the linear perturbation results in [5,22] still hold as the identified errors are higher order terms.

6.1 Evolution equations

The evolution equations for ϕ, ξ and ζ_{ab} are obtained from the projection of $u^a R_{abc}$ as follows:

$u^a N^{bc} R_{abc}$:

$$\diamond\text{Killing frame: } \dot{\phi} = \delta_a\alpha^a + \alpha_a\left(\mathscr{A}^a - a^a\right) -\varepsilon_{ab}\Omega^b\left(a^a - \mathscr{A}^a\right), \qquad (47)$$
$$\diamond\text{ZAMO frame: } \dot{\phi} = \delta_a\alpha^a + \alpha_a\left(\mathscr{A}^a - a^a\right) +\Sigma_a\left(a^a - \mathscr{A}^a\right) - \zeta^{ab}\Sigma_{ab}. \qquad (48)$$

$u^a\varepsilon^{bc}R_{abc}$:

$$\text{Killing frame: } \dot{\xi} = 0 = 2\left(\mathscr{A} - \frac{1}{2}\phi\right)\Omega + \left(a^a + \mathscr{A}^a\right)\left[\Omega_a + \varepsilon_{ab}\alpha^b\right] + \varepsilon_{ab}\delta^a\alpha^b + \mathscr{H}, \qquad (49)$$
$$\diamond\text{ZAMO frame: } \dot{\xi} = 0 = \varepsilon_{ab}\left(a^a + \mathscr{A}^a\right)\left[\alpha^b + \Sigma^b\right] +\varepsilon_{ab}\delta^a\alpha^b + \varepsilon_{ca}\zeta_b{}^c\Sigma^{ab} + \mathscr{H}. \qquad (50)$$

$u^c R_{c\{ab\}}$:

$$\text{Killing frame: } \dot{\zeta}_{\{ab\}} = \Omega\varepsilon_{c\{a}\zeta_{b\}}{}^c + \delta_{\{a}\alpha_{b\}} + \mathscr{A}_{\{a}\alpha_{b\}} -a_{\{a}\alpha_{b\}} + \mathscr{A}_{\{a}\varepsilon_{b\}d}\Omega^d + a_{\{a}\varepsilon_{b\}d}\Omega^d - \varepsilon_{c\{a}\mathscr{H}_{b\}}{}^c, \qquad (51)$$
$$\text{ZAMO frame: } \dot{\zeta}_{\{ab\}} = \left(\mathscr{A} - \frac{1}{2}\phi\right)\Sigma_{ab} - \zeta_{c\{a}\Sigma_{b\}}{}^c +\delta_{\{a}\alpha_{b\}} + \mathscr{A}_{\{a}\alpha_{b\}} - a_{\{a}\alpha_{b\}} - \mathscr{A}_{\{a}\Sigma_{b\}} -a_{\{a}\Sigma_{b\}} - \varepsilon_{c\{a}\mathscr{H}_{b\}}{}^c. \qquad (52)$$

Not all information needed to determine the complete set of 1+1+2 equations is contained in R_{abc}. Hence the 1+1+2 decomposition of the standard 1+3 equations is used to obtain the remaining evolution equations given below. When the 1+3 equations are decomposed, separate equations are obtained for the scalars, 2-vectors and 2-tensors therefore the number of 1+1+2 equations increases.

Vorticity evolution equation (from (14), (15)):

$$\text{Killing frame: } \dot{\Omega} = \frac{1}{2}\varepsilon_{ab}\delta^a\mathscr{A}^b + \Omega_a\alpha^a, \qquad (53)$$
$$\text{ZAMO frame: } 0 = \frac{1}{2}\varepsilon_{ab}\delta^a\mathscr{A}^b. \qquad (54)$$

Shear evolution equation (from (16), (17)):

$$\text{Killing frame: } \dot{\Sigma}_{\{ab\}} = 0 = \delta_{\{a}\mathscr{A}_{b\}} + \mathscr{A}_{\{a}\mathscr{A}_{b\}} -\Omega_{\{a}\Omega_{b\}} + \mathscr{A}\zeta_{ab} - \mathscr{E}_{ab}, \qquad (55)$$
$$\text{ZAMO frame: } \dot{\Sigma}_{\{ab\}} = \delta_{\{a}\mathscr{A}_{b\}} + \mathscr{A}_{\{a}\mathscr{A}_{b\}} -\Sigma_{\{a}\Sigma_{b\}} - 2\Sigma_{\{a}\alpha_{b\}} + \mathscr{A}\zeta_{ab} - \Sigma_{c\{a}\Sigma_{b\}}{}^c - \mathscr{E}_{ab}. \qquad (56)$$

6.2 Mixture of propagation and evolution equations

A mixture of propagation and evolution equations is obtained by either projecting R_{abc} as indicated or by a further decomposition of the 1+3 equations.

$u^a e^b R_{ab\bar{c}} = e^a u^b R_{ab\bar{c}}$:

$$\text{Killing frame: } \hat{\alpha}_{\bar{a}} - \dot{a}_{\bar{a}} = -\left(\frac{1}{2}\phi + \mathscr{A}\right)\alpha_a + \varepsilon_{ab}\Omega^b \times\left(\frac{1}{2}\phi - \mathscr{A}\right) + \zeta_{ab}\left(\varepsilon^{bc}\Omega_c - \alpha^b\right) - \varepsilon_{ab}\mathscr{H}^b, \qquad (57)$$
$$\text{ZAMO frame: } \hat{\alpha}_{\bar{a}} - \dot{a}_{\bar{a}} = -\left(\frac{1}{2}\phi + \mathscr{A}\right)\alpha_a + \Sigma_a\left(\frac{1}{2}\phi - \mathscr{A}\right) +\zeta_{ab}\left(\Sigma^b - \alpha^b\right) - \varepsilon_{ab}\mathscr{H}^b. \qquad (58)$$

⋄ For the sake of completeness we mention that the identity $u^a e^b u^c R_{abc} = -e^a u^b u^c R_{abc}$, as seen in Clarkson [22], is identically satisfied. However, the resulting equation 54 in Clarkson [22] does not appear correctly; we believe there is an error in the calculation.

Raychaudhuri equation (from (12), (13)):

$$\text{Killing frame: } \hat{\mathscr{A}} = -\delta_a\mathscr{A}^a - \mathscr{A}(\mathscr{A}+\phi) + \mathscr{A}^a(a_a - \mathscr{A}_a) -2\Omega^2 - 2\Omega_a\Omega^a, \qquad (59)$$
$$\text{ZAMO frame: } \hat{\mathscr{A}} = -\delta_a\mathscr{A}^a - \mathscr{A}(\mathscr{A}+\phi) + \mathscr{A}^a(a_a - \mathscr{A}_a) +\frac{3}{2}\Sigma^2 + 2\Sigma_a\Sigma^a + \Sigma_{ab}\Sigma^{ab}. \qquad (60)$$

Vorticity evolution equation (from (14), (15)):

$$\text{Killing frame: } \dot{\Omega}_{\bar a} + \frac{1}{2}\varepsilon_{ab}\hat{\mathscr{A}}^b = -\Omega\alpha_a + \frac{1}{2}\varepsilon_{ab} \times \left(-\mathscr{A}a^b + \delta^b\mathscr{A} - \frac{1}{2}\phi\mathscr{A}^b\right) - \frac{1}{2}\varepsilon_{ab}\zeta^{bc}\mathscr{A}_c, \tag{61}$$

$$\text{ZAMO frame: } \frac{1}{2}\varepsilon_{ab}\hat{\mathscr{A}}^b = \frac{1}{2}\varepsilon_{ab}\left(-\mathscr{A}a^b + \delta^b\mathscr{A} - \frac{1}{2}\phi\mathscr{A}^b\right) - \frac{1}{2}\varepsilon_{ab}\zeta^{bc}\mathscr{A}_c. \tag{62}$$

Shear evolution equations (from (16), (17)):

$$\diamond\text{Killing frame: } -\frac{2}{3}\hat{\mathscr{A}} = \frac{2}{3}\mathscr{A}^2 - \frac{1}{3}\phi\mathscr{A} - \frac{2}{3}\Omega^2 - \frac{1}{3}\delta_a\mathscr{A}^a - \frac{2}{3}\mathscr{A}_a a^a - \frac{1}{3}\mathscr{A}_a\mathscr{A}^a + \frac{1}{3}\Omega_a\Omega^a - \mathscr{E}, \tag{63}$$

$$\diamond\text{ZAMO frame: } -\frac{2}{3}\hat{\mathscr{A}} = \frac{2}{3}\mathscr{A}^2 - \frac{1}{3}\phi\mathscr{A} - \frac{1}{3}\delta_a\mathscr{A}^a + \Sigma_a\left(2\alpha^a - \frac{1}{3}\Sigma^a\right) - \frac{2}{3}\mathscr{A}_a a^a - \frac{1}{3}\mathscr{A}_a\mathscr{A}^a + \frac{1}{3}\Sigma_{ab}\Sigma^{ab} - \mathscr{E}. \tag{64}$$

$$\text{Killing frame: } -\frac{1}{2}\hat{\mathscr{A}}_{\bar a} = \frac{1}{2}\delta_a\mathscr{A} + \mathscr{A}_a\left(\mathscr{A} - \frac{1}{4}\phi\right) + \frac{1}{2}\mathscr{A}a_a - \Omega\Omega_a - \frac{1}{2}\zeta_{ab}\mathscr{A}^b - \mathscr{E}_a, \tag{65}$$

$$\text{ZAMO frame: } \dot{\Sigma}_{\bar a} - \frac{1}{2}\hat{\mathscr{A}}_{\bar a} = \frac{1}{2}\delta_a\mathscr{A} + \mathscr{A}_a\left(\mathscr{A} - \frac{1}{4}\phi\right) + \frac{1}{2}\mathscr{A}a_a - \frac{1}{2}\zeta_{ab}\mathscr{A}^b + \Sigma_{ab}\left(\alpha^b - \Sigma^b\right) - \mathscr{E}_a. \tag{66}$$

Additionally the magnetic and electric Weyl evolution equations are listed below.

Electric Weyl evolution equations (from (24), (25)):

$$\text{Killing frame: } \dot{\mathscr{E}} = \varepsilon_{ab}\delta^a\mathscr{H}^b + \mathscr{E}^a\left(2\alpha_a - \varepsilon_{ab}\Omega^b\right) + 2\varepsilon_{ab}\mathscr{A}^a\mathscr{H}^b + \varepsilon_{ab}\mathscr{H}^{bc}\zeta^a{}_c, \tag{67}$$

$$\text{ZAMO frame: } \dot{\mathscr{E}} = \varepsilon_{ab}\delta^a\mathscr{H}^b + \mathscr{E}^a\left(2\alpha_a + \Sigma_a\right) + 2\varepsilon_{ab}\mathscr{A}^a\mathscr{H}^b - \Sigma_{ab}\mathscr{E}^{ab} + \varepsilon_{ab}\mathscr{H}^{bc}\zeta^a{}_c. \tag{68}$$

$$\diamond\text{Killing frame: } \dot{\mathscr{E}}_{\bar a} + \frac{1}{2}\varepsilon_{ab}\hat{\mathscr{H}}^b = \frac{3}{4}\varepsilon_{ab}\delta^b\mathscr{H} + \frac{1}{2}\varepsilon_{bc}\delta^b\mathscr{H}^c{}_a + \frac{3}{4}\mathscr{E}\varepsilon_{ab}\Omega^b + \frac{3}{2}\mathscr{H}\varepsilon_{ab}\mathscr{A}^b - \frac{3}{2}\mathscr{E}\alpha_a - \frac{3}{4}\mathscr{H}\varepsilon_{ab}a^b - \frac{1}{2}\Omega\varepsilon_{ab}\mathscr{E}^b - \frac{1}{4}\phi\varepsilon_{ab}\mathscr{H}^b - \mathscr{A}\varepsilon_{ab}\mathscr{H}^b + \mathscr{E}_{ab}\alpha^b - \frac{1}{2}\mathscr{E}_{ab}\varepsilon^{bc}\Omega_c + \frac{3}{2}\zeta_{ab}\varepsilon^{bc}\mathscr{H}_c - \mathscr{H}_{ab}\varepsilon^{bc}\mathscr{A}_c + \frac{1}{2}\varepsilon_{ab}a^c\mathscr{H}^b{}_c + \varepsilon_{ab}e_c\delta^b\mathscr{H}^c, \tag{69}$$

$$\diamond\text{ZAMO frame: } \dot{\mathscr{E}}_{\bar a} + \frac{1}{2}\varepsilon_{ab}\hat{\mathscr{H}}^b = \frac{3}{4}\varepsilon_{ab}\delta^b\mathscr{H} + \frac{1}{2}\varepsilon_{bc}\delta^b\mathscr{H}^c{}_a + \frac{3}{4}\mathscr{E}\Sigma_a + \frac{3}{2}\mathscr{H}\varepsilon_{ab}\mathscr{A}^b - \frac{3}{2}\mathscr{E}\alpha_a - \frac{3}{4}\mathscr{H}\varepsilon_{ab}a^b - \frac{1}{4}\phi\varepsilon_{ab}\mathscr{H}^b - \mathscr{A}\varepsilon_{ab}\mathscr{H}^b + \frac{3}{2}\Sigma_{ab}\mathscr{E}^b + \frac{3}{2}\mathscr{E}_{ab}\Sigma^b + \mathscr{E}_{ab}\alpha^b + \frac{3}{2}\zeta_{ab}\varepsilon^{bc}\mathscr{H}_c - \mathscr{H}_{ab}\varepsilon^{bc}\mathscr{A}_c + \frac{1}{2}\varepsilon_{ab}a^c\mathscr{H}^b{}_c + \varepsilon_{ab}e_c\delta^b\mathscr{H}^c. \tag{70}$$

$$\text{Killing frame: } \dot{\mathscr{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathscr{H}}_{b\}}{}^c = -\varepsilon_{c\{a}\delta^c\mathscr{H}_{b\}} - \frac{3}{2}\mathscr{H}\varepsilon_{c\{a}\zeta_{b\}}{}^c + \Omega\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c + \varepsilon_{c\{a}\mathscr{H}_{b\}}{}^c\left(\frac{1}{2}\phi + 2\mathscr{A}\right) - 2\alpha_{\{a}\mathscr{E}_{b\}} - \varepsilon_{c\{a}\mathscr{E}_{b\}}\Omega^c + 2\varepsilon_{c\{a}\mathscr{H}_{b\}}\left(a^c - \mathscr{A}^c\right) + \varepsilon_{c\{a}\mathscr{H}_{b\}d}\zeta^{cd}, \tag{71}$$

$$\text{ZAMO frame: } \dot{\mathscr{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathscr{H}}_{b\}}{}^c = -\varepsilon_{c\{a}\delta^c\mathscr{H}_{b\}} - \frac{3}{2}\mathscr{E}\Sigma_{ab} - \frac{3}{2}\mathscr{H}\varepsilon_{c\{a}\zeta_{b\}}{}^c + \varepsilon_{c\{a}\mathscr{H}_{b\}}{}^c\left(\frac{1}{2}\phi + 2\mathscr{A}\right) - 2\alpha_{\{a}\mathscr{E}_{b\}} + 3\Sigma_{\{a}\mathscr{E}_{b\}} + 2\varepsilon_{c\{a}\mathscr{H}_{b\}}\left(a^c - \mathscr{A}^c\right) + 3\Sigma_{c\{a}\mathscr{E}_{b\}}{}^c + \varepsilon_{c\{a}\mathscr{H}_{b\}d}\zeta^{cd}. \tag{72}$$

Magnetic Weyl evolution equations (from (26), (27)):

$$\text{Killing frame: } \dot{\mathscr{H}} = -\varepsilon_{ab}\delta^a\mathscr{E}^b - 2\varepsilon_{ab}\mathscr{A}^a\mathscr{E}^b + \mathscr{H}^a \times \left(2\alpha_a - \varepsilon_{ab}\Omega^b\right) - \frac{1}{2}\varepsilon_{ab}\mathscr{E}^{bc}\zeta^a{}_c, \tag{73}$$

$$\text{ZAMO frame: } \dot{\mathscr{H}} = -\varepsilon_{ab}\delta^a\mathscr{E}^b - 2\varepsilon_{ab}\mathscr{A}^a\mathscr{E}^b + \mathscr{H}^a \times \left(2\alpha_a + \Sigma_a\right) - \Sigma_{ab}\mathscr{H}^{ab} - \frac{1}{2}\varepsilon_{ab}\mathscr{E}^{bc}\zeta^a{}_c. \tag{74}$$

$$\diamond\text{Killing frame: } \dot{\mathscr{H}}_{\bar a} - \frac{1}{2}\varepsilon_{ab}\hat{\mathscr{E}}^b = -\frac{3}{4}\varepsilon_{ab}\delta^b\mathscr{E} - \frac{1}{2}\varepsilon_{bc}\delta^b\mathscr{E}^c{}_a + \frac{3}{4}\mathscr{H}\varepsilon_{ab}\Omega^b - \frac{3}{2}\mathscr{E}\varepsilon_{ab}\mathscr{A}^b - \frac{1}{2}\Omega\varepsilon_{ab}\mathscr{H}^b - \frac{3}{2}\mathscr{H}\alpha_a + \frac{3}{4}\mathscr{E}\varepsilon_{ab}a^b + \frac{1}{4}\phi\varepsilon_{ab}\mathscr{E}^b + \mathscr{A}\varepsilon_{ab}\mathscr{E}^b + \frac{3}{2}\varepsilon_{ab}\zeta^{bc}\mathscr{E}_c + \mathscr{E}_{ab}\varepsilon^{bc}\mathscr{A}_c + \mathscr{H}_{ab}\alpha^b - \frac{1}{2}\mathscr{H}_{ab}\varepsilon^{bc}\Omega_c - \frac{1}{2}\varepsilon_{ab}a^c\mathscr{E}^b{}_c - \varepsilon_{ab}e_c\delta^b\mathscr{E}^c, \tag{75}$$

$$\diamond\text{ZAMO frame: } \dot{\mathscr{H}}_{\bar a} - \frac{1}{2}\varepsilon_{ab}\hat{\mathscr{E}}^b = -\frac{3}{4}\varepsilon_{ab}\delta^b\mathscr{E} - \frac{1}{2}\varepsilon_{bc}\delta^b\mathscr{E}^c{}_a + \frac{3}{4}\mathscr{H}\Sigma_a - \frac{3}{2}\mathscr{E}\varepsilon_{ab}\mathscr{A}^b - \frac{3}{2}\mathscr{H}\alpha_a + \frac{3}{4}\mathscr{E}\varepsilon_{ab}a^b + \frac{1}{4}\phi\varepsilon_{ab}\mathscr{E}^b + \mathscr{A}\varepsilon_{ab}\mathscr{E}^b + \frac{3}{2}\Sigma_{ab}\mathscr{H}^b + \frac{3}{2}\varepsilon_{ab}\zeta^{bc}\mathscr{E}_c + \mathscr{E}_{ab}\varepsilon^{bc}\mathscr{A}_c + \mathscr{H}_{ab}\left(\alpha^b + \frac{3}{2}\Sigma^b\right) - \frac{1}{2}\varepsilon_{ab}a^c\mathscr{E}^b{}_c - \varepsilon_{ab}e_c\delta^b\mathscr{E}^c. \tag{76}$$

$$\diamond\text{Killing frame: } \dot{\mathscr{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathscr{E}}_{b\}}{}^c = \varepsilon_{c\{a}\delta^c\mathscr{E}_{b\}} + \frac{3}{2}\mathscr{E}\varepsilon_{c\{a}\zeta_{b\}}{}^c - \frac{1}{2}\phi\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c - 2\mathscr{A}\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c + \Omega\varepsilon_{c\{a}\mathscr{H}_{b\}}{}^c - \varepsilon_{c\{a}\mathscr{H}_{b\}}\Omega^c + 2\mathscr{E}_{\{a}\varepsilon_{b\}c}a^c - 2\mathscr{E}_{\{a}\varepsilon_{b\}c}\mathscr{A}^c - 2\alpha_{\{a}\mathscr{H}_{b\}} - \varepsilon_{c\{a}\mathscr{E}_{b\}d}\zeta^{cd}, \tag{77}$$

$$\diamond\text{ZAMO frame: } \dot{\mathscr{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathscr{E}}_{b\}}{}^c = \varepsilon_{c\{a}\delta^c\mathscr{E}_{b\}} - \frac{3}{2}\mathscr{H}\Sigma_{ab}$$

$$+\frac{3}{2}\mathscr{E}\varepsilon_{c\{a}\zeta_{b\}}{}^{c} - \frac{1}{2}\phi\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^{c} - 2\mathscr{A}\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^{c} +3\Sigma_{\{a}\mathscr{H}_{b\}} + 2\mathscr{E}_{\{a}\varepsilon_{b\}c}a^{c} - 2\mathscr{E}_{\{a}\varepsilon_{b\}c}\mathscr{A}^{c} + 3\Sigma_{c\{a}\mathscr{H}_{b\}}{}^{c} -2\alpha_{\{a}\mathscr{H}_{b\}} - \varepsilon_{c\{a}\mathscr{E}_{b\}d}\zeta^{cd}. \tag{78}$$

6.3 Propagation equations

Following a similar procedure, the propagation and constraint equations are obtained by either projecting R_{abc} as indicated, by contracting with e^a or by projections involving N^{ab} of the 1+3 constraint equations:

$e^a N^{bc} R_{abc}$:

$$\text{Killing frame: } \hat{\phi} = -\frac{1}{2}\phi^2 + \delta_a a^a - a_a a^a - \zeta_{ab}\zeta^{ab} +2\varepsilon_{ab}\alpha^a\Omega^b + \Omega_a\Omega^a - \mathscr{E}, \tag{79}$$

$$\text{ZAMO frame: } \hat{\phi} = -\frac{1}{2}\phi^2 + \delta_a a^a - a_a a^a -\zeta_{ab}\zeta^{ab} - \Sigma_a\Sigma^a - \mathscr{E}. \tag{80}$$

$e^a\varepsilon^{bc}R_{abc}$:

$$\diamond\text{Killing frame: } \hat{\xi} = 0 = \varepsilon_{ab}\delta^a a^b + 2\alpha_a\Omega^a, \tag{81}$$

$$\diamond\text{ZAMO frame: } \hat{\xi} = 0 = \varepsilon_{ab}\delta^a a^b. \tag{82}$$

$e^a R_{a\{bc\}}$:

$$\text{Killing frame: } \hat{\zeta}_{\{ab\}} = -\phi\zeta_{ab} - \zeta^{c}{}_{\{a}\zeta_{b\}c} + \delta_{\{a}a_{b\}} - a_{\{a}a_{b\}} +2\alpha_{\{a}\varepsilon_{b\}c}\Omega^c - \Omega_{\{a}\Omega_{b\}} - \mathscr{E}_{ab}, \tag{83}$$

$$\text{ZAMO frame: } \hat{\zeta}_{\{ab\}} = -\phi\zeta_{ab} - \zeta^{c}{}_{\{a}\zeta_{b\}c} + \delta_{\{a}a_{b\}} - a_{\{a}a_{b\}} -\Sigma_{\{a}\Sigma_{b\}} - \mathscr{E}_{ab}. \tag{84}$$

Additionally, the divergence equations for the shear, vorticity and the electric and magnetic Weyl parts are written below.

Shear divergence equations (from (18), (19)):

$$\text{Killing frame: } 0 = -\varepsilon_{ab}\delta^a\Omega^b - 2\varepsilon_{ab}\mathscr{A}^a\Omega^b, \tag{85}$$

$$\text{ZAMO frame: } 0 = -\delta_a\Sigma^a + 2\Sigma_a a^a + \Sigma_{ab}\zeta^{ab}. \tag{86}$$

$$\text{Killing frame: } -\varepsilon_{ab}\hat{\Omega}^b = -\varepsilon_{ab}\delta^b\Omega +\varepsilon_{ab}\Omega^b\left(\frac{1}{2}\phi + 2\mathscr{A}\right) + \Omega\varepsilon_{ab}\left(a^b - 2\mathscr{A}^b\right) + \varepsilon_{ab}\zeta^{bc}\Omega_c, \tag{87}$$

$$\text{ZAMO frame: } \hat{\Sigma}_{\bar{a}} = -\frac{3}{2}\phi\Sigma_a - \delta^b\Sigma_{ab} - \zeta_{ab}\Sigma^b +\Sigma_{ab}a^b. \tag{88}$$

Vorticity divergence equations (from (20)):

$$\text{Killing frame: } \quad \hat{\Omega} = -\delta_a\Omega^a + \Omega\left(\mathscr{A} - \phi\right) +\Omega^a\left(a_a + \mathscr{A}_a\right), \tag{89}$$

$$\text{ZAMO frame: } \quad 0 = 0. \tag{90}$$

The projected, symmetric and trace-free part of (21) and (22) according to (B.3):

$$\text{Killing frame: } 0 = -\varepsilon_{c\{a}\delta^c\Omega_{b\}} - \Omega\varepsilon_{c\{a}\zeta_{b\}}{}^{c} - 2\varepsilon_{c\{a}\Omega_{b\}}\mathscr{A}^c -\varepsilon_{c\{a}\mathscr{H}_{b\}}{}^{c}, \tag{91}$$

$$\text{ZAMO frame: } \hat{\Sigma}_{\{ab\}} = \delta_{\{a}\Sigma_{b\}} - \frac{1}{2}\phi\Sigma_{ab} - 2\Sigma_{\{a}a_{b\}} -\Sigma_{c\{a}\zeta_{b\}}{}^{c} - \varepsilon_{c\{a}\mathscr{H}_{b\}}{}^{c}. \tag{92}$$

Electrical Weyl divergence equations (from (28), (29)):

$$\text{Killing frame: } \hat{\mathscr{E}} = -\delta_a\mathscr{E}^a - \frac{3}{2}\phi\mathscr{E} + 3\Omega\mathscr{H} + 2\mathscr{E}_a a^a +3\Omega_a\mathscr{H}^a + \mathscr{E}_{ab}\zeta^{ab}, \tag{93}$$

$$\diamond\text{ZAMO frame: } \hat{\mathscr{E}} = -\delta_a\mathscr{E}^a - \frac{3}{2}\phi\mathscr{E} + 2\mathscr{E}_a a^a +\varepsilon_{ab}\Sigma^{ac}\mathscr{H}_c{}^b + \mathscr{E}_{ab}\zeta^{ab} + \varepsilon_{ab}\Sigma^a\mathscr{H}^b. \tag{94}$$

$$\text{Killing frame: } \hat{\mathscr{E}}_{\bar{a}} = \frac{1}{2}\delta_a\mathscr{E} - \delta^b\mathscr{E}_{ab} - \frac{3}{2}\mathscr{H}\Omega_a - \frac{3}{2}\mathscr{E}a_a -\frac{3}{2}\phi\mathscr{E}_a + 3\Omega\mathscr{H}_a - \zeta_{ab}\mathscr{E}^b + \mathscr{E}_{ab}a^b + 3\mathscr{H}_{ab}\Omega^b, \tag{95}$$

$$\diamond\text{ZAMO frame: } \hat{\mathscr{E}}_{\bar{a}} = \frac{1}{2}\delta_a\mathscr{E} - \delta^b\mathscr{E}_{ab} + \frac{3}{2}\mathscr{H}\varepsilon_{ab}\Sigma^b -\frac{3}{2}\mathscr{E}a_a - \frac{3}{2}\phi\mathscr{E}_a - \zeta_{ab}\mathscr{E}^b + \mathscr{E}_{ab}a^b - \varepsilon_{ab}\Sigma^c\mathscr{H}^b{}_c +\varepsilon_{ab}\Sigma^{bc}\mathscr{H}_c. \tag{96}$$

Magnetic Weyl divergence equations (from (30), (31)):

$$\text{Killing frame: } \hat{\mathscr{H}} = -\delta_a\mathscr{H}^a - \frac{3}{2}\phi\mathscr{H} - 3\Omega\mathscr{E} + 2\mathscr{H}_a a^a -3\Omega_a\mathscr{E}^a + \zeta_{ab}\mathscr{H}^{ab}, \tag{97}$$

$$\diamond\text{ZAMO frame: } \hat{\mathscr{H}} = -\delta_a\mathscr{H}^a - \frac{3}{2}\phi\mathscr{H} + 2\mathscr{H}_a a^a +\zeta_{ab}\mathscr{H}^{ab} - \varepsilon_{ab}\Sigma^a{}_c\mathscr{E}^{bc} -\varepsilon_{ab}\Sigma^a\mathscr{E}^b. \tag{98}$$

$$\text{Killing frame: } \hat{\mathscr{H}}_{\bar{a}} = \frac{1}{2}\delta_a\mathscr{H} - \delta^b\mathscr{H}_{ab} + \frac{3}{2}\mathscr{E}\Omega_a - \frac{3}{2}\mathscr{H}a_a -3\Omega\mathscr{E}_a - \frac{3}{2}\phi\mathscr{H}_a + \mathscr{H}_{ab}a^b - \zeta_{ab}\mathscr{H}^b - 3\mathscr{E}_{ab}\Omega^b, \tag{99}$$

$$\diamond\text{ZAMO frame: } \hat{\mathscr{H}}_{\bar{a}} = \frac{1}{2}\delta_a\mathscr{H} - \delta^b\mathscr{H}_{ab} - \frac{3}{2}\mathscr{E}\varepsilon_{ab}\Sigma^b -\frac{3}{2}\mathscr{H}a_a - \frac{3}{2}\phi\mathscr{H}_a + \mathscr{H}_{ab}a^b - \zeta_{ab}\mathscr{H}^b + \varepsilon_{ab}\Sigma^c\mathscr{E}^b{}_c -\varepsilon_{ab}\Sigma^{bc}\mathscr{E}_c. \tag{100}$$

6.4 Constraint equations

The following constraint equations are obtained by performing contractions on R_{abc} involving ε^{ab}, u^a, e^a and N^{ab}.

$\varepsilon^{ab}u^c R_{abc}$:

$$\text{Killing frame: } \delta_a\Omega^a = \Omega\left(2\mathscr{A} - \phi\right) + \mathscr{H}, \tag{101}$$

$$\text{ZAMO frame: } \varepsilon_{ab}\delta^a \Sigma^b = \varepsilon_{ab}\zeta^{ac}\Sigma^b{}_c + \mathscr{H}. \quad (102)$$

$N^{bc}R_{\bar{a}bc}$:

$$\text{Killing frame: } \frac{1}{2}\delta_a\phi - \delta^b\zeta_{ab} = -\Omega\Omega_a + 2\Omega\varepsilon_{ab}\alpha^b - \mathscr{E}_a, \quad (103)$$

$$\text{ZAMO frame: } \frac{1}{2}\delta_a\phi - \delta^b\zeta_{ab} = -\Sigma_{ab}\Sigma^b - \mathscr{E}_a. \quad (104)$$

$e^a u^c R_{a\bar{b}c}$:

$$\text{Killing frame: } 2\varepsilon_{ab}\delta^b\Omega = \phi\varepsilon_{ab}\Omega^b - 4\Omega\varepsilon_{ab}\mathscr{A}^b + 2\varepsilon_{ab}\zeta^{bc}\Omega_c - 2\varepsilon_{ab}\mathscr{H}^b, \quad (105)$$

$$\diamond\text{ZAMO frame: } 2\delta^b\Sigma_{ab} = -\phi\Sigma_a + 2\zeta_{ab}\Sigma^b - 2\varepsilon_{ab}\mathscr{H}^b. \quad (106)$$

All the above equations have been validated in Maple and GRTensorIII [21].

7 Comments

In Sect. 4 we applied the 1+3 covariant method to the Kerr spacetime for a Killing frame and a ZAMO frame. Analyzing the 1+3 quantities, we immediately note that the expansion scalar (Θ) is zero in both frames. This means that all the timelike observers in the frames we chose are stationary. It is interesting to note that in the Killing frame, the shear quantity is zero and the vorticity quantities are nonzero. On the contrary in the ZAMO frame, the vorticity quantities are zero by definition and the shear quantity is nonzero. The left hand side of Eq. (15) is actually the expression for the curl of acceleration which is equal to zero. This means that the acceleration can be written as a gradient of a scalar in the ZAMO frame. In the Killing frame, Eq. (14) relates the curl of acceleration to the vorticity and in turn Eq. (18) relates the curl of vorticity to the acceleration. This shows a significant relationship between the kinematical quantities, acceleration and vorticity.

In Sect. 6 we decomposed the Kerr spacetime further by applying the 1+1+2 covariant approach to the Kerr spacetime. The 1+1+2 formalism is an extension to the 1+3 formalism and hence the 1+3 result regarding rotation and distortion in the particular frames still holds. In the Killing frame there is zero distortion and in the ZAMO frame there is zero rotation. Further we note that the shear scalar (Σ) and sheet twist (ξ) quantities are also zero in both frames. The value for the sheet acceleration (a_a) is the same for both frames since it entirely depends on the definition of e^a. The purpose of using the 1+1+2 decomposition method is to gain access to and highlight the role of scalars. Constraint Eq. (101) for the Killing frame and (102) for the ZAMO frame are both expressions of the magnetic Weyl scalar $\mathscr{H}$. However in (101) the magnetic component of Weyl is entirely generated by the vorticity. In comparison, in the ZAMO frame (102) it is entirely generated by the distortion (shear). This is physically significant because the magnetic part of the Weyl tensor generates gravitational waves. We can conclude that in the Killing frame, the gravitational wave is a direct consequence of vorticity and in the ZAMO frame, the gravitational wave is a direct consequence of the distortion of spacetime. These results demonstrate that the 1+1+2 formalism also has the potential to excavate new information in spacetimes that aren't necessarily spherically symmetric and this can involve higher order terms. The equations presented in this section help build a framework to study the Kerr spacetime's electromagnetic and kinematic properties.

8 Conclusion

In this paper we have given a complete 1+1+2 semi-tetrad covariant description of the Kerr spacetime outside the horizon. Since there are no matter quantities we explicitly wrote down all the geometrical quantities and the evolution and propagation equations they satisfy. During this process we identified some errors in the equations of the previous 1+1+2 decomposition performed in [5,22] which we corrected in this paper specifically for the Kerr spacetime. However the affected terms in those equations were higher order terms. Hence the linear perturbation results in [5,22] remain unaffected.

As far as we are aware, the 1+1+2 description of the Kerr geometry presented in this paper is the first comprehensive treatment. Furthermore this highlights the role of geometrical variables associated with the timelike congruence and preferred spacelike congruence. That is, when one changes the observers congruence for example from the Killing frame to the ZAMO frame, the geometrical variables are affected. This is reflected in both Tables 1 and 2. This work can be extended in a number of different ways and we are already working on a few of these extensions:

1. As a natural generalisation of the linear perturbation of Schwarzschild geometry, where the Regge–Wheeler tensor was found [5], we can perform a similar but extensive calculation to find the corresponding Teukolsky tensor for rotating geometry.
2. Due to our exploration of the ZAMO frame, details of the geometry of the ergosphere can be investigated.
3. Since our results are valid in the ergosphere, the Penrose process can be applied whereby energy can be extracted from the rotating black hole.
4. The results relating to gravitational waves can give rise to the gravitational wave equation in general.
5. Even though the formalism used in this paper applies to the exterior including the ergosphere, the equations pre-

sented in this paper may be a useful guide in finding interior rotating solutions for an isolated body which is an unsolved problem in astrophysics.

In the future we can use this detailed geometrical description of the Kerr spacetime to find different physical properties of the Kerr spacetime that would have been more difficult to do using the usual coordinate approach or a standard 1+3 decomposition which was also performed in this paper.

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Appendix A: Additional 1+3 definitions

We write down important definitions and identities in the 1+3 formalism. Note that

$$\varepsilon_{abc} = \sqrt{|\det g|}\delta^0{}_{[a}\delta^1{}_b\delta^2{}_c\delta^3{}_{d]}u^d,$$
$$\varepsilon_{abc}\varepsilon^{def} = 3!h^d{}_{[a}h^e{}_bh^f{}_{c]},$$
$$\varepsilon_{abc}\varepsilon^{dec} = 2h^d{}_{[a}h^e{}_{b]}.$$

The 1+3 decomposition of the Riemann curvature tensor is given in [32] and in the context of Kerr is as follows

$$R^{ab}{}_{cd} = 4u^{[a}u_{[c}E^{b]}{}_{d]} + 4h^{[a}{}_{[c}E^{b]}{}_{d]} + 2\varepsilon^{abe}u_{[c}H_{d]e} + 2\varepsilon_{cde}u^{[a}H^{b]e}. \quad (A.1)$$

Appendix B: Additional 1+1+2 definitions

We write down the definitions of important components in the 1+1+2 formalism. Any spacetime 3-vector Φ^a can be irreducibly split into χ, a scalar component along e^a, and a 2-vector χ^a, which is a sheet component orthogonal to e^a, as follows

$$\Phi^a = \chi e^a + \chi^a, \quad (B.1)$$

where $\chi \equiv \Phi_a e^a$, $\chi^a \equiv N^{ab}\Phi_b \equiv \Phi^{\bar{a}}$ and the bar on a particular index denotes projection with N_{ab} on that index such that the vector or tensor lies on the sheet. Similarly we can split a projected, symmetric, trace-free tensor Φ_{ab} into scalar, 2-vector and 2-tensor parts as follows

$$\Phi_{ab} = \Phi_{<ab>} = \chi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\chi_{(a}e_{b)} + \chi_{ab}, \quad (B.2)$$

where the components

$$\chi \equiv e^a e^b \Phi_{ab} = -N^{ab}\Phi_{ab},$$
$$\chi_a \equiv N_a{}^b e^c \Phi_{bc},$$
$$\chi_{ab} \equiv \chi_{\{ab\}} = \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\Phi_{cd}, \quad (B.3)$$

are defined. The curly brackets denote the part of the tensor that is projected, symmetric and trace-free, with respect to e^a.

We mention that in general the dot, hat and delta derivatives do not commute. The commutation relations for any scalar κ are

$$\hat{\dot{\kappa}} - \dot{\hat{\kappa}} = -\mathscr{A}\dot{\kappa} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\kappa} + \left(\Sigma_a + \varepsilon_{ab}\Omega^b - \alpha_a\right)\delta^a\kappa, \quad (B.4)$$

$$\delta_a\dot{\kappa} - (\delta_a\kappa)^{\cdot}_{\perp} = -\mathscr{A}_a\dot{\kappa} + \left(\alpha_a + \Sigma_a - \varepsilon_{ab}\Omega^b\right)\hat{\kappa} + \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\delta_a\kappa + (\Sigma_{ab} + \Omega\varepsilon_{ab})\,\delta^b\kappa, \quad (B.5)$$

$$\delta_a\hat{\kappa} - \left(\widehat{\delta_a\kappa}\right)_{\perp} = -2\varepsilon_{ab}\Omega^b\dot{\kappa} + a_a\hat{\kappa} + \frac{1}{2}\phi\delta_a\kappa + (\zeta_{ab} + \xi\varepsilon_{ab})\,\delta^b\kappa, \quad (B.6)$$

$$\delta_{[a}\delta_{b]}\kappa = \varepsilon_{ab}\left(\Omega\dot{\kappa} - \alpha\hat{\kappa}\right), \quad (B.7)$$

where $\perp$ denotes projection onto the sheet.

Appendix C: Corrections to Clarkson equations

This appendix contains the equations of Clarkson [22] that contained errors and their subsequent corrections in the Kerr formalism. We note once again that the errors were detected in higher order terms hence the results presented in [5,22] remained unaffected. These equations have been checked using the mathematical software Maple and GRTensorIII [21]. The corrected terms have been enclosed in boxes below. We have taken $\Theta = \Sigma = \xi = 0$ which occurs in both frames.

1. Evolution equations (47 and 48)
$u^a N^{bc} R_{abc}$:
Clarkson:

$$\dot{\phi} = \delta_a \alpha^a + \mathscr{A}^a \left(\alpha_a - a_a\right) + \left(a^a - \mathscr{A}^a\right) \left(\Sigma_a - \varepsilon_{ab} \Omega^b\right) -\zeta^{ab} \Sigma_{ab}. \tag{C.1}$$

Correction:

$$\text{Killing frame: } \dot{\phi} = \delta_a \alpha^a + \boxed{\alpha_a \left(\mathscr{A}^a - a^a\right)} -\varepsilon_{ab} \Omega^b \left(a^a - \mathscr{A}^a\right), \tag{C.2}$$

$$\text{ZAMO frame: } \dot{\phi} = \delta_a \alpha^a + \boxed{\alpha_a \left(\mathscr{A}^a - a^a\right)} +\Sigma_a \left(a^a - \mathscr{A}^a\right) - \zeta^{ab} \Sigma_{ab}. \tag{C.3}$$

2. Evolution equation (50)
$u^a \varepsilon^{bc} R_{abc}$:
Clarkson:

$$\dot{\xi} = 0 = 2\Omega \left(\mathscr{A} - \phi\right) + \left(a^a + \mathscr{A}^a\right) \times \left[\Omega_a + \varepsilon_{ab} \left(\alpha^b + \Sigma^b\right)\right] +\varepsilon_{ab} \delta^a \alpha^b - \varepsilon_{ca} \zeta_b{}^c \Sigma^{ab} + \mathscr{H}. \tag{C.4}$$

Correction:

$$\text{ZAMO frame: } \dot{\xi} = 0 = \varepsilon_{ab} \left(a^a + \mathscr{A}^a\right) \left[\alpha^b + \Sigma^b\right] +\varepsilon_{ab} \delta^a \alpha^b \boxed{+\varepsilon_{ca} \zeta_b{}^c \Sigma^{ab}} + \mathscr{H}. \tag{C.5}$$

3. Mixed equations (63 and 64)
Shear evolution equation:
Clarkson:

$$-\frac{2}{3} \hat{\mathscr{A}} = \frac{2}{3} \mathscr{A}^2 - \frac{1}{3} \phi \mathscr{A} - \frac{2}{3} \Omega^2 - \frac{1}{3} \delta_a \mathscr{A}^a + \Sigma_a \times \left(2\alpha^a - \frac{1}{3} \Sigma^a\right) - \frac{2}{3} \mathscr{A}_a a^a + \frac{1}{3} \mathscr{A}_a \mathscr{A}^a +\frac{1}{3} \Omega_a \Omega^a + \frac{1}{3} \Sigma_{ab} \Sigma^{ab} - \mathscr{E}. \tag{C.6}$$

Correction:

$$\text{Killing frame: } -\frac{2}{3} \hat{\mathscr{A}} = \frac{2}{3} \mathscr{A}^2 - \frac{1}{3} \phi \mathscr{A} -\frac{2}{3} \Omega^2 - \frac{1}{3} \delta_a \mathscr{A}^a - \frac{2}{3} \mathscr{A}_a a^a \boxed{-\frac{1}{3} \mathscr{A}_a \mathscr{A}^a} +\frac{1}{3} \Omega_a \Omega^a - \mathscr{E}, \tag{C.7}$$

$$\text{ZAMO frame: } -\frac{2}{3} \hat{\mathscr{A}} = \frac{2}{3} \mathscr{A}^2 - \frac{1}{3} \phi \mathscr{A} - \frac{1}{3} \delta_a \mathscr{A}^a +\Sigma_a \left(2\alpha^a - \frac{1}{3} \Sigma^a\right) - \frac{2}{3} \mathscr{A}_a a^a \boxed{-\frac{1}{3} \mathscr{A}_a \mathscr{A}^a} +\frac{1}{3} \Sigma_{ab} \Sigma^{ab} - \mathscr{E}. \tag{C.8}$$

4. Mixed equations (69 and 70)
Electric Weyl evolution equation:
Clarkson:

$$\dot{\mathscr{E}}_{\bar{a}} + \frac{1}{2} \varepsilon_{ab} \hat{\mathscr{H}}^b = \frac{3}{4} \varepsilon_{ab} \delta^b \mathscr{H} + \frac{1}{2} \varepsilon_{bc} \delta^b \mathscr{H}^c{}_a + \frac{3}{4} \mathscr{E} \Sigma_a +\frac{3}{4} \mathscr{E} \varepsilon_{ab} \Omega^b + \frac{3}{2} \mathscr{H} \varepsilon_{ab} \mathscr{A}^b - \frac{3}{2} \mathscr{E} \alpha_a - \frac{3}{4} \mathscr{H} \varepsilon_{ab} a^b -\frac{1}{2} \Omega \varepsilon_{ab} \mathscr{E}^b - \frac{1}{4} \phi \varepsilon_{ab} \mathscr{H}^b - \mathscr{A} \varepsilon_{ab} \mathscr{H}^b + \frac{3}{2} \Sigma_{ab} \mathscr{E}^b +\frac{3}{2} \mathscr{E}_{ab} \Sigma^b - \mathscr{E}_{ab} \alpha^b - \frac{1}{2} \mathscr{E}_{ab} \varepsilon^{bc} \Omega_c +\frac{1}{2} \zeta_{ab} \varepsilon^{bc} \mathscr{H}_c - \mathscr{H}_{ab} \varepsilon^{bc} \mathscr{A}_c. \tag{C.9}$$

Correction:

$$\text{Killing frame: } \dot{\mathscr{E}}_{\bar{a}} + \frac{1}{2} \varepsilon_{ab} \hat{\mathscr{H}}^b = \frac{3}{4} \varepsilon_{ab} \delta^b \mathscr{H} +\frac{1}{2} \varepsilon_{bc} \delta^b \mathscr{H}^c{}_a + \frac{3}{4} \mathscr{E} \varepsilon_{ab} \Omega^b + \frac{3}{2} \mathscr{H} \varepsilon_{ab} \mathscr{A}^b - \frac{3}{2} \mathscr{E} \alpha_a -\frac{3}{4} \mathscr{H} \varepsilon_{ab} a^b - \frac{1}{2} \Omega \varepsilon_{ab} \mathscr{E}^b - \frac{1}{4} \phi \varepsilon_{ab} \mathscr{H}^b - \mathscr{A} \varepsilon_{ab} \mathscr{H}^b \boxed{+\mathscr{E}_{ab} \alpha^b} - \frac{1}{2} \mathscr{E}_{ab} \varepsilon^{bc} \Omega_c + \boxed{\frac{3}{2} \zeta_{ab} \varepsilon^{bc} \mathscr{H}_c} -\mathscr{H}_{ab} \varepsilon^{bc} \mathscr{A}_c \boxed{+\frac{1}{2} \varepsilon_{ab} a^c \mathscr{H}^b{}_c + \varepsilon_{ab} e_c \delta^b \mathscr{H}^c}, \tag{C.10}$$

$$\text{ZAMO frame: } \dot{\mathscr{E}}_{\bar{a}} + \frac{1}{2} \varepsilon_{ab} \hat{\mathscr{H}}^b = \frac{3}{4} \varepsilon_{ab} \delta^b \mathscr{H} +\frac{1}{2} \varepsilon_{bc} \delta^b \mathscr{H}^c{}_a + \frac{3}{4} \mathscr{E} \Sigma_a + \frac{3}{2} \mathscr{H} \varepsilon_{ab} \mathscr{A}^b - \frac{3}{2} \mathscr{E} \alpha_a -\frac{3}{4} \mathscr{H} \varepsilon_{ab} a^b - \frac{1}{4} \phi \varepsilon_{ab} \mathscr{H}^b - \mathscr{A} \varepsilon_{ab} \mathscr{H}^b + \frac{3}{2} \Sigma_{ab} \mathscr{E}^b +\frac{3}{2} \mathscr{E}_{ab} \Sigma^b \boxed{+\mathscr{E}_{ab} \alpha^b} + \boxed{\frac{3}{2} \zeta_{ab} \varepsilon^{bc} \mathscr{H}_c} -\mathscr{H}_{ab} \varepsilon^{bc} \mathscr{A}_c \boxed{+\frac{1}{2} \varepsilon_{ab} a^c \mathscr{H}^b{}_c + \varepsilon_{ab} e_c \delta^b \mathscr{H}^c}. \tag{C.11}$$

5. Mixed equations (75 and 76)
Magnetic Weyl evolution equation:
Clarkson:

$$\dot{\mathscr{H}}_{\bar{a}} - \frac{1}{2} \varepsilon_{ab} \hat{\mathscr{E}}^b = -\frac{3}{4} \varepsilon_{ab} \delta^b \mathscr{E} - \frac{1}{2} \varepsilon_{bc} \delta^b \mathscr{E}^c{}_a + \frac{3}{4} \mathscr{H} \Sigma_a +\frac{3}{4} \mathscr{H} \varepsilon_{ab} \Omega^b - \frac{3}{2} \mathscr{E} \varepsilon_{ab} \mathscr{A}^b - \frac{3}{2} \mathscr{H} \alpha_a + \frac{3}{4} \mathscr{E} \varepsilon_{ab} a^b +\frac{1}{4} \phi \varepsilon_{ab} \mathscr{E}^b + \mathscr{A} \varepsilon_{ab} \mathscr{E}^b - \frac{1}{2} \Omega \varepsilon_{ab} \mathscr{H}^b + \frac{3}{2} \Sigma_{ab} \mathscr{H}^b$$

$$+\frac{3}{2}\varepsilon_{ab}\zeta^{bc}\mathscr{E}_c+\varepsilon_{ab}\mathscr{A}_c\zeta^{bc}+\mathscr{H}_{ab}\alpha^b$$
$$+\frac{3}{2}\mathscr{H}_{ab}\Sigma^b-\frac{1}{2}\mathscr{H}_{ab}\varepsilon^{bc}\Omega_c. \quad (C.12)$$

Correction:

$$\text{Killing frame: } \dot{\mathscr{H}}_{\bar{a}}-\frac{1}{2}\varepsilon_{ab}\hat{\mathscr{E}}^b=-\frac{3}{4}\varepsilon_{ab}\delta^b\mathscr{E}$$
$$-\frac{1}{2}\varepsilon_{bc}\delta^b\mathscr{E}^c{}_a+\frac{3}{4}\mathscr{H}\varepsilon_{ab}\Omega^b$$
$$-\frac{3}{2}\mathscr{E}\varepsilon_{ab}\mathscr{A}^b-\frac{3}{2}\mathscr{H}\alpha_a+\frac{3}{4}\mathscr{E}\varepsilon_{ab}a^b$$
$$+\frac{1}{4}\phi\varepsilon_{ab}\mathscr{E}^b+\mathscr{A}\varepsilon_{ab}\mathscr{E}^b-\frac{1}{2}\Omega\varepsilon_{ab}\mathscr{H}^b+\frac{3}{2}\varepsilon_{ab}\zeta^{bc}\mathscr{E}_c$$
$$+\boxed{\mathscr{E}_{ab}\varepsilon^{bc}\mathscr{A}_c}+\mathscr{H}_{ab}\alpha^b-\frac{1}{2}\mathscr{H}_{ab}\varepsilon^{bc}\Omega_c$$
$$\boxed{-\frac{1}{2}\varepsilon_{ab}a^c\mathscr{E}^b{}_c-\varepsilon_{ab}e_c\delta^b\mathscr{E}^c}, \quad (C.13)$$

$$\text{ZAMO frame: } \dot{\mathscr{H}}_{\bar{a}}-\frac{1}{2}\varepsilon_{ab}\hat{\mathscr{E}}^b=-\frac{3}{4}\varepsilon_{ab}\delta^b\mathscr{E}$$
$$-\frac{1}{2}\varepsilon_{bc}\delta^b\mathscr{E}^c{}_a+\frac{3}{4}\mathscr{H}\Sigma_a-\frac{3}{2}\mathscr{E}\varepsilon_{ab}\mathscr{A}^b$$
$$-\frac{3}{2}\mathscr{H}\alpha_a+\frac{3}{4}\mathscr{E}\varepsilon_{ab}a^b$$
$$+\frac{1}{4}\phi\varepsilon_{ab}\mathscr{E}^b+\mathscr{A}\varepsilon_{ab}\mathscr{E}^b+\frac{3}{2}\Sigma_{ab}\mathscr{H}^b$$
$$+\frac{3}{2}\varepsilon_{ab}\zeta^{bc}\mathscr{E}_c+\boxed{\mathscr{E}_{ab}\varepsilon^{bc}\mathscr{A}_c}$$
$$+\mathscr{H}_{ab}\left(\alpha^b+\frac{3}{2}\Sigma^b\right)$$
$$\boxed{-\frac{1}{2}\varepsilon_{ab}a^c\mathscr{E}^b{}_c-\varepsilon_{ab}e_c\delta^b\mathscr{E}^c}. \quad (C.14)$$

6. Mixed equations (77 and 78)
 Magnetic Weyl evolution equation:
 Clarkson:

$$\dot{\mathscr{H}}_{\{ab\}}+\varepsilon_{c\{a}\hat{\mathscr{E}}_{b\}}{}^c=\varepsilon_{c\{a}\delta^c\mathscr{E}_{b\}}-\frac{3}{2}\mathscr{H}\Sigma_{ab}+\frac{3}{2}\mathscr{E}\varepsilon_{c\{a}\zeta_{b\}}{}^c$$
$$-\frac{1}{2}\phi\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c-2\mathscr{A}\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c-\Omega\varepsilon_{c\{a}\mathscr{H}_{b\}}{}^c+3\Sigma_{\{a}\mathscr{H}_{b\}}$$
$$-\Omega_{\{a}\varepsilon_{b\}c}\mathscr{H}^c-2\alpha_{\{a}\mathscr{H}_{b\}}+2\mathscr{E}_{\{a}\varepsilon_{b\}c}a^c+2\mathscr{E}_{\{a}\varepsilon_{b\}c}\mathscr{A}^c$$
$$+3\Sigma_{c\{a}\mathscr{H}_{b\}}{}^c-\varepsilon_{c\{a}\mathscr{E}_{b\}d}\zeta^{cd}. \quad (C.15)$$

Correction:

$$\text{Killing frame: } \dot{\mathscr{H}}_{\{ab\}}+\varepsilon_{c\{a}\hat{\mathscr{E}}_{b\}}{}^c=\varepsilon_{c\{a}\delta^c\mathscr{E}_{b\}}$$
$$+\frac{3}{2}\mathscr{E}\varepsilon_{c\{a}\zeta_{b\}}{}^c-\frac{1}{2}\phi\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c-2\mathscr{A}\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c$$
$$\boxed{+\Omega\varepsilon_{c\{a}\mathscr{H}_{b\}}{}^c}$$
$$-\boxed{\varepsilon_{c\{a}\mathscr{H}_{b\}}\Omega^c}-2\alpha_{\{a}\mathscr{H}_{b\}}$$
$$+2\mathscr{E}_{\{a}\varepsilon_{b\}c}a^c\boxed{-2\mathscr{E}_{\{a}\varepsilon_{b\}c}\mathscr{A}^c}-\varepsilon_{c\{a}\mathscr{E}_{b\}d}\zeta^{cd}, \quad (C.16)$$

$$\text{ZAMO frame: } \dot{\mathscr{H}}_{\{ab\}}+\varepsilon_{c\{a}\hat{\mathscr{E}}_{b\}}{}^c=\varepsilon_{c\{a}\delta^c\mathscr{E}_{b\}}$$
$$-\frac{3}{2}\mathscr{H}\Sigma_{ab}+\frac{3}{2}\mathscr{E}\varepsilon_{c\{a}\zeta_{b\}}{}^c-\frac{1}{2}\phi\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c-2\mathscr{A}\varepsilon_{c\{a}\mathscr{E}_{b\}}{}^c$$
$$+3\Sigma_{\{a}\mathscr{H}_{b\}}$$
$$+2\mathscr{E}_{\{a}\varepsilon_{b\}c}a^c\boxed{-2\mathscr{E}_{\{a}\varepsilon_{b\}c}\mathscr{A}^c}$$
$$+3\Sigma_{c\{a}\mathscr{H}_{b\}}{}^c-2\alpha_{\{a}\mathscr{H}_{b\}}-\varepsilon_{c\{a}\mathscr{E}_{b\}d}\zeta^{cd}. \quad (C.17)$$

7. Propagation equations (81 and 82)
 $e^a\varepsilon^{bc}R_{abc}$:
 Clarkson:

$$\hat{\xi}=0=\frac{1}{2}\varepsilon_{ab}\delta^a a^b+\frac{1}{2}\varepsilon_{ab}\Sigma^a a^b+\alpha_a\Omega^a+\frac{1}{2}a_a\Omega^a. \quad (C.18)$$

Correction:

$$\text{Killing frame: } \quad \hat{\xi}=0=\varepsilon_{ab}\delta^a a^b+2\alpha_a\Omega^a, \quad (C.19)$$
$$\text{ZAMO frame: } \quad \hat{\xi}=0=\varepsilon_{ab}\delta^a a^b. \quad (C.20)$$

8. Propagation equation (94)
 Electric Weyl divergence equation:
 Clarkson:

$$\hat{\mathscr{E}}=-\delta_a\mathscr{E}^a-\frac{3}{2}\phi\mathscr{E}+3\Omega\mathscr{H}+2\mathscr{E}_a a^a+3\Omega_a\mathscr{H}^a$$
$$+\varepsilon_{ab}\Sigma^{ac}\mathscr{H}^b{}_c+\mathscr{E}_{ab}\zeta^{ab}. \quad (C.21)$$

Correction:

$$\text{ZAMO frame: } \hat{\mathscr{E}}=-\delta_a\mathscr{E}^a-\frac{3}{2}\phi\mathscr{E}+2\mathscr{E}_a a^a$$
$$+\varepsilon_{ab}\Sigma^{ac}\mathscr{H}_c^b+\mathscr{E}_{ab}\zeta^{ab}\boxed{+\varepsilon_{ab}\Sigma^a\mathscr{H}^b}. \quad (C.22)$$

9. Propagation equation (96)
 Electric Weyl divergence equation:
 Clarkson:

$$\hat{\mathscr{E}}_{\bar{a}}=\frac{1}{2}\delta_a\mathscr{E}-\delta^b\mathscr{E}_{ab}+\mathscr{H}\varepsilon_{ab}\Sigma^b-\frac{3}{2}\mathscr{H}\Omega_a-\frac{3}{2}\mathscr{E}a_a$$
$$-\frac{3}{2}\phi\mathscr{E}_a+3\Omega\mathscr{H}_a-\zeta_{ab}\mathscr{E}^b+\mathscr{E}_{ab}a^b+3\mathscr{H}_{ab}\Omega^b. \quad (C.23)$$

Correction:

$$\text{ZAMO frame: } \hat{\mathscr{E}}_{\bar{a}}=\frac{1}{2}\delta_a\mathscr{E}-\delta^b\mathscr{E}_{ab}\boxed{+\frac{3}{2}\mathscr{H}\varepsilon_{ab}\Sigma^b}$$
$$-\frac{3}{2}\mathscr{E}a_a-\frac{3}{2}\phi\mathscr{E}_a-\zeta_{ab}\mathscr{E}^b$$
$$+\mathscr{E}_{ab}a^b\boxed{-\varepsilon_{ab}\Sigma^c\mathscr{H}^b{}_c+\varepsilon_{ab}\Sigma^{bc}\mathscr{H}_c}. \quad (C.24)$$

10. Propagation equation (98)
Magnetic Weyl divergence equation:
Clarkson:

$$\hat{\mathscr{H}} = -\delta_a \mathscr{H}^a - \frac{3}{2}\phi \mathscr{H} - 3\mathscr{E}\Omega + 2\mathscr{H}_a a^a - 3\Omega_a \mathscr{E}^a + \zeta_{ab}\mathscr{H}^{ab} - \varepsilon_{ab}\Sigma^a{}_c \mathscr{E}^{bc}. \tag{C.25}$$

Correction:

$$\text{ZAMO frame: } \hat{\mathscr{H}} = -\delta_a \mathscr{H}^a - \frac{3}{2}\phi \mathscr{H} + 2\mathscr{H}_a a^a + \zeta_{ab}\mathscr{H}^{ab} - \varepsilon_{ab}\Sigma^a{}_c \mathscr{E}^{bc} \boxed{-\varepsilon_{ab}\Sigma^a \mathscr{E}^b}. \tag{C.26}$$

11. Propagation equation (100)
Magnetic Weyl divergence equation:
Clarkson:

$$\hat{\mathscr{H}_{\bar{a}}} = \frac{1}{2}\delta_a \mathscr{H} - \delta^b \mathscr{H}_{ab} - \frac{3}{2}\mathscr{E}\varepsilon_{ab}\Sigma^b + \frac{3}{2}\mathscr{E}\Omega_a - \frac{3}{2}\mathscr{H} a_a - 3\Omega\mathscr{E}_a - \frac{3}{2}\phi\mathscr{H}_a + \mathscr{H}_{ab}a^b - \zeta_{ab}\mathscr{H}^b - 3\mathscr{E}_{ab}\Omega^b. \tag{C.27}$$

Correction:

$$\text{ZAMO frame: } \hat{\mathscr{H}_{\bar{a}}} = \frac{1}{2}\delta_a \mathscr{H} - \delta^b \mathscr{H}_{ab} - \frac{3}{2}\mathscr{E}\varepsilon_{ab}\Sigma^b - \frac{3}{2}\mathscr{H} a_a - \frac{3}{2}\phi\mathscr{H}_a + \mathscr{H}_{ab}a^b - \zeta_{ab}\mathscr{H}^b \boxed{+\varepsilon_{ab}\Sigma^c \mathscr{E}^b{}_c} \boxed{-\varepsilon_{ab}\Sigma^{bc}\mathscr{E}_c}. \tag{C.28}$$

12. Constraint equation (106)
$e^a u^c R_{a\bar{b}c}$:
Clarkson:

$$2\varepsilon_{ab}\delta^b\Omega + 2\delta^b\Sigma_{ab} = -\phi\left(\Sigma_a - \varepsilon_{ab}\Omega^b\right) - 4\Omega\varepsilon_{ab}\mathscr{A}^b + 2\zeta_{ab}\Sigma^b + 2\varepsilon_{ab}\zeta^{bc}\Omega_c \boxed{+\Sigma_{ab}a^b} - 2\varepsilon_{ab}\mathscr{H}^b. \tag{C.29}$$

Correction:

$$\text{ZAMO frame: } 2\delta^b\Sigma_{ab} = -\phi\Sigma_a + 2\zeta_{ab}\Sigma^b - 2\varepsilon_{ab}\mathscr{H}^b. \tag{C.30}$$

References

1. R.P. Kerr, Phys. Rev. Lett. **11**, 237 (1963)
2. E.T. Newman, R. Penrose, J. Math. Phys. **3**, 566 (1962)
3. J. Ehlers, Akad. Wiss. Lit. Mainz, Abhandl. Math. Nat. Kl. **11**, 793 (1961) [Translation: J. Ehlers, Gen. Relativ. Gravit. **25**, 1225 (1993)]
4. G.F.R. Ellis, Relativistic cosmology in *Proceedings of the International School of Physics "Enrico Fermi", Course 47: General Relativity and Cosmology* (Academic Press, New York, 1971)
5. C.A. Clarkson, R.K. Barrett, Class. Quantum Gravity **20**, 3855 (2003)
6. R. Goswami, G.F.R. Ellis, Class. Quantum Gravity **38**, 085023 (2021)
7. G.F.R. Ellis, M. Bruni, Phys. Rev. D **40**, 1804 (1989)
8. A. Challinor, A. Lasenby, Phys. Rev. D **58**, 023001 (1998)
9. G.F.R. Ellis, J. Math. Phys. **8**, 1171 (1967)
10. J.M. Stewart, G.F.R. Ellis, J. Math. Phys. **9**, 1072 (1968)
11. G.F.R. Ellis, M.A.H. MacCallum, Commun. Math. Phys. **12**, 108 (1969)
12. L. Bianchi, Gen. Relativ. Gravit. **33**, 2157 (2001)
13. S. Singh, G.F.R. Ellis, R. Goswami, S.D. Maharaj, Phys. Rev. D **94**, 104040 (2016)
14. A.M. Nzioki, S. Carloni, R. Goswami, P.K.S. Dunsby, Phys. Rev. D **81**, 084028 (2010)
15. C. Hansraj, R. Goswami, S.D. Maharaj, Gen. Relativ. Gravit. **52**, 63 (2020)
16. R. Goswami, G.F.R. Ellis, Class. Quantum Gravity **38**, 085023 (2021)
17. S. Chandrasekhar, Proc. R. Soc. Lond. A **349**, 571 (1976)
18. J.A. Marck, Proc. R. Soc. Lond. A **385**, 431 (1983)
19. E.G. Kalnins, W. Miller, G.C. Williams, Proc. R. Soc. Lond. A **408**, 1834 (1986)
20. V. Frolov, I. Novikov, *Black Hole Physics: Basic Concepts and New Developments* (Kluwer Academic, New York, 1998)
21. GRTensorIII is a software package developed by P. Musgrave, D. Pollney and K. Lake (2020). It is distributed free of charge at https://github.com/grtensor/grtensor and was run on the mathematical software Maple. Maple 2021 is a trademark of Waterloo Maple Inc
22. C. Clarkson, Phys. Rev. D **76**, 104034 (2007)
23. R.H. Boyer, R.W. Lindquist, J. Math. Phys. **8**, 265 (1967)
24. J. Baines, T. Berry, A. Simpson, M. Visser, Class. Quantum Gravity **38**, 055001 (2020)
25. M. Visser, S. Weinfurtner, Class. Quantum Gravity **22**, 2493 (2005)
26. A.A. Shatskiy, J. Exp. Theor. Phys. **130**, 409 (2020)
27. C. Bambi, A. Càrdenas-Avendaño, T. Dauser, J.A. Garcia, S. Nampalliwar, Astrophys. J. **842**, 76 (2017)
28. D.L. Wiltshire, M. Visser, S.M. Scott, *The Kerr Spacetime: Rotating Black Holes in General Relativity* (Cambridge University Press, Cambridge, 2009)
29. A. Frolov, V. Frolov, Phys. Rev. D **90**, 124010 (2014)
30. R. Maartens, B.A. Bassett, Class. Quantum Gravity **15**, 705 (1998)
31. G.F.R. Ellis, Gen. Relativ. Gravit. **59**, 581 (2009)
32. G.F.R. Ellis, H. van Elst, Cosmological Models, in *Proceedings of the NATO Advanced Study Institute on Theoretical and Observational Cosmology: Cargése Lectures* (Kluwer Academic, Boston, 1998)
33. A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955)
34. P.J. Greenberg, J. Math. Anal. Appl. **30**, 128 (1970)
35. H. van Elst, G.F.R. Ellis, Class. Quantum Gravity **13**, 1099 (1996)
36. G. Betschart, C.A. Clarkson, Class. Quantum Gravity **21**, 5587 (2004)