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$f(Q, T)$ gravity, its covariant formulation, energy conservation and phase-space analysis

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Abstract In the present article we analyze the matter-geometry coupled $f(Q, T)$ theory of gravity. We offer the fully covariant formulation of the theory, with which we construct the correct energy balance equation and employ it to conduct a dynamical system analysis in a spatially flat Friedmann–Lemaître–Robertson–Walker spacetime. We consider three different functional forms of the $f(Q, T)$ function, specifically, $f(Q, T) = \alpha Q + \beta T$, $f(Q, T) = \alpha Q + \beta T^2$, and $f(Q, T) = Q + \alpha Q^2 + \beta T$. We attempt to investigate the physical capabilities of these models to describe various cosmological epochs. We calculate Friedmann-like equations in each case and introduce some phase space variables to simplify the equations in more concise forms. We observe that the linear model $f(Q, T) = \alpha Q + \beta T$ with $\beta = 0$ is completely equivalent to the GR case without cosmological constant Λ. Further, we find that the model $f(Q, T) = \alpha Q + \beta T^2$ with $\beta \neq 0$ successfully depicts the observed transition from decelerated phase to an accelerated phase of the universe. Lastly, we find that the model $f(Q, T) = Q + \alpha Q^2 + \beta T$ with $\alpha \neq 0$ represents an accelerated de-Sitter epoch for the constraints $\beta < -1$ or $\beta \geq 0$.

Contents

1 Introduction

In the last two decades, the standard ΛCDM model, where the General Relativity has been manifested as the geometry of the background, provides an excellent fit to several cosmological observations. Nevertheless, the model is still struggling in the discrepancy of the value of the cosmological constant Λ. Thus, the standard theory of relativity may not be the ultimate theory to address the dark energy and the dark matter issues. This motivates a search for other theoretical cosmological scenarios that can describe cosmic acceleration with observational compatibility. General Relativity and its extensions, such as $f(R)$ gravity, $f(R, G)$ gravity, $f(R, T)$ gravity among others, were the curvature-based theory of gravitation formulated and extensively studied in the past [1,2]. Another promising way to acquire the modified theory of gravitation is to formulate gravity on a flat spacetime geometry depending solely either on the torsion or the non-metricity, the first is known as metric teleparallel theories and the second as symmetric teleparallel theories. Nester and Yo [3] proposed the latter and due to its dependence on the dark sector, Jimenez et al. extended it to formulate the $f(Q)$ gravity [4] such that the late-time acceleration could be demonstrated from the additional geometric components. In the recent past, a tremendous amount of works were carried out in this $f(Q)$ theories [5–28].

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Very recently, a matter-geometry coupling in the form of $f(Q, T)$ theories were proposed [29] in which the Lagrangian was represented by any viable function of the non-metricity scalar Q, and the trace T of the energy-momentum tensor. Harko [30] argued that this dependence can be caused by exotic imperfect fluids or quantum phenomena. Recently, several cosmological and astrophysical aspects of $f(Q, T)$ gravity have been tested, for instance, Energy conditions [31], Baryogenesis [32], Cosmological inflation [33], Reconstruction of $f(Q, T)$ lagrangian [34], Static spherically symmetric wormholes [35], Constraint on the effective equation of state [36], Observational constraints on $f(Q, T)$ gravity models [37], and Cosmological perturbations [38]. However, except the introductory article, most other studies were primarily conducted to contact observational datasets and not much theoretical investigation was carried out in this gravity theory which is still at its infancy at best. On a closer look, it is noticed that a covariant formulation was eluded so far, and there is some missing terms in the energy balance equation.

This motivated us to derive the covariant formulation and the energy balance equation of the $f(Q, T)$ gravity. In addition, we present the asymptotic behavior of some cosmological $f(Q, T)$ models by utilizing the dynamical system technique. This approach is quite efficient in describing the asymptotic behavior of non-linear modified gravity models. Several cosmological models of the modified gravity have been tested by utilizing the dynamical system technique [39–43]. One can investigate the asymptotic behavior of the cosmological model by analyzing the nature of critical points obtained by solving an autonomous system of first-order ordinary differential equations. The most important feature of any cosmological model is to have late-time stable solutions that depict the late-time behavior of the model.

The manuscript is organized as follows: in Sect. 2, we present the mathematical formulation of $f(Q, T)$ gravity. Then in Sect. 3, we derive the covariant formula for $f(Q, T)$ gravity. In Sect. 4, we derive the corrected version of the energy-balance equation, both from the original metric field equation as well as from the novel covariant form of it. Further in Sect. 5, we investigate the asymptotic behavior of some cosmological $f(Q, T)$ models with the help of dynamical system analysis. Finally in Sect. 6, we summarize the obtained results.

2 The mathematical formulation

In $f(Q, T)$-gravity theory, the background framework of the spacetime is the torsion free teleprallel geometry, i.e., $R^\rho{}_{\sigma\mu\nu} = 0$ and $T^\rho_{\mu\nu} = 0$. The difference between the associated connection $\Gamma^\lambda{}_{\mu\nu}$ and the Levi-Civita connection $\mathring{\Gamma}^\lambda{}_{\mu\nu}$

is known as the disformation tensor

$$L^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \mathring{\Gamma}^\lambda{}_{\mu\nu}, \tag{1}$$

where $Q_{\lambda\mu\nu} := \nabla_\lambda g_{\mu\nu}$ is the non-metricity tensor. It follows that

$$L^\lambda{}_{\mu\nu} = \frac{1}{2}(Q^\lambda{}_{\mu\nu} - Q_\mu{}^\lambda{}_\nu - Q_\nu{}^\lambda{}_\mu).$$

In addition, we define the non-metricity scalar

$$Q = Q_{\lambda\mu\nu}P^{\lambda\mu\nu} = -\frac{1}{2}Q_{\lambda\mu\nu}L^{\lambda\mu\nu}$$
$$+\frac{1}{4}Q_\lambda Q^\lambda - \frac{1}{2}Q_\lambda \tilde{Q}^\lambda, \tag{2}$$

where

$$P^\lambda{}_{\mu\nu} := \frac{1}{4}\Big(-2L^\lambda{}_{\mu\nu} + Q^\lambda g_{\mu\nu}$$
$$-\tilde{Q}^\lambda g_{\mu\nu} - \frac{1}{2}\delta^\lambda_\mu Q_\nu - \frac{1}{2}\delta^\lambda_\nu Q_\mu\Big), \tag{3}$$

is the superpotential tensor. Here we have used these two traces of non-metricity tensor: $Q_\lambda = Q_{\lambda\mu}{}^\mu$ and $\tilde{Q}_\lambda = Q_{\nu\lambda}{}^\nu$. The action of $f(Q, T)$-gravity is given by

$$S = \int \left[\frac{1}{2\kappa}f(Q, T) + \mathcal{L}_M\right]\sqrt{-g}\, d^4x$$

where g is the determinant of the metric tensor, $\mathcal{L}_M$ is the matter Lagrangian and T is the trace of the stress energy tensor $T_{\mu\nu}$, which is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}}. \tag{4}$$

The variation of the action with respect to the metric, gives the metric field equation

$$\frac{2}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}f_Q P^\lambda{}_{\mu\nu}\right) - \frac{1}{2}fg_{\mu\nu} + f_T(T_{\mu\nu} + \Theta_{\mu\nu})$$
$$+ f_Q(P_{\nu\rho\sigma}Q_\mu{}^{\rho\sigma} - 2P_{\rho\sigma\mu}Q^{\rho\sigma}{}_\nu) = \kappa T_{\mu\nu}, \tag{5}$$

where $f_{(\cdot)}$ represents the partial derivative of f with respect to $(\cdot)$ and

$$\Theta_{\mu\nu} = \frac{g^{\alpha\beta}\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}. \tag{6}$$

Furthermore, varying the action with respect to the connection, we obtain the following connection field equation

$$\nabla_\mu \nabla_\nu(2\sqrt{-g}f_Q P^{\mu\nu}{}_\lambda - \kappa H_\lambda{}^{\mu\nu}) = 0, \tag{7}$$

where

$$H_\lambda{}^{\mu\nu} = \frac{\sqrt{-g}}{2\kappa}f_T\frac{\delta T}{\delta\Gamma^\lambda{}_{\mu\nu}} + \frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta\Gamma^\lambda{}_{\mu\nu}},$$

is the hypermomentum tensor density.

Noticing that (5) is valid only in the coincident gauge coordinate [4], it is essential to express it in its covariant form in the following section which firstly is independent of the choice of coordinate systems and secondly provides an relatively straightforward manner in identifying the effective energy density and pressure.

3 Covariant formulation of $f(Q, T)$

In the literature of cosmological application of symmetric teleparallel theory, we commonly observe the use of the so-called "coincident gauge", that is, to identify a coordinate system whereby the connection vanishes and covariant derivatives reduce to partial derivatives. This is common in both isotropic spatially flat FLRW spacetime as well as in Bianchi type anisotropic spacetimes. As described in [20,44], this sometime poses a major issue when we attempt to indulge into investigation of some other spacetimes using the same vanishing connection. Most of the time, the system is not consistent unless we force the non-metricity scalar Q to be a constant, even worse, $Q = 0$. To alleviate this problem, a fully-covariant formulation is very useful to employ non-vanishing connections into the game. In this section, we introduce the much-awaited covariant formulation of the $f(Q, T)$ theory.

Let us begin from the curvature free and torsion free constraints providing

$$\mathring{R}_{\mu\nu} + \mathring{\nabla}_\alpha L^\alpha{}_{\mu\nu} - \mathring{\nabla}_\nu \tilde{L}_\mu + \tilde{L}_\alpha L^\alpha{}_{\mu\nu} - L_{\alpha\beta\nu} L^{\beta\alpha}{}_\mu = 0, \tag{8}$$

$$\mathring{R} + \mathring{\nabla}_\alpha (L^\alpha - \tilde{L}^\alpha) - Q = 0. \tag{9}$$

As mentioned before, the coincident gauge is chosen in which $\Gamma^\lambda{}_{\mu\nu} = 0$ or $\mathring{\Gamma}^\lambda{}_{\mu\nu} = -L^\lambda{}_{\mu\nu}$. Then we have

$$\partial_\lambda \sqrt{-g} = -\sqrt{-g}\, \tilde{L}_\lambda. \tag{10}$$

Furthermore, we derive

$$\frac{2}{\sqrt{-g}} \partial_\lambda \left(\sqrt{-g} f_Q P^\lambda{}_{\mu\nu} \right) + f_Q (P_{\nu\rho\sigma} Q_\mu{}^{\rho\sigma} - 2P_{\rho\sigma\mu} Q^{\rho\sigma}{}_\nu)$$

$$= 2(\nabla_\lambda f_Q) P^\lambda{}_{\mu\nu} + 2f_Q (\mathring{\nabla}_\lambda P^\lambda{}_{\mu\nu} - L_{\alpha\beta\mu} P^{\beta\alpha}{}_\nu$$
$$- L_{\alpha\beta\mu} P_\nu{}^{\beta\alpha} + L_{\nu\alpha\beta} P^{\alpha\beta}{}_\mu)$$

$$= 2(\nabla_\lambda f_Q) P^\lambda{}_{\mu\nu} + 2f_Q (\mathring{\nabla}_\lambda P^\lambda{}_{\mu\nu} - \tilde{L}_\alpha L^\alpha{}_{\nu\mu} + L_{\alpha\beta\nu} L^{\beta\alpha}{}_\mu). \tag{11}$$

On the other hand, using (3), (8) and (9), we obtain

$$2\mathring{\nabla}_\alpha P^\alpha{}_{\mu\nu} = -\mathring{\nabla}_\alpha L^\alpha{}_{\mu\nu} + \frac{\mathring{\nabla}_\alpha (L^\alpha - \tilde{L}^\alpha)}{2} g_{\mu\nu}$$
$$+ \frac{\mathring{\nabla}_\nu \tilde{L}_\mu + \mathring{\nabla}_\mu \tilde{L}^\nu}{2}$$

$$= \mathring{R}_{\mu\nu} + \frac{Q - \mathring{R}}{2} g_{\mu\nu} + \tilde{L}_\alpha L^\alpha{}_{\mu\nu} - L_{\alpha\beta\nu} L^{\beta\alpha}{}_\mu. \tag{12}$$

Combining (11) and (12) we obtain

$$\frac{2}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} f_Q P^\lambda{}_{\mu\nu}) + f_Q (P_{\nu\rho\sigma} Q_\mu{}^{\rho\sigma} - 2P_{\rho\sigma\mu} Q^{\rho\sigma}{}_\nu)$$

$$= 2(\nabla_\lambda f_Q) P^\lambda{}_{\mu\nu} + f_Q \left(\mathring{R}_{\mu\nu} - \frac{\mathring{R} - Q}{2} g_{\mu\nu} \right). \tag{13}$$

Finally, the metric field equation can be rewritten covariantly as

$$f_Q \mathring{G}_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(Qf_Q - f) + f_T(T_{\mu\nu} + \Theta_{\mu\nu})$$
$$+ 2(\nabla_\lambda f_Q) P^\lambda{}_{\mu\nu} = \kappa T_{\mu\nu} \tag{14}$$

where $\mathring{G}_{\mu\nu}$ is the Einstein tensor corresponding to the Levi-Civita connection. We define the effective stress energy tensor as

$$\kappa T^{\text{eff}}_{\mu\nu} = \kappa T_{\mu\nu} - f_T(T_{\mu\nu} + \Theta_{\mu\nu}) - \frac{1}{2} g_{\mu\nu}(Qf_Q - f)$$
$$- 2(f_{QQ} \nabla_\lambda Q + f_{QT} \nabla_\lambda T) P^\lambda{}_{\mu\nu}. \tag{15}$$

In the present paper, we consider a perfect fluid type spacetime, whose stress energy tensor takes the form

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu \tag{16}$$

to which the matter Lagrangian can be taken as $\mathcal{L}_M = p$, where ρ, p and u^μ denote the energy density, pressure and four velocity of the fluid respectively. It follows that

$$\Theta_{\mu\nu} = pg_{\mu\nu} - 2T_{\mu\nu} \tag{17}$$

4 Energy conservation in $f(Q, T)$ theory

The $f(Q, T)$ theory is not compatible with the energy conservation criterion, so an energy momentum balance equation was offered in [29]. Unfortunately, it seems to be not derived correctly as two crucial terms were noticed missing, particularly while applying the covariant derivative $\mathring{\nabla}_\mu$ to

$$\frac{2}{\sqrt{-g}} \nabla_\alpha (\sqrt{-g} f_Q P^{\alpha\mu}{}_\nu) \quad \text{and} \quad f_T T^\mu{}_\nu.$$

The corrected energy-momentum balance equation should read as

$$\mathring{\nabla}_\mu T^\mu{}_\nu = \frac{1}{\kappa - f_T}$$

$$\times \left\{ \mathring{\nabla}_\mu (f_T \Theta^\mu{}_\nu) + (\nabla_\mu f_T) T^\mu{}_\nu - \frac{1}{2} f_T \nabla_\nu T + \frac{\kappa}{\sqrt{-g}} \nabla_\alpha \nabla_\mu H_\nu{}^{\alpha\mu} \right\}. \tag{18}$$

In what follows, we briefly discuss the derivation of (18). Let us begin with the field equation (5) of type (1, 1)

$$\kappa T^\mu_{\ \nu} - f_T(\Theta^\mu_{\ \nu} + T^\mu_{\ \nu}) + \frac{1}{2}f\delta^\mu_{\ \nu} = f_Q P^{\mu\rho\sigma}Q_{\nu\rho\sigma}$$
$$+ \frac{2}{\sqrt{-g}}\nabla_\lambda(\sqrt{-g}f_Q P^{\lambda\mu}_{\ \ \nu}). \tag{19}$$

The divergence of the preceding equation gives

$$(\kappa - f_T)\mathring{\nabla}_\mu T^\mu_{\ \nu} - (\nabla_\mu f_T)T^\mu_{\ \nu} - \mathring{\nabla}_\nu(f_T\Theta^\mu_{\ \nu}) + \frac{1}{2}\nabla_\nu f$$
$$= (\nabla_\lambda f_Q)P^{\mu\rho\sigma}Q_{\nu\rho\sigma}) + \mathring{\nabla}_\mu\left\{\frac{2}{\sqrt{-g}}\nabla_\lambda(\sqrt{-g}f_Q P^{\lambda\mu}_{\ \ \nu})\right\}. \tag{20}$$

To simplify the above equation, we explicitly expand the two terms in the right hand side as

$$\mathring{\nabla}_\mu(f_Q P^{\mu\rho\sigma}Q_{\nu\rho\sigma}) = Q_{\nu\rho\sigma}(\nabla_\mu - \tilde{L}_\mu)(f_Q P^{\mu\rho\sigma})$$
$$+ f_Q P^{\mu\rho\sigma}(\nabla_\mu Q_{\nu\rho\sigma} + L^\beta_{\ \nu\mu}Q_{\beta\rho\sigma})$$
$$= -2L_{\sigma\nu\rho}(\nabla_\mu - \tilde{L}_\mu)(f_Q P^{\mu\rho\sigma})$$
$$+ f_Q P^{\mu\rho\sigma}(\mathring{\nabla}_\nu Q_{\mu\rho\sigma} - 2L^\beta_{\ \rho\nu}Q_{\mu\beta\sigma}), \tag{21}$$

$$\mathring{\nabla}_\mu\left\{\frac{2}{\sqrt{-g}}\nabla_\lambda(\sqrt{-g}f_Q P^{\lambda\mu}_{\ \ \nu})\right\}$$
$$= (\nabla_\mu - \tilde{L}_\mu)\left\{\frac{2}{\sqrt{-g}}\nabla_\lambda(\sqrt{-g}f_Q P^{\lambda\mu}_{\ \ \nu})\right\}$$
$$+ \frac{2L^\beta_{\ \nu\mu}}{\sqrt{-g}}\nabla_\lambda(\sqrt{-g}f_Q P^{\lambda\mu}_{\ \ \beta})$$
$$= \frac{2}{\sqrt{-g}}\nabla_\mu\nabla_\lambda(\sqrt{-g}f_Q P^{\lambda\mu}_{\ \ \nu})$$
$$+ 2L^\beta_{\ \nu\rho}(\nabla_\mu - \tilde{L}\mu)(f_Q P^{\mu\rho}_{\ \ \beta}), \tag{22}$$

where we have used the relation (10). In addition, we have the relations

$$2L^\beta_{\ \nu\rho}(\nabla_\mu - \tilde{L}\mu)(f_Q P^{\mu\rho}_{\ \ \beta})$$
$$= L_{\sigma\nu\rho}(\nabla_\mu - \tilde{L}\mu)(f_Q P^{\mu\rho\sigma}) + L^\beta_{\ \nu\rho}(f_Q P^{\mu\rho}_{\ \ \beta})Q_{\mu\sigma\beta} \tag{23}$$

$$\nabla_\nu Q = \mathring{\nabla}_\nu(P^{\mu\rho\sigma}Q_{\mu\rho\sigma}) = 2P^{\mu\rho\sigma}\mathring{\nabla}_\nu Q_{\mu\rho\sigma}. \tag{24}$$

Finally, (18) can be obtained after substituting (7) and the above relations into (20).

Now that we offer the fully covarinat formulation of the field equation (14), we can directly use the Bianchi identity to derive another equivalent form of the energy momentum balance equation by taking divergence of (14)

$$(\kappa - f_T)\mathring{\nabla}_\mu T^\mu_{\ \nu} - (\nabla_\mu f_T)T^\mu_{\ \nu}$$
$$- \mathring{\nabla}_\nu(f_T\Theta^\mu_{\ \nu}) + \frac{f_T}{2}\nabla_\nu Q$$

$$= (\nabla_\lambda f_Q)\left(\mathring{G}^\lambda_{\ \nu} + \frac{Q}{2}\delta^\lambda_{\ \nu} + 2\mathring{\nabla}_\mu P^{\lambda\mu}_{\ \ \nu}\right)$$
$$+ 2(\mathring{\nabla}_\mu\mathring{\nabla}_\lambda f_Q)P^{\lambda\mu}_{\ \ \nu}. \tag{25}$$

Note that (18) and (25) of the energy-balance in $f(Q,T)$ theory are identical once the affine connection field equation (7) are taken into account.

5 Cosmological application

In this section we explore some cosmological applications of the previously obtained covariant formulation and energy balance equation. For this purpose we consider the spatially flat homogeneous and isotropic FLRW spacetime with the line element given in Cartesian coordinates by

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]. \tag{26}$$

For simplicity, we utilise the usual vanishing affine connection in this coincident gauge choice to obtain the non-metricity scalar as

$$Q = -6H^2. \tag{27}$$

Using (14) and (26), we obtain the following Friedmann-like equations.

$$(\kappa + f_T)\rho + f_T p = \frac{f}{2} + 6H^2 f_Q, \tag{28}$$

$$\kappa p = -\frac{f}{2} - 6H^2 f_Q - \frac{\partial}{\partial t}(2Hf_Q). \tag{29}$$

When particularly $f_T = -\kappa$, we obtain the equation

$$\frac{\partial}{\partial t}(2Hf_Q) = 0, \tag{30}$$

whose solution is given as

$$f(Q,T) = \alpha\sqrt{-Q} - \kappa T, \tag{31}$$

where α is a constant. As this solution does not include GR, we conclude that this is not an adequate model. In what follows, we consider only $f_T \neq -\kappa$. Using (28) and (29), we obtain

$$\kappa\rho = \frac{f}{2} + 6H^2 f_Q + \frac{f_T}{\kappa + f_T}\frac{\partial}{\partial t}(2Hf_Q). \tag{32}$$

From the corrected energy balance Eq. (25) derived in the last section, we can write the continuity relation as

$$-\dot{\rho} - 3H(\rho + p) = -\frac{1}{\kappa}\frac{\partial}{\partial t}\left(\frac{f}{2} + 6H^2 f_Q\right)$$

$$-\frac{1}{\kappa + f_T}\left(\frac{\dot{f}_T}{\kappa + f_T} - 3H\right)\frac{\partial}{\partial t}(2Hf_Q)$$

$$-\frac{1}{\kappa}\frac{f_T}{\kappa + f_T}\frac{\partial^2}{\partial t^2}(2Hf_Q). \tag{33}$$

In particular, for pressureless dust era ($p = 0$) it reduces to

$$\dot{\rho} + 3H\rho = \frac{1}{\kappa + f_T}\left\{\frac{\dot{f}}{2} + 12\dot{H}Hf_Q + 6H^2\dot{f}_Q\right.$$

$$\left. + \left(3H - \frac{\dot{f}_T}{\kappa + f_T}\right)\left(\frac{f}{2} + 6H^2 f_Q\right)\right\}. \tag{34}$$

On the other hand, we can derive the effective energy density and pressure equations using (15)

$$\kappa\rho^{\text{eff}} = (\kappa + f_T)\rho + f_T p - \frac{f}{2} - 3H^2 f_Q, \tag{35}$$

$$\kappa p^{\text{eff}} = \kappa p + \frac{f}{2} + 3H^2 f_Q + 2H\dot{f}_Q. \tag{36}$$

Now that we pull the necessary things together, in the following subsections we are going to examine the dynamical behavior of some well known cosmological $f(Q,T)$ models by incorporating the phase space approach.

5.1 Linear $f(Q,T)$ model

We consider the following linear $f(Q,T)$ model

$$f(Q,T) = \alpha Q + \beta T \tag{37}$$

Here α and β are free model parameters. The cosmological implications of the considered model have been investigated in [29]. We are going to investigate the asymptotic behavior of the model. The Friedmann equations (29) and (32) corresponding to our linear $f(Q,T)$ model, for the dust case, becomes

$$(2\kappa + 3\beta)\rho = 6\alpha H^2 \tag{38}$$

$$2\dot{H} + 3H^2 = \frac{\beta}{2\alpha}\rho, \tag{39}$$

and the Eq. (34) becomes

$$\dot{\rho} + \frac{6(\kappa + \beta)}{(2\kappa + 3\beta)}H\rho = 0 \tag{40}$$

We define the following phase space variables

$$x = \frac{(2\kappa + 3\beta)\rho}{6\alpha H^2} \text{ and } y = \frac{1}{\frac{H_0}{H} + 1} \tag{41}$$

Then we have constraints $x = 1$ and $0 \leq y \leq 1$. Now we define $N = d\ln(a)$, then corresponding to our linear $f(Q,T)$ model we obtained the following autonomous system by using (38)–(40),

$$x' = \frac{dx}{dN} = \frac{3\beta}{(2\kappa + 3\beta)}x(1-x) \tag{42}$$

$$y' = \frac{dy}{dN} = \frac{3}{2}\left[\frac{\beta x}{(2\kappa + 3\beta)} - 1\right]y(1-y) \tag{43}$$

Now by the definition of deceleration and EoS parameter, we have for $x = 1$

$$q = -1 + \frac{3(\kappa + \beta)}{(2\kappa + 3\beta)} \text{ and } \omega = -1 + \frac{2(\kappa + \beta)}{(2\kappa + 3\beta)} \tag{44}$$

The nature of the critical points obtained by solving the above autonomous equations with $\kappa = 1$, are presented below in the Table 1.

The asymptotic behaviour of our linear $f(Q,T)$ model corresponding to the case $\beta = 0$ with $\kappa = 1$, presented below in Fig. 1.

For the case $\beta = 0$, the obtained critical points are P(1,1) and Q(1,0) with corresponding eigenvalues $0, \frac{3}{2}$ and $0, -\frac{3}{2}$ respectively. From Fig. 1, it is evident that the trajectories are emerging from the unstable past attractor P(1,1) and converging to future attractor Q(1,0). Hence, point P(1,1) is a source, whereas point Q(1,0) is a sink. From Eq. (44), we obtained $q = \frac{1}{2}$ and $\omega = 0$ for both P and Q. Therefore, we can conclude that our model $f(Q,T) = \alpha Q + \beta T$ with $\beta = 0$ remains lies in the matter dominated epoch and cannot describe the accelerated era as well as the initial singularity. Hence, it is completely equivalent to the GR case without cosmological constant Λ.

From the Eq. (44), one can notice that the linear model $f(Q,T) = \alpha Q + \beta T$ cannot describes the transition from decelerated to accelerated era for any parameter values.

5.2 Non-linear $f(Q,T)$ model

5.2.1 Model-I

We consider the following non-linear $f(Q,T)$ model

$$f(Q,T) = \alpha Q + \beta T^2 \tag{45}$$

Here α and $\beta \neq 0$ are free model parameters. The cosmological implications of the considered model have been investigated in [29]. We are going to investigate the asymptotic behavior of the model. The Friedmann equations (29) and (32) corresponding to our non-linear $f(Q,T)$ model, for the dust case, becomes

$$(2\kappa - 5\beta\rho)\rho = 6\alpha H^2 \tag{46}$$

$$2\dot{H} + 3H^2 = -\frac{\beta}{2\alpha}\rho^2 \tag{47}$$
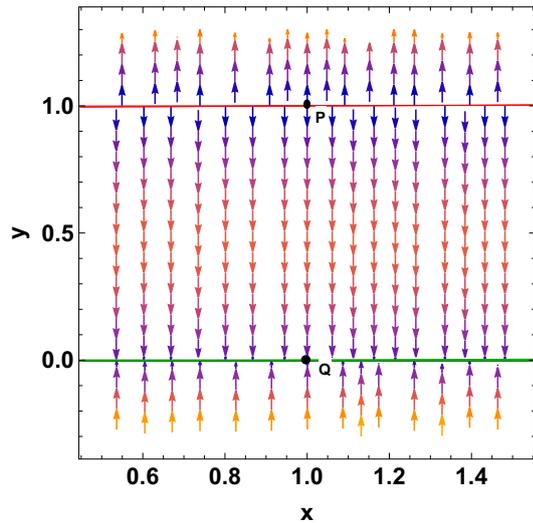
and the Eq. (34) becomes

$$\dot{\rho} + \frac{3(\kappa - 2\beta\rho)}{(\kappa - 5\beta\rho)}H\rho = 0 \tag{48}$$

We define the following phase space variables

$$x = \frac{\kappa\rho}{3\alpha H^2}, \quad y = \frac{1}{\frac{H_0}{H} + 1} \text{ and } z = -\frac{5\beta\rho^2}{6\alpha H^2} \tag{49}$$

Table 1 Table shows the critical points and their behavior corresponding to the model $f(Q, T) = \alpha Q + \beta T$

Critical points (x_c, y_c)	Eigenvalues λ_1 and λ_2	Nature of critical point	q	ω
$P(1, 1)$	$-\frac{3\beta}{(2+3\beta)}$ and $\frac{3(1+\beta)}{(2+3\beta)}$	Stable for $-1 < \beta < -\frac{2}{3}$	$\frac{1}{2+3\beta}$	$-\frac{\beta}{2+3\beta}$
$Q(1, 0)$	$-\frac{3(1+\beta)}{(2+3\beta)}$ and $-\frac{3\beta}{(2+3\beta)}$	Stable for $\beta < -1$ or $\beta \geq 0$	$\frac{1}{2+3\beta}$	$-\frac{\beta}{2+3\beta}$

Fig. 1 Phase plots corresponding to the case $\beta = 0$ with $\kappa = 1$

Then we have constraints $x + z = 1$ and $0 \leq y \leq 1$. Now corresponding to our non-linear $f(Q, T)$ model we obtained the following autonomous system with respect to the variable $N = dln(a)$, by using (46)–(48) and the constrain $x + z = 1$,

$$x' = \frac{dx}{dN} = \frac{3x(x-1)(x+4)}{5(x-2)} \tag{50}$$

$$y' = \frac{dy}{dN} = \frac{3}{10}y(y-1)(x+4) \tag{51}$$

Now by the definition of deceleration and EoS parameter, we have

$$q = \frac{1}{10}(3x + 2) \text{ and } \omega = \frac{1}{5}(x - 1) \tag{52}$$

The nature of the critical points obtained by solving the above autonomous equations with $\kappa = 1$, are presented below in the Table 2.

The asymptotic behaviour of our non-linear $f(Q, T)$ model with $\kappa = 1$, presented below in Fig. 2.

From Fig. 2, it is evident that the critical points A(0,0) and B(1,1) are saddle points, whereas the critical point C(1,0) indicates a stable matter dominated epoch. The critical point D(0,1) is a source while the critical point E(-4,y), for any y, is a sink representing a future attractor. The trajectories emerging from the D(0,1), indicating a decelerated epoch, are then converging to the point E(-4,y) representing an accelerated epoch. Thus we conclude that our non-linear f(Q,T) model

efficiently describes the observed transition from decelerated phase to an accelerated phase of the universe. The considered model $f(Q, T) = \alpha Q + \beta T^2$ with $\beta \neq 0$ behaves like standard ΛCDM model, and hence this model may represent an alternative to the ΛCDM.

5.2.2 Model-II

We consider the following non-linear $f(Q, T)$ model

$$f(Q, T) = Q + \alpha Q^2 + \beta T \tag{53}$$

Here $\alpha \neq 0$ and β are free model parameters. The astrophysical implications of the considered model have been investigated in [35]. We are going to examine the cosmological behavior of the model. The Friedmann equations (29) and (32) corresponding to this non-linear $f(Q, T)$ model, for the dust case, becomes

$$(2\kappa + 3\beta)\rho = 6H^2(1 - 18\alpha H^2) \tag{54}$$

$$2\dot{H}(1 - 36\alpha H^2) + 3H^2(1 - 18\alpha H^2) = \frac{\beta}{2}\rho \tag{55}$$

and the Eq. (34) becomes

$$\dot{\rho} + \frac{6(\kappa + \beta)}{(2\kappa + 3\beta)}H\rho = 0 \tag{56}$$

We define the following phase space variables

$$x = \frac{(2\kappa + 3\beta)\rho}{6H^2(1 - 18\alpha H^2)} \text{ and } y = \frac{1}{\frac{H_0}{H} + 1} \tag{57}$$

Then we have constraints $x = 1$ and $0 \leq y \leq 1$. Now we define $N = dln(a)$, then corresponding to our non-linear $f(Q, T)$ model we obtained the following autonomous system by using (54)–(56),

$$x' = \frac{dx}{dN} = \frac{3\beta}{(2\kappa + 3\beta)}x(1 - x) \tag{58}$$

$$y' = \frac{dy}{dN} = \frac{3(\kappa + \beta)}{(2\kappa + 3\beta)}\frac{[(1-y)^2 - \bar{\alpha}y^2]}{[(1-y)^2 - 2\bar{\alpha}y^2]}y(y - 1) \tag{59}$$

Here $\bar{\alpha} = 18\alpha H_0^2$.

Again, by the definition of deceleration and EoS parameter, we have

$$q = -1 + \frac{3(\kappa + \beta)[(1-y)^2 - \bar{\alpha}y^2]}{(2\kappa + 3\beta)[(1-y)^2 - 2\bar{\alpha}y^2]} \tag{60}$$

$$\omega = -1 + \frac{2(\kappa + \beta)[(1-y)^2 - \bar{\alpha}y^2]}{(2\kappa + 3\beta)[(1-y)^2 - 2\bar{\alpha}y^2]} \tag{61}$$

Table 2 Table shows the critical points and their behavior corresponding to the model $f(Q, T) = \alpha Q + \beta T^2$

Critical Points (x_c, y_c)	Eigenvalues λ_1 and λ_2	Nature of critical point	q	ω
$A(0, 0)$	$-\frac{6}{5}$ and $\frac{6}{5}$	Saddle	$\frac{1}{5}$	$-\frac{1}{5}$
$B(1, 1)$	-3 and $\frac{3}{2}$	Saddle	$\frac{1}{2}$	0
$C(1, 0)$	-3 and $-\frac{3}{2}$	Stable	$\frac{1}{2}$	0
$D(0, 1)$	$\frac{6}{5}$ and $\frac{6}{5}$	Unstable	$\frac{1}{5}$	$-\frac{1}{5}$
$E(-4, y)$	-2 and 0	Stable	-1	-1
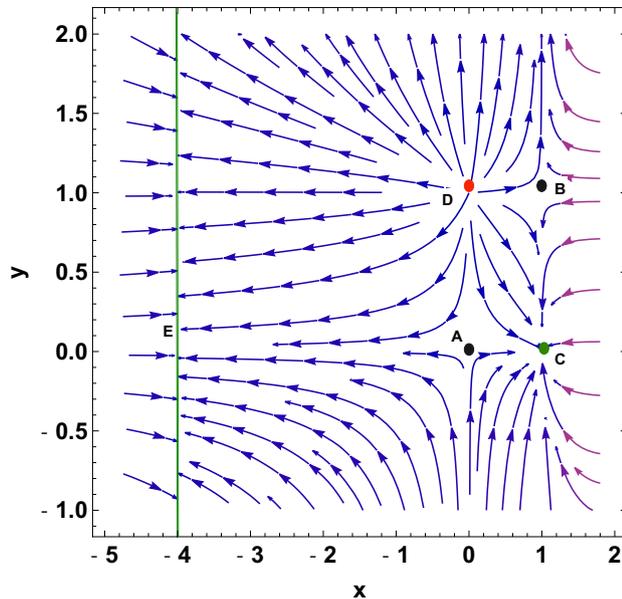
Fig. 2 Phase plot corresponding to the model $f(Q, T) = \alpha Q + \beta T^2$ with $\kappa = 1$

The nature of the critical points obtained by solving the above autonomous equations with $\kappa = 1$, are presented below in the Table 3.

The asymptotic behaviour of our non-linear $f(Q, T)$ model corresponding to the case $\beta = 0$ with $\kappa = 1$ and $\bar{\alpha} = 1$, presented below in Fig. 3.

For the case $\beta = 0$ with $\kappa = 1$ and $\bar{\alpha} = 1$, the obtained critical points are A(1,0), B(1,1), and C(1,0.5). From Table 3 and Fig. 3, it is evident that the critical points A(1,0) and C(1,0.5) with corresponding eigenvalues $-\frac{3}{2}, 0$ and $0, -3$ are stable and representing the matter dominated decelerated

epoch and accelerated de-Sitter epoch respectively, whereas the point B(1,1) with eigenvalues $\frac{3}{4}$, 0 is unstable.

6 Concluding remarks

In the present article we have examined the matter-geometry coupling in the form of the $f(Q, T)$ theory which shows promising development and rising research interest in the cosmological sector. We have derived an equivalent covariant formulation of the theory which yields an explicit comparison with the GR. The non-conservation of the energy-momentum tensor in this theory evokes a generally non-zero covariant vector which has been obtained from the divergence of this covariant formula of the field equation, and by using the Bianchi identity. In account of this result, we have computed the modified continuity relation, specific to this theory. The covariant formulation of the theory has also helped us to efficiently express the effective pressure and energy terms. Further, we have considered three different functional forms of the $f(Q, T)$ function, specifically, $f(Q, T) = \alpha Q + \beta T$, $f(Q, T) = \alpha Q + \beta T^2$, and $f(Q, T) = Q + \alpha Q^2 + \beta T$. These $f(Q, T)$ functions frequently appear in the literature. We have made an attempt to investigate the physical capabilities of the studied models to describe various cosmological epochs. We have incorporated the dynamical system technique to investigate the asymptotic behavior of the considered $f(Q, T)$ models. The obtained outcomes for the assumed linear and two non-linear $f(Q, T)$ models have been presented in Tables 1, 2 and 3, and the corresponding phase plots in Fig. 1, 2 and 3. Naturally, we have observed that the linear model $f(Q, T) = \alpha Q + \beta T$ with $\beta = 0$ is completely equivalent to the GR case without cosmological constant Λ. Further, we have found that the model $f(Q, T) = \alpha Q + \beta T^2$

Table 3 Table shows the critical points and their behavior corresponding to the model $f(Q, T) = Q + \alpha Q^2 + \beta T$

Critical Points (x_c, y_c)	Eigenvalues λ_1 and λ_2	Nature of critical point	q	ω
$A(1, 0)$	$-\frac{3(1+\beta)}{(2+3\beta)}$ and $-\frac{3\beta}{(2+3\beta)}$	Stable for $\beta < -1$ or $\beta \geq 0$	$\frac{1}{2+3\beta}$	$-\frac{\beta}{2+3\beta}$
$B(1, 1)$	$-\frac{3\beta}{(2+3\beta)}$ and $\frac{3(1+\beta)}{2(2+3\beta)}$	Stable for $-1 < \beta < -\frac{2}{3}$	$-\frac{1+3\beta}{2(2+3\beta)}$	$-\frac{1+2\beta}{(2+3\beta)}$
$C(1, \frac{1}{1+\sqrt{\alpha}}), \bar{\alpha} > 0$	$-\frac{3\beta}{(2+3\beta)}$ and $-\frac{6(1+\beta)}{(2+3\beta)}$	Stable for $\beta < -1$ or $\beta \geq 0$	-1	-1
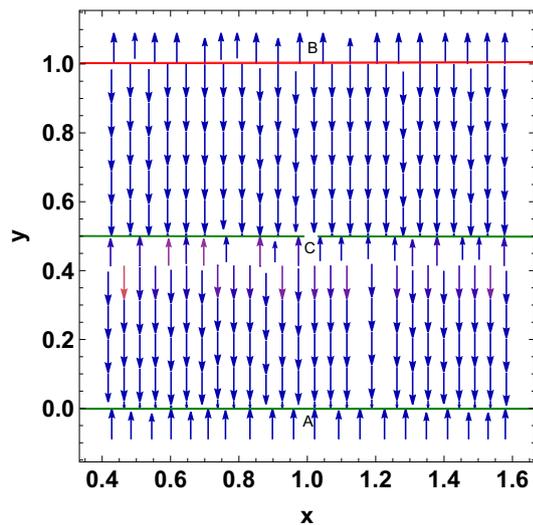
Fig. 3 Phase plots corresponding to the case $\beta = 0$ with $\kappa = 1$ and $\bar{\alpha} = 1$

with $\beta \neq 0$ successfully describes the observed transition from decelerated phase to an accelerated phase of the universe and hence the considered non-linear model I behaves like standard ΛCDM model. Moreover, our obtained results agrees with the cosmological implications of the same model found in [29]. Lastly, we found that the non-linear model II $f(Q, T) = Q + \alpha Q^2 + \beta T$ with $\alpha \neq 0$ represents an accelerated de-Sitter epoch for the constraint $\beta < -1$ or $\beta \geq 0$. Moreover, a stable matter dominated epoch with an accelerated de-Sitter type epoch can be obtained in the absence of energy momentum scalar term T i.e., for the case $\beta = 0$. From the basis of our findings we can conclude that both considered non-linear models can efficiently predicts the de-Sitter type expansion of the universe and may represent a viable geometric alternative to dark energy.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and so no observational data was required.]

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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