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Charged Gauss–Bonnet black holes supporting non-minimally coupled scalar clouds: analytic treatment in the near-critical regime

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Abstract Recent numerical studies have revealed the physically intriguing fact that charged black holes whose charge-to-mass ratios are larger than the critical value $(Q/M)_{\text{crit}} = \sqrt{2(9+\sqrt{6})/5}$ can support hairy matter configurations which are made of scalar fields with a non-minimal negative coupling to the Gauss–Bonnet invariant of the curved spacetime. Using *analytical* techniques, we explore the physical and mathematical properties of the composed charged-black-hole-nonminimally-coupled-linearized-massless-scalar-field configurations in the near-critical $Q/M \gtrsim (Q/M)_{\text{crit}}$ regime. In particular, we derive an analytical resonance formula that describes the charge-dependence of the dimensionless coupling parameter $\bar{\eta}_{\text{crit}} = \bar{\eta}_{\text{crit}}(Q/M)$ of the composed Einstein–Maxwell-nonminimally-coupled-scalar-field system along the *existence-line* of the theory, a critical border that separates bald Reissner–Nordström black holes from hairy charged-black-hole-scalar-field configurations. In addition, it is explicitly shown that the large-coupling $-\bar{\eta}_{\text{crit}}(Q/M) \gg 1$ analytical results derived in the present paper for the composed Einstein–Maxwell-scalar theory agree remarkably well with direct numerical computations of the corresponding black-hole-field resonance spectrum.

1 Introduction

Early no-hair theorems [1–7], which were mainly motivated by Wheeler's no-hair conjecture for black holes [8,9], have revealed the physically interesting fact that black holes with spatially regular horizons cannot support static matter configurations which are made of scalar fields.

However, recent studies [10–14] (see also [15–18]) have explicitly demonstrated that the no-hair conjecture can be violated in generalized Einstein-scalar field theories whose actions are characterized by a direct non-trivial coupling $f(\phi)\mathcal{G}$ between the scalar field ϕ and the Gauss–Bonnet invariant $\mathcal{G}$ of the curved spacetime. In particular, it has been revealed in the physically important works [10–14] that spatially regular matter configurations which are made of non-minimally coupled scalar fields can be supported in asymptotically flat black-hole spacetimes[19–25].[1]

The physically intriguing phenomenon of spontaneous scalarization of black holes in generalized Einstein–Gauss–Bonnet-scalar field theories [10–18] owe its existence to the presence of an effective spatially-dependent mass term of the linearized form $-\bar{\eta}\mathcal{G}$, which reflects the direct non-trivial coupling between the scalar field and the Gauss–Bonnet curvature invariant, in the Klein–Gordon wave equation of the supported scalar configurations [see Eq. (10) below]. The dimensionless physical parameter $\bar{\eta}$ of the composed field theory controls the strength of the direct non-minimal interaction between the scalar field and the spatially-dependent Gauss–Bonnet invariant of the curved spacetime.

The spontaneous scalarization phenomenon of *charged* black holes in composed Einstein–Maxwell–Gauss–Bonnet-scalar-field theories has been explored, using numerical techniques, in the physically interesting works [26,27] (see [15,16,28–33] for recent studies of the spontaneous scalarization phenomenon of asymptotically flat spinning black holes). Intriguingly, it has been revealed in [26,27] that the composed

[1] A physically interesting related phenomenon, in which matter configurations that are made of scalar fields with a direct non-trivial (non-minimal) coupling to the Maxwell electromagnetic invariant are supported in charged black-hole spacetimes, has been revealed in [19,20] (see also [21–24] and references therein).

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charged-black-hole-nonminimally-coupled-scalar-field system is characterized by a charge-dependent *existence-line* $\bar{\eta} = \bar{\eta}(\bar{Q})$[2] which marks the onset of the spontaneous scalarization phenomenon in the generalized Einstein–Maxwell-scalar field theory. In particular, for a given value of the black-hole electric charge $\bar{Q}$, the critical existence line marks the boundary between bald Reissner–Nordström black holes and composed charged-black-hole-nonminimally-coupled-scalar-field hairy configurations.

The charge-dependent critical existence-line $\bar{\eta} = \bar{\eta}(\bar{Q})$ of the generalized Einstein–Maxwell-scalar theory is composed of charged Reissner–Nordström black holes that support scalar 'clouds' [34–36], spatially regular matter configurations which are made of the non-trivially coupled linearized scalar fields. The characteristic existence-line of the physical system is universal in the sense that different non-trivially coupled Einstein–Maxwell-scalar field theories that share the same weak-field functional behavior $f(\phi) = 1 + \bar{\eta}\phi^2/2 + O(\phi^4)$ [26,27] of the coupling function are characterized by the same critical boundary between bald and hairy black-hole spacetimes.

Interestingly, the numerical results presented in [26, 27] have revealed that the spontaneous scalarization phenomenon of black holes in composed Einstein–Maxwell-scalar field theories with *negative* values of the non-minimal coupling parameter $\bar{\eta}$ may be induced by the electric charge of the supporting black hole. In particular, one finds that, in the $\bar{\eta} < 0$ regime, the onset of the spontaneous scalarization phenomenon is marked by the dimensionless black-hole electric charge [26,27]

$$\bar{Q}_{\text{crit}} = \frac{\sqrt{2(9+\sqrt{6})}}{5} \ . \tag{1}$$

Only charged black holes with $\bar{Q} \geq \bar{Q}_{\text{c}}$ can support non-minimally coupled spatially regular scalar clouds in the negative coupling $\bar{\eta} < 0$ regime. Intriguingly, it has been demonstrated numerically in [26,27] that, in the near-critical regime $\bar{Q}/\bar{Q}_{\text{crit}} \to 1^+$, the hairy charged black holes are characterized by the large-coupling asymptotic relation

$$-\bar{\eta} \to \infty \quad \text{for} \quad \bar{Q}/\bar{Q}_{\text{crit}} \to 1^+ \ . \tag{2}$$

The main goal of the present paper is to study, using *analytical* techniques, the physical and mathematical properties of the composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-scalar-field cloudy configurations in the near-critical regime $\bar{Q} \gtrsim \bar{Q}_{\text{crit}}$. In particular, a remarkably compact WKB resonance formula, which describes the charge-dependence $\bar{\eta} = \bar{\eta}(\bar{Q})$ of the characteristic critical existence-line of the composed Einstein–Maxwell–Gauss–Bonnet-nonminimally-coupled-massless-scalar-field theory in the dimensionless large-coupling $-\bar{\eta} \gg 1$ regime, will be derived.

Interestingly, we shall explicitly show below that the analytically derived near-critical resonance formula of the present paper [see Eq. (40) below] provides a simple *analytical* explanation for the *numerically* observed [26,27] monotonically decreasing functional behavior $|\bar{\eta}| = |\bar{\eta}(\bar{Q})|$ of the critical existence-line that characterizes, in the $\bar{\eta} < 0$ regime, the composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-linearized-scalar-field configurations.

2 Description of the system

We explore the physical and mathematical properties of spatially regular scalar 'clouds' (linearized scalar field configurations) which are supported by charged Reissner–Nordström black holes. The supported massless scalar fields are characterized by a direct non-trivial (non-minimal) coupling to the Gauss–Bonnet invariant of the curved charged spacetime. The action of the composed Einstein–Maxwell–Gauss–Bonnet-nonminimally-coupled-massless-scalar-field system is given by the expression [27][3]

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{4}R - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi + f(\phi)\mathcal{G}\Big] \ , \tag{3}$$

where

$$\mathcal{G} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \tag{4}$$

is the Gauss–Bonnet curvature invariant.

Following [13,14,19], we assume that the scalar function $f(\phi)$, which controls the non-trivial direct coupling of the scalar field to the Gauss–Bonnet curvature invariant, is characterized by the leading order universal functional behavior

$$f(\phi) = \frac{1}{2}\eta\phi^2 \tag{5}$$

in the weak-field regime. As discussed in [13,14,19], the functional behavior (5) of the scalar coupling function in the weak scalar field regime guarantees that the Einstein-matter field equations are satisfied by the familiar scalarless black-hole solutions of general relativity (in our case, the

[2] Here $\bar{Q} \equiv Q/M$, where $\{M, Q\}$ are respectively the ADM mass and electric charge of the black-hole spacetime. We shall assume, without loss of generality, the relation $\bar{Q} > 0$ for the black-hole electric charge.

[3] We use natural units in which $8\pi G = c = \hbar = 1$.

Reissner–Nordström black-hole spacetime) in the $\phi \to 0$ limit. The physical parameter η,[4] which may take either positive or negative values [26,27], determines the strength of the direct (non-minimal) interaction between the spatially regular massless scalar field configurations and the Gauss–Bonnet invariant (4) of the charged curved spacetime.

The supporting charged Reissner–Nordström black-hole spacetime is characterized by the line element [37,38][5]

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{6}$$

where the metric function Δ is given by the functional expression

$$\Delta \equiv r^2 - 2Mr + Q^2\ . \tag{7}$$

The physical parameters $\{M, Q\}$ are respectively the mass and electric charge of the central supporting Reissner–Nordström black hole. The horizon radii of the curved black-hole spacetime (6) are determined by the zeros of the metric function Δ:

$$r_{\pm} = M \pm (M^2 - Q^2)^{1/2}\ . \tag{8}$$

The radially-dependent Gauss–Bonnet invariant, which characterizes the curved Reissner–Nordström black-hole spacetime (6), is given by the expression [27]

$$\mathcal{G}_{\text{RN}}(r; M, Q) = \frac{8}{r^8}\big(6M^2r^2 - 12MQ^2r + 5Q^4\big)\ . \tag{9}$$

A variation of the action (3) with respect to the scalar field yields the generalized Klein–Gordon equation [27]

$$\nabla^{\nu}\nabla_{\nu}\phi = \mu^2_{\text{eff}}\phi \tag{10}$$

for the non-minimally coupled scalar field with the radially-dependent effective mass term

$$\mu^2_{\text{eff}}(r; M, Q) = -\eta\mathcal{G}\ . \tag{11}$$

This effective mass term reflects the direct (non-minimal) coupling of the scalar field ϕ to the Gauss–Bonnet invariant $\mathcal{G}$ [see Eq. (9)] of the charged curved spacetime.

Interestingly, one finds that, in the regime [27][6]

$$Q \geq Q_{\text{crit}} = M \cdot \frac{\sqrt{2(9+\sqrt{6})}}{5}\ , \tag{12}$$

the charged-dependent Gauss–Bonnet curvature invariant becomes negative in the exterior region

$$r_+ \leq r \leq \Big(1 + \frac{1}{\sqrt{6}}\Big) \cdot \frac{Q^2}{M} \tag{13}$$

of the charged black-hole spacetime. Thus, the spatially-dependent effective mass term (11) with $\eta < 0$ may become negative in the interval (13). As we shall explicitly prove below, this intriguing property of the coupled Einstein–Maxwell-nonminimally-coupled-scalar-field system (3) allows the central charged Reissner–Nordström black hole (6) to support bound-state linearized cloudy configurations of the non-minimally coupled scalar field.

Using the functional field decomposition[7]

$$\phi(r, \theta, \varphi) = \sum_{lm} R_{lm}(r)Y_{lm}(\theta, \varphi)\ , \tag{14}$$

one obtains from Eq. (10) the differential equation [27],[8][9]

$$\frac{d}{dr}\Big(\Delta\frac{dR}{dr}\Big) - l(l+1)R + \eta\Big(\frac{48M^2}{r^4} - \frac{96MQ^2}{r^5} + \frac{40Q^4}{r^6}\Big)R = 0 \tag{15}$$

for the radial part of the linearized non-minimally coupled scalar field in the supporting curved black-hole spacetime (6).

The ordinary differential equation (15) determines the spatial behavior of the static non-minimally coupled linearized massless scalar field configurations in the supporting charged Reissner–Nordström black-hole spacetime (6). Following [26,27], we shall assume that the scalar eigenfunction $\psi(r)$ of the bound-state field configurations is spatially well behaved with the physically motivated boundary conditions [26,27]

$$\psi(r = r_{\text{H}}) < \infty; \quad \psi(r \to \infty) \to 0 \tag{16}$$

at the black-hole horizon and at spatial infinity.

In the next section we shall study, using analytical techniques, the physical and mathematical properties of the negatively coupled charged-black-hole-nonminimally-coupled-linearized-massless-scalar-field configurations that characterize the composed Einstein–Maxwell-scalar field theory

[4] The coupling parameter η of the composed field theory (3) has the dimensions of length^2.

[5] Here (t, r, θ, ϕ) are the familiar Schwarzschild spacetime coordinates.

[6] Note that the critical electric charge (12) corresponds to the critical radius $r^c_+ = M \cdot [(4+\sqrt{6})/5]$ of the black-hole outer horizon [see Eq. (8)].

[7] Here $Y_{lm}(\theta, \varphi)$ are the spherical harmonic functions whose angular eigenvalues are given by the familiar expression $l(l+1)$ with the property $l \geq |m|$.

[8] For brevity, we shall henceforth omit the angular field parameters $\{l, m\}$.

[9] Note that the non-minimal coupling parameter η of the composed Einstein–Maxwell–Gauss–Bonnet-nonminimally-coupled-massless-scalar-field theory (3) is related to the non-minimal coupling parameter λ of [27] by the relation $\eta = \lambda^2/4$ [see Eq. (15) and Eq. (3.27) of [27]].

(3). In particular, we shall explicitly prove that the composed Reissner–Nordström-black-hole-scalar-field cloudy configurations in the $Q > Q_{\rm crit}$ regime [see Eq. (12)] with $\eta < 0$ owe their existence to the presence of an effective near-horizon binding potential well [see Eq. (31) below].

3 Composed Reissner–Nordström-black-hole-nonminimally-coupled-scalar-field configurations: a WKB analysis

In the present section we shall determine the charge-dependent functional behavior $\eta = \eta(Q/M)$ of the critical existence-line, which characterizes the composed charged-black-hole-nonminimally-coupled-scalar-field cloudy configurations, in the dimensionless large-coupling regime

$$-\bar{\eta} \equiv -\frac{\eta}{M^2} \gg 1 \; . \tag{17}$$

Defining the new radial eigenfunction

$$\psi \equiv rR \tag{18}$$

and using the coordinate $y(r)$, which is defined by the radial differential relation[10]

$$dy = \frac{r^2}{\Delta}\cdot dr \; , \tag{19}$$

one obtains from Eq. (15) the Schrödinger-like radial equation

$$\frac{d^2\psi}{dy^2} - V\psi = 0 \; , \tag{20}$$

where the radially-dependent potential $V[r(y)]$ of the composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-scalar-field system is given by the functional expression

$$V(r; M, Q, l, \bar{\eta}) = h(r)\Big[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\Big] + V_{\rm GB}(r) \tag{21}$$

with [see Eq. (7)]

$$h(r) \equiv \frac{\Delta}{r^2} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \; . \tag{22}$$

The presence of the non-trivial term

$$V_{\rm GB}(r; M, Q) = -\bar{\eta}\cdot\frac{h(r)}{r^2}\Big(\frac{48M^4}{r^4} - \frac{96M^3Q^2}{r^5} + \frac{40M^2Q^4}{r^6}\Big) \tag{23}$$

in the effective interaction potential (21) is a direct consequence of the non-trivial (non-minimal) coupling between the Gauss–Bonnet curvature invariant (9) of the charged spacetime (6) and the supported scalar field.

We shall now prove that, in the dimensionless large-coupling regime (17), the ordinary differential equation (20), which characterizes the composed Einstein–Maxwell-scalar field theory (3), is amenable to an analytical treatment. In particular, a standard second-order WKB analysis of the Schrödinger-like ordinary differential equation (20) yields the familiar quantization condition [39–41]

$$\int_{y_{t-}}^{y_{t+}} dy\sqrt{-V(y; M, Q, \bar{\eta})} = \big(n + \tfrac{1}{2}\big)\cdot\pi \; ; \quad n = 0, 1, 2, \ldots \tag{24}$$

for the discrete spectrum $\{\bar{\eta}(M, Q; n\}_{n=0}^{n=\infty}$ of the dimensionless coupling parameter which characterizes the composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-scalar-field cloudy configurations. Here $n \in \{0, 1, 2, \ldots\}$ is the discrete resonant parameter of the physical system. The critical existence-line of the field theory is determined by the fundamental $n = 0$ mode. The integration limits $\{y_{t-}, y_{t+}\}$ in the WKB integral relation (24) are the classical turning points of the effective binding potential (21). Using Eq. (19), one finds that the WKB relation (24) can be expressed in the integral form

$$\int_{r_{t-}}^{r_{t+}} dr\frac{\sqrt{-V(r; M, Q, \bar{\eta})}}{h(r)} = \big(n + \tfrac{1}{2}\big)\cdot\pi \; ; \quad n = 0, 1, 2, \ldots . \tag{25}$$

Defining the dimensionless physical parameters

$$Q \equiv Q_{\rm crit}\cdot(1 + \epsilon); \quad \epsilon \geq 0 \tag{26}$$

and

$$r \equiv r_+\cdot(1 + x); \quad x \geq 0 \; , \tag{27}$$

one obtains the near-critical ($\epsilon \ll 1$) near-horizon ($x \ll 1$) relations [see Eqs. (8), (12), and (22)]

$$\frac{r_+}{M} = \frac{4 + \sqrt{6}}{5}\cdot\big(1 - \sqrt{6}\cdot\epsilon\big) + O(\epsilon^2) \; , \tag{28}$$

[10] Note that the differential relation (19) maps the semi-infinite regime $[r_+, \infty]$ of the Schwarzschild radial coordinate r into the infinite regime $[-\infty, \infty]$ of the new radial coordinate y.

$$\frac{r}{M}=\frac{r_+\cdot(1+x)}{M}=\frac{4+\sqrt{6}}{5}\cdot\big(1+x-\sqrt{6}\cdot\epsilon\big)+O(x^2,\epsilon^2,x\epsilon)\ , \tag{29}$$

and

$$h(r)=(\sqrt{6}-2)\cdot x+O(x^2,\epsilon^2,x\epsilon)\ . \tag{30}$$

Substituting the relations (28), (29), and (30) into Eqs. (21) and (23), one finds the (rather cumbersome) near-critical near-horizon expression

$$M^2\frac{V(r)}{h(r)} =\Big\{\frac{11-4\sqrt{6}}{2}\cdot l(l+1)+\frac{19\sqrt{6}-46}{2} -\bar{\eta}\Big[\big(15204\sqrt{6}-37236\big)\cdot x-\big(16752-6828\sqrt{6}\big)\cdot \epsilon\Big]\Big\}\cdot[1+O(x,\epsilon)] \tag{31}$$

for the effective interaction potential of the composed black-hole-massless-scalar-field cloudy configurations.

We shall henceforth consider composed charged -Reissner–Nordström-black-hole-massless-scalar-field cloudy configurations in the dimensionless large-coupling regime [see Eqs. (17), (26), and Eq. (38) below]

$$-\bar{\eta}\epsilon\gg1\ , \tag{32}$$

in which case one finds from Eqs. (30) and (31) the remarkably compact leading order functional expression

$$M^2\frac{V(r)}{[h(r)]^2}=\bar{\eta}\cdot6(1396-569\sqrt{6})\cdot\Big[(2+\sqrt{6})\cdot\frac{\epsilon}{x}-1\Big] \tag{33}$$

for the effective binding potential of the composed charged-black-hole-scalar-field system.

Taking cognizance of Eqs. (25), (27), and (33), one finds the resonance condition

$$\frac{r_+}{M}\cdot\int_0^{(2+\sqrt{6})\cdot\epsilon}dx\sqrt{-\bar{\eta}\cdot6(1396-569\sqrt{6})\cdot\Big[(2+\sqrt{6})\cdot\frac{\epsilon}{x}-1\Big]} =\big(n+\frac{1}{2}\big)\cdot\pi;\quad n=0,1,2,\ldots \tag{34}$$

for the composed black-hole-field system. The WKB integral relation (34) can be expressed in the compact mathematical form

$$\epsilon\sqrt{-\bar{\eta}}\cdot2\sqrt{6}\sqrt{16+\sqrt{6}}\int_0^1dz\sqrt{\frac{1}{z}-1} =\Big(n+\frac{1}{2}\Big)\cdot\pi;\quad n=0,1,2,\ldots \tag{35}$$

where

$$z\equiv\frac{1}{2+\sqrt{6}}\cdot\frac{x}{\epsilon}\ . \tag{36}$$

Using the integral relation

$$\int_0^1dz\sqrt{\frac{1}{z}-1}=\frac{\pi}{2}\ , \tag{37}$$

one finds from (35) the remarkably simple discrete resonance formula[11]

$$\bar{\eta}(\epsilon;n)=-\frac{1}{\epsilon^2}\cdot\frac{1}{96+6\sqrt{6}}\cdot\Big(n+\frac{1}{2}\Big)^2;\quad n=0,1,2,\ldots, \tag{38}$$

which characterizes the composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-linearized -massless-scalar-field cloudy configurations in the dimensionless large-coupling $-\bar{\eta}\gg1$ regime (or equivalently, in the near-critical $\epsilon\ll1$ regime).

The discrete resonance spectrum (38) of the composed Einstein–Maxwell–Gauss–Bonnet-nonminimally-coupled -scalar-field system (3) can be expressed in the dimensionless form [see Eqs. (1) and (26)]

$$\bar{Q}(\bar{\eta};n)=\frac{\sqrt{2(9+\sqrt{6})}}{5} +\frac{1}{\sqrt{-\bar{\eta}}}\cdot\frac{1}{\sqrt{138-7\sqrt{6}}}\cdot\Big(n+\frac{1}{2}\Big) \quad n=0,1,2,\ldots \tag{39}$$

where $\bar{Q}\equiv Q/M$.

4 Numerical confirmation

In the present section we shall test the accuracy of the analytically derived large-coupling resonance spectrum (38), which characterizes the composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-linearized-scalar-field cloudy configurations. The charge-dependent resonance spectrum of the black-hole-field system has recently been computed numerically in [27].

[11] It is important to point out that, in accord with the assumed strong inequality (32), one finds from Eq. (38) the characteristic scaling relation $\bar{\eta}\epsilon=O(\epsilon^{-1})\gg1$ for the dimensionless coupling parameter of the composed Einstein–Maxwell–Gauss–Bonnet-nonminimally-coupled-massless-scalar-field theory in the near-critical $\epsilon\ll1$ regime. In particular, from the analytically derived resonance formula (38) one finds that, for the fundamental ($n=0$) resonant mode, the assumed strong inequality (32) is satisfied in the near-critical regime $\epsilon\ll[4(96+6\sqrt{6})]^{-1}$, or equivalently in the dimensionless large-coupling regime $-\bar{\eta}\gg4(96+6\sqrt{6})$.

Table 1 Composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-linearized-massless-scalar-field cloudy configurations. We present, for various values of the coupling parameter $\lambda \equiv 2\sqrt{|\bar{\eta}|}$ [27] (see footnote 9) of the theory the dimensionless ratio $\mathcal{R}(\lambda) \equiv \epsilon^{\text{analytical}}/\epsilon^{\text{numerical}}$ between the analytically calculated value of the dimensionless critical parameter ϵ [as calculated directly from the resonance formula (38) for the fundamental $n = 0$ mode] and the corresponding exact values of the critical parameter as computed numerically in [27]. One finds that, in the large-coupling $\lambda \gg 1$ regime of the composed Reissner–Nordström-black-hole-scalar-field cloudy configurations, the agreement between the analytically derived resonance formula (38) and the corresponding numerically computed resonance spectrum of [27] is remarkably good (see footnote 12)

λ	5	10	15	20	25
$\mathcal{R}(\lambda) \equiv \frac{\epsilon^{\text{analytical}}}{\epsilon^{\text{numerical}}}$	1.168	1.065	1.049	1.019	1.016

In Table 1 we present, for various values of the dimensionless coupling parameter $\lambda \equiv 2\sqrt{|\bar{\eta}|}$ (see footnote 9) used in [27], the charge-dependent ratio $\mathcal{R}(\lambda) \equiv \epsilon^{\text{analytical}}/\epsilon^{\text{numerical}}$ between the analytically calculated value of the dimensionless critical parameter ϵ [as calculated directly from the analytically derived large-coupling resonance formula (38)] and the corresponding exact (numerically computed [27]) values of the critical parameter. The data presented in Table 1 for the cloudy Reissner–Nordström-black-hole-scalar-field configurations reveals the fact that the agreement between the *analytically* derived resonance formula (38) and the corresponding *numerically* computed resonance spectrum of [27] is remarkably good in the large-coupling $\lambda \gg 1$ regime [12] of the Einstein–Maxwell-scalar field theory (3).

5 Summary and discussion

Asymptotically flat black holes with spatially regular horizons can support bound-state matter configurations which are made of scalar fields with a direct (non-minimal) coupling to the Gauss–Bonnet invariant of the curved spacetime [10–18].

The spontaneous scalarization phenomenon of charged black holes in composed Einstein–Maxwell–Gauss–Bonnet-scalar field theories has recently been studied numerically in the physically important works [26,27]. In particular, it has been revealed in [26,27] that a charge-dependent critical existence-line $\bar{\eta}_{\text{crit}} = \bar{\eta}_{\text{crit}}(Q/M)$ separates bald Reissner–Nordström black-hole spacetimes from the composed charged-black-hole-nonminimally-coupled-massless-scalar-field hairy configurations of the Einstein–Maxwell-scalar theory (3), [13] where the dimensionless physical parameter $\bar{\eta}$ quantifies the strength of the direct non-trivial interaction between the supported scalar field and the Gauss–Bonnet curvature invariant.

Interestingly, it has been demonstrated [26,27] that, in the negative coupling $\bar{\eta} < 0$ regime, the composed charged-black-hole-linearized-scalar-field cloudy configurations that sit on the critical existence-line of the Einstein–Maxwell-scalar field theory (3) are restricted to the dimensionless charge regime $\bar{Q} > \bar{Q}_{\text{crit}} = \sqrt{2(9+\sqrt{6})/5}$ [see Eq. (1)]. In particular, the numerical results presented in [26,27] provide important evidence for an intriguing divergent functional behavior $-\bar{\eta}(\bar{Q}) \to \infty$ of the non-minimal coupling parameter of the theory along the existence-line of the system in the near-critical limit $\bar{Q}/\bar{Q}_{\text{crit}} \to 1^{+}$.

In the present paper we have used *analytical* techniques in order to explore the physical and mathematical properties of the composed charged-Reissner–Nordström-black-hole-nonminimally-coupled-massless-scalar-field hairy configurations of the Einstein–Maxwell-scalar theory (3) in the large-coupling $-\bar{\eta}(\bar{Q}) \gg 1$ regime. In particular, using a WKB procedure we have derived the analytical formula (38) for the discrete resonant spectrum of the dimensionless coupling parameter $\bar{\eta}(\bar{Q})$ which characterizes the composed charged-black-hole-scalar-field cloudy configurations in the near-critical $Q \gtrsim Q_{\text{crit}}$ regime.

The analytically derived discrete WKB resonance formula (38) yields the remarkably compact charge-dependent functional relation [14]

$$\bar{\eta}(\epsilon) = -\frac{1}{\epsilon^2} \cdot \frac{1}{4(96+6\sqrt{6})} \tag{40}$$

for the critical existence-line of the composed Einstein–Maxwell–Gauss–Bonnet-nonminimally-coupled-massless-scalar field theory (3), where $0 \leq \epsilon = (\bar{Q} - \bar{Q}_{\text{crit}})/\bar{Q}_{\text{crit}} \ll 1$ [see Eqs. (1) and (26)] is the dimensionless distance of the system from the exact critical ($-\bar{\eta} \to \infty$) configuration.

It is worth stressing the fact that the physical significance of the critical existence-line (40) in the Einstein–Maxwell-scalar field theory (3) stems from the fact that it separates, in the large-coupling $-\bar{\eta} \gg 1$ regime, bald Reissner–Nordström black holes from hairy charged-black-

[12] Interestingly, from the data presented in Table 1 one finds that the agreement between the analytically derived resonance formula (38) and the interesting numerical results presented in [27] for the cloudy black-hole-field configurations is quite good already in the $\bar{\eta}\epsilon = O(1)$ regime.

[13] The critical existence-line, which characterizes the composed Einstein–Maxwell–Gauss–Bonnet-scalar field theory (3), corresponds to spatially regular non-minimally coupled linearized scalar field configurations (scalar clouds) that are supported by central charged Reissner–Nordström black holes.

[14] Note that the critical existence-line (40) of the composed Einstein–Maxwell–Gauss–Bonnet-scalar field theory (3) corresponds to the fundamental ($n = 0$) mode of the analytically derived discrete resonance spectrum (38).

hole-nonminimally-coupled-massless-scalar-field configurations. From the analytically derived resonance formula (40) one finds, in accord with the physically interesting numerical results presented in [26,27], that the dimensionless coupling parameter $|\bar{\eta}(\bar{Q})|$ of the theory (3) is a monotonically decreasing function of the black-hole electric charge $\bar{Q}$ in the negative coupling $\bar{\eta} < 0$ regime.

Finally, it is interesting to point out that our results provide a simple *analytical* explanation for the physically intriguing *numerical* observation originally made in [26,27], according to which the charge-dependent coupling parameter $\bar{\eta}(\bar{Q})$ of the Einstein–Maxwell-scalar field theory (3) diverges along the critical existence line of the system in the $\bar{Q} \to \bar{Q}_{\text{crit}}$ limit. In particular, the analytically derived resonance formula (40) reveals the fact that the coupling parameter $\bar{\eta}(\bar{Q})$, which characterizes the cloudy black-hole-field configurations, diverges quadratically in the near-critical $\epsilon \to 0$ limit.

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