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Cosmological dynamics of Cuscuta–Galileon gravity

Sirachak Panpanich[a], **Kei-ichi Maeda**[b]

Department of Pure and Applied Physics, Graduate School of Advanced Science and Engineering, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan

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Abstract We study cosmological dynamics of the Cuscuta–Galileon gravity with a potential term by using the dynamical system approach. This model is galileon generalization of the cuscuton gravity where we add a potential term to the theory in order to obtain the radiation and matter dominated eras. The exponential potential can provide the sequence of the thermal history of the Universe correctly, i.e. starting from radiation dominance, passing through matter dominant era, and then approaching de Sitter expansion stage. This model has no ghosts and the Laplacian instability for both scalar and tensor perturbations. We also discuss the observational constraints on the model parameters. It turns out that the model actually has three degrees of freedom unlike the original cuscuton theory.

1 Introduction

Many modified gravity models require additional degrees of freedom (d.o.f.) besides two tensor gravitational degrees of freedom to explain an accelerated expansion of the Universe [1,2]. For example, Horndeski theories [3–5] using a scalar field have three d.o.f., generalized Proca theories [6] using a vector field have five d.o.f., and massive gravity using a massive tensor field has five d.o.f in the case of de Rham–Gabadadze–Tolley massive gravity [7,8]. However, until now, a fifth force or deviation from General Relativity in the solar system scale has not been detected [9]. Therefore, they require screening mechanisms to hide their additional degrees of freedom [10–20].

Recently there is development on modification of the gravitational theories which propagate only two gravitational degrees of freedom. The cuscuton gravity model was first proposed in [21–23], which can be regarded as the low-energy Horava–Lifshitz theory [23]. Some extension was also found in minimally modified gravity (MMG) [24–29] and extended cuscuton gravity [30,31]. The minimally modified gravity is a construction of Hamiltonian of the gravitational theory which provides only two d.o.f., while the extended cuscuton is a generalization of an original cuscuton theory in the context of the beyond Horndeski theories [32]. In these models the scalar field turns out to be nondynamical because either the second-order time derivatives of a scalar field are absent in the equations of motion (MMG), or we can eliminate them after linear combination of the equations of motion and the Friedmann equation (extended cuscuton).

Besides above two classes of theories, the Cuscuta–Galileon gravity which is a simple galileon generalization of the original cuscuton gravity, was proposed [33]. In the original cuscuton gravity, there exists a caustic singularity, which shows lacking predictability. Hence adding a galileon-like kinetic term in the original cuscuton theory, they discuss a simple extended model (the Cuscuta–Galileon theory), which can avoid the formation of caustic singularities in flat space-time [33].

In this work we investigate cosmological dynamics of the Cuscuta–Galileon gravity. We include a potential term because without a potential term, such a model does not provide a viable cosmological model just as the same as the original cuscuton theory [21,22]. In fact we find that radiation dominant and matter dominant eras do not exist as we will show in Appendix A. In the original cuscuton theory, adding a quadratic potential, we obtain the ΛCDM model. Therefore, we imitate this idea by adding a potential term to the Cuscuta–Galileon action, and then investigate cosmological dynamics of the model with an appropriate potential term whether it provides a consistent cosmic evolution or not.

This paper is organized as follows. In Sect. 2 we derive basic equations of the model. In Sect. 3 we study the cosmological dynamics by using the dynamical system approach where we consider a scalar potential in two cases: an expo-

[a] e-mail: sirachakp@aoni.waseda.jp (corresponding author)
[b] e-mail: maeda@waseda.jp

nential potential and an inverse power-law potential. In Sect. 4 we use the Hamiltonian formalism to investigate number degrees of freedom of the Cuscuta–Galileon model rigorously. It turns out that the Cuscuta–Galileon gravity in fact has three d.o.f. which leads to tendency that the model is not in a subclass of the extended cuscuton gravity, but rather in a subclass of the Horndeski theories. In Sect. 5.1, we solve autonomous equations of the model numerically and show evolution of density parameters and equation of state parameters. We check ghosts and Laplacian instabilities in Sect. 5.2. Lastly, Sect. 6 is devoted to conclusions.

2 Action and basic equations

We start at an action of the Cuscuta–Galileon gravity as Ref. [33] in curved space-time with a potential term,

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm PL}^2 R + a_2\sqrt{-X} + a_3 \ln\Big(-\frac{X}{\Lambda^4}\Big)\Box\phi - V(\phi)\Big] + S_M(g_{\mu\nu}, \psi_M), \quad (2.1)$$

where R is the Ricci scalar, $M_{\rm PL}$ is the reduced Planck mass, g is the determinant of the metric $g_{\mu\nu}$, and ψ_M is a fermion field. X is defined as $X \equiv g^{\mu\nu}\partial_\mu\phi\partial_\mu\phi$. a_2, a_3, and Λ are constants with dimension of mass squared, mass, and mass, respectively. We consider up to cubic order to satisfy the constraint from the gravitational waves observations, GW170817 [34–37], and add the potential term in order to obtain radiation dominated and matter dominated eras according to the thermal history of the Universe. Without the potential term, the theory provides only the de Sitter expansion as shown in Appendix A.

We consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric and a homogeneous scalar field as

$$ds^2 = -N(t)^2dt^2 + a(t)^2\delta_{ij}dx^i dx^j, \quad \phi = \phi(t). \quad (2.2)$$

Substituting the metric into the above action, and then varying with respect to N, a, and ϕ, after setting $N = 1$ we find

$$3M_{\rm PL}^2H^2 - \rho_m - \rho_r - V(\phi) + 6a_3H\dot{\phi} = 0, \quad (2.3)$$
$$3M_{\rm PL}^2H^2 + 2M_{\rm PL}^2\dot{H} + P_m + P_r - V(\phi) + a_2|\dot{\phi}| + 2a_3\ddot{\phi} = 0, \quad (2.4)$$
$$18a_3H^2 + 6a_3\dot{H} - V_{,\phi} - 3a_2H\,\mathrm{sgn}(\dot{\phi}) = 0, \quad (2.5)$$

where ρ_m, P_m, ρ_r, and P_r are densities and pressures of nonrelativistic matter (or matter for abbreviation) and radiation, respectively. H is the Hubble parameter, an upper dot means the derivative with respect to time, "$_{,\phi}$" denotes the partial derivative with respect to ϕ, and $\mathrm{sgn}(\dot{\phi})$ is the sign of $\dot{\phi}$. Equations (2.3) and (2.4) are the Friedmann equations, and Eq. (2.5) is the equation of motion of the scalar field. Although there is no second-order time derivatives of the scalar field in the equation of motion, there is ambiguity because these three basic equations are not independent as shown in Appendix B which allows us to write independent equations for $\ddot{\phi}$ and H. Therefore in order to know exact number degrees of freedom we have to perform the Hamiltonian analysis which will be given in Sect. 4.

From the Friedmann equations we can define density and pressure of the scalar field as

$$\rho_\phi = V(\phi) - 6a_3H\dot{\phi}, \quad (2.6)$$
$$P_\phi = a_2|\dot{\phi}| + 2a_3\ddot{\phi} - V(\phi). \quad (2.7)$$

Combination of the Friedmann equations and the equation of motion, we obtain the energy conservation equations:

$$\dot{\rho}_m + 3H\rho_m = 0, \quad (2.8)$$
$$\dot{\rho}_r + 4H\rho_r = 0, \quad (2.9)$$
$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0, \quad (2.10)$$

where we assume that the nonrelativistic matter is pressureless, $P_m \approx 0$, while the pressure of radiation is $P_r = \rho_r/3$.

In the next section we will use dynamical system approach to study cosmological dynamics of the Cuscuta–Galileon gravity.

3 Dynamical system

3.1 Autonomous equations

We introduce dimensionless variables as follows

$$x_1 \equiv \frac{V}{3M_{\rm PL}^2H^2}, \quad x_2 \equiv \frac{a_2|\dot{\phi}|}{M_{\rm PL}^2H^2}, \quad x_3 \equiv \frac{2a_3\dot{\phi}}{M_{\rm PL}^2H}, \quad x_4 \equiv \frac{\rho_r}{3M_{\rm PL}^2H^2}, \quad \lambda \equiv \frac{M_{\rm PL}^2V_{,\phi}}{a_3V}. \quad (3.1)$$

Thus the first Friedmann equation (2.3) can be written as

$$\Omega_m = 1 - x_1 + x_3 - x_4, \quad (3.2)$$

where $\Omega_m \equiv \rho_m/3M_{\rm PL}^2H^2$ is a density parameter of the nonrelativistic matter. This is a constraint equation where dynamics of the matter density parameter can be realized via variables x_1, x_3, and x_4. Also, density parameters of the radiation and the scalar field are

$$\Omega_r \equiv \frac{\rho_r}{3M_{\rm PL}^2H^2} = x_4, \quad \Omega_\phi = x_1 - x_3. \quad (3.3)$$

Taking derivative with respect to the e-foldings number, $N \equiv \ln a$, we find a set of autonomous equations:

$$\frac{dx_1}{dN} = \frac{1}{2}\lambda x_1 x_3 - 2x_1\frac{\dot{H}}{H^2}, \quad (3.4)$$

Table 1 The fixed points, the density parameters, and the equation of state parameters of the Cuscuta–Galileon with the exponential potential

Fixed point	x_1	x_2	x_3	x_4	Ω_m	Ω_r	Ω_ϕ	w_ϕ	$w_{\rm eff}$
(a)	0	-3	-1	0	0	0	1	-1	-1
(b)	0	0	-1	0	0	0	1	1	1
(c)	$\frac{2}{\lambda}$	0	$-\frac{8}{\lambda}$	$1-\frac{10}{\lambda}$	0	$1-\frac{10}{\lambda}$	$\frac{10}{\lambda}$	$\frac{1}{3}$	$\frac{1}{3}$
(d)	$\frac{3}{\lambda}$	0	$-\frac{6}{\lambda}$	0	$1-\frac{9}{\lambda}$	0	$\frac{9}{\lambda}$	0	0
(e)	$-1+\frac{12}{\lambda}$	0	$-2+\frac{12}{\lambda}$	0	0	0	1	$-3+\frac{\lambda}{3}$	$-3+\frac{\lambda}{3}$

$$\frac{dx_2}{dN} = \frac{x_2}{x_3}\left(3 - x_4 + 3x_1 - \lambda x_1 - x_2 - \frac{2x_2}{x_3}\right) -2x_2\frac{\dot{H}}{H^2}, \tag{3.5}$$

$$\frac{dx_3}{dN} = 3 - x_4 + 3x_1 - \lambda x_1 - x_2 - \frac{2x_2}{x_3} - x_3\frac{\dot{H}}{H^2}, \tag{3.6}$$

$$\frac{dx_4}{dN} = -4x_4 - 2x_4\frac{\dot{H}}{H^2}, \tag{3.7}$$

$$\frac{d\lambda}{dN} = \frac{1}{2}\lambda^2 x_3(\Gamma - 1), \tag{3.8}$$

and

$$\frac{\dot{H}}{H^2} = -3 + \frac{1}{2}\lambda x_1 + \frac{x_2}{x_3}, \quad \Gamma \equiv \frac{V V_{,\phi\phi}}{V_{,\phi}^2}.$$

We have used the second Friedmann equation (2.4), the equation of motion (2.5), and the continuity equation of radiation (2.9) to obtain these autonomous equations.

Effective equation of state parameter and equation of state parameter of the scalar field are defined as

$$w_{\rm eff} = \frac{P_{\rm total}}{\rho_{\rm total}} = 1 - \frac{2x_2}{3x_3} - \frac{1}{3}\lambda x_1, \tag{3.9}$$

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{w_{\rm eff} - \frac{1}{3}x_4}{x_1 - x_3}. \tag{3.10}$$

Next we will consider the Cuscuta–Galileon model in two cases: an exponential potential and an inverse power-law potential.

3.2 Fixed points

3.2.1 Exponential potential

If the potential is an exponential form, the λ is a constant. We then have only 4 autonomous Eqs. (3.4)–(3.7) with 4 parameters. Integrating the definition of λ in the Eq. (3.1), we find

$$V(\phi) = V_0 e^{a_3\lambda\phi/M_{\rm PL}^2}, \tag{3.11}$$

where V_0 is a constant. Setting $dx_1/dN = dx_2/dN = dx_3/dN = dx_4/dN = 0$, we find five fixed points as Table 1.

In this work we are interested in the case $V(\phi) \geq 0$ (i.e., $V_0 > 0$) and the expanding universe, $H > 0$, thus the $x_1 \geq 0$. The x_2 and the x_3 can be positive or negative values depending on the signs of a_2, a_3, and $\dot{\phi}$. The $x_4 \geq 0$ because $\rho_r \geq 0$. These conditions lead to constraints on λ of some fixed points. The fixed point (c) requires $\lambda \geq 10$ to satisfy $x_4 \geq 0$, whereas the fixed point (d) and (e) require $\lambda > 0$ and $\lambda \leq 12$ to satisfy $x_1 \geq 0$, respectively.

Considering the equation of state parameters and the density parameters, we find that only the fixed point (c) can be the radiation dominated epoch. Rigorously, this fixed point is the ϕ-radiation dominated epoch because Ω_ϕ does not vanish but keeps constant. The energy density of the scalar field decreases in the same way as that of radiation. We still call it as the radiation dominated epoch for simplicity.

Although the scalar field component is not negligible at early time, the model is not one of the early dark energy models because w_ϕ is not less than $-1/3$ in the deep radiation dominated era. By the same reason only the fixed point (d) can be the matter-dominated epoch (or rigorously ϕ-matter dominated epoch). The fixed point (b) cannot be the dark energy dominated epoch because w_ϕ is not less than $-1/3$. The fixed point (e) requires $\lambda < 8$ to provide the accelerated expansion. However, it is contradictory to the constraint on the fixed point (c). We adopt the Big Bang Nucleosysthesis (BBN) constraint on the quintessence model, $\Omega_\phi|_{\rm BBN} < 0.045$ [38], then the scalar field density parameter of the fixed point (c), $\Omega_\phi = 10/\lambda$, leads to

$$\lambda > 222.22. \tag{3.12}$$

Therefore the dark energy dominated epoch corresponds only to the fixed point (a) which is the de Sitter fixed point because $\Omega_\phi = 1$ and $w_\phi = -1$.

3.2.2 Inverse power-law potential

Considering the inverse power-law potential as the following form

$$V(\phi) = \frac{M^{4+n}}{\phi^n}, \tag{3.13}$$

where M is a constant with dimension of mass, and $n > 0$. We find $\Gamma = (n+1)/n$ or $(\Gamma - 1) = 1/n$. In this case λ is not

Table 2 The fixed points, the density parameters, and the equation of state parameters of the Cuscuta–Galileon with the inverse power-law potential for any integer n

Fixed point	x_1	x_2	x_3	x_4	λ	Ω_m	Ω_r	Ω_ϕ	w_ϕ	$w_{\rm eff}$
(f)	$1+x_3$	$3x_3$		0	0	0	0	1	-1	-1
(g)	0	-3	-1	0	0	0	0	1	-1	-1
(h)	0	0	-1	0	0	0	0	1	1	1

a constant, we then need to solve the Eq. (3.8) along with the previous 4 autonomous Eqs. (3.4)–(3.7). Setting $dx_1/dN = dx_2/dN = dx_3/dN = dx_4/dN = d\lambda/dN = 0$, we find three fixed points as shown in Table 2.

The fixed points (f) and (g) are possible to be the dark energy dominated epoch just as a conventional quintessence model. The point (h) describes the stiff-matter universe and then does not match with any thermal history of the Universe. Since the autonomous system of the inverse power-law potential does not provide the radiation dominated and matter dominated eras, we will no longer consider this case for the rest of this paper.

In the next subsection we will check stability of the fixed points of the Cuscuta–Galileon with the exponential potential.

3.3 Stability of fixed points

In order to discuss the roles of the above fixed points in the history of the universe, we have to discuss the stability of the fixed points. We then perturb the variables around fixed points.

The 4 autonomous Eqs. (3.4)–(3.7) with 4 parameters are described as

$$\frac{dx_1}{dN} = \mathcal{A}(x_1, x_2, x_3, x_4),$$
$$\frac{dx_2}{dN} = \mathcal{B}(x_1, x_2, x_3, x_4),$$
$$\frac{dx_3}{dN} = \mathcal{C}(x_1, x_2, x_3, x_4),$$
$$\frac{dx_4}{dN} = \mathcal{D}(x_1, x_2, x_3, x_4).$$

Considering the linear perturbation around the fixed points, $x \to x^{(\rm FP)} + \delta x$, we obtain the first order coupled differential equations:

$$\frac{d}{dN}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\\ \delta x_4\end{pmatrix} = \mathcal{M}\begin{pmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\\ \delta x_4\end{pmatrix}, \tag{3.14}$$

where the matrix $\mathcal{M}$ depends on the fixed points as

$$\mathcal{M} = \left.\begin{pmatrix}\frac{\partial\mathcal{A}}{\partial x_1} & \frac{\partial\mathcal{A}}{\partial x_2} & \frac{\partial\mathcal{A}}{\partial x_3} & \frac{\partial\mathcal{A}}{\partial x_4}\\ \frac{\partial\mathcal{B}}{\partial x_1} & \frac{\partial\mathcal{B}}{\partial x_2} & \frac{\partial\mathcal{B}}{\partial x_3} & \frac{\partial\mathcal{B}}{\partial x_4}\\ \frac{\partial\mathcal{C}}{\partial x_1} & \frac{\partial\mathcal{C}}{\partial x_2} & \frac{\partial\mathcal{C}}{\partial x_3} & \frac{\partial\mathcal{C}}{\partial x_4}\\ \frac{\partial\mathcal{D}}{\partial x_1} & \frac{\partial\mathcal{D}}{\partial x_2} & \frac{\partial\mathcal{D}}{\partial x_3} & \frac{\partial\mathcal{D}}{\partial x_4}\end{pmatrix}\right|_{x_1^{(\rm FP)}, x_2^{(\rm FP)}, x_3^{(\rm FP)}, x_4^{(\rm FP)}}.$$

The eigen functions of the Eq. (3.14) are given by

$$\delta x_i^{(a)} \propto e^{\mu^{(a)} N}, \quad (a = 1, \cdots 4) \tag{3.15}$$

where $\mu^{(a)}$ are the eigenvalues of the matrix $\mathcal{M}$.

If all eigenvalues are negative, we find a stable fixed point. In the case of complex eigenvalues, if all real parts are negative, the fixed point is a stable spiral point, whereas if all of them are positive, the fixed point is an unstable point or unstable spiral point for complex eigenvalues. If at least one eigenvalue but not all is positive (or gives a positive real part), the fixed point is a saddle point.

We summarize all eigenvalues of the fixed points in the Table 1:

$$\text{(a)}: \Big(-4, -3, -3, -\frac{\lambda}{2}\Big),$$
$$\text{(b)}: \Big(3,\ 3,\ 2,\ 6 - \frac{\lambda}{2}\Big),$$
$$\text{(c)}: \Big(2,\ 1, -\frac{1}{2} \pm \frac{\sqrt{41 - 4\lambda}}{2}\Big),$$
$$\text{(d)}: \Big(\frac{3}{2}, -1, -\frac{3}{4} \pm \frac{\sqrt{3(75 - 8\lambda)}}{4}\Big),$$
$$\text{(e)}: \Big(\lambda - 10,\ \frac{\lambda}{2} - 6,\ \lambda - 9,\ \frac{\lambda}{2} - 3\Big).$$

Consequently, the fixed point (a) is a stable fixed point, the fixed point (b), (c), and (d) are saddle points, and the fixed point (e) is an unstable point. Remind that the fixed point (b) does not relate to any thermal history of the Universe, and the fixed point (e) requires $\lambda < 8$ to give the accelerated expansion. However, since we need the fixed point (c) to be the radiation dominated epoch, it must satisfy the condition, $\lambda > 222.22$, from the BBN constraint. Therefore, if we start from the fixed point (c), the cosmological sequence is

$$\text{(c)} \to \text{(d)} \to \text{(a)}.$$

4 Degrees of freedom

As shown in Appendix B, we find the dynamical equation for the scalar field in a homogeneous field in FLRW universe. Hence, first we have to check the degree of freedom for the present model.

In this section we will use the Hamiltonian formalism to find degrees of freedom. According to the Refs. [39–41] the action (2.1) can be written in the Arnowitt–Deser–Misner (ADM) form as

$$S = \int dt d^3x N\sqrt{h}\Big[\frac{1}{2}M_{\rm PL}^2\left({}^3R + K_{ij}K^{ij} - K^2\right) + \left(\frac{a_2|\dot{\phi}|}{N} - V(\phi)\right) + \left(-\frac{2a_3|\dot{\phi}|}{N} + C\right)K\Big], \tag{4.1}$$

where ${}^3R$ is the three-dimensional Ricci scalar, K_{ij} is the extrinsic curvature, $K \equiv K_{ij}h^{ij}$, and C is a constant. Note that in this section we will not consider contribution from the matter field.

In the ADM Language the fundamental variables are N, N^i, and h^{ij} where they are the lapse function, the shift vector, and the three-dimensional metric, respectively. Following calculations in Ref. [42] we choose the unitary gauge, $\phi = \phi(t)$, then the scalar field is merely time. Hence we have only 10 fundamental variables, whose conjugate momenta are

$$\pi_N = \frac{\partial\mathcal{L}}{\partial\dot{N}} = 0, \quad \pi_i = \frac{\partial\mathcal{L}}{\partial\dot{N}^i} = 0, \tag{4.2}$$

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} = \frac{1}{2}M_{\rm PL}^2\sqrt{h}\left(K^{ij} - h^{ij}K\right) + \frac{1}{2}\sqrt{h}\left(-\frac{2a_3|\dot{\phi}|}{N} + C\right)h^{ij}. \tag{4.3}$$

Using the Legendre transformation, the Hamiltonian of the Cuscuta–Galileon gravity is given by

$$H = \int d^3x\left(\mathcal{H} + N^i\mathcal{H}_i + \lambda_N\pi_N + \lambda^i\pi_i\right), \tag{4.4}$$

where λ_N and λ^i are Lagrange multipliers, and

$$\mathcal{H} = N\sqrt{h}\Big[\frac{2}{M_{\rm PL}^2}\left(\frac{\pi^{ij}\pi_{ij}}{h} - \frac{\pi^2}{2h}\right) - \frac{1}{2}M_{\rm PL}^2\,{}^3R - \left(\frac{a_2|\dot{\phi}|}{N} - V(\phi)\right) + \frac{\pi}{M_{\rm PL}^2\sqrt{h}}\left(-\frac{2a_3|\dot{\phi}|}{N} + C\right) - \frac{3}{4M_{\rm PL}^2}\left(-\frac{2a_3|\dot{\phi}|}{N} + C\right)^2\Big], \tag{4.5}$$

$$\mathcal{H}_i = -2h_{ik}D_j\pi^{kj}. \tag{4.6}$$

π is a trace of the π_{ij}, and D_j is the three-dimensional covariant derivative. Although the form of $\mathcal{H}_i$ is the same as in GR, it is not a first-class constraint because the $\mathcal{H}$ is not a linear function of N (see Ref. [42]). In order to obtain $\mathcal{H}_i$ as the first-class constraint we need to add additional terms which vanish weakly to the Hamiltonian as

$$\bar{\mathcal{H}}_i \equiv \mathcal{H}_i + \pi_N\partial_i N. \tag{4.7}$$

Using conservation of the 4 primary constraints, $\pi_N = 0, \pi_i = 0$, we find secondary constraints as

$$0 = \frac{d\pi_N}{dt} = -\frac{\partial H}{\partial N} \approx -\frac{\partial\mathcal{H}}{\partial N} \equiv \mathcal{C} \;\rightarrow\; \mathcal{C} \approx 0, \tag{4.8}$$

$$0 = \frac{d\pi_i}{dt} = -\frac{\partial H}{\partial N^i} \approx \bar{\mathcal{H}}_i \;\rightarrow\; \bar{\mathcal{H}}_i \approx 0. \tag{4.9}$$

The notation $\approx$ means the weak equality, i.e. it is the equality on the constraint surface in phase space.

We can check whether these constraints are first-class or second-class by using the Poisson bracket which is given by

$$\{F, G\} \equiv \int d^3y\Big[\frac{\delta F}{\delta N(y)}\frac{\delta G}{\delta\pi_N(y)} - \frac{\delta F}{\delta\pi_N(y)}\frac{\delta G}{\delta N(y)} + \frac{\delta F}{\delta N^i(y)}\frac{\delta G}{\delta\pi_i(y)} - \frac{\delta F}{\delta\pi_i(y)}\frac{\delta G}{\delta N^i(y)} + \frac{\delta F}{\delta h_{ij}(y)}\frac{\delta G}{\delta\pi^{ij}(y)} - \frac{\delta F}{\delta\pi^{ij}(y)}\frac{\delta G}{\delta h_{ij}(y)}\Big]. \tag{4.10}$$

Therefore, we find

$$\{\pi_i(x), \pi_N(x')\} = 0, \tag{4.11}$$

$$\{\pi_i(x), \bar{\mathcal{H}}_j(x')\} = 0, \tag{4.12}$$

$$\{\pi_i(x), \mathcal{C}(x')\} = 0, \tag{4.13}$$

$$\{\bar{\mathcal{H}}_i[f^i], \bar{\pi}_N[\varphi]\} = \int d^3y\pi_N f^i\partial_i\varphi \approx 0, \tag{4.14}$$

$$\{\bar{\mathcal{H}}_i[f^i], \mathcal{C}[\varphi]\} = \int d^3y\mathcal{C}f^i\partial_i\varphi \approx 0, \tag{4.15}$$

$$\{\pi_N(x), \mathcal{C}(x')\} = \frac{\partial^2\mathcal{H}}{\partial N^2}\delta(x - x'). \tag{4.16}$$

Some Poisson brackets we use the smeared constraint form defined as

$$\bar{\mathcal{H}}_i[f^i] \equiv \int d^3x f^i(x)\bar{\mathcal{H}}_i(x), \tag{4.17}$$

$$\bar{\pi}_N[\varphi] \equiv \int d^3x\varphi(x)\pi_N(x), \tag{4.18}$$

$$\mathcal{C}[\varphi] \equiv \int d^3x\varphi(x)\mathcal{C}(x). \tag{4.19}$$

The Poisson brackets are vanished except the last one because the $\mathcal{H}$ of the Cuscuta–Galileon model gives

$$\frac{\partial^2\mathcal{H}}{\partial N^2} \neq 0. \tag{4.20}$$

As a result, we have 10 variables which correspond to 20 dimensions in phase space with 8 constraints where $\pi_i, \bar{\mathcal{H}}_i$ are the first-class constraints, and $\pi_N, \mathcal{C}$ are the second-class constraints. Consequently, the number degrees of freedom of the Cuscuta–Galileon gravity is given by

$$\begin{aligned}\text{d.o.f.} &= \frac{1}{2}(\text{variables} \times 2 - \text{1st class} \times 2 - \text{2nd class}) \\ &= \frac{1}{2}(20 - 6 \times 2 - 2) \\ &= 3.\end{aligned} \tag{4.21}$$

We then find that the Cuscuta–Galileon gravity has three d.o.f. instead of two. As a result, the present model is neither included in MMG nor a subclass of the extended cuscuton gravity, but it is rather in a subclass of the Horndeski theories.

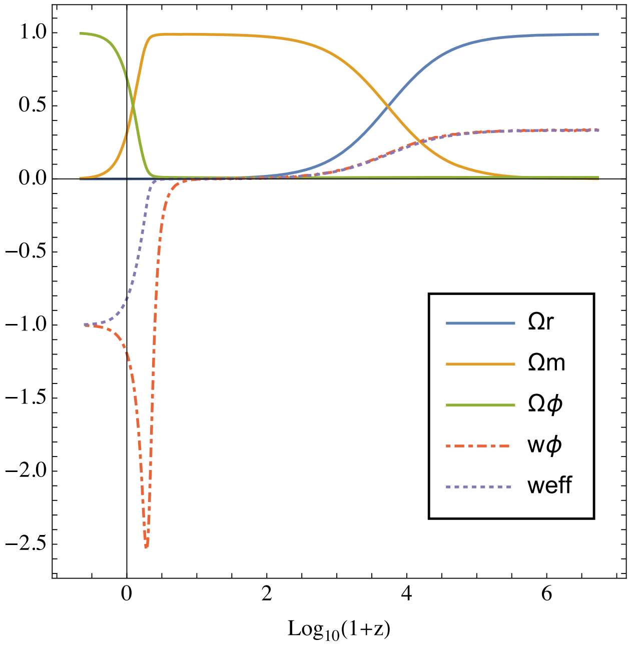

Fig. 1 The evolution of the density parameters and the equation of state parameters where we set $x_1 - 0.002 = 1 \times 10^{-6}$, $x_2 = -1 \times 10^{-13}$, $x_3 + 0.008 = 1 \times 10^{-6}$, and $x_4 = 0.989$ at $\log_{10}(1+z) = 6.73$

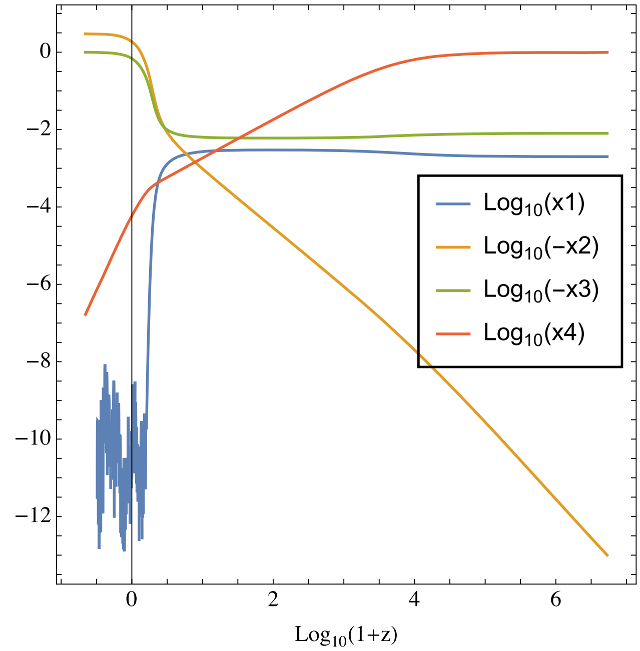

Fig. 2 The evolution of the dynamical parameters where the initial conditions are the same as the Fig. 1

5 Cosmic evolution in the Cuscuta–Galileon theory

5.1 Numerical solution

In this section, solving the autonomous Eqs. (3.4)–(3.7) of the Cuscuta–Galileon with the exponential potential numerically, we discuss how the Universe evolves in the present model. We set $\lambda = 10^3$ and choose initial conditions near the fixed point (c) (i.e., starting from the radiation dominated epoch). The evolution of the density parameters and the equation of state parameters according to the Eqs. (3.2), (3.3), (3.9), and (3.10) are shown as Fig. 1.

Figure 1 reveals that the evolution of the Cuscuta–Galileon with the exponential potential corresponds to the thermal history of the Universe correctly. This result is consistent with the stability analysis on the fixed points. The $w_\phi = 1/3$ in the radiation dominated era, and then it is around zero in the matter dominated era. However, before approaching the de Sitter fixed point, the w_ϕ crosses the cosmological constant boundary, $w_\Lambda = -1$, and then approaching -1 at late time. The large negative value can be understood by considering the evolution plot of the dynamical parameters as Fig. 2.

According to the Fig. 2, the x_1 and the x_4 tend to zero around the end of the matter dominated epoch, then the Eq. (3.10) becomes $w_\phi \simeq -w_{\rm eff}/x_3$. Since $|x_3| < 1$, we find $|w_\phi| > |w_{\rm eff}|$, and at late time the $|x_3| \to 1$, thus $|w_\phi| \simeq |w_{\rm eff}|$. Therefore, we obtain a large negative value of the w_ϕ around the end of the matter dominated era, and then it approaches to the $w_{\rm eff}$ at late time.

Note that the fine-tuning parameters are the amount of the radiation component, x_4, and the ratio of velocity of the scalar field and the Hubble parameter squared, x_2, in the radiation dominant to have long enough the matter dominant epoch. If x_4 is larger or x_2 is more negative, the matter dominant era will be shorter. It is then inconsistent with observations that the age of matter-radiation equality is around $z \approx 3300$. The other parameters are more flexible, for example, x_1 and x_3 can be around 10^{-5} from the fixed point (c), we still obtain the sequence of the cosmic evolution properly. However, there is a small oscillations on the value of w_ϕ in this case.

5.2 Ghosts and Laplacian instability

As shown in Sect. 4, this model includes an additional degrees of freedom (a scalar field) in the present model. Then we have to check whether there exists no ghost or Laplacian instability in our cosmic evolution. Since there are three degrees of freedom, we expect that there exist scalar perturbations as well as tensor perturbations.

According to Refs. [39–41,43–45] the second order action of the tensor perturbations is

$$S_2^T = \int d^4x \frac{a^3}{4} L_S \Big[\dot{\gamma}_{ij}^2 - c_T^2 \frac{(\partial_k \gamma_{ij})^2}{a^2}\Big], \tag{5.1}$$

where γ_{ij} is the tensor perturbations which satisfies transverse and traceless conditions, L_S relates to the action in the background level, and c_T^2 is a sound speed squared in the tensor mode which also relates to the action in the background level. In order to avoid ghosts the coefficient in front of the term $\dot{\gamma}_{ij}^2$ must be positive, thus we need the $L_S > 0$. Similarly, we require the $c_T^2 > 0$ to avoid the Laplacian instability.

For simplicity we use notations as the Ref. [43] where they correspond to the action of the Cuscuta–Galileon (2.1) as follows

$$\mathcal{W}_1 = M_{\rm PL}^2, \tag{5.2}$$

$$\mathcal{W}_2 = 2M_{\rm PL}^2 H + 2\dot{\phi} a_3 = (2 + x_3) M_{\rm PL}^2 H, \tag{5.3}$$

$$\mathcal{W}_3 = -9M_{\rm PL}^2 H^2 - 18 H\dot{\phi} a_3 = (-9 - 9x_3) M_{\rm PL}^2 H^2, \tag{5.4}$$

$$\mathcal{W}_4 = M_{\rm PL}^2, \tag{5.5}$$

where $L_S \equiv \mathcal{W}_1/2$ and $c_T^2 \equiv \mathcal{W}_4/\mathcal{W}_1$. We then find

$$L_S = \frac{1}{2} M_{\rm PL}^2 > 0, \quad c_T^2 = 1. \tag{5.6}$$

Consequently, the Cuscuta–Galileon gravity satisfies the no ghosts and no Laplacian instability conditions of the tensor mode.

For the scalar perturbations it is similar to the tensor perturbations. In order to avoid the ghosts we require

$$Q_S \equiv \frac{\mathcal{W}_1(4\mathcal{W}_1\mathcal{W}_3 + 9\mathcal{W}_2^2)}{3\mathcal{W}_2^2} = \frac{3x_3^2}{(2+x_3)^2} > 0, \tag{5.7}$$

and the sound speed squared in the scalar mode must be greater than zero to avoid the Laplacian instability:

$$c_S^2 \equiv \frac{3\Big(-2\mathcal{W}_1^2\dot{\mathcal{W}}_2 + 2\mathcal{W}_1^2\mathcal{W}_2 H - \mathcal{W}_2^2\mathcal{W}_4 - 2\mathcal{W}_1^2\Big((1+w_m)\rho_m + (1+w_r)\rho_r\Big)\Big)}{\mathcal{W}_1(4\mathcal{W}_1\mathcal{W}_3 + 9\mathcal{W}_2^2)} = \frac{2x_2 - x_3(8+x_3)}{3x_3^2} > 0. \tag{5.8}$$

Other conditions involving the existence of nonrelativistic matter and radiation fluids are automatically satisfied when we choose forms of the k-essence type perfect fluid as the Ref. [40].

Figure 3 reveals that the Cuscuta–Galileon gravity has no ghosts and the Laplacian instability in the scalar mode. In matter dominant and radiation dominant the sound speed squared is greater than unity because $|x_3| \ll 1$ in the denominator of the Eq. (5.8), while x_2 is about zero.

The results of the scalar perturbations reveal that there is a scalar degree of freedom propagating in this model, which is consistent with the analysis in the previous section.

We can construct a viable cosmological model in the Cuscuta–Galileon gravity theory. We show the cosmological evolution from radiation dominated era to de Sitter expansion stage via matter dominated era.

5.3 Observational constraints

However if we look into the detail, we find that the Cuscuta–Galileon gravity may not satisfy observations by several reasons. First, the Planck 2018 results [46] reveal that the dark energy equation of state parameter is $w_{\rm DE} = -1.028 \pm 0.031$, it is consistent with the cosmological constant, while the Cuscuta–Galileon gives $w_\phi = -1.196$ at $\Omega_m = 0.315$. Second, there is a large amount of the dark energy component comparing to the cosmological constant in the matter and radiation dominated eras as Fig. 4.

From the Lambda-Cold Dark Matter (ΛCDM) model, the density parameter of the cosmological constant is given by

$$\Omega_\Lambda(z) = \frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}}, \tag{5.9}$$

where $\Omega_m^{(0)} = 0.315$, $\Omega_r^{(0)} = 9 \times 10^{-5}$, and $\Omega_\Lambda = 0.685$ according to the Planck 2018 results. In the Fig. 4 we find that the $\Omega_\phi \sim \mathcal{O}(10^{-2})$ in the matter and radiation dominated epochs, whereas Ω_Λ is utterly small, for instance, at the last scattering surface, $z \approx 1090$, the $\Omega_\Lambda \sim \mathcal{O}(10^{-9})$. It is obvious that the model is different from the ΛCDM model. Therefore, the Cuscuta–Galileon gravity is likely not to satisfy the observations which prefer the ΛCDM model, such as the Cosmic Microwave Background (CMB) observations. Although rigorous calculations and global fitting with observational data are required, they are beyond the scope of this paper. Lastly, if we increase the λ in order to obtain the lower Ω_ϕ, such as $\lambda = 10^4$, we find $\Omega_\phi \sim \mathcal{O}(10^{-3})$, the equation of state parameter of the scalar field will be more negative and more deviate from the observational value because $|x_3| \sim \mathcal{O}(10^{-3}) \ll 1$ around the end of the matter dominated epoch.

6 Conclusions

In this work we study cosmological dynamics of the Cuscuta–Galileon gravity. The model was proposed in the Ref. [33] as galileon generalization of the cuscuton model which is

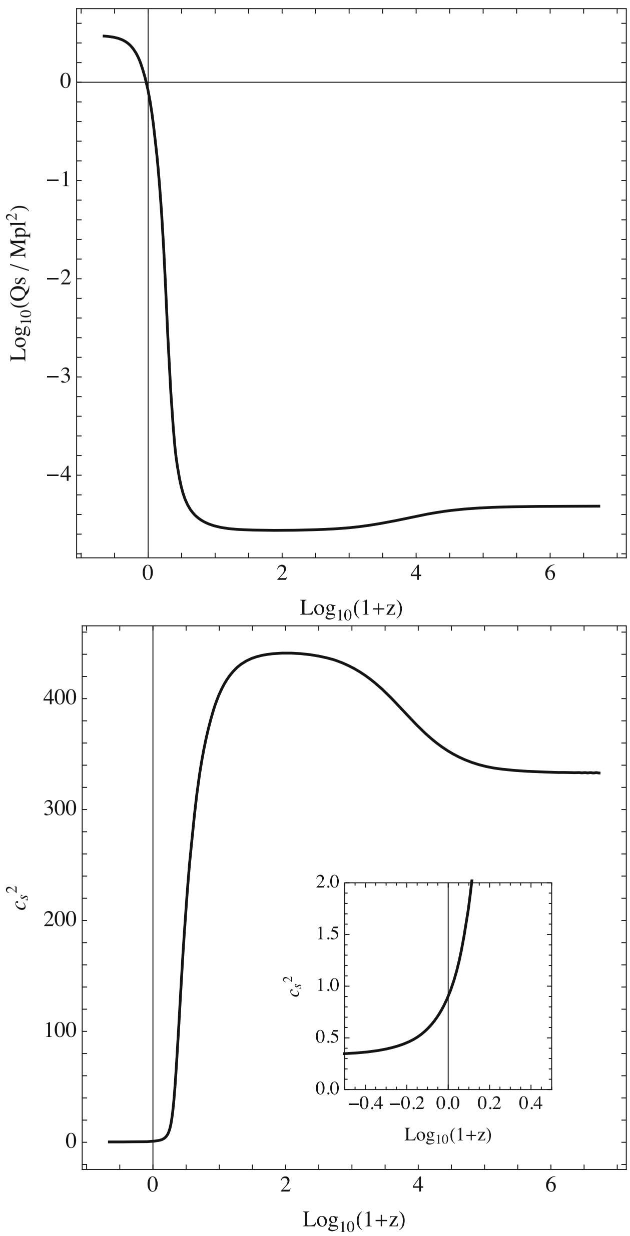

Fig. 3 The evolutions of the Q_S as Eq. (5.7) in log scale and the c_S^2 as Eq. (5.8) where the initial conditions are the same as the Fig. 1

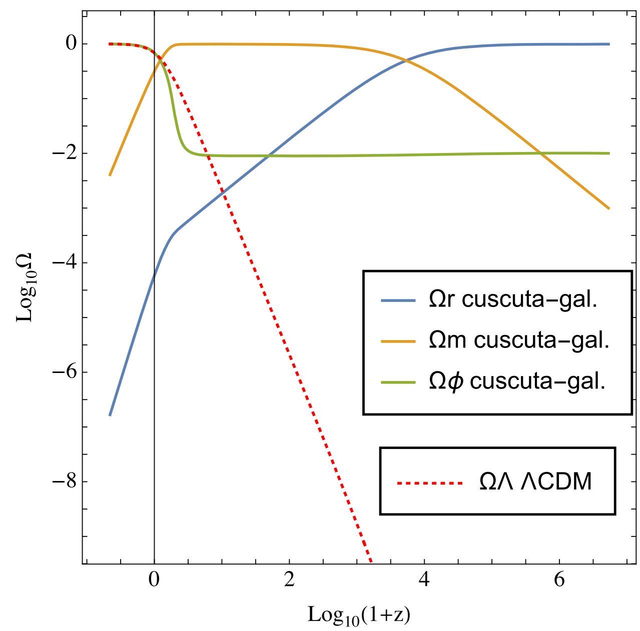

Fig. 4 The evolution of the density parameters in log scale where the initial conditions are the same as the Fig. 1

free from the caustic singularities in flat space-time. In the case without a potential term the equation of motion of the Cuscuta–Galileon does not depend on a scalar field, finding that there exists only the de Sitter expansion under the flat FLRW background. Thus in order to obtain the radiation and matter dominated eras we need to add a potential term, for which we consider two cases: an exponential potential and an inverse power-law potential. Using the dynamical system approach and studying stability of fixed points of the autonomous system, we find that only the exponential potential case can provide a proper sequence of the thermal history of the Universe successfully.

Even though there is no second-order time derivatives in the equation of motion, the results of the scalar perturbation reveal that there is a scalar degree of freedom propagating in this model. This is confirmed by using the Hamiltonian analysis where we find that the Cuscuta–Galileon gravity actually has three degrees of freedom and belongs to a subclass of Horndeski theories. In order to discuss the similar Cuscuta–Galileon theory with only two d.o.f., we have to include an additional kinetic term, which was discussed in [47].

In the perturbation level, the conditions for avoidance of ghosts and the Laplacian instability in the tensor mode are automatically satisfied by the form of the action. In the scalar mode we find that there is no ghost and Laplacian instabilities in the present cosmological model.

However, the detail numerical analysis reveals that there appears a large amount of the dark energy component in the matter and radiation dominated eras comparing to that in the ΛCDM model. Therefore, the present Cuscuta–Galileon gravity may not satisfy the observational constraints.

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Appendix A: Cuscuta–Galileon gravity without a potential term

Considering the action of the Cuscuta–Galileon gravity as Ref. [33] in curved space-time up to cubic order:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm PL}^2 R + a_2\sqrt{-X} + a_3 \ln\left(-\frac{X}{\Lambda^4}\right)\Box\phi\right] + S_M(g_{\mu\nu}, \psi_M), \tag{A1}$$

Substituting the flat FLRW metric, $ds^2 = -N(t)^2dt^2 + a(t)^2\delta_{ij}dx^i dx^j$, into the above action, and choosing the unitary gauge, $\phi = \phi(t)$. Varying the action with respect to ϕ, after setting $N = 1$ the equation of motion of the scalar field is given by

$$6a_3H^2 + 2a_3\dot{H} - a_2 H \mathrm{sgn}(\dot{\phi}) = 0. \tag{A2}$$

Since the above equation depends on H only, we can integrate it directly. The evolution of the Hubble parameter is

$$H(t) = \frac{a_2 \mathrm{sgn}(\dot{\phi})}{6a_3 - e^{-(a_2\mathrm{sgn}(\dot{\phi})(t+C)/2a_3)}}, \tag{A3}$$

where C is a constant of integration. If $a_2\mathrm{sgn}(\dot{\phi})/2a_3 > 0$, we find

$$\lim_{t\to\infty} H(t) = \frac{a_2\mathrm{sgn}(\dot{\phi})}{6a_3} = \text{constant}. \tag{A4}$$

Then, we obtain the de Sitter solution at late time. If $a_2\mathrm{sgn}(\dot{\phi})/2a_3 < 0$, we find

$$\lim_{t\to\infty} H(t) = 0. \tag{A5}$$

This is the static universe solution; however, this solution is contradict with observations. We thus accept only the de Sitter solution.

Using the dynamical system approach as the Sect. 3, if the Cuscuta–Galileon model does not have a potential term, then the $x_1 = 0$ and the λ is undefined. Therefore we have only 3 autonomous Eqs. (3.5)–(3.7) with 3 parameters. Setting $dx_2/dN = dx_3/dN = dx_4/dN = 0$, we find fixed points as Table 3.

Table 3 The fixed points, the density parameters, and the equation of state parameters of the Cuscuta–Galileon without a potential

Fixed point	x_2	x_3	x_4	Ω_m	Ω_r	Ω_ϕ	w_ϕ	$w_{\rm eff}$
(i)	-3	-1	0	0	0	1	-1	-1
(j)	0	-1	0	0	0	1	1	1

The fixed point (i) can be the dark energy dominated epoch, whereas the point (j) does not match with any thermal history of the Universe. Then we obtain only the de Sitter expansion in the Cuscuta–Galileon gravity without a potential term. This result is consistent with the analytic solution (A4).

Appendix B: Two independent equations

There are three basic equations, but they are not independent. For example, taking the time derivative of Eq. (2.3) and eliminating $\ddot{\phi}$ by use of Eq. (2.4) and the equations of $\dot{\rho}_i$ ($i = m, r$), i.e.,

$$\dot{\rho}_i + 3H(\rho_i + P_i) = 0,$$

we obtain Eq. (2.5).

In fact we obtain the following two independent equations:

$$\begin{aligned}\ddot{\phi} &- \frac{3a_3}{M_{PL}^2}\dot{\phi}^2 + \left(3\dot{\phi} + \frac{a_2}{2a_3^2}M_{PL}^2\mathrm{sgn}(\dot{\phi})\right)\\ &\times\sqrt{\left(\frac{a_3}{M_{PL}^2}\dot{\phi}\right)^2 + \frac{1}{3M_{PL}^2}(\rho_m + \rho_r + V)}\\ &-\frac{1}{2a_3}(\rho_m - P_m + \rho_r - P_r + 2V)\\ &+\frac{M_{PL}^2}{6a_3^2}V_{,\phi} = 0,\end{aligned} \tag{B1}$$

$$H \equiv \frac{\dot{a}}{a} = -\frac{a_3}{M_{PL}^2}\dot{\phi} + \sqrt{\left(\frac{a_3}{M_{PL}^2}\dot{\phi}\right)^2 + \frac{1}{3M_{PL}^2}(\rho_m + \rho_r + V)}. \tag{B2}$$

Equation (B1) is the second order differential equation for ϕ, while Eq. (B2) is the first differential equation for a. P_m , ρ_m and P_r, ρ_r are given by a scale factor a as

$$P_m = 0,\ \ \rho_m \propto a^{-3}, \tag{B3}$$

and

$$P_r = \frac{\rho_r}{3},\ \ \rho_r \propto a^{-4}. \tag{B4}$$

Hence once we know the initial values of ϕ, $\dot{\phi}$, ρ_m, ρ_r and a, we find the time evolution of those variables.

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