



# Covariant Wick rotation: action, entropy, and holonomies

Raghvendra Singh<sup>1,a</sup> , Dawood Kothawala<sup>2,b</sup>

<sup>1</sup> Institute of Mathematical Sciences, Homi Bhabha National Institute (HBNI), IV Cross Road, C. I. T. Campus, Taramani, Chennai 600 113, India

<sup>2</sup> Centre for Strings, Gravitation and Cosmology, Department of Physics, Indian Institute of Technology Madras, Chennai 600 036, India

Received: 3 December 2022 / Accepted: 19 February 2023 / Published online: 6 March 2023  
© The Author(s) 2023

**Abstract** Given an arbitrary Lorentzian metric  $g_{ab}$  and a nowhere vanishing, timelike vector field  $u^a$ , one can construct a class of metrics  $\widehat{g}_{ab}$  which have Euclidean signature in a specific domain, with a transition to Lorentzian regime occurring on some hypersurface  $\Sigma$  orthogonal to  $u^a$ . Geometry associated with  $\widehat{g}_{ab}$  has been shown to yield some remarkable insights for classical and quantum gravity. In this work, we focus on studying the implications of this geometry for thermal effects in curved spacetimes and compare and contrast the results with those obtained through conventional Euclidean methods. We show that the expression for entropy computed using  $\widehat{g}_{ab}$  for simple field theories and Lanczos–Lovelock actions differ from Wald entropy by additional terms depending on extrinsic curvature. We also compute the holonomy associated with loops lying partially or wholly in the Euclidean regime in terms of extrinsic curvature and acceleration and compare it with the well-known expression for temperature.

## 1 Introduction

The conventional method of Wick rotation, which involves the transformation  $t \rightarrow it$  is known to be problematic when applied to the metric tensor itself since the procedure does not always produce *real* Euclidean metrics, and the interpretation of *imaginary* part of the metric is quite ambiguous [1–4]. The flat spacetime provides us with a preferred choice of the time coordinate, i.e., the one used by inertial observers, but there is no such preferred choice available in a general curved spacetime. Moreover, the transformation  $t \rightarrow it$  is not covariant as it stands. However, for the interpretation of physical effects usually associated with Euclideanization, such as thermal properties of horizons and tunneling amplitudes, it

is desirable to have manifest covariance. The above issues are best demonstrated in non-stationary and stationary metrics with off-diagonal “time-space” components. Such oddities are easily illustrated with a simple example of the de Sitter metric in two different coordinate systems. In the *positive spatial curvature slicing*, the conventional Wick rotation yields

$$ds^2 = -d\tau^2 + \cosh^2 \tau d\Omega_3^2 \longrightarrow d\tau^2 + \cos^2 \tau d\Omega_3^2 \\ = d\Omega_4^2 \quad (1)$$

allowing us to consider  $\tau$  as angular coordinate with  $\tau \in [-\pi/2, \pi/2]$  [5–7]. On the other hand, in the *negative spatial curvature slicing*, the analytic continuation yields

$$ds^2 = -d\tau^2 + \sinh^2 \tau d\mathbb{H}_3^2 \longrightarrow d\tau^2 - \sin^2 \tau d\mathbb{H}_3^2 \quad (2)$$

with  $d\mathbb{H}_3^2$  the line-element on a unit hyperboloid. The resultant metric has signature (3, 1)! It should be clear that conventional Wick rotation through imaginary time does not guarantee any unique structure for the corresponding geometry. As mentioned above, many of the above oddities and ambiguities are tied to a lack of manifest covariance in the standard analytic continuation of the time coordinate. A covariant alternative to Wick rotation can indeed be given if one introduces an observer field  $u^a$ , which is essentially a non-vanishing timelike field associated with the original Lorentzian spacetime  $(M, g_{ab})$ . Let  $\lambda$  be the parameter along  $u^a$ , and consider the class of metrics

$$\widehat{g}^{ab} = g^{ab} - \Theta u^a u^b, \quad (3)$$

with an arbitrary function,  $\Theta$ , that smoothly goes from  $\Theta = -2$  to  $\Theta = 0$ , with the signature of  $\widehat{g}^{ab}$  going from Euclidean to Lorentzian respectively. We take  $\widehat{g}^{ab}$  as the candidate metric with a Euclidean regime for  $\Theta < -1$  and Lorentzian regime for  $\Theta > -1$  while being degenerate for  $\Theta = -1$ . We call the co-dimension one hypersurface defined by  $\Theta = -1$

<sup>a</sup> e-mail: [raghvendra@imsc.res.in](mailto:raghvendra@imsc.res.in) (corresponding author)

<sup>b</sup> e-mail: [dawood@iitm.ac.in](mailto:dawood@iitm.ac.in)

as  $\Sigma_0$ . The above formalism was given in [1,2], motivated essentially by observation in Hawking and Ellis [8] (which corresponds to purely Euclidean metrics with  $\Theta = -2$ ). It goes beyond the conventional constructions which aim to obtain Euclidean counterparts of Lorentzian geometries because it describes geometries with both Euclidean and Lorentzian regimes. Several new features arise in the above formalism, which is not present in the conventional Wick rotation, including terms that have compact support on  $\Sigma_0$ . We refer the reader to [1] for a more detailed discussion relevant from the context of Euclidean quantum gravity and [2] for a discussion on how it results in a Euclidean action with an interesting mathematical structure. We can immediately apply this construction to the examples (1) and (2) discussed above, which should already highlight the key features and differences from the conventional case. For both cases, choose  $u^a = (1, 0, 0, 0)$  as the direction field. It is then easy to show that the metric  $\widehat{g}^{ab}$ , for both the examples, is a well-behaved Euclidean metric (it is, in fact, the same as  $g_{ab}$  except that  $-\widehat{g}^{00} = 1 + \Theta$ ). References [1,2,4] have studied the geometric aspects of curvature associated with geodesic congruences (characterizing freely falling frames) in well-known spacetimes [1] and the implications for Euclidean action and quantum gravity [2].

Given that Euclidean methods have most prominently been used in the study of thermal properties associated with the presence of horizons, in this paper, we analyze these thermal aspects using the above class of Euclidean metrics. More specifically, we shall focus on the computation of Euclidean entropy *à la* the physically well-motivated definition of Visser [9], and follow it up with an analysis of how the existence of a hypersurface  $\Sigma_0$  (across which Euclidean to Lorentzian transition occurs) affects the *holonomy* associated with certain loops. This holonomy is related to surface gravity for cases when  $\Sigma_0$  is chosen close to a horizon. This latter analysis is closely related to the recent work by Samuel [10], except for certain differences which we highlight. As will become evident, our analysis and results differ from the conventional Euclidean ones [5–7] through terms involving extrinsic curvature, which only sometimes vanish and might contribute non-trivially for non-stationary foliations. Motivated by a recent analysis by Samuel [10] based on similar considerations, we also analyze the compelling case of *holonomy* associated with loops that cross the hypersurface  $\Sigma_0$ . This holonomy is related to twice the surface gravity for cases when  $\Sigma_0$  is chosen close to a horizon.

The paper is structured as follows: In Sect. 2, we study Euclidean entropy within our formalism and present precise results for non-gravitational (scalar and electromagnetic) and gravitational (Einstein–Hilbert and the more general Lanczos–Lovelock) theories. In Sect. 3, we compute the holonomies associated with loops and evaluate them for the case of spacetime with horizons. Finally, in Sect. 4, we sum-

marize and briefly discuss the implications of our results. Since the results derived require a general expression for the curvature and its concomitants associated with  $\widehat{g}_{ab}$ , we present several of these in Appendix A. The expressions for the Kretschmann invariant and the Weyl tensor are of particular relevance, which might be of interest for further work along these lines. Before moving on to the main results, we would like to briefly highlight the following relevant features of the expressions presented in the Appendix: (i) These expressions make it clear that the behaviour of curvature scalars such as Ricci, Kretschmann, etc., do not exhibit any singularity in the limit  $\Theta = -1$  – the transition surface, although the metric is degenerate there. This absence of any curvature singularities is certainly reassuring. (However, note that this does not guarantee that quantities constructed from additional vector/tensor fields and their derivatives would remain finite.) (ii) For static spacetimes, the curvature scalars come out to be the same in both regimes (67), this explains the fact of how the usual Wick rotated results match with our formalism.

## 2 Euclidean actions and entropy

In this section, we will investigate the structure of Euclidean entropy within our formalism. As we shall see, this leads to non-trivial results that match standard ones for static cases but differ in more generic cases.

Our analysis and derivation will be tailored to the arguments sketched in Visser [9]. The basic idea there is physically well motivated and yields an expression for entropy which matches with Wald entropy [11,12] for a class of Lagrangians of the form  $L(g_{ab}, R_{abcd})$ . We summarise below some key facts of the original argument that will be directly relevant to our analysis here and refer the reader to Visser’s original work [9] for an extended discussion. Let  $L_E$  be the Euclidean Lagrangian constructed from  $L$  by Wick rotation,  $t \rightarrow it$ , which is well defined for static spacetimes. Let  $t_{ab}$  be the “stress–energy” tensor defined by

$$I = \int L \sqrt{-g} d^4x, \\ \delta_g I = -\frac{1}{2} \int t_{ab} \delta g^{ab} \sqrt{-g} d^4x. \quad (4)$$

Therefore, the object  $t_{ab}$  is the conventional metric stress–energy tensor if  $L$  is the matter Lagrangian. However, one may define  $t_{ab}$  similarly for gravitational Lagrangians as well, in which case we will obtain

$$t_{ab} = -2E_{ab} \quad (\text{gravitational Lagrangian}), \quad (5)$$

where  $E_{ab}$  represents the gravitational equation of motion tensor; for example, for Einstein–Hilbert Lagrangian,  $E_{ab} = (16\pi G)^{-1}G_{ab}$ . Given these definitions, the key observa-

tion made in [9] is that the difference between  $t_{ab}u^a u^b$  and  $-L_E$  is a measure of entropy contributed by the fields with Lagrangian  $L$ . Although the discussion in [9] separated out the Einstein–Hilbert part, as we will show below, this is not necessary.<sup>1</sup>

In this section, we will use the above setup and check how it works when the Euclidean regime is defined by the  $\Theta < -1$  domain of the metric  $\widehat{g}_{ab}$ . We will see that, in general, the entropy obtained by using the above method, but computed using our formalism of covariant Wick rotation, differs from known results due to the presence of foliation-dependent terms associated with the extrinsic curvature of  $\Sigma$ , which does not vanish in general (although we recover standard results when  $\Theta = -2$ , as expected). In particular, such terms will contribute, for instance, in the presence of non-stationary horizons and may have a physically relevant role in considerations such as the generalized second law.

To proceed with the calculation, we define, following [9], the so called “anomalous” entropy (density) as

$$S_{\text{anomalous}} = t_{ab}u^a u^b + L_E. \tag{6}$$

The tag “anomalous” was used in [9] since, as mentioned above, that work focussed on *deviations* from the Bekenstein–Hawking entropy  $S_{\text{BH}} = A/4$  in Einstein–Hilbert theory. We will keep the tag, but as we will see, there is, in fact, no need to separate out the Einstein–Hilbert part. We will analyze the above expression for some well-known Lagrangians, both gravitational and non-gravitational, and thereby deduce the structure of resultant entropy.

### 2.1 Scalar field theory

We start with the simplest example of a scalar field theory in curved space time, with the Lagrangian and the stress–energy tensor given by standard expressions

$$L = -\frac{1}{2}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi), \tag{7}$$

$$t_{ab} = \partial_a\phi \partial_b\phi - (1/2)g_{ab} \left( g^{ij}\partial_i\phi\partial_j\phi \right) - g_{ab}V(\phi). \tag{8}$$

The above Lagrangian, for metric  $\widehat{g}$ , becomes

$$\begin{aligned} \widehat{L} &= -\frac{1}{2}\widehat{g}^{ab}\nabla_a\phi\nabla_b\phi - V(\phi) \\ &= L + \frac{1}{2}\Theta (u^a\partial_a\phi)^2. \end{aligned} \tag{9}$$

<sup>1</sup> There are some crucial sign differences to take note of when comparing our definition with the one in [9]. The relative sign differences appear in: (i) the definition of  $L$ , (ii) the relation between  $L_E$  and  $L$ , and (iii) the identification of  $t_{ab}u^a u^b$  with  $\rho$  and  $t_{ab}$  with  $G_{ab}$ . Overall, the entropy expressions we quote must be multiplied by  $(-1)$  to give the proper entropy *a la* Visser [9].

From the given expressions, it trivially follows that

$$t_{ab}u^a u^b + \widehat{L}_{\Theta=-2} = 0. \tag{10}$$

Therefore, if we define  $L_E = \widehat{L}_{\Theta=-2}$ , we get

$$S_{\text{anomalous}} = 0. \tag{11}$$

Establishing the above analysis for more general scalar field Lagrangians is straightforward. However, it must be clear that unless there are higher derivative terms and/or curvature couplings, the extrinsic curvature terms will not explicitly appear in the final result.

### 2.2 Electromagnetic field theory

For the EM field, the Lagrangian and the stress–energy tensor are

$$L = -(1/4)g^{ac}g^{bd} F_{ab}F_{cd}, \tag{12}$$

$$t_{ab} = -F_{am}F_b^m + Lg_{ab}. \tag{13}$$

For the metric  $\widehat{g}$ , the Lagrangian becomes

$$\begin{aligned} \widehat{L} &= -(1/4)\widehat{g}^{ac}\widehat{g}^{bd} F_{ab}F_{cd} \\ &= L - \frac{1}{2}\Theta F_{am}F_b^m u^a u^b. \end{aligned} \tag{14}$$

Once again, if we define  $L_E = \widehat{L}_{\Theta=-2}$ , we get

$$S_{\text{anomalous}} = 0. \tag{15}$$

### 2.3 Vector field theory

The apparent triviality of the results above for scalar and electromagnetic cases is, in fact, deceptive, as can be illustrated by considering a more non-trivial example of a general vector field theory with Lagrangian and corresponding stress energy tensor

$$L_{\text{vector}} = g^{ab}g_{mn}\nabla_a V^m\nabla_b V^n, \tag{16}$$

$$\begin{aligned} t_{ab} &= -2(g_{mn}\nabla_a V^m\nabla_b V^n + g^{mn}\nabla_m V_a\nabla_n V_n) \\ &\quad + g_{ab}L_{\text{vector}}. \end{aligned} \tag{17}$$

The Euclidean Lagrangian  $L_E = \widehat{L}_{\Theta=-2}$ , where  $\widehat{L} = \widehat{g}^{ab}\widehat{g}_{mn}\nabla_a V^m\nabla_b V^n$ , and the anomalous entropy for this action becomes,

$$\begin{aligned} S_{\text{anomalous}} &= 8(a t_m V^m)^2 + 6t_l V^l a_i \nabla_{\mathbf{u}} V^i + 4(t_m \nabla_{\mathbf{u}} V^m)^2 \\ &\quad + 7K^l_m K_{ln} V^m V^n + 6K_j^a V^j t_m \nabla_a V^m. \end{aligned} \tag{18}$$

We must then conclude, along the lines of [9], that in this case, the “anomalous entropy” will generically be non-zero even for static spacetime and must be interpreted as the correction to the “(1/4) area” for spacetimes with horizons.

### 2.4 Einstein–Hilbert

We now apply the same method as above to gravitational Lagrangians, starting with the Einstein–Hilbert action  $\mathcal{L} = (16\pi G)^{-1}R$ . As stated in the introductory paragraph of this section, in this case,  $t_{ab} = -2G_{ab}/(16\pi G)$  and the Lagrangian  $\widehat{\mathcal{L}} = \widehat{R}/(16\pi G)$ , where  $G$  is the universal gravitational constant, we will use absolute units for the calculation purpose. Using the Eq. (53) explicitly given in the Appendix A and standard differential geometric identities, it is easy to prove that entropy density has additional foliation-dependent terms:

$$\begin{aligned} (16\pi G)S_{\text{anomalous}} &= t_{ab}u^a u^b + L_E \\ &= -2G_{ab}u^a u^b + \widehat{R}, \\ &= 2R_{ab}u^a u^b + 2K_{mn}K^{mn} \\ &\quad - 4\nabla_m a^m - 2K^2, \end{aligned} \tag{19}$$

where  $L_E$  is Euclidean Lagrangian constructed from Einstein–Hilbert Lagrangian by covariant Wick rotation. For static spacetime above expression reduce to

$$(16\pi G)S_{\text{anomalous}} = -2R_{ab}u^a u^b = -2\nabla_m a^m. \tag{20}$$

We get the associated entropy for static spacetime perceived by the accelerated congruence after integrating the above equation

$$(16\pi G)S_{\text{anomalous}} = \int S_{\text{anomalous}} \sqrt{g_E} d^4x = \frac{A}{4}. \tag{21}$$

The time integral in Eq. (21) reduces to multiplication by  $\beta$ (inverse temperature). Since only the spatial components of  $a^m$  are non-zero, the divergence becomes a three-dimensional one over  $\Sigma$ , which is converted to integration over its boundary  $\partial\Sigma$ . Using the Gauss divergence theorem, the Eq. (21) becomes,

$$(8\pi G)S_{\text{anomalous}} = \beta \int_{\partial\Sigma} \sqrt{\sigma} d^2x (N n_\mu a^\mu). \tag{22}$$

Where  $n_\mu$  is the normal to the surface  $\partial\Sigma$  and  $N$  is the lapse function, we have removed the minus sign according to the convention stated in footnote 1.  $N$  tends to zero if the boundary  $\partial\Sigma$  is a standard black hole horizon, and the quantity  $(N n_\mu a^\mu)$  will tend to a constant surface gravity  $\kappa$  and the using  $\beta\kappa = 2\pi$ , we get

$$S_{\text{anomalous}} = S_{BH} = A/4. \tag{23}$$

$A$  is the area of the horizon. Our formalism reproduces the usual  $A/4$  entropy law only for static spacetimes, for which our covariant alternative to Wick rotation reduces to the usual

Wick rotation. It must be evident that, in general, additional terms will involve extrinsic curvature.

### 2.5 Lanczos–Lovelock gravity

One of the most direct higher curvature generalizations of the Einstein–Hilbert Lagrangian is the so-called Lanczos–Lovelock (LL) Lagrangians, which become non-trivial in  $D > 4$  and share several features of the Einstein–Hilbert Lagrangian. In particular, yielding equations of motion are second order despite the appearance of higher curvature terms in the Lagrangian. These features arise from the extraordinary structure of these Lagrangians, reviewed at length in [13]. We refer the reader to this review for the derivation of various identities that we will use below.

In  $D$ -dimensions, the LL Lagrangian is given by the sum:

$$L = \sum_m c_m L_m. \tag{24}$$

$$L_m^{(D)} = \frac{1}{16\pi} \frac{1}{2^m} \delta_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} R_{a_1 b_1}^{c_1 d_1} \dots R_{a_m b_m}^{c_m d_m}, \tag{25}$$

where the tensor appearing in the right-hand side of the equation (25) is the completely antisymmetric determinant tensor defined as:

$$\delta_{j c_1 d_1 \dots c_m d_m}^{i a_1 b_1 \dots a_m b_m} = \det \begin{bmatrix} \delta_j^i & \delta_{c_1}^i & \dots & \delta_{d_m}^i \\ \delta_j^{a_1} & & & \\ \vdots & & \delta_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} & \\ \delta_j^{b_m} & & & \end{bmatrix} \tag{26}$$

for  $m \geq 0$ . The lowest order terms,  $m = 0, 1$  correspond to the cosmological constant and the Einstein–Hilbert action, respectively, as can be easily seen by expanding the alternating determinant. For  $m = 1$ ,  $L_1 = (16\pi)^{-1}R$ , the factor of  $16\pi$  in the definition of  $L_m$  essentially changes the right-hand side of equations of motion from the conventional  $8\pi T_{ab}$  to  $(1/2)T_{ab}$ . The equations of motion for a generic LL Lagrangian  $L = \sum_m c_m L_m$  are given by the following two equivalent forms:

$$E_b^a = \sum_m c_m E_{b(m)}^a = \frac{1}{2} T_b^a,$$

where

$$\begin{aligned} E_{j(m)}^i &= \frac{1}{16\pi} \frac{m}{2^m} \delta_{j d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} R_{a_1 b_1}^{i d_1} \dots R_{a_m b_m}^{c_m d_m} - \frac{1}{2} \delta_j^i L_m \\ &= -\frac{1}{2} \frac{1}{16\pi} \frac{1}{2^m} \delta_{j c_1 d_1 \dots c_m d_m}^{i a_1 b_1 \dots a_m b_m} R_{a_1 b_1}^{c_1 d_1} \dots R_{a_m b_m}^{c_m d_m}. \end{aligned} \tag{27}$$

We may now proceed with our computations in the following two steps:

1. Compute the Euclidean LL Lagrangian: This is easily done by replacing  $R_{cd}^{ab} \rightarrow \widehat{R}_{cd}^{ab}$  in the Eq. (25) above.
2. Compute  $E_j^i t_i u^j$ .

3. Compute the difference between the above two quantities, hence compute  $S_{\text{anomalous}}$ .

Right at the outset, it is obvious that the resultant expression will differ from Wald entropy [11, 12] due to the additional terms involving the extrinsic curvature tensor  $K_{ab}$ . We will discuss these terms momentarily. Before that, let us consider the trivial case of  $K_{ab} = 0$ , applicable to, say, the case of static Killing horizons. Using one of the relations given in the Appendix A (67)

$$\widehat{R}_{cd}^{ab} = R_{cd}^{ab} \tag{28}$$

The Euclidean version of the Lagrangian is:

$$\widehat{L}_m^{(D)} = L_m^{(D)}. \tag{29}$$

One can obtain anomalous entropy for the case of  $K_{ab} = 0$ ,  $S_{\text{anomalous}} = \widehat{L}_m^{(D)} - 2E_{00}$ , where  $E_{00} := E^a_b t_a u^b$ .

$$S_{\text{anomalous}} = -2\mathcal{R}_{00} \tag{30}$$

where  $\mathcal{R}_{ab}$  is defined by  $E_b^a = \mathcal{R}_b^a - (1/2)L\delta_b^a$  and  $\mathcal{R}_{00} = \mathcal{R}_b^a t_a u^b$ . It is obvious that  $\mathcal{R}_{ab}$  is the analog of the Ricci tensor for LL models and reduces to it for  $m = 1$ . The above expression is known to give correct entropy that matches with Wald entropy for Lovelock gravity [11, 14].

Let us now derive the general entropy relation for LL Lagrangian with non-vanishing extrinsic curvature. Substituting the Eq. (51) into the Eq. (25). The  $m$ th order LL Lagrangian for D- spacetime dimensions becomes

$$\widehat{L}_m^{(D)} = L_m^{(D)} + L_K + L_{\partial K} \tag{31}$$

$$L_K = \alpha \delta_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} \sum_{r=1}^m (-2\Theta)^r \binom{m}{r} K_{[a_1}^{c_1} K_{b_1]}^{d_1} \dots K_{[a_r}^{c_r} K_{b_r]}^{d_r} R_{a_{r+1} b_{r+1}}^{c_{r+1} d_{r+1}} \dots R_{a_m b_m}^{c_m d_m} \tag{32}$$

$$L_{\partial K} = +4\Theta \alpha m \delta_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} u^{[c_m} \nabla_{[a_m} K_{b_m]}^{d_m]} (R_{a_1 b_1}^{c_1 d_1} \dots R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}} + \sum_{r=1}^{m-1} (-2\Theta)^r \binom{m-1}{r} \times K_{[a_1}^{c_1} K_{b_1]}^{d_1} \dots K_{[a_r}^{c_r} K_{b_r]}^{d_r} R_{a_{r+1} b_{r+1}}^{c_{r+1} d_{r+1}} \dots R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}}).$$

The above equation gives Euclidean LL Lagrangian for  $\Theta = -2$ . We write the final expression for  $m$ th order anomalous entropy by using the Eqs. (31–33) and (27).

$$S_{\text{anomalous}} = -2\mathcal{R}_{00} + S_K + S_{\partial K} \tag{34}$$

$$S_K = \alpha \delta_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} \sum_{r=1}^m 4^r \binom{m}{r}$$

$$\times K_{[a_1}^{c_1} K_{b_1]}^{d_1} \dots K_{[a_r}^{c_r} K_{b_r]}^{d_r} R_{a_{r+1} b_{r+1}}^{c_{r+1} d_{r+1}} \dots R_{a_m b_m}^{c_m d_m} \tag{35}$$

$$S_{\partial K} = -8\alpha m \delta_{c_1 d_1 \dots c_m d_m}^{a_1 b_1 \dots a_m b_m} u^{[c_m} \nabla_{[a_m} K_{b_m]}^{d_m]} (R_{a_1 b_1}^{c_1 d_1} \dots R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}} + \sum_{r=1}^{m-1} 4^r \binom{m-1}{r} \times K_{[a_1}^{c_1} K_{b_1]}^{d_1} \dots K_{[a_r}^{c_r} K_{b_r]}^{d_r} R_{a_{r+1} b_{r+1}}^{c_{r+1} d_{r+1}} \dots R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}}), \tag{36}$$

where  $\alpha = \frac{1}{16\pi} \frac{1}{2^m}$ ,  $\binom{m}{r} = \frac{m!}{r!(m-r)!}$ . This entropy relation is much more general in the sense that it contains additional terms apart from the term that gives the Bekenstein–Hawking entropy for static Killing horizons in four dimensions. For future work, it would be interesting to compare the terms we obtain with similar terms arising in other approaches to computing entropy. The closest to ours seems to be the approach sketched in [15]. (Similar terms also appear, for instance, in the discussion of holographic entanglement entropy – see [16–18]. However, there does not seem to be any obvious connection between our analysis and these approaches.) One distinctive feature of the additional terms in our expression for entropy is the presence of terms with derivatives of extrinsic curvature.

### 3 Holonomy of closed loops

It has long been known that thermal effects associated with horizons can be understood in terms of holonomy about certain loops in the Euclidean spacetime, obtained by setting  $t \rightarrow it$ , for a chosen time coordinate  $t$ . For example, for Rindler horizons in flat spacetime,  $t$  is chosen to be the proper time of an accelerated observer, while in Schwarzschild, it is the time coordinate that appears in the standard form of the metric. More recently, in [10], it was shown that demanding the holonomy of null curves in the Euclidean spacetime to be trivial indeed gives the standard temperature associated with these spacetimes. Motivated by this, we aim to study the holonomy of a special class of loops in spacetimes given by  $\widehat{g}_{ab}$ , particularly when the loop crosses the transition surface  $\Sigma_0$  so that part of it lies in the Euclidean domain. Our setup a priori does not seem to bear any direct relation to the work in [10], although it is in a similar spirit. Moreover, there might be a curious connection that should be apparent from the final result and comments presented at the end of this section. Since accelerated observers play the central role as far as thermal effects are concerned, we need to consider  $a^i \neq 0$ . We note that for a small rectangle with sides given by  $u^i$  and  $S^m := a^m/|a|$ , the area form associated with the loop is  $\Sigma^{mn} = u^{[m} S^{n]}$ .

### 3.1 Loops in Euclidean regime

It is straightforward to compute holonomy about such loops as mentioned above by using the expression for the Christoffel connection  $\widehat{\Gamma}_{bc}^a$  given in Appendix A. We will discuss this in the next section, but before proceeding, we analyze the standard expression for change of a vector, say  $X^i$ , about such a loop in terms of the curvature tensor. This analysis should give a rough idea about the additional terms that might arise due to  $\Theta$  and  $\dot{\Theta}$  terms in the curvature tensor:  $\widehat{\delta X^i} = \widehat{R}^i_{bcd} X^b \Sigma^{cd} \delta u \delta s$ , where  $\delta u$  and  $\delta s$  are parameters along  $u^i$  and  $S^i$  respectively. From the previously established identities, it is easy to see that

$$\begin{aligned} \widehat{R}^i_{bcd} X^b \Sigma^{cd} &= R^i_{bcd} X^b \Sigma^{cd} + \Theta \left( -R_{abcd} u^a u^i \Sigma^{cd} \right. \\ &+ u^i \nabla_b |a| - \frac{1}{1 + \Theta} g^{ai} t_b \nabla_a |a| + F t_b u^i \nabla_u |a| \\ &+ S_b u^i |a|^2 - \frac{1}{1 + \Theta} t_b S^i |a|^2 \left. \right) X^b \\ &+ \frac{\dot{\Theta}}{2} \left( K_{bm} u^i S^m - \frac{1}{1 + \Theta} t_b K^i_m S^m \right) X^b. \end{aligned} \tag{37}$$

The above expression simplifies considerably in static spacetimes if one chooses  $u^i$  in the direction of the timelike Killing vector. Using various standard identities (see, for example, [19]), the above expression then reduces to

$$\begin{aligned} \widehat{R}^i_{bcd} X^b \Sigma^{cd} &= R^i_{bcd} X^b \Sigma^{cd} \\ &- \underbrace{F S^i t_b X^b (|a|^2 + S^m \nabla_m |a|)}_{\text{additional term}} \\ &\text{(Static Killing Foliation)} \end{aligned} \tag{38}$$

where  $F = \Theta/(1 + \Theta)$  and  $F = 2$  in the Euclidean regime with  $\Theta = -2$ . The additional term above, which depends purely on acceleration, is worth exploring further in some physically relevant spacetimes. Let us consider static spherically symmetric spacetime, described by the standard line element

$$ds^2 = -B(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2d\Omega^2, \tag{39}$$

where  $B(r)$  is an arbitrary function such that  $B'(r)$  vanish at infinity and  $B(r)$  has a zero at some finite radius:  $B(r_0) = 0$ . In this case, the previous expression reduces to

$$\widehat{R}^i_{bcd} X^b \Sigma^{cd} = R^i_{bcd} X^b \Sigma^{cd} + (F/2) \left( S^i t_b X^b \right) {}^{(2)}R_{t,r}, \tag{40}$$

where  ${}^{(2)}R_{t,r} = -B''(r)$  is the curvature scalar of the two dimensional space  $\theta, \phi = \text{constant}$ .

We will now highlight a possible connection of the additional term above with the relationship between Euclidean holonomy and temperature, particularly with the discussion in [10]. Let us choose our vector  $X^i$  to be  $u^i$  and imagine moving this vector about a loop in the Euclidean domain ( $\Theta = -2, F = +2$ ) defined by a rectangular region in the  $t - r$  plane bounded by  $t = t_1, t = t_1 + \beta, r = r_0, r = b$ . Here,  $\beta > 0$  is a constant parameter and consider  $b > r_0$  to be some large radius (below, we assume  $b \rightarrow \infty$ ). The area measure of such a loop is  $dt dr$  (the  $B(r)$  factor cancels out), and the integration of the last term in Eq. (40) gives,

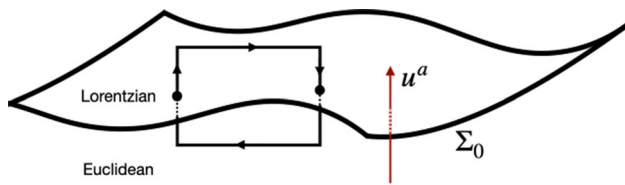
$$S_i \delta X^i = -\beta B'(r_0). \tag{41}$$

This is an instructive result. For spacetimes of the above form (39), the quantity  $B'(r_0) = 2\kappa$  where  $\kappa$  is the surface gravity of the horizon defined by  $B(r_0) = 0$ . The RHS above is, therefore, of magnitude  $2\beta\kappa$ . Now, the surface gravity at the horizon is  $2\pi T_H$ , therefore  $\beta B'(r_0) = 2\pi$  provided one chooses  $\beta = (2T_H)^{-1}$ . The extra factor 2 is directly related to the identification of the period of Euclidean time ( $\pi$  vs.  $2\pi$ ) and has been noted before in a different context in [20]. It has also been recently discussed in the semi-classical consideration of Hawking radiation in terms of quantum tunneling [21, 22].

The above analysis, though suggestive, there are following points that make it restrictive: First, let us point out that while the last term Eq. (40) has been written in a nice geometric interpretation (with no approximations made), the connection we have highlighted with surface gravity and the range of time integration  $\beta$  depends on the choice of the vector and the loop. It is unclear how to interpret Eq. (40) for a generic case. Second, the expression for change of vector in terms of Riemann tensor holds only for small loops, but we have here taken  $b \rightarrow \infty$  so that the contribution from the  $r = b$  vanishes. Essentially, what we have given is an interpretation for the contribution of this term due to the presence of the horizon at  $r = r_0$ . The last point we note is that the discussion above is tied to static horizons, and it is important to repeat it for stationary horizons; this would require generalizing the whole analysis to the case when  $u^a$  is not hypersurface orthogonal. Some results useful for this are given in Appendix C, but the Riemann tensor seems more challenging to obtain, and we leave it for future work.

### 3.2 Loops straddling the transition surface

As a more interesting case, we now comment on loops that straddle the transition surface  $\Sigma_0$ , so that part of these loops lies in the Euclidean regime; see Fig. 1.



**Fig. 1** Loops straddling the transition surface  $\Sigma_0$

While using an analysis similar to the one in the preceding section, one must be careful since the metric  $\hat{g}$  is degenerate on  $\Sigma_0$ ; therefore, the area measure of the loop needs to be appropriately defined. However, a more immediate analysis can be presented in terms of the connection itself, which is given in Appendix A.

Let us choose our vector  $X^i$  to be such that  $X^i t_i = 0$  everywhere in the region of interest and, similarly, let  $s^i$  be a properly normalized vector orthogonal to  $u^i$ . Imagine parallel transporting  $X^i$  about the loop in Fig. 1, whose legs are defined by tangents  $u^i$  and  $s^i$ . Then, we can estimate the change in the vector using the expression for the connection, which reads (see Eq. (50) in (Appendix A):

$$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc} + F \left[ (1 + \Theta) u^a K_{(bc)} - a^a t_b t_c \right] - (1/2) \dot{F} (1 + \Theta) t_b t_c u^a. \tag{42}$$

Above the surface,  $\Theta = 0 = F$ , while  $\Theta = -2, F = +2$  below the surface. Therefore, the legs of the loop tangential to the surface will give different contributions to the change in vector, and the additional contribution from the Euclidean domain is easily shown to yield

$$t_i \delta X^i = 2 \left( K_{ab} s^a s^b \right) \delta s, \tag{43}$$

where  $\delta s$  is the parameter along  $s^i$ . Although instructive, we cannot say anything further about a generic interpretation of the above result. Moreover, we have assumed that the contribution of the legs normal to the surface can be made arbitrarily small (say, by letting  $\delta u \rightarrow 0$ ). However, since the metric is becoming degenerate on  $\Sigma_0$ , how to handle the divergent  $(1 + \Theta)^{-1}$  terms is not very clear. At best, we can evaluate the above quantity in a simple spacetime such as the one in Eq. 39 with a suitable choice of  $u^a$  and see if it yields anything sensible. For this purpose, we consider the region  $r < r_0$  of this spacetime and describe this in new coordinate  $\tilde{t} = r, \tilde{r} = t$ , in which the metric becomes

$$ds^2 = -\frac{1}{\tilde{B}(\tilde{t})} d\tilde{t}^2 + \tilde{B}(\tilde{t}) d\tilde{r}^2 + \tilde{t}^2 d\Omega^2 \tag{44}$$

where  $\tilde{B}(\tilde{t}) = -B(\tilde{t})$  and  $\tilde{t} < r_0$ . Thus, for Schwarzschild, we will have  $\tilde{B}(\tilde{t}) = r_0/\tilde{t} - 1$ . As before, we focus on the two-dimensional plane with  $\theta, \phi = \text{constant}$ . A trivial computation then gives

$$K_{\tilde{r}\tilde{r}} = \frac{1}{2} \sqrt{\tilde{B}} \frac{\partial \tilde{B}}{\partial \tilde{t}}, \quad \delta s = \sqrt{\tilde{B}} \delta r, \tag{45}$$

$$K_{ab} s^a s^b \delta s = \left( \frac{1}{2} \frac{\partial \tilde{B}}{\partial \tilde{t}} \right) \delta r. \tag{46}$$

If we choose the transition surface as  $\tilde{t}_0 = r_0 - \epsilon$  and evaluate everything at  $\tilde{t} = r_0$ , it is obvious that  $\partial \tilde{B} / \partial \tilde{t} |_{\tilde{t}_0} = -2\kappa$  and the expression for change of vector now becomes  $t_i \delta X^i = -2\kappa \beta$  with  $\delta r = \beta$ . This is the same as what we had obtained in the previous section (the minus sign is easy to understand since here, the time coordinate  $\tilde{t}$  decreases from  $r_0$  to 0 as we go into the Euclidean regime).

In this section, we have sought to demonstrate a fascinating connection between holonomies about loops in space(time)s with distinct Euclidean and Lorentzian regimes. We emphasize that the calculations presented above are rigorous, including the factor of two. Our analysis shows that one can naturally extract quantities such as temperature by working within the covariant formulation for Euclideanization without considering complex values of the time coordinate, very much in the spirit of [10].

#### 4 Discussion and conclusions

In this work, we study the mathematical setup of covariant Wick rotation with the aim of studying thermal effects in curved spacetimes within this setup, and comparing and contrasting the results with those obtained through conventional Euclidean methods. In this section, we will summarize the key results of this paper:

1. We have obtained the expression for entropy for Einstein as well as the more general Lanczos–Lovelock class of gravitational actions, along the lines of generic arguments in [9]. Our results reproduce the Wald entropy formula for stationary foliations [11, 12], but otherwise have additional interesting terms that depend on derivatives of extrinsic curvature. To the best of our knowledge, such terms do not arise in any other method for computing entropy. In particular, even the conventional Bekenstein–Hawking entropy in Einstein gravity gets corrected by these additional terms.
2. Having discussed the concept of entropy using our formalism, we turned to explore the concept of temperature using our Euclidean method. To understand how our approach might work when used to study thermal effects associated with accelerated or black hole horizons, we computed the holonomy of some chosen vectors about the certain class of curves, including ones that straddle the transition surface separating Euclidean and Lorentzian domains. Interestingly, the result comes close to the standard expression for surface gravity, except for a factor 2 ambiguity.

3. The appendix of this paper contains explicit mathematical expressions for the curvature tensor, the Weyl tensor and Kretschmann scalar corresponding to the class of Euclidean geometries studied here. From the expression for the Ricci scalar, Appendix (53), we conclude that (i) the boundary terms to Einstein–Hilbert action are independent of the acceleration of the observers. (ii) Except when the spacetime foliation has vanishing extrinsic curvature, there is no valid reason to consider  $R$  (or  $-R$ ) to be the Euclidean Lagrangian. The additional terms will not only affect classical geometrical variables, but they may also affect quantum mechanically since the Euclidean action appears explicitly in the phase of the saddle point approximation to a system’s ground state wave function [23].

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors’ comment: There are no data associated with this manuscript.]

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Funded by SCOAP<sup>3</sup>. SCOAP<sup>3</sup> supports the goals of the International Year of Basic Sciences for Sustainable Development.

### Appendix A: The curvature tensors associated with $\widehat{g}$

**Conventions:** Latin indices run over  $0, \dots, n$  while Greek indices  $\alpha, \beta, \dots$ , etc. run over  $1, \dots, n$ . Except when indicated otherwise, we choose the units with  $c = 1, \hbar = 1$ . Lorentzian metric signature is  $(-, +, +, +)$  and Euclidean metric signature is  $(+, +, +, +)$ . Our convention for the extrinsic curvature of a hyper-surface is  $K_{ab} = \nabla_a u_b + u_a a_b$ .

**Useful formulae:** We list some useful formulae associated with the class of metrics studied in this paper. The expressions below extend the ones given in [1] to include the acceleration of  $u^a$ .

$$\widehat{g}^{ab} = g^{ab} - \Theta u^a u^b, \quad \widehat{g}_{ab} = g_{ab} + F t_a t_b, \\ g_{ab} u^b = t_a, \quad g_{ab} u^a u^b = -1. \tag{47}$$

$$F = \frac{\Theta}{1 + \Theta}, \quad \dot{F} = \frac{\dot{\Theta}}{(1 + \Theta)^2}. \tag{48}$$

$$\nabla_a F = -\dot{F} t_a, \quad \nabla_a \dot{F} = -\ddot{F} t_a - \dot{F} a_a. \tag{49}$$

Here  $u^m$  is the observer’s velocity vector,  $a^m$  is the observer’s acceleration vector, i.e.,  $a^m = u^i \nabla_i u^m$  and  $a_m = g_{mn} a^n$ . The Christoffel connection is given by

$$\widehat{\Gamma}^a_{bc} = \Gamma^a_{bc} + C^a_{bc}, \\ C^a_{bc} = F \left[ (1 + \Theta) u^a K_{(bc)} - a^a t_b t_c \right] \\ - (1/2) \dot{F} (1 + \Theta) t_b t_c u^a. \tag{50}$$

**The curvature tensors associated with  $\widehat{g}$ :** It is a lengthy, though straightforward exercise to compute the various geometrical quantities associated with the metric  $\widehat{g}_{ab}$  (47). We listed out the useful formulae above. Using them, we present the curvature tensor and its concomitants associated with  $\widehat{g}_{ab}$ .

The Riemann tensor turns out to be

$$\widehat{R}_{ab}{}^{cd} = R_{ab}{}^{cd} + 2\Theta \left( -u^{[c} R_{abm}{}^{d]m} - K_{[a}{}^d K_{b]}{}^c \right. \\ \left. + 2t_{[a} a_{b]} a^{[c} u^{d]} + 2u^{[c} (\nabla_{[a} a^{d]}) t_{b]} \right) + 2\dot{\Theta} u^{[c} K_{[a}{}^{d]} t_{b]}. \tag{51}$$

We may similarly write down the expressions for Ricci and Einstein tensors and the Ricci scalar. We quote the final expressions below:

$$\widehat{R}^a_c = (1 + \Theta) R^a_c - \Theta \left( {}^{(3)}R^a_c - t_c C^a + t_c a^b K_b^a - a^a a_c \right. \\ \left. - g^{la} h_l^r \nabla_r a_c + u^a t_c \nabla_b a^b \right) \\ + (1/2) \dot{\Theta} (\pi_c^a + K \delta_c^a), \tag{52}$$

$$\widehat{R} = (1 + \Theta) R + \Theta \left( -{}^{(3)}R + 2\nabla_b a^b \right) + \dot{\Theta} K, \tag{53}$$

$$\widehat{G}^a_c = (1 + \Theta) G^a_c - \Theta \left( {}^{(3)}G^a_c + (1/2) {}^{(3)}R u^a t_c - t_c C^a \right. \\ \left. - t_c a^b K_b^a - a^a a_c + u^a t_c \nabla_b a^b - g^{la} h_l^r \nabla_r a_c \right) \\ + (1/2) \dot{\Theta} \pi_c^a, \tag{54}$$

where we have used Gauss–Codazzi and Gauss–Weingarten equations,  $C^m = D_a K^{am} - D^m K$ , with  $D_m$  the natural covariant derivative that acts on tangent vectors to the hypersurfaces  $\Sigma_t$ , and  $\pi_b^a = K_b^a - K h_b^a$ ,  $h_{ab}$  being the induced metric on  $\Sigma_t$ .

We also quote a few alternate forms for the Riemann tensor and its concomitants; that help simplify certain expressions

$$\widehat{R}^i_{bcd} = R^i_{bcd} + 2\Theta \left( K^i_{[c} K_{d]b} - 2t_{[c} a_{d]} u^i a_b \right. \\ \left. + \frac{2}{1 + \Theta} (t_b t_{[c} \nabla^i a_{d]} - u^i t_{[c} \nabla_b a_{d]} - a^i t_b a_{[c} t_{d]}) \right. \\ \left. - F u^i t_b t_{[c} \nabla_u a_{d]} - F u^i t_{[c} \nabla_b a_{d]} \right) \\ - \frac{\dot{\Theta}}{1 + \Theta} \left( u^i t_{[c} K_{d]b} - t_{[c} K_{d]}^i t_b + \Theta t_{[c} K_{d]b} u^i \right). \tag{55}$$



Above form is useful in the calculation for Eqs. (37) and (59). A few steps using Gauss–Codazzi equations allow us to write the curvature tensors completely in terms of the extrinsic curvature and its derivative.

$$\widehat{R}_{ab}{}^{cd} = R_{ab}{}^{cd} + 2\Theta \left( 2u^{[c}\nabla_{[a}K_{b]}{}^{d]} - K_{[a}{}^d K_{b]}{}^c \right) + 2\dot{\Theta}u^{[c}K_{[a}{}^{d]}t_{b]}, \tag{56}$$

$$\widehat{R}_a{}^c = R_a{}^c + \Theta \left( u^c\nabla_a K - u^c\nabla_b K_a{}^b - u^b\nabla_a K_b{}^c + u^b\nabla_b K_a{}^c - K_a{}^b K_b{}^c + K K_a{}^c \right) + \frac{\dot{\Theta}}{2}(K_a{}^c - t_a u^c K), \tag{57}$$

$$\widehat{R} = R + \Theta(2\nabla_{\mathbf{u}}K + K_{mn}K^{mn} + K^2) + \dot{\Theta}K. \tag{58}$$

Next, we discuss quantities of direct physical significance that can be immediately constructed from the above expressions. In particular, we quote the expressions for the tidal part of the Riemann tensor, Kretschmann scalar, and the Weyl tensor associated with  $\widehat{g}$ . These expressions were not given in the closed form in previous literature but are expected to be of prominent significance from the physical interpretation of the geometry described by  $\widehat{g}$ .

**Tidal tensor:** From the above, we can immediately write down the components of the Tidal part of the Riemann tensor, defined by  $E^i{}_d := R^i{}_{bcd}u^b u^c$

$$\widehat{E}^i{}_d = E^i{}_d + F \left( g^{ai}\nabla_a a_d + u^i\nabla_{\mathbf{u}}a_d - t_d a_c K^{ic} - a^i a_d \right) + \frac{\dot{\Theta}}{1 + \Theta} K^i{}_d, \tag{59}$$

where  $F = \Theta/(1 + \Theta)$  and  $\nabla_{\mathbf{u}}a_d = u^k\nabla_k a_d$ . Let us consider  $\xi^i$  be a vector orthogonal to  $t_i$  (dual of  $u^i$  given in Appendix A), so that  $\xi^i t_i = 0$ . This vector could, for example, represent deviation between members of the congruence  $u^i$ . From the above expression for tidal tensor, it immediately follows that

$$\widehat{\mathcal{A}}^i = \widehat{E}^i{}_d \xi^d = \mathcal{A}^i + F \left( g^{ai}\xi^d\nabla_a a_d + \xi^d u^i\nabla_{\mathbf{u}}a_d - a^i \xi^d a_d \right) + \frac{\dot{\Theta}}{1 + \Theta} K^i{}_d \xi^d, \tag{60}$$

where  $\mathcal{A}^i = E^i{}_d \xi^d$ . The component of  $\widehat{\mathcal{A}}^i$  orthogonal to  $u^i$  is then given by  $\widehat{\mathcal{A}}^i_{\perp} = \widehat{\mathcal{A}}^i + (\widehat{\mathcal{A}}^k t_k)u^i$ , and quickly checking that  $\widehat{\mathcal{A}}^k t_k = \mathcal{A}^k t_k$ , we obtain

$$\widehat{\mathcal{A}}^i_{\perp} = \mathcal{A}^i_{\perp} + \frac{\Theta}{1 + \Theta} \left( h^{ai}\xi^d\nabla_a a_d - a^i \xi^d a_d \right) + \frac{\dot{\Theta}}{1 + \Theta} K^i{}_d \xi^d, \tag{61}$$

where  $h^{ai} = g^{ai} + u^a u^i$  is the standard projector. The astute reader would have noticed that the quantity  $\mathcal{A}^i_{\perp}$  we have constructed above is precisely the deviation acceleration associated with the congruence when  $a^i = 0$ . One needs to consider the Fermi acceleration for an accelerated congruence, which can be easily done, but we skip it. What is worth noticing here is that in the Euclidean regime ( $\Theta = -2, F = 2$ ), for non-geodesic congruences, there is already an additional term in the deviation acceleration solely due to the signature change of the metric. Of course, to extract a direct physical measure of this acceleration, one must properly consider the normalization of vectors in the Euclidean sector. However, this is straightforward, and we do not state it here.

**Kretschmann scalar:**

$$\widehat{S} = S + \Theta \left( 8R_{ab}{}^{cd}u^{[a}\nabla_{[c}K_{d]}{}^{b]} - 4R_{ab}{}^{cd}K_{[c}{}^b K_{d]}{}^a \right) + 4\Theta^2((\nabla_{\mathbf{u}}K_b{}^d)(\nabla_{\mathbf{u}}K_b{}^d) + 2(\nabla_{\mathbf{u}}K_b{}^d)K^{ad}K_{ba} + K^{db}K_{cd}K^{ac}K_{ba} + \frac{1}{2}(K_{mn}K^{mn})^2 - \frac{1}{2}K^{cb}K_{ac}K^{da}K_{bd}) + 2\Theta\dot{\Theta} \left( K_c{}^b\nabla_{\mathbf{u}}K_b{}^c + K^{ac}K_{ba}K_c{}^b \right) + 2\dot{\Theta}u^{[a}K_{[c}{}^{b]}R_{ab}{}^{cd} + \dot{\Theta}^2 K^{bd}K_{bd}. \tag{62}$$

**Weyl tensor:** Writing the expression for Weyl tensor is much more tedious, though we write four dimensional Weyl tensor using the Eqs. (51, 52) as follows,

$$\widehat{W}_{abcd} = W_{abcd} + \Theta \left( \frac{1}{3}(K^2 - K_{mn}K^{mn} + R + 2\nabla_m a^m - 2R_{mn}u^m u^n)(g_{a[c}g_{d]b} + 2Fg_{[a[c}t_{d]}t_{b]}) + \frac{2R}{3(1 + \Theta)}g_{[a[c}t_{d]}t_{b]} - \frac{2}{1 + \Theta}t_{[a}t_{[c}R_{d]b]} - 2Kg_{[a[c}K_{d]b]} - 2g_{[a[c}u^m\nabla_m K_{d]b]} + \frac{2}{1 + \Theta}g_{[a[c}t_{d]}t_{b]}\nabla_l a^l - 2a^m(K_{m[c}g_{d][a}t_{b]} + K_{m[a}g_{b][c}t_{d]}) - 2FKt_{[a}t_{[c}K_{d]b]} - 2Ft_{[a}t_{[c}n^l\nabla_l K_{d]b]} - 2K_{d[a}K_{b]c} + \frac{8}{1 + \Theta}t_{[c}(\nabla_{[a}a_{d]})t_{b]} + \frac{8}{1 + \Theta}t_{[d}t_{[a}a_{b]}a_{c]} \right) - \dot{\Theta} \left( -\frac{K}{3}g_{a[c}g_{d]b} - \frac{2\Theta + 3}{3(1 + \Theta)}Kg_{[a[c}t_{d]}t_{b]} + g_{[a[c}K_{d]b]} + \frac{2 + \Theta}{1 + \Theta}K_{[a[c}t_{d]}t_{b]} \right), \tag{63}$$

where we have used anti symmetric index notation e.g.  $K_{[a[c}t_{d]}t_{b]} = -\frac{1}{4}(K_{ac}t_d t_b - K_{ad}t_c t_b + K_{bd}t_c t_a - K_{bc}t_d t_a)$ . The above expression for the Weyl tensor looks complicated,

so we try to write it in a simpler form by the following expression,

$$\begin{aligned} \widehat{W}_{ab}{}^{cd} &= W_{ab}{}^{cd} + \Theta(4u^{[c}\nabla_{[a}K_{b]}^{d]} \\ &\quad - 2K_{[a}^dK_{b]}^c - 2\delta_{[a}^{[c}u^{d]}\nabla_{b]}K + 2\delta_{[a}^{[c}u^{d]}\nabla_mK_b^m \\ &\quad + 2\delta_{[a}^{[c}u^m\nabla_{b]}K_m^{d]} - 2\delta_{[a}^{[c}u^m\nabla_mK_{b]}^{d]} + 2\delta_{[a}^{[c}K_{b]}^mK_m^{d]} \\ &\quad - 2\delta_{[a}^{[c}K_{b]}^{d]}K \\ &\quad + \frac{1}{3}\delta_{[a}^c\delta_{b]}^d(2u^m\nabla_mK + K_{mn}K^{mn} + K^2)) \\ &\quad + \dot{\Theta}\left(2u^{[c}K_{[a}^{d]}t_{b]} + \delta_{[a}^{[c}t_{b]}u^{d]}K - \delta_{[a}^{[c}K_{b]}^{d]} + \frac{1}{3}K\delta_{[a}^c\delta_{b]}^d\right). \end{aligned} \tag{64}$$

The above expression clearly shows that, generally, a conformally flat geometry  $g$  will not be mapped to a conformally flat  $\widehat{g}$ . The additional terms are characterized by extrinsic curvature of the hypersurfaces orthogonal to  $u^a$ . It will be interesting to understand the consequences of this property, specifically in the context of early universe cosmology. From this point of view, let us consider the illustrative example of the standard FLRW geometry

$$ds^2 = -dt^2 + a^2(t)d\Omega_{(k)}^2 \quad (k = -1, 0, 1), \tag{65}$$

where  $a(t)$  is scale factor. We choose as our congruence the vector field is  $t_m = -\partial_m t$ . A quick calculation gives  $K_m^n = \nabla_m u^n = (\dot{a}/a)h_m^n$ ,  $K = 3\dot{a}/a$  and plugging this in Eq. (64) gives

$$\widehat{W}_{ab}{}^{cd} = W_{ab}{}^{cd} = 0. \tag{66}$$

The above result would most easily be obtained by writing down  $\widehat{g}$  and noticing that it is easily put in a conformally flat form. However,  $\widehat{W}_{ab}{}^{cd}$  will be non-vanishing in the Euclidean regime of FLRW for an arbitrary  $u^a$ . As stated above, it will be interesting to extract the physical significance of this in the context of quantum cosmology.

**A note on static spacetime:** From the Eqs. (56–58), (62) and (64), It is easy to conclude that the following identities hold for static spacetime:

$$\begin{aligned} \widehat{R}_{ab}{}^{cd} &= R_{ab}{}^{cd}, \quad \widehat{R}_c^a = R_c^a, \quad \widehat{R} = R, \\ \widehat{G}_c^a &= G_c^a, \quad \widehat{S} = S, \quad \widehat{W}_{ab}{}^{cd} = W_{ab}{}^{cd}. \end{aligned} \tag{67}$$

### Appendix B: Examples

We first discuss the example where the metric is time-independent. In this case, our results match the expected Wick rotation results. Then, we illustrate the time-dependent

case, where there is no straightforward way to apply Wick rotation while keeping the spacetime metric real.

#### B.1 Accelerated observers in anti-de sitter space

A similar example can be illustrated for de-sitter space, though we are considering the accelerated observers in Anti-de sitter space. The embedding equation of Anti-de Sitter space in flat 5-dimensional space is,

$$-(z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 - (z^4)^2 = -\ell^2. \tag{68}$$

Global coordinates are provided by writing the general solution to the equation as,

$$z^0 = \ell \cosh \rho \sin \tau, \quad z^\alpha = \ell \omega^\alpha \sinh \rho, \quad z^4 = \ell \cosh \rho \cos \tau, \tag{69}$$

where  $\delta_{\alpha\beta}\omega^\alpha\omega^\beta = 1$ . Then one finds the metric

$$ds^2 = \ell^2(-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2), \tag{70}$$

with  $0 \leq \rho < \infty$  and  $-\infty < \tau < +\infty$ . Let us choose the spacetime foliation by the observers whose tangent vectors are always in the direction of the global timelike Killing vector,  $u^a = (\frac{1}{\ell \cosh \rho}, 0, 0, 0)$ . These are clearly accelerated observers with  $a_m = (0, \tanh \rho, 0, 0)$ . We write Ricci tensor, Ricci scalar, and Einstein tensor, respectively, as follows, using  $\nabla_m a^m = \frac{3}{\ell^2}$ ,  ${}^{(3)}R = -\frac{6}{\ell^2}$ ,  $K = 0$ .

$$\widehat{R}_c^a = -\frac{3}{\ell^2}\delta_c^a; \quad \widehat{R} = -\frac{12}{\ell^2}; \quad \widehat{G}_c^a = \frac{3}{\ell^2}\delta_c^a. \tag{71}$$

One can notice that all the quantities of the above equation are  $\Theta$  independent; the reason behind it is that the velocity vector is always in the direction of the timelike Killing vector. In general, for orthogonal foliation, if we choose the velocity vector in the direction of the timelike Killing vector, the foliation will always be extrinsically flat [19]. Moreover, the Eq. (67) will hold for extrinsically flat foliations. Another thing to notice is that the Euclidean metric for (70) is again maximally symmetric.

#### B.2 Accelerated observers in time-dependent spherically symmetric spacetime

Any spherically symmetric metric can locally be expressed in the following form

$$ds^2 = \gamma_{AB}(x^A)dx^A dx^B + \tilde{r}^2(x^A)d\Omega^2, \quad A, B \in \{0, 1\}. \tag{72}$$

There exist special fiducial observers called Kodama observers in any time-dependent spherically symmetric metric [24,25]. Given the metric (72), it is possible to introduce the Kodama vector field  $\mathbf{k}$ , and those components are

$$k^A(x) = \frac{1}{\sqrt{-\gamma}} \varepsilon^{AB} \partial_B \tilde{r}, \quad k^\theta = k^\phi = 0. \tag{73}$$

From the above Eq. (73), we conclude that the Kodama observers are characterized by the condition  $\tilde{r} = C(\tilde{r}_0)$ , where  $C$  is constant. Furthermore, the remarkable corresponding conserved current is  $J^a = G^a_b k^b$  [26].

Let us consider an example of metric (72) by considering the following metric of de Sitter space for a comoving observer,

$$ds^2 = -dt^2 + e^{2Ht} (dr^2 + r^2 d\Omega_2^2). \tag{74}$$

Consider the observers (Kodama Observers) stay at a fixed distance from its cosmological horizon and move along the trajectory  $re^{Ht} = C$  (where  $C$  is constant) with four-velocity in the direction of Kodama vector  $k^a = (-1, Hr, 0, 0)$ ,

$$u^a = \frac{k^a}{|\mathbf{k}|} = \frac{k^a}{\sqrt{1 - H^2 C^2}}. \tag{75}$$

These observers foliation space(time) into orthogonal hypersurfaces with acceleration,

$$a_a = \left( \frac{H^3 C^2}{H^2 C^2 - 1}, \frac{H^2 C^2 r^{-1}}{H^2 C^2 - 1}, 0, 0 \right). \tag{76}$$

One can calculate the curvature tensor and its concomitants possessed by  $\hat{g}$  by using Eqs. (51–64). We write the following

$$\hat{R} = 12H^2, \quad G_{ab} u^a u^b = 3H^2. \tag{77}$$

There is a locally conserved current  $J^a$  in terms of the Einstein tensor and the Kodama vector,

$$\hat{J}^a = \hat{G}^a_b k^b = (3H^2, -3H^3 r, 0, 0). \tag{78}$$

By using the relation (47), We write the metric as

$$ds^2 = - \left( 1 - \frac{F}{1 - H^2 C^2} \right) dt^2 + \frac{2F H^2 C^2 r^{-1}}{1 - H^2 C^2} dt dr + e^{2Ht} \left( \left( 1 + \frac{F H^2 C^2}{1 - H^2 C^2} \right) dr^2 + r^2 d\Omega_2^2 \right), \tag{79}$$

where  $F = \Theta / (1 + \Theta)$ , this gives the Euclidean metric with real entries for  $F = 2$  ( $\Theta = -2$ ). Contrary to this, the usual Wick rotation gives the complex metric.

### Appendix C: The case of non-hypersurface orthogonal $u^a$

Let us now consider arbitrary foliation. In this case, the basic definition of  $F, \dot{F}$  remains the same as above, but the key differences arise in the gradient of various functions

$$\nabla_a \Theta = -\dot{\Theta} t_a + f a_a, \quad \nabla_a F = -\dot{F} t_a + f' a_a, \tag{80}$$

$$\nabla_m \dot{F} = f' A_m + f' a_b \nabla_m u^b - (\dot{F} - \dot{f}') a_m - \ddot{F} t_m, \tag{81}$$

where  $f$  is some smooth scalar and  $f'$  and  $A_m$  are given as follows

$$f' = \frac{f}{(1 + \Theta)^2}, \quad A_m = \nabla_u a_m. \tag{82}$$

And the difference in Christoffel connection  $\dot{C}^a_{bc}$  for Nonorthogonal case is given in terms of  $C^a_{bc}$  of Orthogonal foliation,

$$\begin{aligned} \dot{C}^a_{bc} = C^a_{bc} - F K^a_{(b} t_{c)} + F K^a_{(b} t_{c)} \\ + f' (a_{(b} t_{c)} u^a (1 + \Theta) - a^a t_{b} t_{c}). \end{aligned} \tag{83}$$

Unlike the orthogonal case,  $K_{mn}$  is not symmetric here i.e.  $K_{mn} = K_{(mn)} + w_{mn}$ . We write Ricci scalar associated by  $\hat{g}$  in terms of quantities associated with  $g$ , assuming that the function  $\Theta$  changes are in the direction of the observer's tangent vector and their direction of acceleration.

$$\begin{aligned} \hat{R} = R + \Theta \left( K^2 + \nabla_m a^m + \nabla_n K + F w_{ab} w^{ab} - R_{ac} u^a u^c \right) \\ + \dot{\Theta} K + \frac{f'}{2} \left( 2(1 + \Theta) \nabla_m a^m + (1 + \Theta)^2 a^2 + a^2 \right. \\ \left. + a^2 f' - a^2 \Theta^2 \right) + a^m (1 + \Theta) \nabla_m f'. \end{aligned} \tag{84}$$

### References

1. D. Kothawala, Action and observer dependence in Euclidean quantum gravity. *Class. Quantum Gravity* **35**, 03LT01 (2018). [arXiv:1705.02504](https://arxiv.org/abs/1705.02504)
2. D. Kothawala, Euclidean action and the Einstein tensor. *Phys. Rev. D* **97**, 124062 (2018). [arXiv:1802.07055](https://arxiv.org/abs/1802.07055)
3. M. Visser, How to Wick rotate generic curved spacetime. [arXiv:1702.05572](https://arxiv.org/abs/1702.05572)
4. A. Baldazzi, R. Percacci, V. Skrinjar, Wicked metrics. *Class. Quantum Gravity* **36**, 105008 (2019). [arXiv:1811.03369](https://arxiv.org/abs/1811.03369)
5. G.W. Gibbons, S.W. Hawking (eds.), *Euclidean Quantum Gravity* (World Scientific, Singapore, 1993)
6. S.W. Hawking, W. Israel, *General Relativity: An Einstein Centenary Survey* (Cambridge University Press, Cambridge, 1979)
7. S.W. Hawking, W. Israel, *300 Years of Gravitation* (Cambridge University Press, Cambridge, 1987)
8. S.W. Hawking, G.F.R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973)

9. M. Visser, Dirty black holes: entropy versus area. *Phys. Rev. D* **48**, 583 (1993). [arXiv:hep-th/9303029](#)
10. J. Samuel, Wick rotation in the tangent space. *Class Quantum Gravity* **33**, 015006 (2016). [arXiv:1510.07365](#)
11. R.M. Wald, Black hole entropy is Noether charge. *Phys. Rev. D* **48**, 3427–3431 (1993). [arXiv:gr-qc/9307038](#)
12. V. Iyer, R. Wald, Some properties of the Noether charge and a proposal for dynamical black hole entropy. *Phys. Rev. D* **50**, 846–864 (1994). [arXiv:gr-qc/9403028](#)
13. T. Padmanabhan, D. Kothawala, Lanczos–Lovelock models of gravity. *Phys. Rep.* **531**, 115 (2013). [arXiv:1302.2151](#)
14. T. Padmanabha, Surface density of spacetime degrees of freedom from equipartition law in theories of gravity. *Phys. Rev. D* **81**, 124040 (2010). [arXiv:1003.5665](#)
15. D.V. Fursaev, A. Patrushev, S.N. Solodukhin, Distributional geometry of squashed cones. *Phys. Rev. D* **88**(4), 044054 (2013). [arXiv:1306.4000](#)
16. X. Dong, Holographic entanglement entropy for general higher derivative gravity. *JHEP* **01**, 044 (2014). [arXiv:1310.5713](#)
17. L.-Y. Hung, R.C. Myers, M. Smolkin, On holographic entanglement entropy and higher curvature gravity. *JHEP* **04**, 025 (2011). [arXiv:1101.5813](#)
18. P. Bueno, J. Camps, A.V. López, Holographic entanglement entropy for perturbative higher-curvature gravities. *JHEP* **04**, 145 (2021). [arXiv:2012.14033](#)
19. F. Dahia, P.J. Felix da Silva, Static observers in curved spaces and non-inertial frames in Minkowski spacetime. *Gen. Relativ. Gravit.* **43**, 269–292 (2011). [arXiv:1004.3937](#)
20. G. 't Hooft, Ambiguity of the equivalence principle and Hawking's temperature. *J. Geom. Phys.* **1**, 45–52 (1984)
21. E.T. Akhmedov, V. Akhmedova, D. Singleton, Hawking temperature in the tunneling picture. *Phys. Lett. B* **642**, 124–128 (2006). [arXiv:hep-th/0608098](#)
22. P. Mitra, Hawking temperature from tunnelling formalism. *Phys. Lett. B* **648**, 240–242 (2007). [arXiv:hep-th/0611265](#)
23. J.B. Hartle, S.W. Hawking, Wave function of the Universe. *Phys. Rev. D* **28**, 2960 (1983)
24. R. Casadio, S. Chiodini, A. Orlandi, G. Acquaviva, R. Di Criscienzo, L. Vanzo, On the Unruh effect in de Sitter space. *Mod. Phys. Lett. A* **26**, 2149–2158 (2011). [arXiv:1011.3336](#)
25. G. Abreu, M. Visser, Kodama time: geometrically preferred foliations of spherically symmetric spacetimes. *Phys. Rev. D* **82**, 044027 (2010). [arXiv:1004.1456v3](#)
26. H. Kodama, Conserved energy flux for the spherically symmetric system and the backreaction problem in the black hole evaporation. *Prog. Theor. Phys.* **63**, 1217 (1980)