



Analytical investigations on non-minimally coupled scalar fields outside neutral reflecting shells

Yan Peng^a

School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China

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Abstract We study the existence of scalar fields outside neutral reflecting shells. We consider static massive scalar fields non-minimally coupled to the Gauss–Bonnet invariant. We analytically investigated properties of scalar fields through the scalar field equation. In the small scalar field mass regime, we derive a compact resonance formula for the allowed masses of scalar fields in the composed scalar field and shell configurations.

1 Introduction

The famous black hole no hair theorem has attracted a lot of attention from physicists and mathematicians for decades. If true, it states that asymptotically flat black holes cannot support static scalar field hairs outside horizons, for progress see references [1–19] and reviews [20, 21]. This no hair property was usually attributed to the existence of absorbing boundary conditions at black hole horizons. So it is of great interest to examine whether such no hair behavior can appear in horizonless spacetimes.

In the horizonless spacetime, Hod recently proved a new type no hair theorem that static scalar field hairs cannot exist outside asymptotically flat neutral reflecting stars [22, 23]. Furthermore, such no hair theorem also holds for systems constructed with static scalar fields and neutral horizonless reflecting shells [24, 25]. In the asymptotically dS spacetimes, the static scalar field hair also cannot exist outside neutral horizonless reflecting stars [26]. And further studies showed that such no scalar hair theorem is a general property for horizonless objects with reflecting boundary conditions [27–32].

However, asymptotically flat black holes can support static scalar fields non-minimally coupled to electromagnetic Maxwell fields, which violates the spirit of no hair

theorems [33, 34]. Similarly, Hod proved that scalar fields can exist outside charged horizonless reflecting shells when considering non-minimally couplings between static scalar fields and electromagnetic Maxwell fields [35]. In particular, Hod derived a remarkably compact resonance formula for the allowed masses of the spatially regular scalar fields supported by charged shell configurations. In the background of neutral horizonless reflecting shells, exterior scalar fields also can exist when including non-minimally couplings between scalar fields and the Gauss–Bonnet invariant [36]. Inspired by analysis in [35], we plan to carry out an analytical investigation on configurations composed of scalar fields and neutral reflecting shells in the scalar-Gauss–Bonnet gravity.

This work is organized as follows. We firstly introduce the system with scalar fields non-minimally coupled to the Gauss–Bonnet invariant outside neutral horizonless reflecting shells. Then we shall derive a remarkably compact resonance formula for the allowed masses of scalar fields supported by neutral reflecting shells. We will summarize our main results in the last section.

2 Scalar field equations in the reflecting shell background

We consider scalar fields non-minimally coupled to the Gauss–Bonnet invariant. The Lagrangian density of the scalar-Gauss–Bonnet gravity is given by [37–43]

$$\mathcal{L} = R - |\nabla_\alpha \Psi|^2 - \mu^2 \Psi^2 + f(\Psi) \mathcal{R}_{GB}^2. \quad (1)$$

Here R is the Ricci scalar curvature, Ψ is the scalar field with mass μ and $f(\Psi) \mathcal{R}_{GB}^2$ describes the coupling, where \mathcal{R}_{GB}^2 is the Gauss–Bonnet invariant [39, 40]

$$\mathcal{R}_{GB}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \quad (2)$$

^ae-mail: yanpengphy@163.com (corresponding author)

and $f(\Psi)$ is a general function of Ψ . When neglecting matter fields' backreaction on the metric, the Gauss–Bonnet invariant term is $\mathcal{R}_{GB}^2 = \frac{48M^2}{r^6}$. We put the function $f(\Psi)$ in a simple form $f(\Psi) = \eta\Psi^2$ in the linear limit, where η is the coupling strength parameter. We emphasize that this choice of the coupling function is physically motivated since it guarantees the existence of hairless solutions.

The spherically symmetric spacetime is characterized by the curved line element [40]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3}$$

The radial coordinate depending function $g(r)$ is the metric solution. The angular coordinates are labeled as θ and ϕ respectively. And the radius of the shell is defined as r_s .

With variation methods, we obtain the scalar field equation [37–43]

$$(\nabla^\alpha \nabla_\alpha - \mu^2 + \eta \mathcal{R}_{GB}^2)\Psi = 0. \tag{4}$$

We choose to study a stationary massive scalar field in the form

$$\Psi(r, \theta, \phi) = \sum_{lm} \frac{\psi_{lm}(r)}{r} S_{lm}(\theta) e^{im\phi} \tag{5}$$

with l, m representing the integer harmonic parameters. $S_{lm}(\theta)$ is the angular scalar eigenfunction with the eigenvalue $l(l + 1)$, where l is the spherical harmonic index [35]. For simplicity, we label the characteristic radial function $\psi_{lm}(r)$ as $\psi(r)$.

The Eq. (4) and field decomposition (5) yield the ordinary differential equation

$$\left[\frac{d^2}{dr^2} - \mu^2 - \frac{l(l + 1)}{r^2} + \frac{48\eta M^2}{r^6} \right] \psi = 0. \tag{6}$$

Here we take the metric outside shells in the flat spacetime limit with $g = 1$ and $g' = 1$. In particular, the present model includes a coupling of the scalar field to the Gauss–Bonnet invariant but the metric functions used in Eq. (6) are valid for a flat spacetime. Thus, the present model is only a toy-model.

At the shell radius, the scalar field satisfies reflecting surface conditions

$$\psi(r_s) = 0. \tag{7}$$

At the infinity, the scalar field is spatially regular, leading to the vanishing condition

$$\psi(\infty) = 0. \tag{8}$$

3 Analytical formula for the allowed masses of scalar fields

We investigate on properties of scalar fields outside reflecting shells through the ordinary differential equation (6). The equation can be analytically studied in two regions $r \ll 1/\mu$ and $r \gg (\sqrt{\eta}M)^{1/2}$. In the small scalar field mass regime $\mu r \ll 1$, we can analyze the system in the overlapping region $(\sqrt{\eta}M)^{1/2} \ll r \ll 1/\mu$ [35]. In the overlapping region, we apply matching methods to obtain a formula of the masses of the non-minimally coupled scalar fields outside reflecting shells.

In the limit $r \ll 1/\mu$, the ordinary differential equation (6) can be expressed as

$$\left[\frac{d^2}{dr^2} - \frac{l(l + 1)}{r^2} + \frac{48\eta M^2}{r^6} \right] \psi = 0. \tag{9}$$

The general mathematical solution of the Eq. (9) is

$$\begin{aligned} \psi(r) = & A_1 r^{\frac{1}{2}} J_{\frac{1}{4} + \frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r^2} \right) \\ & + A_2 r^{\frac{1}{2}} Y_{\frac{1}{4} + \frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r^2} \right), \end{aligned} \tag{10}$$

where $J_\nu(z)$ and $Y_\nu(z)$ are Bessel functions of the first and second kinds respectively. And the coefficients A_1 and A_2 are normalization constants.

In the limit $z \rightarrow 0$, the Bessel functions asymptotically behave as s (see Eqs. 9.1.7 and 9.1.9 of [44])

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1 + \nu)} \cdot [1 + O(z^2)], \tag{11}$$

$$Y_\nu(z) = -\frac{\Gamma(\nu)}{\pi(z/2)^\nu} \cdot [1 + O(z^2)]. \tag{12}$$

In the radial overlap region $(\sqrt{\eta}M)^{1/2} \ll r \ll 1/\mu$, there is the relation $\frac{\sqrt{\eta}M}{r^2} \ll 1$ and solution (10) of the scalar field equation (9) can be expressed as

$$\begin{aligned} \psi(r) = & A_1 \frac{(\sqrt{3}\sqrt{\eta}M)^{\frac{1}{2}l + \frac{1}{4}}}{\Gamma(\frac{1}{2}l + \frac{5}{4})} r^{-l} \\ & - A_2 \frac{\Gamma(\frac{1}{2}l + \frac{1}{4})}{\pi (\sqrt{3}\sqrt{\eta}M)^{\frac{1}{2}l + \frac{1}{4}}} r^{l+1}. \end{aligned} \tag{13}$$

In the limit $r \gg (\sqrt{\eta}M)^{1/2}$, the ordinary differential equation (6) can be approximated by

$$\left[\frac{d^2}{dr^2} - \mu^2 - \frac{l(l + 1)}{r^2} \right] \psi = 0. \tag{14}$$

The general mathematical solution of (14) can be expressed by the Bessel functions in the form

$$\psi(r) = B_1 r^{\frac{1}{2}} J_{l+\frac{1}{2}}(i\mu r) + B_2 r^{\frac{1}{2}} Y_{l+\frac{1}{2}}(i\mu r), \tag{15}$$

where the coefficients B_1 and B_2 are normalization constants.

According to behaviors (11) and (12), Eq. (15) can be mathematically expressed as

$$\psi(r) = B_1 \frac{(i\mu/2)^{l+\frac{1}{2}}}{\Gamma(l+\frac{3}{2})} \cdot r^{l+1} - B_2 \frac{\Gamma(l+\frac{1}{2})}{\pi(i\mu/2)^{l+\frac{1}{2}}} \cdot r^{-l} \tag{16}$$

for the scalar field in the overlap region $(\sqrt{\eta}M)^{1/2} \ll r \ll 1/\mu$. Therefore, two analytically derived mathematical expressions (13) and (16) for the scalar field $\psi(r)$ are both valid in the intermediate radial region

$$(\sqrt{\eta}M)^{1/2} \ll r \ll 1/\mu. \tag{17}$$

This fact allows us to match the solutions (13) and (16) in the region (17) and obtain relations between coefficients A_i and B_i as

$$B_1 = -A_2 \frac{\Gamma(\frac{1}{2}l + \frac{1}{4}) \Gamma(l + \frac{3}{2})}{\pi} \cdot \left(-\frac{4}{\sqrt{3}\sqrt{\eta}\mu^2 M} \right)^{\frac{1}{2}l + \frac{1}{4}}, \tag{18}$$

$$B_2 = -A_1 \frac{\pi}{\Gamma(\frac{1}{2}l + \frac{5}{4}) \Gamma(l + \frac{1}{2})} \cdot \left(-\frac{\sqrt{3}\sqrt{\eta}\mu^2 M}{4} \right)^{\frac{1}{2}l + \frac{1}{4}}. \tag{19}$$

In the large-argument ($z \rightarrow \infty$), the Bessel functions behave as (see Eqs. 9.2.1 and 9.2.2 of [44])

$$\sqrt{\eta}\mu^2 M = \frac{4}{\sqrt{3}} (-1)^{\frac{1}{2}l+1} \left[\frac{\Gamma(l+\frac{1}{2}) \Gamma(l+\frac{3}{2}) \Gamma(\frac{1}{2}l+\frac{1}{4}) \Gamma(\frac{1}{2}l+\frac{5}{4}) J_{\frac{1}{4}+\frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} \right)}{\pi^2 Y_{\frac{1}{4}+\frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} \right)} \right]^{\frac{2}{2l+1}}. \tag{26}$$

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \cdot \left[\cos\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \right] \cdot \left[1 + O(z^{-1}) \right], \tag{20}$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} \cdot \left[\sin\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \right] \cdot \left[1 + O(z^{-1}) \right], \tag{21}$$

and the solution (15) has the asymptotic functional behavior

$$\psi(r \rightarrow \infty) = B_1 \sqrt{\frac{2}{i\pi\mu}} \cos\left(i\mu r - \frac{1}{2}l\pi - \frac{1}{2}\pi\right) + B_2 \sqrt{\frac{2}{i\pi\mu}} \sin\left(i\mu r - \frac{1}{2}l\pi - \frac{1}{2}\pi\right). \tag{22}$$

The asymptotic behavior (22) and the boundary condition (8) yield the relation

$$B_2 = i B_1. \tag{23}$$

With relations (18), (19) and (23), we derive the relation between A_i in the form

$$\left(\frac{3\eta\mu^4 M^2}{16} \right)^{\frac{1}{2}l + \frac{1}{4}} = i \left[\frac{\Gamma(l+\frac{1}{2}) \Gamma(l+\frac{3}{2}) \Gamma(\frac{1}{2}l+\frac{1}{4}) \Gamma(\frac{1}{2}l+\frac{5}{4})}{\pi^2} \right] \times \frac{A_2}{A_1}. \tag{24}$$

We investigate on properties of scalar fields in the regime $\mu r \ll 1$. So there is the relation $\mu r_s \ll 1$. The expression (10) holds at the shell surface r_s satisfying $\mu r_s \ll 1$. The boundary condition (7) and the solution (10) yield expression of A_2/A_1 as

$$\frac{A_2}{A_1} = -\frac{J_{\frac{1}{4}+\frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} \right)}{Y_{\frac{1}{4}+\frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} \right)}. \tag{25}$$

Substituting the ratio A_2/A_1 of (25) into relation (24), we arrive at the resonance equation

The relation (17) yields the relation

$$\sqrt{\eta}\mu^2 M \ll 1. \tag{27}$$

According to Eq. (26), the relation (27) implies

$$\frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} \approx j_{\frac{1}{4}+\frac{1}{2}l,n}, \tag{28}$$

where $\{j_{\nu,n}\}_{n=1}^{\infty}$ are positive zeros of the Bessel function $J_\nu(x)$.

The Bessel functions can be expanded around the positive zeros in the form (see Eq. 9.1.27d of [44])

$$J_{\frac{1}{4}+\frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} \right) = -J_{\frac{5}{4}+\frac{1}{2}l} \left(j_{\frac{1}{4}+\frac{1}{2}l,n} \right) \cdot \Delta_n \cdot [1 + O(\Delta_n)] \tag{29}$$

and

$$Y_{\frac{1}{4}+\frac{1}{2}l} \left(\frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} \right) = Y_{\frac{1}{4}+\frac{1}{2}l} \left(j_{\frac{1}{4}+\frac{1}{2}l,n} \right) \cdot [1 + O(\Delta_n)], \tag{30}$$

where Δ_n is the small quantity defined as

$$\Delta_n = \frac{2\sqrt{3}\sqrt{\eta}M}{r_s^2} - j_{\frac{1}{4}+\frac{1}{2}l,n} \ll 1. \tag{31}$$

Substituting the expansions (29) and (30) into (26), one obtains the formula

$$\sqrt{\eta}\mu^2 M = \frac{4}{\sqrt{3}}(-1)^{\frac{1}{2l+1}} \left[\frac{\Gamma(l+\frac{1}{2})\Gamma(l+\frac{3}{2})\Gamma(\frac{1}{4}l+\frac{1}{4})\Gamma(\frac{1}{4}l+\frac{5}{4})J_{\frac{5}{4}+\frac{1}{2}l}(j_{\frac{1}{4}+\frac{1}{2}l,n})}{\pi^2 Y_{\frac{1}{4}+\frac{1}{2}l}(j_{\frac{1}{4}+\frac{1}{2}l,n})} \Delta_n \right]^{\frac{2}{2l+1}}, \tag{32}$$

which shows that the allowed values of scalar field masses are discrete.

4 Conclusions

We studied the existence of scalar fields outside neutral reflecting shells in the flat spacetime limit. We considered static massive scalar fields non-minimally coupled to the Gauss–Bonnet invariant. We investigated on properties of scalar fields outside reflecting shells through the ordinary differential equation (6). The system is amenable to an analytical analysis in two regions $r \ll 1/\mu$ and $r \gg (\sqrt{\eta}M)^{1/2}$, where η is the coupling parameter, M is the shell mass and μ is the scalar field mass. In the small scalar field mass regime $\mu r \ll 1$, we can analyze the system in the overlapping region $(\sqrt{\eta}M)^{1/2} \ll r \ll 1/\mu$. In the overlapping region, we applied matching methods to derive a remarkably compact resonance formula (32) for the allowed masses of the supported spatially regular scalar fields outside neutral horizonless reflecting shells.

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