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Radiating stars and Riccati equations in higher dimensions

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Abstract The objective of this study is to investigate spherically symmetric radiating stars undergoing gravitational collapse, in higher dimensional general relativity, inclusive of acceleration, expansion, shear, an electromagnetic field and a cosmological constant. Methods that can be used to obtain exact solutions to the boundary condition with/without a linear equation state are studied. Two distinct approaches are investigated. In the first approach, the boundary condition is expressed as a Riccati equation in terms of one of the dependent variables, and restrictions are placed to obtain new exact solutions. In the second approach, transformations that map the boundary condition into a new Riccati equation are investigated. The resulting new transformed equation is solved, by placing restrictions on the coefficients, to obtain new exact models. Special properties of the transformation are shown when appropriate restrictions on the parameters of the transformation are placed. This allows the order of the boundary condition to be reduced from a second order partial differential equation into a first order partial differential equation. The versatility of the transformation on other equations is exhibited when new solutions to the system of equations consisting of both the boundary condition and equation of state are obtained. When the dimension is set to four, some known solutions are recovered. It is shown that horizons can be identified by using a special case of the transformation. Our results elucidates the importance of the use of transformations that map the coordinates of differential equations into new and different coordinate systems.

1 Introduction

Only approximately 4.9% of the known universe can be explained with known physical theories, and on the quantum scale the theories of general relativity and quantum physics do not reconcile [1]. As a result there is an abundance of research that focuses on modified gravity theories in efforts to better understand the universe. Some modified gravity theories, such as the Kaluza-Klein theory [2], Lovelock gravity theories [3] and higher dimensional Einstein gravity theories, involve introducing extra dimensions to the three spatial and one temporal dimensions in general relativity. The inclusion of extra dimensions has been inspired and influenced by string theory which requires more than four dimensions. A summary of some of the modified gravity theories can be found in [4]. Higher dimensions are likely to have existed in the early universe and compactification has led to the present observed four dimensions. There is no conclusive evidence that additional dimensions survive in the late universe. However, it is known that higher dimensions affect the geometry of spacetime and the matter distribution. For a recent study of the energy conditions in higher dimensions see the treatment of Brassel et al. [5], which shows the effect on physical quantities in higher dimensional astrophysical fluids. As a result of this, it is plausible to consider all possible scenarios until there is conclusive observational evidence. There are current research projects that attempt to discover additional dimensions. Examples of these experiments include the Compact Muon Solenoid (CMS) experiment [6] and A Toroidal Large Hadron Collider ApparatuS (ATLAS) experiment [7] at the Large Hadron Collider in CERN. These two experiments search for “Z” and “W” like particles, gravitons and quantum black holes.

The number of dimensions affects physical features of astrophysical objects in higher dimensional general relativity. Arbañil and Malheiro [8] studied the effect extra dimensions have on equilibrium configurations and radial pulsations of

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compact objects, Harko and Mak [9] investigated the effect dimensionality has on the upper limits of radius and mass in a charged anisotropic fluid sphere, and Burikham et al. [10] showed that for a stable spherically symmetric compact object in higher dimensions there is a minimum and maximum mass to radius ratio. In this paper we will look at a model of a spherically symmetric radiating star undergoing gravitational collapse in higher dimensional general relativity. Studying effects of higher dimensions on radiating stars should reveal insights as to how dimensions affect certain features, such as the gravitational potential functions, in higher dimensional general relativity. Maharaj and Brassel [11] showed that the gravitational collapse of a radiating object is affected by dimension. Our interest is in modelling a radiating star in higher dimensions.

There are three fundamental components that are important in the modelling of radiating stars: the interior spacetime, the exterior spacetime and the junction condition. The Vaidya metric [12] models the exterior spacetime of a radiating star and has several applications in astrophysics. It was extended to higher dimensions by Iyer and Vishveshwara [13]. The Vaidya metric in higher dimensions was extended to include an electromagnetic field by Chatterjee et al. [14], and extended to include the cosmological constant by Saa [15]. The interior manifold for superdense stars in higher dimensions was studied by Patel and Singh [16].

Static stars have no heat flux since the pressures at the boundary are zero. Static stars in higher dimensional general relativity with a Vaidya-Tikekar metric were studied by Paul [17] and Chattopadhyay et al. [18]. The Vaidya exterior caters for non-static stars when heat flux is present. The presence of heat flux complicates the model, as this results in a boundary condition that relates the pressure and heat flux. This boundary condition has to be solved to complete the model of the radiating star. There are few models of the boundary condition with a Vaidya exterior for stars in higher dimensions. Studies of the boundary condition in higher dimensions with the presence of heat flux is an underdeveloped area. The junction condition in four dimensions for radiating stars exclusive of shear was first obtained by Santos [19]. The junction condition in higher dimensions for a shear-free interior line element exclusive of charge with a Vaidya exterior was studied by various authors [20–22]. Maharaj and Brassel [11] obtained the junction condition in higher dimensions for a charged shearing composite matter interior spacetime with a generalised Vaidya exterior spacetime. The generalised Vaidya solution has been studied in other alternative gravity theories such as Lovelock gravity [23,24]. The boundary condition for stars with heat flux in some higher dimensional modified theories of gravity such as $F(r,t)$ gravity, Gauss-Bonnet gravity or Lovelock gravity are currently unknown [25]. Our main interest is to model a radiating star with barotropic matter in the interior. The inclusion of scalar fields would lead to a new model with additional scalar wave equations. This would describe boson stars, quark or quark-diquark stars [26]. This is an interesting problem to consider, and we will pursue this in future work.

Higher dimensional general relativity reduces to four dimensional general relativity when constraints on dimensions are imposed. There are several distinct approaches that were used to obtain exact solutions to the four dimensional junction condition which include: performing a Lie symmetry analysis on the junction condition [27–34], treating the junction condition as a Riccati equation [35–37], and using transformations to transform the junction condition to different coordinates that results in the junction condition been expressed as a new, simpler, Riccati equation [38–42].

The main intention of this paper is to obtain exact solutions to the boundary condition of a star with a shearing interior line element and a Vaidya exterior in higher dimensions. There are currently no solutions to this boundary condition. Searching the databases of Scopus, arxiv and google scholar using the parameters “higher dimensions” and “boundary condition” or “higher dimensions” and “junction condition”, we find no papers that presented solutions of the boundary condition with shear. This adds to the novelty of the study as finding such solutions has physical significance. There are some solutions to the shear-free models with a Vaidya exterior in higher dimensions. These simpler shear-free cases are contained in [20–22]. The physical significance is enhanced by introducing a linear equation of state to the model. There are currently no known solutions to the boundary condition that admit an equation of state, in such stars inclusive and exclusive of shear in higher dimensions. Searching the databases of Scopus, arxiv and google scholar using the parameters “higher dimensions”, “boundary condition” and “equation of state” or “higher dimensions”, “junction condition” and “equation of state” we find no such solutions.

The boundary condition we are solving has not been solved before; therefore any method to obtain solutions to such an equation will be new. Here, we use a systematic approach to solve this equation, by using two methods that involve treating the equation as a Riccati equation. It is important to note that both of these methods are successful in obtaining exact solutions on the different analytic versions of the four dimensional boundary condition. The first method we use is fairly simple, and is included in this paper to highlight the importance of the second method we use. In the first approach, we express the higher dimensional boundary condition as a Riccati equation; and in the second method we use a transformation, introduced by Naidoo et al [42], to get a *new* simpler Riccati equation. There were numerous transformations used on the different analytic forms of the four dimensional boundary condition to simplify and obtain exact solutions to the equation. However, we only need to utilise the transformation by Naidoo et al [42], as all the transformations previously used [38–41] are contained therein. The transformation has special features when used on the four dimensional boundary condition; we aim to investigate how these features are translated through higher dimensions. This allows us to obtain insights into the higher dimensional boundary condition, which highlights the significance of this approach. The transformation has not been used on any of the higher dimensional boundary conditions before, and further adds to the novelty of this study. We also obtain solutions to the boundary condition that admits a linear equation of state using the transformation. By using approaches established earlier to solve the four dimensional boundary condition, to solve the higher dimensional boundary condition we are able to identify how successful these methods are in obtaining solutions to different types of equations. The effect dimension has on these approaches is clearly evident.

This paper is divided as follows: in Sect. 2 we describe our model for the radiating star in higher dimensions. We express the higher dimensional boundary condition as a Riccati equation, and obtain solutions in Sect. 3. In Sect. 4 we apply the transformation listed in [43] to the higher dimensional junction condition. We obtain exact solutions to the transformed equations by restricting the boundary to a linear equation in Sect. 5. In Sect. 6 we obtain exact solutions by restricting the boundary condition to a Bernoulli equation. In Sect. 7 we show how we can reduce the order of the junction condition from a second order partial differential to a first order partial differential equation by placing restrictions on arbitrary parameters. In Sect. 8 we introduce an equation of state and obtain exact solutions to the system of partial differential equations consisting of both an equation of state and junction condition in higher dimensional general relativity. We show how the presence of horizons may be identified by using a special case of the transformation in Sect. 9. Finally, in Sect. 10, we provide a discussion of the results obtained in this paper.

2 Model

The interior line element of a spherically symmetric n-dimensional spacetime is given by

$$ds^2 = -A^2 \mathrm{d}t^2 + B^2 \mathrm{d}r^2 + Y^2 \mathrm{d}\Omega^2_{n-2}, \tag{1}$$

where A, B and Y are arbitrary functions of r and t, and

$$\mathrm{d}\Omega^2_{n-2} = \sum_{i=1}^{n-2} \left(\prod_{j=1}^{i-1} \sin^2(\theta_j) \right) (\mathrm{d}\theta_i)^2 . \tag{2}$$

When $Y = rB$ the shear-free line element is regained. The comoving fluid n-velocity

$$u^a = \left(\frac{1}{A}, 0, \ldots, 0 \right), \tag{3}$$

is a timelike and unit vector. Then, we obtain the fluid n-acceleration as

$$\dot{u}^a = \left(0, \frac{A_r}{AB^2}, 0, \ldots, 0 \right), \tag{4}$$

the expansion scalar as

$$\Theta = \frac{1}{A} \left(\frac{B_t}{B} + (n-2) \frac{Y_t}{Y} \right), \tag{5}$$

and the magnitude of the shear scalar as

$$\sigma = \frac{1}{\sqrt{n-1}} \frac{1}{A} \left(\frac{Y_t}{Y} - \frac{B_t}{B} \right), \tag{6}$$

where subscripts denote partial differentiation. We note that the vorticity vanishes.

The energy momentum tensor $\boldsymbol{T}$ that describes the interior matter field is given by

$$T_{ab} = (\rho + p) u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} + E_{ab}, \tag{7}$$

where ρ represents the density, p represents the isotropic pressure, $\boldsymbol{q}$ represents the heat flux, $\boldsymbol{\pi}$ represents the anisotropic stress and the tensor $\boldsymbol{E}$ represents the electromagnetic field tensor. The heat flux $\boldsymbol{q}$ and anisotropic stress tensor $\boldsymbol{\pi}$ can be defined respectively as

$$q^a = \left(0, \frac{1}{B} q, 0, \ldots, 0 \right), \tag{8a}$$

$$\pi_{ab} = \left(p_\parallel - p_\perp \right) \left(n_a n_b - \frac{1}{n-1} h_{ab} \right), \tag{8b}$$

where $p_\parallel$, $p_\perp$, $\boldsymbol{n}$ and $\boldsymbol{h}$ represent the radial pressure, tangential pressure, unit spacelike vector and the projection tensor, respectively. The isotropic pressure can be written in terms of the radial and tangential pressures via

$$p = \frac{1}{n-1} \left(p_\parallel + (n-2) p_\perp \right). \tag{9}$$

The pressure is isotropic when $p_\parallel = p_\perp = p$.

The Einstein–Maxwell equations can be expressed as [11]

$$G_{ab} = \kappa_n T_{ab} - \Lambda g_{ab}, \tag{10a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{10b}$$

$$F^{ab}{}_{;b} = \mathcal{A}_{n-2} J^a, \tag{10c}$$

where the tensors $\boldsymbol{G}$ and $\boldsymbol{F}$ are the Einstein tensor and Faraday tensor respectively, and $\boldsymbol{J}$ represents the current. Note that we have included the cosmological constant Λ. The coupling constant κ_n is given by

$$\kappa_n = \frac{2\,(n-2)\,\pi^{\frac{n-1}{2}}}{(n-3)\left(\frac{n-1}{2}-1\right)!}, \tag{11}$$

and $\mathcal{A}_{n-2}$ represents the surface of the $n-2$ sphere which is given by

$$\mathcal{A}_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}, \tag{12}$$

where Γ represents the gamma function. The Faraday tensor and electromagnetic current can be respectively defined as

$$F_{ab} = \Phi_{b;a} - \Phi_{a;b}, \tag{13a}$$
$$J^a = \zeta u^a, \tag{13b}$$

where Φ_a is the electromagnetic n-potential, and ζ is the proper charge density. The electromagnetic field tensor $\boldsymbol{E}$ in n dimensions can be written as

$$E_{ab} = \frac{1}{\mathcal{A}_{n-2}}\left(F_a{}^c F_{bc} - \frac{1}{4}F^{cd}\,F_{cd}\,g_{ab}\right). \tag{14}$$

In spherical symmetry the n-potential can be chosen as

$$\Phi_a = (\varphi(r,t), 0, \ldots, 0)\,. \tag{15}$$

Then Maxwell's equations (10b) and (10c) give the constraints

$$\varphi_{rr} - \left(\frac{A_r}{A} + \frac{B_r}{B} - (n-2)\,\frac{Y_r}{Y}\right)\varphi_r = \mathcal{A}_{n-2}\zeta A B^2, \tag{16a}$$
$$\varphi_{rt} - \left(\frac{A_t}{A} + \frac{B_t}{B} - (n-2)\,\frac{Y_t}{Y}\right)\varphi_r = 0. \tag{16b}$$

The system (16) can be solved to obtain

$$\varphi_r = \frac{AB}{Y^{n-2}}Q, \tag{17a}$$
$$Q = \mathcal{A}_{n-2}\int^r \zeta B Y^{n-2}\mathrm{d}r, \tag{17b}$$

where Q is a function of r and gives the total charge within the star. The Einstein–Maxwell field equations, with the cosmological constant Λ in n dimensions, are given by

$$\begin{aligned}\kappa_n\rho = {} & \frac{n-2}{A^2}\frac{B_tY_t}{BY} - \Lambda + \frac{(n-2)\,(n-3)}{2} \times \left(\frac{1}{Y^2} + \frac{Y_t^2}{A^2Y^2}\right) \\ & -\frac{n-2}{B^2}\left(\frac{Y_{rr}}{Y} + \frac{n-3}{2}\frac{Y_r^2}{Y^2} - \frac{B_rY_r}{BY}\right) - \frac{\kappa_n}{2\mathcal{A}_{n-2}}\frac{Q^2}{Y^{2n-4}},\end{aligned} \tag{18a}$$

$$\begin{aligned}\kappa_n\left(p + \frac{n-2}{n-1}\Delta\right) = {} & \frac{n-2}{A^2}\left(\frac{A_tY_t}{AY} - \frac{Y_{tt}}{Y} - \frac{n-3}{2}\frac{Y_t^2}{Y^2}\right) \\ & + \frac{n-2}{B^2}\left(\frac{n-3}{2}\frac{Y_r^2}{Y^2} + \frac{A_rY_r}{AY}\right) \\ & - \frac{(n-2)\,(n-3)}{2}\frac{1}{Y^2} + \Lambda \\ & + \frac{\kappa_n}{2\mathcal{A}_{n-2}}\frac{Q^2}{Y^{2n-4}},\end{aligned} \tag{18b}$$

$$\begin{aligned}\kappa_n\left(p - \frac{1}{n-1}\Delta\right) = {} & \frac{1}{B^2}\left(\frac{A_{rr}}{A} - \frac{A_rB_r}{AB} + (n-3)\,\frac{A_rY_r}{AY} - (n-3)\,\frac{B_rY_r}{BY} + (n-3)\,\frac{Y_{rr}}{Y}\right) \\ & - \frac{1}{A^2}\left(\frac{B_{tt}}{B} - \frac{A_tB_t}{AB} - (n-3)\,\frac{A_tY_t}{AY} + (n-3)\,\frac{Y_{tt}}{Y}\right) - \frac{(n-3)\,(n-4)}{2Y^2} \\ & \times\left(\frac{Y_t^2}{A^2} - \frac{Y_r^2}{B^2} + 1\right) \\ & - \frac{\kappa_n}{2\mathcal{A}_{n-2}}\frac{Q^2}{Y^{2n-4}} + \Lambda,\end{aligned} \tag{18c}$$

$$\kappa_n q = (2-n)\,\frac{1}{AB} \times\left(\frac{B_tY_r}{BY} + \frac{A_rY_t}{AY} - \frac{Y_{rt}}{Y}\right), \tag{18d}$$

$$\zeta = \frac{1}{\mathcal{A}_{n-2}}\frac{Q_r}{BY^{n-2}}, \tag{18e}$$

where we have introduced the degree of anisotropy Δ given by

$$\Delta = p_{\parallel} - p_{\perp}. \tag{19}$$

The generalised Vaidya metric in n dimensions describes the exterior spacetime and is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2}{n-3}\frac{m(v,\bar{r})}{\bar{r}^{n-3}}\right)\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}\bar{r} + \bar{r}^2\mathrm{d}\Omega_{n-2}^2, \tag{20}$$

where $m(\bar{r})$, v and $\bar{r}$ represents the mass at infinity, retarded time and radial coordinates respectively. The quantity $m(v,\bar{r})$ is the mass function. When $m(v,\bar{r}) = m(v)$, the generalised Vaidya exterior reduces to the Vaidya exterior. The matching of the interior metric (1) to the exterior metric (20) was completed by Maharaj and Brassel [11] for a composite matter distribution. For this case the generalised external atmosphere contains a null string fluid component. Another physically interesting case corresponds to the situation where the exterior atmosphere contains the electromagnetic charge Q and the cosmological constant Λ in n dimensions. For this case the mass function in n dimensions is given by

$$m(v,\bar{r}) = m(v) - \frac{\kappa_n Q^2}{2\,(n-2)\,\mathcal{A}_{n-2}\bar{r}^{n-3}} + \frac{(n-3)\,\Lambda\bar{r}^{n-1}}{(n-2)\,(n-1)}. \tag{21}$$

All cases in four dimensions with Q and Λ studied previously are contained in the mass function (21). We have extended the treatment in [11] to also include the cosmological constant in higher dimensions. Then matching the interior and exterior metrics and extrinsic curvatures at the boundary of the star, with the Vaidya exterior, yields the junction condition as

$$(p)_\Sigma = (q)_\Sigma\,, \tag{22}$$

where the hypersurface Σ represents the boundary of the star.

We substitute (18) into (22) to express the junction condition as the partial differential equation

$$\begin{aligned}&(ABY)^{-1}\Big((n-2)\Big(-2A^2Y\,(Y_r\,(A_r+B_t)\\&-BY_{rt}) + AB\Big(B\Big((n-3)Y_t^2+2YY_{tt}\Big)\\&-2A_rYY_t) - 2A_tB^2YY_t + (n-3)A^3\Big(B^2\\&-Y_r^2\Big)\Big)\Big) + (n-3)^{-1}\Big(2A^2BY\Big((n-2)Q^2Y^{4-2n}\\&+2\Lambda(n-3)\Big)\Big) = 0.\end{aligned} \tag{23}$$

There are currently no known solutions to the boundary condition (23) when $n > 4$. It is clear that the presence of charge Q and cosmological constant Λ in the boundary condition (23) affects the evolution of the radiating star together with the dimension n. There are various types of radiating models that are contained in the generalised junction condition (23). By placing appropriate restrictions the following types of models can be regained from (24): geodesic, neutral matter, charged matter, matter with the cosmological constant Λ, and the general matter distribution (7). In addition, when we set $n = 4$ we regain the four dimensional junction condition. In particular, when $n = 4$ we regain the results of Naidoo et al [42].

In our investigations we have incorporated charge and the cosmological constant. Charge is an important quantity especially in the early stages of stellar evolution. Abebe and Maharaj [28] used a Lie symmetry analysis to study the effect of charge on the four dimensional junction condition. Models for the charged composite matter junction condition in four dimensions were also studied by Maharaj and Brassel [44], who later extended this study to include higher dimensions [11]. Introducing charge to a stellar model is difficult to model as it involves adding Maxwell equations to the Einstein field equations. The presence of charge in the junction condition affects the rate of gravitational collapse. The effect charge has on a higher dimensional charged shear-free relativistic fluids with heat flux can be found in [45]. Charge can be removed from the n dimensional junction condition by setting $Q = 0$ to obtain the neutral junction condition.

In 1998 it was independently observed by two different experiments, the Supernova Cosmology Project [46] and The High-Z Supernova Search Team [47], that the universe is expanding at an accelerating rate. The accelerated expansion of the universe implies that the cosmological constant Λ, which represents the background energy density of spacetime, on a cosmological scale could be a strictly positive number [48]. A negative cosmological constant could exist on the astronomical scale, and an example of this is the anti-de Sitter spacetime. The inclusion of a cosmological constant can modify the features of certain types of stars. For example, in four dimensions, it was shown by Largani and Álvarez-Castillo [49] that different values of the cosmological constant affect the mass features of twin compact stars. Afifah and Sulaksono [50] showed, using numerical methods, that the value of the cosmological constant is inversely proportional to the mass of neutron stars. Gibbons et al. [51] investigated higher dimensional rotating black holes with a cosmological constant. The cosmological constant can be related to the concept of dark energy, which is a repulsive force. It is therefore important to consider both cases for the cosmological constant, $\Lambda = 0$ and $\Lambda \neq 0$.

We consider the general case for a radiating star incorporating all dimensions $n \geq 4$, the charge Q and the cosmological constant Λ. The junction condition in higher dimensional general relativity (24) can be reduced to the classic four dimensional junction condition in general relativity when $n = 4$.

3 Original Riccati equation

We express (23) in the compact form

$$B_t - \mathcal{L}_1B^2 - \mathcal{L}_2B - \mathcal{L}_3 = 0, \tag{24}$$

where

$$\mathcal{L}_1 = \frac{1}{2Y_r}\left(-\frac{2A_tY_t}{A^2} + \frac{1}{AY}\left((n-3)Y_t^2 + 2YY_{tt}\right) + \frac{A}{Y}\left(n-3-\frac{Q^2Y^{6-2n}}{n-3} - \frac{2\Lambda Y^2}{n-2}\right)\right), \tag{25a}$$

$$\mathcal{L}_2 = \frac{1}{AY_r}\left(AY_{rt} - A_rY_t\right), \tag{25b}$$

$$\mathcal{L}_3 = -A_r - \frac{(n-3)AY_r}{2Y}. \tag{25c}$$

In the form (24) we note that the junction condition is a *Riccati equation* in the potential B. Expressing the equation as a Riccati equation allows us to identify restrictions that can be placed resulting in either a linear equation, Bernoulli equation or a simpler Riccati equation. We obtain these restrictions by setting either $\mathcal{L}_1$, $\mathcal{L}_2$ or $\mathcal{L}_3$ to zero in (24). The studies conducted by [35–37] treated the different analytical versions of the four dimensional boundary condition as Riccati equations in one of the dependent variable and obtained exact solutions using the approach described above.

3.1 Linear equation

When $\mathcal{L}_1 = 0$ (24) is a linear equation in B. We set $\mathcal{L}_1 = 0$ and solve to obtain

$$A = (n-3)\sqrt{(n-2)(n-1)}Y_t Y^n \left(-(n-2)(n-1)Q^2Y^6 + (n-3)^2 Y^n\left((n-2)(n-1)\mathcal{R}_1 Y^3 + 2\Lambda Y^{n+2} - ((n-2)(n-1)Y^n)\right)\right)^{-1/2}, \tag{26}$$

where $\mathcal{R}_1$ is an arbitrary function of r. We substitute (26) into (24) and solve to find

$$B = \exp\left(\int_1^t \wp_1 \mathrm{d}w\right)\left(\int_1^t \wp_2 \mathrm{d}\bar{w} + \mathcal{R}_2\right), \tag{27}$$

where w and $\bar{w}$ are dummy variables, $\mathcal{R}_2$ is an arbitrary function of r and

$$\begin{aligned}\wp_1 = &-\Big(2Y_r\Big((n-2)(n-1)Q^2Y^6 + (n-3)^2Y^n\Big(Y^n\Big(-2\Lambda Y^2 + n^2 - 3n + 2\Big) - (n-2)(n-1)\mathcal{R}_1Y^3\Big)\Big)\Big)^{-1} \\ &\times\Big(YY_w\Big((n-3)^2(n-2)(n-1)Y^{n+1}\left(\mathcal{R}_1'Y - (n-3)\mathcal{R}_1Y_r\right) + 2(n-2)(n-1)QY^4 \\ &\times\left((n-3)QY_r - Q'Y\right) + 4\Lambda(n-3)^2Y_rY^{2n}\Big)\Big),\end{aligned} \tag{28a}$$

$$\begin{aligned}\wp_2 = &\Big(2\Big((n-2)(n-1)Q^2Y^6 + (n-3)^2Y^n \times\Big(Y^n\Big(-2\Lambda Y^2 + n^2 - 3n + 2\Big) - (n-2) \\ &\times(n-1)\mathcal{R}_1Y^3\Big)\Big)\Big(-(n-2)(n-1)Q^2Y^6 + (n-3)^2Y^n\Big((n-2)(n-1)\mathcal{R}_1Y^3 + 2\Lambda Y^{n+2} \\ &-((n-2)(n-1)Y^n)\Big)\Big)^{1/2}\Big)^{-1}((n-3) \times\sqrt{n-2}\sqrt{n-1}Y^{n-1}\Big((n-3)^2Y^n \\ &\times\Big(Y^n\Big(2YY_{r\bar{w}}\Big(2\Lambda Y^2 - n^2 + 3n - 2\Big) - Y_{\bar{w}}Y_r((n-3)(n-2)(n-1) - 2\Lambda(n-5)Y^2)\Big)\Big) + (n-2)(n-1)Y^3 \\ &\times\left(Y_{\bar{w}}\left(2(n-3)\mathcal{R}_1Y_r - \mathcal{R}_1'Y\right) + 2\mathcal{R}_1YY_{r\bar{w}}\right)\Big) + 2(n-2)(n-1)QQ'Y_{\bar{w}}Y^7 - (n-2) \\ &\times(n-1)Q^2Y^6\left(3(n-3)Y_{\bar{w}}Y_r + 2YY_{r\bar{w}}\right)\Big) \times\exp\left(-\int_1^{\bar{w}}\wp_1\mathrm{d}w\right)\Big).\end{aligned} \tag{28b}$$

In (28a) Y is a function of r and w, and in (28b) Y is a function of r and $\bar{w}$. The solution set (26) and (27) is a new solution to the higher dimensional boundary condition (24). When $n = 4$ we recover the results of Mahomed et al. [52].

3.2 Bernoulli equation

A Bernoulli equation can be obtained by setting $\mathcal{L}_3 = 0$. We set $\mathcal{L}_3 = 0$ and solve to obtain

$$-A_r - \frac{(n-3)AY_r}{2Y} = 0. \tag{29}$$

We solve (29) to get

$$A = \mathcal{T}_1 Y^{\frac{3-n}{2}}, \tag{30}$$

where $\mathcal{T}_1$ is an arbitrary function of t. We substitute (30) into (24) and solve to obtain

$$B = \frac{Y_r Y^{\frac{n-3}{2}}}{-\int_1^t \wp_3 \mathrm{d}w + \mathcal{R}_2}, \tag{31}$$

where w is a dummy variable and

$$\wp_3 = \frac{\mathcal{T}_1\left(-2\Lambda Y^2 + n^2 - 5n + 6\right)}{2(n-2)Y} + \frac{Q^2\mathcal{T}_1Y^{5-2n}}{6-2n} + \frac{Y^{n-4}\left((n-3)\mathcal{T}_1Y_w^2 + Y\left(\mathcal{T}_1Y_{ww} - Y_w\mathcal{T}_1'\right)\right)}{\mathcal{T}_1{}^2}. \tag{32}$$

In (32) Y is a function of r and w, and $\mathcal{T}_1$ is a function of w. The solution set (30) and (31) is a new solution to the higher dimensional boundary condition (24). When $n = 4$ we recover the results of Mahomed et al. [52].

3.3 Special Riccati equation

A special Riccati equation can be obtained by setting $\mathcal{L}_2 = 0$. We set $\mathcal{L}_2 = 0$ and solve to find

$$A = \mathcal{T}_1 Y_t, \tag{33}$$

where $\mathcal{T}_1$ is an arbitrary function of t. We set $Q = \Lambda = 0$ and let

$$Y = \mathcal{R}_1\mathcal{T}_2, \tag{34}$$

where $\mathcal{R}_1$ is an arbitrary function of r and $\mathcal{T}_2$ is an arbitrary function of t. Then (24) has the form

$$B_t - \mathcal{L}_1 B^2 - \mathcal{L}_3 = 0, \tag{35}$$

where

$$\mathcal{L}_1 = \frac{(n-3)\left(\mathcal{T}_1^3+\mathcal{T}_1\right)\mathcal{T}_2' - 2\mathcal{T}_2\mathcal{T}_1'}{2\mathcal{T}_1^2\mathcal{T}_2^2\mathcal{R}_1'}, \tag{36a}$$

$$\mathcal{L}_3 = -\frac{1}{2}(n-1)\mathcal{T}_1\mathcal{R}_1'\mathcal{T}_2'. \tag{36b}$$

We can solve (35) for specific values of $\mathcal{T}_1$. We set $\mathcal{T}_1 = 1$ in (35) and solve to obtain

$$B = \frac{\mathcal{T}_2\mathcal{R}_1'\wp_4}{2(n-3)}, \tag{37}$$

where

$$\wp_4 = \frac{2\sqrt{n-3}\sqrt{n-1}\sqrt{2(n-4)n+7}\mathcal{R}_2}{\sqrt{(n-3)(n-1)}\left(\mathcal{T}_2^{\sqrt{2(n-4)n+7}}+\mathcal{R}_2\right)} - \frac{\sqrt{n-3}\sqrt{n-1}\sqrt{2(n-4)n+7}}{\sqrt{(n-3)(n-1)}} + 1, \tag{38}$$

where $\mathcal{R}_2$ is an arbitrary function of r. The solution set (33), (34) and (37), when $\Lambda = Q = 0$ and $\mathcal{T}_1 = 1$, is a new solution to (24) when $n \geq 4$.

4 Transformed Riccati equation

Transformations can be useful as they can provide a different coordinate system in which a differential equation under study may be simplified. Here, we use the transformation

$$H = \left(\alpha\frac{Y_r}{B} + \beta\frac{Y_t}{A}\right)\mathcal{F} + \mathcal{G}, \tag{39}$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions of r, t, A and Y, first used in [42] for four dimensions but now applied in the n dimensional case. The parameters α and β are arbitrary. We have chosen to use the transformation (39) since the higher dimensional junction condition (23) can be expressed as the Riccati equation (24). The transformations included in [38–42] (which are all contained in (39)) transform the four dimensional junction condition, which is also a Riccati equation in one of the gravitational potential functions, into a new Riccati equation which yielded new solutions to the Einstein and Einstein–Maxwell field equations. We will show that these transformations have a similar effect on the higher dimensional junction condition (24). We rewrite (39) in terms of B to obtain

$$B = -\frac{\alpha A Y_r \mathcal{F}}{\beta Y_t \mathcal{F} + A(\mathcal{G} - H)}. \tag{40}$$

We substitute (40) into (24) to obtain the *new Riccati equation* in H as

$$H_t - \mathcal{L}_4 H^2 - \mathcal{L}_5 H - \mathcal{L}_6 = 0, \tag{41}$$

where

$$\mathcal{L}_4 = \frac{1}{2\alpha Y Y_r \mathcal{F}}\left(2A_r Y + (n-3)AY_r\right), \tag{42a}$$

$$\mathcal{L}_5 = \frac{1}{\alpha A Y Y_r \mathcal{F}}\Big(A\left(Y_r\left(\alpha Y\left(A_t\mathcal{F}_A + Y_t\mathcal{F}_Y + \mathcal{F}_t\right) - \beta(n-3)Y_t\mathcal{F}\right) - 2A_r Y\mathcal{G}\right) + (\alpha - 2\beta)A_r Y Y_t \mathcal{F} - (n-3)A^2 Y_r \mathcal{G}\Big), \tag{42b}$$

$$\begin{aligned}\mathcal{L}_6 = \frac{1}{2\alpha}\Bigg(&2\Bigg(\mathcal{G}\left(\frac{\beta(n-3)\mathcal{F}_t}{Y} - \frac{1}{\mathcal{F}}\left(\alpha\left(A_t\mathcal{F}_A + Y_t\mathcal{F}_Y + \mathcal{F}_t\right)\right)\right) + \frac{A_r G^2}{Y_r\mathcal{F}} + \alpha\left(A_t\mathcal{G}_A + Y_t\mathcal{G}_Y + \mathcal{G}_t\right)\Bigg)\\ &- \frac{1}{A}\Bigg(\frac{2(\alpha-2\beta)A_r Y_t \mathcal{G}}{Y_r} + \frac{(\alpha-\beta)\mathcal{F}\left((n-3)(\alpha+\beta)Y_t^2 + 2\alpha Y Y_{tt}\right)}{Y}\Bigg)\\ &+ \frac{2(\alpha-\beta)Y_t\mathcal{F}\left(\alpha A_t Y_r - \beta A_r Y_t\right)}{A^2 Y_r}\\ &+ \frac{A}{(n-3)Y\mathcal{F}}\left(\alpha^2\mathcal{F}^2\left(Q^2 Y^{6-2n} + (n-3)\left(\frac{2\Lambda Y^2}{n-2} - n + 3\right)\right) + (n-3)^2\mathcal{G}^2\right)\Bigg).\end{aligned} \tag{42c}$$

We note that (24) and (41) are both Riccati equations. Equation (41) is the higher dimensional generalisation of the boundary condition generated by Naidoo et al [42]. When $n = 4$ we regain the Riccati equation of [42]. The Riccati equation (41) is the fundamental equation of interest in this paper. The master equation (41) possesses two distinctive features. The first feature is that (41) contains only one second order term, Y_{tt}, while Eq. (24) has the two second order terms Y_{rt} and Y_{tt}. Our transformation (39) removed the term Y_{rt} thereby simplifying (24). The second feature is the insertion of new terms containing $\mathcal{F}_A$, $\mathcal{F}_Y$, $\mathcal{G}_A$, and $\mathcal{G}_Y$, which change the appearance of the Riccati equation (24). The appearance of the new terms allows for the generation of new exact solutions. It is interesting that these two features of the transformation are maintained regardless of dimension n. This is true even though the transformation (39) has no dependence on the dimension n. This suggests the existence of a geometric property of the differential equation (24) that does not change with dimension n. The use of the transformation (39) allowed us to obtain a *new* Riccati equation.

5 Linear equation: $\mathcal{L}_4 = 0$

We set

$$\frac{2A_r}{Y_r\mathcal{F}}+\frac{(n-3)A}{Y\mathcal{F}}=0, \tag{43}$$

to obtain

$$A=\mathcal{T}_1 Y^{\frac{3-n}{2}}, \tag{44}$$

where $\mathcal{T}_1$ is a function of t. Due to the relationship (44), we can write $\mathcal{F}$ and $\mathcal{G}$ as

$$\mathcal{F}=\mathcal{F}(r,t,Y), \tag{45a}$$

$$\mathcal{G}=\mathcal{G}(r,t,Y), \tag{45b}$$

to ensure the solutions we obtain are not implicit. We substitute (44) and (45) into (41) to obtain the linear equation

$$H_t-\mathcal{L}_5 H-\mathcal{L}_6=0, \tag{46}$$

where

$$\mathcal{L}_5=\frac{1}{\mathcal{F}}\left(Y_t\mathcal{F}_Y+\mathcal{F}_t\right)-\frac{1}{2Y}(n-3)Y_t, \tag{47a}$$

$$\begin{aligned}\mathcal{L}_6=\mathcal{G}_t+\frac{1}{2}&\left(\frac{1}{(n-3)(n-2)\mathcal{T}_1{}^2}\left(\mathcal{F}Y^{-\frac{5}{2}(n+1)}\right.\right.\\&\times\left(\alpha(n-2)Q^2\mathcal{T}_1{}^3Y^9+(n-3)Y^{2n}\right.\\&\times\left(\alpha\mathcal{T}_1{}^3Y^3\left(2\Lambda Y^2-(n-3)(n-2)\right)\right.\\&-2(n-2)(\alpha-\beta)Y^n\left((n-3)\mathcal{T}_1Y_t^2\right.\\&\left.\left.\left.\left.\left.+Y\left(\mathcal{T}_1Y_{tt}-Y_t\mathcal{T}_1'\right)\right)\right)\right)\right)-\frac{1}{\mathcal{F}}\left(2\left(Y_t\mathcal{F}_Y\right.\right.\\&\left.\left.+\mathcal{F}_t\right)\mathcal{G}\right)+\frac{1}{Y}\left(Y_t\left(2Y\mathcal{G}_Y\right.\right.\\&\left.\left.\left.+(n-3)\mathcal{G}\right)\right)\right).\end{aligned} \tag{47b}$$

We solve (46) to obtain

$$H=\mathcal{F}Y^{\frac{3-n}{2}}\left(\int_1^t\wp_5\mathrm{d}w+\mathcal{R}_1\right), \tag{48}$$

where w is a dummy variable and

$$\begin{aligned}\wp_5=&-\frac{1}{2(n-2)Y}\left(\alpha\mathcal{T}_1\left(-2\Lambda Y^2+n^2-5n+6\right)\right)\\&+\frac{1}{2(n-3)}\left(\alpha Q^2\mathcal{T}_1Y^{5-2n}\right)\\&+\frac{1}{\mathcal{T}_1{}^2}\left((\alpha-\beta)Y^{n-4}\left(Y\left(Y_w\mathcal{T}_1'-\mathcal{T}_1Y_{ww}\right)\right.\right.\\&\left.\left.-(n-3)\mathcal{T}_1Y_w^2\right)\right)-\frac{1}{\mathcal{F}^2}\left(\left(Y_w\mathcal{F}_Y+\mathcal{F}_w\right)\mathcal{G}Y^{\frac{n-3}{2}}\right)\\&+\frac{1}{2\mathcal{F}}\left(Y^{\frac{n-5}{2}}\left(2Y\left(Y_w\mathcal{G}_Y+\mathcal{G}_w\right)\right.\right.\\&\left.\left.+(n-3)Y_w\mathcal{G}\right)\right).\end{aligned} \tag{49}$$

In the above, note that $\mathcal{R}_1$ is a function of r, and Y is a function of r and w, $\mathcal{T}_1$ is a function of w, and $\mathcal{F}$ and $\mathcal{G}$ are functions of r, w and Y.

We have to reverse the transformation (39), and transform the solution (48) back to the original coordinate system. This is done by using (40), (44), (45), and (48) to express the gravitational potential function B as

$$\begin{aligned}B=-\alpha\mathcal{T}_1Y_r\mathcal{F}Y^{\frac{n+3}{2}}&\left(\mathcal{F}\left(\beta Y_tY^n\right.\right.\\&\left.\left.-\mathcal{T}_1Y^3\left(\int_1^t\wp_1\mathrm{d}w+\mathcal{R}_1\right)\right)\right.\\&\left.+\mathcal{T}_1\mathcal{G}Y^{\frac{n+3}{2}}\right)^{-1}.\end{aligned} \tag{50}$$

It is important to note that the original coordinate system is the generic coordinate system that most researchers use. The n dimensional solution reduces to the four dimensional solution of Naidoo et al. [42] when $n=4$ in (44) and (50). The earlier result of Mahomed et al. [41] is regained when $n=4$, $\mathcal{F}=1$ and $\mathcal{G}=0$. An important feature of the transformation (39) is observed in the B gravitational potential functions (31) and (50). The restrictions and gravitational potential functions (29) and (44) are the same, however the gravitational potential function B given in (31) and (50) are different. The same restriction under different coordinates has resulted in different solution sets. This is a direct consequence of the complexity of the transformation (39) which involves arbitrary functions of the potential functions A and Y. The use of the transformation has revealed new insights as it allowed us to obtain a different gravitational potential function B for the same gravitational potential function A.

The results obtained in this section can be summarised in the following theorem:

Theorem 1 *When (44) holds, the higher dimensional boundary condition for a general relativistic radiating star with charge and cosmological constant reduces to a linear equation with general solution given by (48) and* $\mathcal{F}=\mathcal{F}(r,t,Y)$ *and* $\mathcal{G}=\mathcal{G}(r,t,Y)$.

We note that explicit forms for the potentials A and B have been provided. A suitable choice of the potential Y and the charge Q yields functional forms for A and B.

Corollary 1.1 *Known solutions with accelerating particle trajectories in dimension* $n=4$ *are contained in this class of models.*

6 Bernoulli equation: $\mathcal{L}_6=0$

We impose the restriction

$$\begin{aligned}\frac{1}{2\alpha}&\left(2\left(\mathcal{G}\left(\frac{\beta(n-3)\mathcal{F}_t}{Y}\right.\right.\right.\\&\left.\left.-\frac{1}{\mathcal{F}}\left(\alpha\left(A_t\mathcal{F}_A+Y_t\mathcal{F}_Y+\mathcal{F}_t\right)\right)\right)\right.\\&\left.+\frac{A_rG^2}{Y_r\mathcal{F}}+\alpha\left(A_t\mathcal{G}_A+Y_t\mathcal{G}_Y+\mathcal{G}_t\right)\right)\end{aligned}$$

$$-\frac{1}{A}\left(\frac{2(\alpha-2\beta)A_rY_t\mathcal{G}}{Y_r}\right.$$
$$\left.+\frac{(\alpha-\beta)\mathcal{F}\left((n-3)(\alpha+\beta)Y_t^2+2\alpha YY_{tt}\right)}{Y}\right)$$
$$+\frac{2(\alpha-\beta)Y_t\mathcal{F}\left(\alpha A_tY_r-\beta A_rY_t\right)}{A^2Y_r}$$
$$+\frac{A}{(n-3)Y\mathcal{F}}\left(\alpha^2\mathcal{F}^2\left(Q^2Y^{6-2n}\right.\right.$$
$$\left.+(n-3)\left(\frac{2\Lambda Y^2}{n-2}-n+3\right)\right)$$
$$\left.\left.+(n-3)^2\mathcal{G}^2\right)\right)=0, \qquad (51)$$

on (41) to obtain the Bernoulli equation

$$H_t-\mathcal{L}_4H^2-\mathcal{L}_5H=0, \qquad (52)$$

where

$$\mathcal{L}_4=\frac{1}{2\alpha YY_r\mathcal{F}}\left(2A_rY+(n-3)AY_r\right), \qquad (53a)$$
$$\mathcal{L}_5=\frac{1}{\alpha AYY_r\mathcal{F}}\left(A\left(Y_r\left(\alpha Y\left(A_t\mathcal{F}_A+Y_t\mathcal{F}_Y+\mathcal{F}_t\right)\right.\right.\right.$$
$$\left.\left.-\beta(n-3)Y_t\mathcal{F}\right)-2A_rY\mathcal{G}\right)$$
$$\left.+(\alpha-2\beta)A_rYY_t\mathcal{F}-\left((n-3)A^2Y_r\mathcal{G}\right)\right). \qquad (53b)$$

We cannot obtain the general solution to (51). However we can obtain solutions to (51) by imposing further restrictions (See later.). We can solve (52) to obtain

$$H=\frac{\exp\left(\int_1^t\wp_6\mathrm{d}w\right)}{-\int_1^t\exp\left(\int_1^{\bar{w}}\wp_6\mathrm{d}w\right)\wp_7\mathrm{d}\bar{w}+\mathcal{R}_1}, \qquad (54)$$

where w and $\bar{w}$ are dummy variables and

$$\wp_6=\frac{1}{\alpha AYY_r\mathcal{F}}\left(A\left(Y_r\left(\alpha Y\left(A_w\mathcal{F}_A+Y_w\mathcal{F}_Y+\mathcal{F}_w\right)\right.\right.\right.$$
$$\left.\left.-\beta(n-3)Y_wF\right)-2A_rY\mathcal{G}\right)$$
$$\left.+(\alpha-2\beta)A_rYY_w\mathcal{F}-\left((n-3)A^2Y_r\mathcal{G}\right)\right), \qquad (55a)$$
$$\wp_7=\frac{1}{2\alpha YY_r\mathcal{F}}\left(2A_rY+(n-3)AY_r\right), \qquad (55b)$$

and $\mathcal{R}_1$ is a function of r. In (55a) A and Y are functions of r and w, and $\mathcal{F}$ and $\mathcal{G}$ are functions of r, w, $A(r,w)$ and $Y(r,w)$. In (55b) A and Y are functions of r and $\bar{w}$, and $\mathcal{F}$ is a function of r, $\bar{w}$, $A(r,\bar{w})$ and $Y(r,\bar{w})$. We use (40) and (54) to obtain the potential

$$B=\frac{-\alpha AY_r\mathcal{F}}{\beta Y_t\mathcal{F}+A\left(\mathcal{G}-\frac{\exp\left(\int_1^t\wp_6\mathrm{d}w\right)}{-\int_1^t\exp\left(\int_1^{\bar{w}}\wp_6\mathrm{d}w\right)\wp_7\mathrm{d}\bar{w}+\mathcal{R}_1}\right)}, \qquad (56)$$

subject to the restriction (51).

The n dimensional solution reduces to the four dimensional solution by Naidoo et al. [42] when $n=4$ in (56). It is still necessary to show that the condition (51) is integrable, as will be demonstrated below.

6.1 Solution I

We set

$$\mathcal{F}=\mathcal{F}(r,t), \qquad (57a)$$
$$\mathcal{G}=0, \qquad (57b)$$
$$\beta=0, \qquad (57c)$$
$$Y=(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{k_3}, \qquad (57d)$$

where $\mathcal{R}_2$ and $\mathcal{R}_3$ are arbitrary functions of r, $\mathcal{T}_1$ is an arbitrary function of t, and k_1, k_2 and k_3 are arbitrary constants. We use the restrictions in (57) to write the condition (51) as

$$2k_1k_3\mathcal{R}_2\mathcal{T}_1'A_t(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{2k_3-1}$$
$$+A^3\left(\frac{1}{n-3}\left(Q^2(k_1\mathcal{R}_2\mathcal{T}_1\right.\right.$$
$$\left.+k_2\mathcal{R}_3)^{6k_3}\left((k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{k_3}\right)^{-2n}\right)$$
$$\left.+\frac{1}{n-2}\left(2\Lambda(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{2k_3}\right)-n+3\right)$$
$$+k_1k_3\mathcal{R}_2A(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{2k_3-2}\left(k_1(-nk_3+k_3\right.$$
$$\left.+2)\mathcal{R}_2\mathcal{T}_1'^2-2\mathcal{T}_1''(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)\right). \qquad (58)$$

We solve (58) to obtain

$$A=\left(\left(-(n-3)^2(n-2)(n-1)k_1^2k_3^2\mathcal{R}_2^2\mathcal{T}_1'^2\right.\right.$$
$$\times(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{nk_3}$$
$$\left.\times\left(\left((k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{k_3}\right)^{2n}\right)^{\frac{1}{2}}\right)$$
$$\times\left((k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{k_3+2}\left((n-2)(n-1)Q^2\right.\right.$$
$$\times(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{(n+3)k_3}$$
$$+(n-3)^2\left((k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{k_3}\right)^{2n}$$
$$\times\left(-2\Lambda(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{(n-1)k_3}\right.$$
$$+(n-2)(n-1)\left((n_1\mathcal{R}_2\mathcal{T}_1+k_2R_3)^{(n-3)k_3}\right.$$
$$\left.\left.\left.\left.-k_1^2k_3^2\mathcal{R}_2^2\mathcal{R}_4\right)\right)\right)\right)^{-\frac{1}{2}}, \qquad (59)$$

where $\mathcal{R}_4$ is a function of r. We use (39), (54), (57) and (59) to express the gravitational potential function B as

$$B=\alpha k_3\mathcal{F}(k_1\mathcal{R}_2\mathcal{T}_1+k_2\mathcal{R}_3)^{k_3-1}\left(k_1\mathcal{T}_1\mathcal{R}_2'\right.$$
$$\left.+k_2\mathcal{R}_3'\right)\exp\left(-\int_1^t\wp_8\mathrm{d}w\right)$$

$$\times\left(-\int_1^t \exp\left(\int_1^{\bar{w}} \wp_8 \mathrm{d}w\right)\wp_9 \mathrm{d}\bar{w} + \mathcal{R}_4\right), \quad (60)$$

where

$$\begin{aligned}\wp_8 = {} & \frac{\mathcal{F}_w}{\mathcal{F}} - \big(2(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)\big(k_1\mathcal{R}_2'\mathcal{T}_1 \\ & +k_2\mathcal{R}_3'\big)\Big(-(n-2)(n-1)Q^2(k_1\mathcal{R}_2\mathcal{T}_1 \\ & +k_2\mathcal{R}_3)^{(n+3)k_3} + (n-3)^2\,((k_1\mathcal{R}_2\mathcal{T}_1 \\ & +k_2\mathcal{R}_3)^{k_3}\Big)^{2n}\Big(2\Lambda(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n-1)k_3} \\ & -(n-2)(n-1)\,((n_1\mathcal{R}_2\mathcal{T}_1 \\ & +n_2\mathcal{R}_3)^{(n-3)k_3} - k_1^2k_3^2\mathcal{R}_2^2\mathcal{R}_4\big)\Big)\Big)\Big)^{-1} \\ & \times\Big(k_1\mathcal{T}_1'\Big((n-3)^2\Big((k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{k_3}\Big)^{2n} \\ & \times\big(\mathcal{R}_2\,\big(n_2\mathcal{R}_3'\,(4\Lambda(k_1\mathcal{R}_2\mathcal{T}_1 \\ & +k_2\mathcal{R}_3)^{(n-1)k_3} + (n-2)(n-1)\,(2(k_3 \\ & -1)(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n-3)k_3} + k_1^2k_3^2(-nk_3 \\ & +k_3+2)\mathcal{R}_2^2\mathcal{R}_4\big)\Big) + (n-2)(n-1)k_1k_3\mathcal{T}_1 \\ & \times\Big(2\mathcal{R}_2'(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n-3)k_3} \\ & +k_1^2k_3\mathcal{R}_2^2\,((-nk_3+k_3+2)\mathcal{R}_4\mathcal{R}_2' + \mathcal{R}_2\mathcal{R}_4')\Big)\Big) \\ & +k_2\mathcal{R}_3\Big(2\mathcal{R}_2'(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n-3)k_3} \\ & \times\Big(-2\Lambda(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{2k_3} + (n-3)n + 2\Big) \\ & +(n-2)(n-1)k_1^2k_3^2\mathcal{R}_2^3\mathcal{R}_4'\Big)\Big) - 2(n-2)(n-1) \\ & \times Q\mathcal{R}_2Q'(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n+3)k_3+1} \\ & +2(n-2)(n-1)Q^2(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n+3)k_3} \\ & \times\big(\mathcal{R}_2'((n-2)k_1k_3\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3) \\ & +k_2((n-2)k_3-1)\mathcal{R}_2\mathcal{R}_3')\big)\big), \end{aligned} \quad (61a)$$

$$\begin{aligned}\wp_9 = {} & \Big(2\alpha k_3\mathcal{F}\big(k_1\mathcal{R}_2'\mathcal{T}_1 + k_2\mathcal{R}_3'\big)\Big((k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{k_3+2} \\ & \times\Big((n-2)(n-1)Q^2(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n+3)k_3} \\ & +(n-3)^2\Big((k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{k_3}\Big)^{2n} \\ & \times\Big(-2\Lambda(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n-1)k_3} \\ & +(n-2)(n-1)\,((k_1\mathcal{R}_2\mathcal{T}_1 \\ & +k_2\mathcal{R}_3)^{(n-3)k_3} - k_1^2k_3^2\mathcal{R}_2^2\mathcal{R}_4\big)\Big)\Big)\Big)^{3/2}\Big)^{-1} \\ & \times\Big((k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^2\Big(-(n-3)^2(n-2)(n \\ & -1)k_1^2k_3^2\mathcal{R}_2^2\mathcal{T}_1'^2(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{nk_3} \\ & \times\Big((k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{k_3}\Big)^{2n}\Big)^{1/2} \\ & \times\Big(\frac{1}{\mathcal{R}_2}\Big((n-3)^2\Big((k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{k_3}\Big)^{2n} \\ & \times\big(\mathcal{R}_2\,\big(k_2\mathcal{R}_3'\,(-2\Lambda((n-3)k_3-2)(k_1\mathcal{R}_2\mathcal{T}_1 \\ & +k_2\mathcal{R}_3)^{(n-1)k_3} + (n-2)(n-1) \\ & \times\Big(((n-1)k_3-2)(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n-3)k_3} \\ & +2k_1^2k_3^2(-nk_3+2k_3+1)\mathcal{R}_2^2\mathcal{R}_4\Big)\Big) \\ & +k_1k_3\mathcal{T}_1\big(-2\Lambda(n-3)\mathcal{R}_2'(k_1\mathcal{R}_2\mathcal{T}_1 \\ & +k_2\mathcal{R}_3)^{(n-1)k_3} + (n-2)(n-1)\,((n \\ & -1)\mathcal{R}_2'(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{(n-3)k_3} \\ & +k_1^2k_3\mathcal{R}_2^2\,\big(2(-nk_3+2k_3+1)\mathcal{R}_4\mathcal{R}_2' \\ & +\mathcal{R}_2\mathcal{R}_4)))) + k_2\mathcal{R}_3\,\big(2\mathcal{R}_2'(k_1\mathcal{R}_2\mathcal{T}_1 \\ & +k_2\mathcal{R}_3)^{(n-3)k_3}\Big(-2\Lambda(k_1\mathcal{R}_2\mathcal{T}_1 + k_2\mathcal{R}_3)^{2k_3} \\ & +(n-3)n+2) + (n-2)(n \\ & -1)k_1^2k_3^2\mathcal{R}_2^3\mathcal{R}_4'\big)\Big)\Big) - 2(n-2)(n-1) \\ & \times QQ'(k_1\mathcal{R}_2\mathcal{T}_1 + k_2R_3)^{(n+3)k_3+1} \\ & +\frac{1}{\mathcal{R}_2}\Big((n-2)(n-1)Q^2(k_1\mathcal{R}_2\mathcal{T}_1 \\ & +n_2\mathcal{R}_3)^{(n+3)k_3}\big(\mathcal{R}_2'((3n-7)k_1k_3\mathcal{R}_2\mathcal{T}_1 \\ & +2k_2\mathcal{R}_3) + k_2((3n-7)k_3-2)\mathcal{R}_2\mathcal{R}_3')\big)\big)\big). \end{aligned} \quad (61b)$$

In (61a) Y and $\mathcal{F}$ are functions of r and w; in (61b) Y and $\mathcal{F}$ are functions of r and $\bar{w}$.

This n dimensional solution reduces to the four dimensional solution of Naidoo et al. [42] when $n = 4$ in (59) and (60). The earlier result of Thirukkanesh and Maharaj [38] arises when $n = 4$, $k_1 = k_2 = 1$, $\mathcal{T}_1 = t$ and $k_3 = \frac{2}{3}$.

6.2 Solution II

We set

$$\mathcal{F} = \mathcal{F}(r, t, Y), \quad (62a)$$

$$\mathcal{G} = 0, \quad (62b)$$

$$\beta = 0, \quad (62c)$$

in (51) to obtain the restriction

$$\begin{aligned} & 2A_tY_t + A\left(\frac{1}{Y}\left(A^2\left(\frac{Q^2Y^{6-2n}}{n-3} + \frac{2\Lambda Y^2}{n-2}\right.\right.\right. \\ & \left.\left.\left. -n+3\right) - (n-3)Y_t^2\right) - 2Y_{tt}\right) = 0. \end{aligned} \quad (63)$$

We solve (63) to obtain

$$\begin{aligned} A = {} & (n-3)\sqrt{(n-2)(n-1)}Y_tY^n\,(-(n-2)(n \\ & -1)Q^2Y^6 + (n-3)^2Y^n\,((n-2)(n-1) \\ & \times\mathcal{R}_2Y^3 + 2\Lambda Y^{n+2} - ((n-2) \end{aligned}$$

$$\times(n-1)Y^n)))^{-\frac{1}{2}}, \tag{64}$$

where $\mathcal{R}_2$ is a function of r. We use (40), (54), (62) and (64) to express the gravitational potential B as

$$B = \alpha Y_r \mathcal{F} \exp\left(-\int_1^t \wp_{10} \mathrm{d}w\right) \times \left(-\int_1^t \exp\left(\int_1^{\bar{w}} \wp_{10} \mathrm{d}w\right) \wp_{11} \mathrm{d}\bar{w} + \mathcal{R}_2\right), \tag{65}$$

where

$$\begin{aligned}\wp_{10} = &\Big(2Y_r\Big((n-2)(n-1)Q^2Y^6 + (n-3)^2Y^n \\ &\Big(Y^n\left(-2\Lambda Y^2 + n^2 - 3n + 2\right) - (n-2)(n \\ &-1)\mathcal{R}_2Y^3\Big)\Big)\Big)^{-1}\Big(Y_w\Big((n-3)^2(n-2)(n \\ &-1)Y^{n+2}\left((n+3)\mathcal{R}_2Y_r + \mathcal{R}_2'Y\right) - 2(n-2) \\ &\times(n-1)QY^5\left(Q'Y + 3QY_r\right) - 2(n \\ &-3)^2Y_rY^{2n-1}\left((n-2)(n-1)n - 2\Lambda(n\right. \\ &\left.+1)Y^2\right)\Big)\Big) + \frac{1}{\mathcal{F}}\left(Y_w\mathcal{F}_Y + \mathcal{F}_w\right) \\ &+ \frac{nY_w}{Y} + \frac{Y_{rw}}{Y_r},\end{aligned} \tag{66a}$$

$$\begin{aligned}\wp_{11} = &\Big(2\alpha Y_r\mathcal{F}\Big((n-2)(n-1)Q^2Y^6 + (n-3)^2Y^n \\ &\times\Big(Y^n\left(-2\Lambda Y^2 + n^2 - 3n + 2\right) - (n-2)(n \\ &-1)\mathcal{R}_2Y^3\Big)\Big)\Big(-(n-2)(n-1)Q^2Y^6 \\ &+(n-3)^2Y^n\Big((n-2)(n-1)\mathcal{R}_2Y^3 \\ &+2\Lambda Y^{n+2} - \left((n-2)(n-1)Y^n\right)\Big)\Big)^{1/2}\Big)^{-1} \\ &\times\Big((n-3)\sqrt{(n-2)(n-1)}Y^{n-1}\Big((n-3)^2Y^n \\ &\times\Big(Y^n\Big(2YY_{r\bar{w}}\left(-2\Lambda Y^2 + n^2 - 3n + 2\right) \\ &+Y_{\bar{w}}Y_r\left((n-3)(n-2)(n-1)\right. \\ &\left.-2\Lambda(n-5)Y^2\right)\Big) + (n-2)(n-1)Y^3 \\ &\times\left(Y_{\bar{w}}\left(\mathcal{R}_2'Y - 2(n-3)\mathcal{R}_2Y_r\right) - 2\mathcal{R}_2YY_{r\bar{w}}\right)\Big) \\ &-2(n-2)(n-1)QQ'Y_{\bar{w}}Y^7 + (n-2)(n-1) \\ &\times Q^2Y^6\left(3(n-3)Y_{\bar{w}}Y_r + 2YY_{r\bar{w}}\right)\Big)\Big).\end{aligned} \tag{66b}$$

The n dimensional solution reduces to the four dimensional case of Naidoo et al. [42] when $n = 4$ in (64) and (65). Observe that the result of Mahomed et al. [41] can be regained when $n = 4$. Their potential A is obtained when $n = 4$ in (64), and in addition their potential B is regained when $n = 4$ and $\mathcal{F} = 1$ in (65).

The condition (51) would be difficult to obtain without the use of the transformation (39), due to the dependence of the arbitrary functions $\mathcal{F}$ and $\mathcal{G}$. The restriction $\mathcal{L}_6$ is different from $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$: hence setting $\mathcal{L}_6 = 0$ results in a new restriction not obtained by setting either $\mathcal{L}_1$, $\mathcal{L}_2$ or $\mathcal{L}_3$ to zero. This new restriction produces new solutions. The restriction (51) has to be solved to obtain solutions to (41). The transformation (51) allowed us to easily identify new restrictions that leads to simplification by treating (41) as a Riccati equation.

The results obtained in this section can be summarised in the following.

Theorem 2 *When the higher dimensional boundary condition for a general relativistic radiating star with charge and a cosmological constant in higher dimensions is restricted to the Bernoulli equation (52) it can be solved in general with $\mathcal{F} = \mathcal{F}(r, t, A, Y)$ and $\mathcal{G} = \mathcal{G}(r, t, A, Y)$. The potential B is found explicitly and the potentials A and Y satisfy a constraint equation which can be solved in terms of charge Q and the cosmological constant Λ.*

Corollary 2.1 *Known solutions when $n = 4$ are contained as special cases with $\mathcal{F} = \mathcal{F}(r, t)$ and $\mathcal{G} = \mathcal{G}(r, t)$ for accelerating particles.*

7 Reduction of order

The boundary condition (23) (or its equivalent (24)) is a nonlinear equation with second order partial derivatives. Clearly it is desirable if a first order equation is attainable from (23) as in the treatments of Ivanov [39,40], Mahomed et al. [41] and others. In this section we explore a transformation that reduces the order of (24) from a second order equation to a first order equation. This is done by removing the Y_{rt} and Y_{tt} terms from (24) in n dimensions.

We set $\alpha = \beta = 1$ in (39) to obtain

$$H = \left(\frac{Y_t}{A} + \frac{Y_r}{B}\right)\mathcal{F} + \mathcal{G}. \tag{67}$$

From (67) we obtain the explicit form of the potential

$$B = -\frac{AY_r\mathcal{F}}{Y_t\mathcal{F} + A\mathcal{G} - AH}. \tag{68}$$

We substitute (68) into (24) to obtain a first order equation, which we express as

$$H_t - \mathcal{L}_4H^2 - \mathcal{L}_5H - \mathcal{L}_6 = 0, \tag{69}$$

where

$$\mathcal{L}_4 = \frac{1}{2YY_r\mathcal{F}}\left(2A_rY + (n-3)AY_r\right), \tag{70a}$$

$$\begin{aligned}\mathcal{L}_5 = &\frac{1}{AYY_r\mathcal{F}}\left(A\left(Y_r\left(Y\left(A_t\mathcal{F}_A + Y_t\mathcal{F}_Y + \mathcal{F}_t\right)\right.\right.\right. \\ &\left.\left.-(n-3)Y_t\mathcal{F}\right) - 2A_rY\mathcal{G}\right) - A_rYY_t\mathcal{F}\end{aligned}$$

$$-\left((n-3)A^2Y_r\mathcal{G}\right)\Big), \tag{70b}$$

$$\mathcal{L}_6 = \mathcal{G}\left(\frac{1}{Y}\left((n-3)Y_t\right) - \frac{1}{\mathcal{F}}\left(A_t\mathcal{F}_A + Y_t\mathcal{F}_Y + \mathcal{F}_t\right)\right) + \frac{A_r G^2}{Y_r\mathcal{F}} + A_t\mathcal{G}_A + \frac{A_r Y_t\mathcal{G}}{AY_r} + \frac{1}{2(n-3)Y\mathcal{F}}\left(A\left(\mathcal{F}^2\left(Q^2Y^{6-2n} + (n-3)\left(\frac{2\Lambda Y^2}{n-2} - n + 3\right)\right) + (n-3)^2\mathcal{G}^2\right)\right) + Y_t\mathcal{G}_Y + \mathcal{G}_t. \tag{70c}$$

The equation (69) is a *new Riccati equation* in H in which the second order terms Y_{rt} and Y_{tt} have been eliminated. Thus the transformation (67) reduces the order of (24) from a second order equation partial differential equation into a first order partial differential equation. This is an interesting feature as Lie symmetries can be used to reduce the order of ordinary differential equations [53]. There are currently no known methods for reducing the order of partial differential equations. It is therefore interesting that the ad hoc transformation reduces the order of the partial differential equation (24) in four and higher dimensions. This suggests that there is a property yet to be identified in the geometric structure of the partial differential equation that is persistent, regardless of dimension that allows the transformation (67) to reduce the order of (24). This is a remarkable result as the reduction of order of partial differential equations is a relatively unexplored area of research.

When we set $n = 4$, $\mathcal{F} = 1$ and $\mathcal{G} = 0$ we regain the Ivanov [40] result. When $n = 4$ the case of Naidoo et al. [42] arises. Given the dependence of our functions $\mathcal{F} = \mathcal{F}(r,t,A,Y)$ and $\mathcal{G} = \mathcal{G}(r,t,A,Y)$, we have generalised the Ivanov [40] transformation and maintained the reduction to a first order equation. We now demonstrate that the first order equation (69) admits exact solutions in n dimensions.

7.1 Bernoulli equation

We impose the restriction

$$\mathcal{G}\left(\frac{1}{Y}\left((n-3)Y_t\right) - \frac{1}{\mathcal{F}}\left(A_t\mathcal{F}_A + Y_t\mathcal{F}_Y + \mathcal{F}_t\right)\right) + \frac{A_r\mathcal{G}^2}{Y_r\mathcal{F}} + A_t\mathcal{G}_A + \frac{A_rY_t\mathcal{G}}{AY_r} + \frac{1}{2(n-3)Y\mathcal{F}} \times\left(A\left(\mathcal{F}^2\left(Q^2Y^{6-2n} + (n-3)\left(\frac{2\Lambda}{n-2}Y^2 - n + 3\right)\right) + (n-3)^2\mathcal{G}^2\right)\right) + Y_t\mathcal{G}_Y + \mathcal{G}_t = 0, \tag{71}$$

on (69) to obtain the Bernoulli equation

$$H_t - \mathcal{L}_4H^2 - \mathcal{L}_5H = 0, \tag{72}$$

where

$$\mathcal{L}_4 = \frac{1}{2YY_r\mathcal{F}}\left(2A_rY + (n-3)AY_r\right), \tag{73a}$$

$$\mathcal{L}_5 = \frac{1}{AYY_r\mathcal{F}}\Big(A\left(Y_r\left(Y\left(A_t\mathcal{F}_A + Y_t\mathcal{F}_Y + \mathcal{F}_t\right) - (n-3)Y_t\mathcal{F}\right) - 2A_rY\mathcal{G}\right) - A_rYY_t\mathcal{F} - \left((n-3)A^2Y_r\mathcal{G}\right)\Big). \tag{73b}$$

We cannot obtain the general solution to the restriction (71); however, particular solutions to (71) do exist (See later.). We solve (72), subject to the restriction (71), to obtain

$$H = \frac{\exp\left(\int_1^t \wp_{12}\mathrm{d}w\right)}{-\int_1^t \exp\left(\int_1^{\bar{w}}\wp_{12}\mathrm{d}w\right)\wp_{13}\mathrm{d}\bar{w} + \mathcal{R}_1}. \tag{74}$$

In (74) we have defined

$$\wp_{12} = \frac{1}{AYY_r\mathcal{F}}\Big(A\left(Y_r\left(Y\left(A_w\mathcal{F}_A + Y_w\mathcal{F}_Y + \mathcal{F}_w\right) - (n-3)Y_wF\right) - 2A_rY\mathcal{G}\right) - A_rYY_w\mathcal{F} - \left((n-3)A^2Y_r\mathcal{G}\right)\Big), \tag{75a}$$

$$\wp_{13} = \frac{1}{2YY_r\mathcal{F}}\left(2A_rY + (n-3)AY_r\right), \tag{75b}$$

and w and $\bar{w}$ are dummy variables and $\mathcal{R}_1$ is an arbitrary function of r. In (75a) $\mathcal{F}$ and $\mathcal{G}$ are functions of r, w, $A(r,w)$, and $Y(r,w)$. In (75b) $\mathcal{F}$ is a function of r, $\bar{w}$, $A(r,\bar{w})$ and $Y(r,\bar{w})$. We utilise (68) and (74) to express the potential function B as

$$B = -\frac{AY_r\mathcal{F}}{Y_t\mathcal{F} + A\left(\mathcal{G} - \frac{\exp\left(\int_1^t\wp_{12}\mathrm{d}w\right)}{-\int_1^t\exp\left(\int_1^{\bar{w}}\wp_{12}\mathrm{d}w\right)\wp_{13}\mathrm{d}\bar{w}+\mathcal{R}_1}\right)}. \tag{76}$$

The n dimensional solution reduces to the four dimensional solution by Naidoo et al. [42] when $n = 4$ in (76).

We now show that exact solutions to restriction (71) exist. We set

$$\mathcal{F} = \mathcal{F}(r,t,A,Y), \tag{77a}$$

$$\mathcal{G} = 0, \tag{77b}$$

in the restriction (71) to obtain the algebraic equation

$$Q^2Y^{6-2n} + (n-3)\left(\frac{2\Lambda Y^2}{n-2} - n + 3\right) = 0. \tag{78}$$

The general solution to (78) is unknown. However, specific solutions to (78) can be obtained by placing appropriate restrictions. When $n = 4$ then Y can be found explicitly as shown by Naidoo et al. [42]. If $\Lambda = 0$ then (78) can be solved for all n to give

$$Y = \left(\frac{(n-3)^2}{Q^2}\right)^{\frac{1}{6-2n}}. \tag{79}$$

When $\Lambda \neq 0$ we cannot solve (78) in general for all values of n; however, the particular spacetime dimension $n = 5$ leads to

$$\frac{4}{3}\Lambda Y^6 - 4Y^4 + Q^2 = 0. \tag{80}$$

Equation (80) is a cubic equation in Y^2. We solve (80) to obtain

$$Y^2 = \frac{1}{2}\Lambda\left(\sqrt[3]{-3\Lambda^2 Q^2 + \sqrt{9\Lambda^4 Q^4 - 48\Lambda^2 Q^2} + 8} + \frac{4}{\sqrt[3]{-3\Lambda^2 Q^2 + \sqrt{9\Lambda^4 Q^4 - 48\Lambda^2 Q^2} + 8}} + 2\right). \tag{81}$$

This result can be summarised in the following.

Theorem 3 *The higher dimension boundary condition with charge and a cosmological constant can be written as a first order differential equation in all potential functions, if we apply the transformation (which is a generalisation of Ivanov's horizon function* [40]*)*

$$H = \left(\frac{Y_t}{A} + \frac{Y_r}{B}\right)\mathcal{F} + \mathcal{G}, \tag{82}$$

where $\mathcal{F} = \mathcal{F}(r, t, A, Y)$ *and* $\mathcal{G} = \mathcal{G}(r, t, A, Y)$.

Corollary 3.1 *Known transformations, where the boundary condition is a first order equation, with* $n = 4$, $\mathcal{F} = 1$ *and* $\mathcal{G} = 0$ *are recovered as special cases.*

8 Equation of state

There are nine dependent variables, ρ, $p_\parallel$, $p_\perp$, q, ζ, A, B, Y and Q, in the system of Einstein–Maxwell field equations (18) with five equations. However (18e) allocates a relationship between ζ and Q. Thus if one of the ζ or Q is specified then the number of dependent variables in (18) is reduced to seven: ρ, $p_\parallel$, $p_\perp$, q, A, B and Y, with four Eqs. (18a), (18b), (18c) and (18d). The condition for isotropic pressure (9) and the boundary condition (22) are imposed on the system of Einstein–Maxwell field equations (18) which increases the number of equations to six. Since the number of dependent variables are not equal to the number of equations, the system is open. Additional equations are required to close the system of Einstein–Maxwell field equations (18). This can be done by specifying values for A, B or Y which is done when we solve the differential equation representing the boundary condition (24). Another way to close the Einstein–Maxwell field equations (18) is to add an equation of state.

Equations of state add to the physical reasonableness of theoretical models, as they relate to observations. There are different types of equations of state that add to the physical relevance of stellar models by dictating a relationship between the pressure and density. Some of the different types of equation of states in astrophysical scenarios can be found in [54]. An equation of state in an astrophysical scenario, relating the radial pressure and energy density, can be expressed as

$$p_\parallel = p_\parallel(\rho). \tag{83}$$

We only consider $p_\parallel$ in (83) as the tangential pressure $p_\perp$ in stars is small [55]. In this paper we supplement the boundary condition in higher dimensions (24) with the linear equation of state given by

$$p_\parallel = \nu\rho, \tag{84}$$

where ν is an arbitrary constant. The equation of state (84) is the simplest form (83) can take when pressure is non-zero. The value of the parameter ν is required to be carefully chosen, as it describes the matter type of the model. We construct the nonlinear partial differential that describes a linear equation of state by substituting the pressure and density expressions given by Eqs. (18a) and (18b) respectively, from the system of Einstein–Maxwell field equations, into the linear equation of state (84) to obtain

$$\begin{aligned}
&-\frac{2(n-3)A_t B^2 Y_t}{A} - 2(n-3)A_r A Y_r \\
&+\frac{1}{BY}\left(A^2\left(2\nu(n-3)B_r Y Y_r + (\nu+1)B^3\right.\right. \\
&\times\left((n-3)\left(-\frac{2\Lambda Y^2}{n-2} + n - 3\right) - Q^2 Y^{6-2n}\right) \\
&+(n-3)B\left(-\left((\nu+1)(n-3)Y_t^2\right)\right. \\
&\left.\left.\left.-2\nu Y Y_{rr}\right)\right)\right) + \frac{1}{Y}(n-3)B\left(2Y\left(\nu B_t Y_t + B Y_{tt}\right)\right. \\
&\left.+(\nu+1)(n-3)B Y_t^2\right) = 0.
\end{aligned} \tag{85}$$

Few solutions are known in the spacetime dimension $n = 4$, where *both* the boundary condition (24) and linear equation of state (85) are satisfied in the presence of charge Q and cosmological constant Λ. Some results in this regard are presented in Naidoo et al. [42]. In the absence of Λ or Q particular models of the boundary condition were obtained by Govinder and Govender [31], Abebe et al. [34] and Abebe and Maharaj [28,56] with a barotropic equation of state. The presence of Q and Λ leads to more complex gravitational interactions. Equations (24) and (85) are generalisations of the result of Naidoo et al. [42] to higher dimensions. Note that

Table 1 Physical features for the boundary condition with an equation of state $p_{\|} = \nu\rho$ for Model I

Matter variables

$$\rho = (n-2)\pi^{\frac{1}{2}-\frac{n}{2}}(n-3)^{\frac{2\nu}{(\nu-1)(n-3)}+1}(\nu(n-2))^{\frac{2}{\nu-1}}\Gamma\left(\frac{n-1}{2}\right)Q^{\frac{\nu-(\nu+1)n+3}{(\nu-1)(n-3)}}\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{\frac{2}{\nu-1}}(2(\nu-1))^{-1}$$

$$p_{\|} = \nu^{\frac{\nu+1}{\nu-1}}(n-2)\pi^{\frac{1}{2}-\frac{n}{2}}(n-3)^{\frac{2\nu}{(\nu-1)(n-3)}+1}\Gamma\left(\frac{n-1}{2}\right)Q^{\frac{\nu-(\nu+1)n+3}{(\nu-1)(n-3)}}\left((n-2)\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)\right)^{\frac{2}{\nu-1}}(2(\nu-1))^{-1}$$

$$p_{\perp} = (n-3)^3\pi^{\frac{1}{2}-\frac{n}{2}}\Gamma\left(\frac{n-3}{2}\right)\left(\frac{Q}{n-3}\right)^{-\frac{2}{n-3}}\left((n-2)\left(-(n-3)^{\frac{2}{(\nu-1)(n-3)}}\right)(\nu(n-2))^{\frac{2}{\nu-1}}Q^{-\frac{2(n-2)}{(\nu-1)(n-3)}-1}\right.$$
$$\left.\times\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{\frac{2}{\nu-1}}+2n-6\right)(8(2-n))^{-1}$$

$$q = \pi^{\frac{1}{2}-\frac{n}{2}}\frac{n-3}{2}!(n-3)^{\frac{-\nu+(\nu-1)n+3}{(\nu-1)(n-3)}}(\nu(n-2))^{\frac{\nu+1}{\nu-1}}Q^{\frac{\nu-(\nu+1)n+3}{(\nu-1)(n-3)}}\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{\frac{2}{\nu-1}}(2(\nu-1))^{-1}$$

Kinematical quantities

$$\dot{u}^a = (n-3)^{-\frac{\nu-\nu n+n-3}{(\nu-1)(n-3)}}(\nu(n-2))^{\frac{2}{\nu-1}}(\nu(n-1)+n-3)Q^{-\frac{2(\nu+n-3)}{(\nu-1)(n-3)}}\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{\frac{2}{\nu-1}}(2(\nu-1)Q')^{-1}$$

$$\theta = (n-3)^{\frac{\nu}{(\nu-1)(n-3)}}(\nu(n-2))^{\frac{\nu}{\nu-1}}Q^{\frac{\nu-(\nu+1)n+3}{2(\nu-1)(n-3)}}\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{\frac{1}{\nu-1}}(1-\nu)^{-1}$$

$$\sigma = (n-3)^{\frac{\nu}{(\nu-1)(n-3)}}(\nu(n-2))^{\frac{\nu}{\nu-1}}Q^{\frac{\nu-(\nu+1)n+3}{2(\nu-1)(n-3)}}\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{\frac{1}{\nu-1}}\left((1-\nu)\sqrt{n-1}\right)^{-1}$$

the charge Q is a function of r only, and the functions A, B and Y are functions of r and t. An added complication is the appearance of the parameter n in higher dimensions. There are currently no known solutions to the boundary condition (24) that admits an equation of state (85) when $n > 4$.

In spite of the issues mentioned above it is possible to obtain exact solutions satisfying both (24) and (85). For convenience we set $\Lambda = 0$ and present two classes of exact solutions. They are valid for all spacetime dimensions $n \geq 4$. The solutions found have been obtained by considering special cases of the transformation (39) on (85) and on special cases of (41) to obtain solutions.

8.1 Model I

The line element is given by

$$\mathrm{d}s^2 = -\left(\nu^{-\frac{\nu}{\nu-1}}(n-3)^{-\frac{\nu}{(\nu-1)(n-3)}}(n-2)^{-\frac{\nu}{\nu-1}}\right.$$
$$\times\mathcal{T}_1 Q^{\frac{\nu(n-1)+n-3}{2(\nu-1)(n-3)}}$$
$$\left.\times\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{-\frac{\nu}{\nu-1}}\right)^2\mathrm{d}t^2$$
$$+\left(\nu^{\frac{1}{1-\nu}}(n-3)^{\frac{2\nu-\nu n+n-3}{(\nu-1)(n-3)}}(n-2)^{\frac{1}{1-\nu}}\right.$$
$$\times Q'Q^{-\frac{-5\nu+(\nu-3)n+9}{2(\nu-1)(n-3)}}$$
$$\left.\times\left(\int_1^t \mathcal{T}_1(w)\mathrm{d}w\right)^{\frac{1}{1-\nu}}\right)^2\mathrm{d}r^2$$
$$+\left((n-3)^{\frac{1}{3-n}}Q^{\frac{1}{n-3}}\mathrm{d}\phi^2\right), \tag{86}$$

where $\mathcal{T}_1 = \mathcal{T}_1(t)$. In this case $\nu \neq 1$ and we have a linear equation of state. This model is also new when $n = 4$. The kinematical quantities and matter variables corresponding to (86) are given in Table 1.

8.2 Model II

The metric has the form

$$\mathrm{d}s^2 = -\left(\frac{1}{\sqrt{Q}\mathcal{J}_1}\right)^2\mathrm{d}t^2$$
$$+\left((n-3)^{\frac{1}{3-n}-1}Q^{\frac{1}{n-3}-\frac{1}{2}}Q'\mathcal{J}_1^{-1}\right)^2\mathrm{d}r^2$$
$$+\left((n-3)^{\frac{1}{3-n}}Q^{\frac{1}{n-3}}\right)^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right), \tag{87}$$

where $\mathcal{J}_1 = \mathcal{J}_1\left(t-(n-3)^{\frac{1}{3-n}}Q^{\frac{1}{n-3}+1}(n-2)^{-1}\right)$. In this case $\nu = 1$, and the equation of state is stiff. This model is also new when $n = 4$. The relevant kinematical quantities and matter variables to (87) are given in Table 2.

9 The horizon function

The horizon function of Ivanov [40] is regained when we substitute

$$\alpha = \beta = 1, \tag{88a}$$

Table 2 Physical features for the boundary condition with an equation of state $p_{\parallel} = \rho$

Matter variables

$$\rho = \frac{1}{4}(n-3)^{\frac{1}{n-3}+2}\pi^{\frac{1}{2}-\frac{n}{2}}\Gamma\left(\frac{n-3}{2}\right)Q^{\frac{1}{3-n}}\mathcal{J}_1\mathcal{J}_1'$$
$$p_{\parallel} = \frac{1}{2}(n-3)^{\frac{1}{n-3}+1}\pi^{\frac{1}{2}-\frac{n}{2}}\Gamma\left(\frac{n-1}{2}\right)Q^{\frac{1}{3-n}}\mathcal{J}_1\mathcal{J}_1'$$
$$p_{\perp} = (n-3)^{\frac{2}{n-3}+3}\pi^{\frac{1}{2}-\frac{n}{2}}\Gamma\left(\frac{n-1}{2}\right)Q^{-\frac{2}{n-3}}(2(2-n))^{-1}$$
$$q = \frac{1}{2}(n-3)^{\frac{1}{n-3}+1}\pi^{\frac{1}{2}-\frac{n}{2}}\Gamma\left(\frac{n-1}{2}\right)Q^{\frac{1}{3-n}}\mathcal{J}_1\mathcal{J}_1'$$

Kinematical quantities

$$\dot{u}^a = (n-3)^{\frac{1}{n-3}+1}Q^{\frac{1}{3-n}}\mathcal{J}_1\left(2Q\mathcal{J}_1' - (n-3)^{\frac{1}{n-3}+1}Q^{\frac{1}{3-n}}\mathcal{J}_1\right)(2Q')^{-1}$$
$$\theta = -\sqrt{Q}\mathcal{J}_1'$$
$$\sigma = -\sqrt{\frac{Q}{n-1}}\mathcal{J}_1'$$

$$\mathcal{F} = 1, \tag{88b}$$
$$\mathcal{G} = 0, \tag{88c}$$

in (39) to obtain

$$H = \frac{Y_r}{B} + \frac{Y_t}{A}. \tag{89}$$

Ivanov [40] showed that the presence of a horizon is implied in the four dimensional boundary condition when H approaches zero. We can extend this to higher dimensions. When $Q = \Lambda = 0$ the mass function can be expressed as

$$m(v) = \frac{n-3}{2}Y^{n-3}\left(1 + \frac{Y_t}{A^2} - \frac{Y_t^2}{B^2}\right). \tag{90}$$

Using (89) we express (90) as

$$\frac{2}{n-3}\frac{m(v)}{Y^{n-3}} - 1 = \frac{2Y_r}{A}H - H^2. \tag{91}$$

As H approaches zero the presence of a horizon is implied, as this results in a singularity in the line element (21).

10 Discussion

There are currently no known solutions to the higher dimensional shearing boundary condition (24). There is clearly a need to obtain solutions to (24), which may be shearing, in higher dimensions to understand the gravitational dynamics. We investigated two methods to solve the higher dimensional boundary condition (23) with Riccati equations been an underlying theme. In the first approach we rewrote (23) as the Riccati equation (24). This allowed us to place restrictions on $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ which allowed us to express (24) as a linear, Bernoulli or simpler Riccati equation. This simplification allowed us to solve the resultant equations to obtain exact solutions to (24). When $n = 4$ some of the results obtained using this approach reduce to results obtained by Mahomed et al. [52]. In the second approach we studied the effects the generalised transformation (39) has on the n dimensional boundary condition (24). The transformation mapped the n dimensional boundary condition (24) into a *new Riccati equation* (41). When $n = 4$ we regain the results of Naidoo et al [42] and earlier investigations. The transformation (39) introduced a dependence on the new functions $\mathcal{F}(r, t, A, Y)$ and $\mathcal{G}(r, t, A, Y)$. We obtained exact solutions to the n dimensional boundary condition that depend on the functions $\mathcal{F}$ and $\mathcal{G}$ in Sects. 4, 5, 6 and 7, by placing restrictions on $\mathcal{L}_4$ and $\mathcal{L}_6$. We obtained different solution sets to the boundary condition (24) using two different approaches. Our treatment has highlighted the role of Riccati equations in describing radiating stars in higher dimensional general relativity.

There are interesting features of (67) that highlight the importance of using transformations to solve differential equations. The importance of the transformation is seen when:

- The same restriction (30) and (31) result in the different solutions (31) and (50) which both satisfy (24).
- Setting $\mathcal{L}_6$ to zero results in a new restriction that cannot be obtained by setting $\mathcal{L}_1$, $\mathcal{L}_2$ or $\mathcal{L}_3$ to zero, allowing for new solutions.
- The transformation (67) reduces the order of the boundary condition (24) from a second order nonlinear partial differential equation into a first order nonlinear partial differential Eq. (69).
- It allowed for solutions to be obtained that admitted an equation of state.
- Horizons can be identified when the limit of H approaches zero in the transformation (89).

We have related the pressure to the energy density by including a linear equation of state $p_{\parallel} = \nu\rho$. We obtained new exact solutions to the boundary condition (24) that admit a linear equation of state, using special cases of the transformation (39). The use of the transformation to obtain solutions to the system of equations consisting of (24) and (85) shows the versatility of the transformation. The line elements can be written explicitly. The physical features (that provide a description for the kinematical quantities and matter variables) for each of the two models are listed in Tables 1 and 2. It can be seen in Tables 1 and 2 how dimension n, charge Q and the equation of state parameter ν affects the evolution of the different models of the radiating star.

There are currently no known methods to reduce the order of partial differential equations. The transformation (67) reduces the order of the partial differential Eq. (24). It will be interesting to identify what geometric property of the transformation allows it to behave in such a way. This will provide insight into how one can reduce the order of partial differential equations. We will investigate the geometric properties of (67) in future studies. Riccati equations frequently arise in spherically symmetric stars; it would also be interesting to investigate systems with axial and cylindrical symmetry to determine if Riccati or other standard nonlinear equations arise.

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