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All basic quantizations of $D = 3$, $N = 1$ Lorentz supersymmetry

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Abstract By the supersymmetrization of a simple algebraic technique proposed in Lukierski and Tolstoy (Eur Phys J C 77:226, 2017) we obtain the complete classification of all basic (nonisomorphic) quantum deformations for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2; \mathbb{C})$ and its pseudoreal and real forms in terms of the classical r-matrices. In particular, we prove that pseudoreal compact form has only one quantum deformation (standart q-analog), and the $D = 3$, $N = 1$ Lorentz supersymmetry, which is the noncompact real form of $\mathfrak{osp}(1|2; \mathbb{C})$, has four different Hopf-algebraic quantum deformations: two standard q-analogs, and two (Jordanian and super-Jordanian) twist deformations. All basic Hopf-algebraic quantum deformations are presented in the explicit form.

1 Introduction

The search for quantum gravity is linked with studies of noncommutative space-times and quantum deformations of space-time symmetries. The considerations of simple dynamical models in quantized gravitational background (see e.g. [2–4]) indicate that the presence of quantum gravity effects generates noncommutativity of space-time coordinates, and as well the well-known Lie-algebraic space-time symmetries (e.g. Euclidean, Lorentz, Poincare) are modified into respective quantum symmetries, described by noncommutative Hopf algebras, named quantum deformations or quantum groups [5]. Therefore, studding all aspects of the quantum deformations in details is an important issue in particular in the search of quantum gravity models and their quantum superextensions (SUGRA).

For classifications, constructions and applications of quantum Hopf deformations of an universal enveloping algebra $U(\mathfrak{g})$ of a Lie superalgebra $\mathfrak{g}$ ($\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, as a linear

space, with a linear $\mathbb{Z}_2$-grading function $\deg(\cdot)$: $\deg(\mathfrak{g}_a) = a \in \{\bar{0}, \bar{1}\}$) Lie bisuperalgebras $(\mathfrak{g}, \delta)$ play an essential role (in analogy to the case of Lie algebras (see e.g. [5,6] and [7,8])), where the *cobracket* δ is a linear super-skew-symmetric map:

$$\mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} = \mathfrak{g}_{\bar{0}} \wedge \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \wedge \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{0}} \wedge \mathfrak{g}_{\bar{1}}, \quad (1.1)$$

which conserves the grading function $\deg(\cdot)$:

$$\delta(\mathfrak{g}_{\bar{0}}) \in \mathfrak{g}_{\bar{0}} \wedge \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \wedge \mathfrak{g}_{\bar{1}}, \quad \delta(\mathfrak{g}_{\bar{1}}) \in \mathfrak{g}_{\bar{0}} \wedge \mathfrak{g}_{\bar{1}}. \quad (1.2)$$

provided that

$$\deg(\mathfrak{g}_a \wedge \mathfrak{g}_b) = \deg(\mathfrak{g}_a) + \deg(\mathfrak{g}_b) = a + b = c \mod 2 \quad (1.3)$$

for $a, b, c \in \{\bar{0}, \bar{1}\}$. In the general case $x = x_{\bar{0}} + x_{\bar{1}}$ we use the linearity of $\delta(x)$ in the argument x. It should be noted also that the superskew-symmetric bilinear space $\mathfrak{g} \wedge \mathfrak{g}$ is defined as follows:

$$x_a \wedge y_b := x_a \otimes y_b - (-1)^{ab} y_b \otimes x_a \quad (1.4)$$

for all homogeneous elements $x_a \in g_a$, $y_b \in g_b$ ($a, b \in \{\bar{0}, \bar{1}\}$). In case of general elements $x = x_{\bar{0}} + x_{\bar{1}}$ and $y = y_{\bar{0}} + y_{\bar{1}}$ we use the bilinearity of $x \wedge y$.

Moreover the cobracket δ satisfies the relations consisted with the superbracket $[\![\cdot, \cdot]\!]$ in the Lie superalgebra $\mathfrak{g}$:

$$\delta([\![x, y]\!]) = [\![\Delta_0(x), \delta(y)]\!] + [\![\delta(x), \Delta_0(y)]\!]$$
$$= [x, \delta(y)]_S + [\delta(x), y]_S \quad (x, y \in \mathfrak{g}), \quad (1.5)$$

$$(\delta \otimes \mathrm{id})\delta(x_a) + \mathrm{grcycle} = 0, \quad (1.6)$$

for any homogeneous element $x_a \in g_a$, $a \in \{\bar{0}, \bar{1}\}$. Here $\Delta_0(\cdot)$ is a trivial (non-deformed) coproduct

$$\Delta_0(x) = x \otimes 1 + 1 \otimes x, \quad (1.7)$$

and the symbol $[\cdot, \cdot]_S$ means the graded Schouten bracket (3.2). The first relation (1.5) is a condition of the 1-cocycle

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and the second one (1.6) is the co-Jacobi identity. The Lie bisuperalgebra $(\mathfrak{g}, \delta)$ is a correct infinitesimalization of the quantum Hopf deformation of $U(\mathfrak{g})$ and the operation δ is an infinitesimal part of difference between a coproduct Δ and an opposite coproduct $\tilde{\Delta}$ in the Hopf algebra, $\delta(x) = h^{-1}(\Delta - \tilde{\Delta}) \bmod h$, where h is a deformation parameter. Any two Lie bialgebras $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}, \delta')$ are isomorphic (equivalent) if they are connected by a $\mathfrak{g}$-automorphism φ satisfying the condition

$$\delta(x) = (\varphi \otimes \varphi)\delta'(\varphi^{-1}(x)) \tag{1.8}$$

for any $x \in \mathfrak{g}$.

Of our special interest here are the quasitriangular Lie bisuperalgebras $(\mathfrak{g}, \delta_{(r)}) := (\mathfrak{g}, \delta, r)$, where the cobracket $\delta_{(r)}$ is given by the classical r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ as follows:

$$\delta_{(r)}(x) = [\![\Delta_0(x), r]\!] = [x, r]_S. \tag{1.9}$$

Because the co-bracket $\delta_{(r)}$ conserves the grading value then from (1.9) we see that the r-matrix r is even one, $\deg(r) = \bar{0}$, i.e.

$$r \in \mathfrak{g}_{\bar{0}} \wedge \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \wedge \mathfrak{g}_{\bar{1}}. \tag{1.10}$$

Moreover it is easy to see also from (1.8) and (1.9) that *two quasitriangular Lie bisuperalgebras $(\mathfrak{g}, \delta_{(r)})$ and $(\mathfrak{g}, \delta_{(r')})$ are isomorphic iff the classical r-matrices r and r' are isomorphic, i.e.*

$$(\varphi \otimes \varphi)r' = r. \tag{1.11}$$

Therefore for a classification of all nonequivalent quasitriangular Lie bisuperalgebras $(\mathfrak{g}, \delta_{(r)})$ of the given Lie superalgebra $\mathfrak{g}$ we need to find all nonequivalent (nonisomorphic) classical r-matrices. Because nonequivalent quasitriangular Lie bisuperalgebras uniquely determine non-equivalent quasitriangular quantum deformations (Hopf algebras) of $U(\mathfrak{g})$ therefore the classification of all nonequivalent quasitriangular Hopf superalgebras is reduced to the classification of all nonequivalent classical r-matrices.

Let $\mathfrak{g}^* := (\mathfrak{g}, *)$ be a real or pseudoreal,[1] form of a classical complex Lie superalgebra $\mathfrak{g}$, where $*$ is an antilinear involutive or semiinvolutive antiautomorphism of $\mathfrak{g}$, then *the bisuperalgebra $(\mathfrak{g}^*, \delta_{(r)})$ is real iff the classical r-matrix r is $*$-anti-real ($*$-anti-Hermitian).*[2] Indeed, the condition of $*$-reality for the bisuperalgebra $(\mathfrak{g}^*, \delta)$ means that

$$\delta(x)^{*\otimes *} = \delta(x^*). \tag{1.12}$$

Applying this condition to the relations (1.9) we obtain that

$$r^{*\otimes *} = -r, \tag{1.13}$$

i.e. the r-matrix r is $*$-anti-Hermitian.

Recently in the paper [1] there were investigated the quantum deformations of the complex Lie algebra $\mathfrak{sl}(2; \mathbb{C}) \simeq \mathfrak{o}(3; \mathbb{C})$ and its real forms $\mathfrak{su}(2) \simeq \mathfrak{o}(3)$, $\mathfrak{sl}(2; \mathbb{R}) \simeq \mathfrak{o}(2, 1)$ and $\mathfrak{su}(1, 1) \simeq \mathfrak{o}(2, 1)$. Namely, firstly it was obtained the complete classifications of the nonequivalent (nonisomorphic) classical r-matrices (bialgebras) for all these Lie algebras and then Hopf deformations corresponding to these bialgebras were presented in explicit form. In particular, it was shown that $D = 3$ Lorentz symmetry $\mathfrak{o}(2, 1)$ ($\simeq \mathfrak{sl}(2; \mathbb{R}) \simeq \mathfrak{su}(1, 1)$) has two standard q-deformations and one Jordanian.

In this work we would like to present some superanalog of these results, namely we first give the complete classifications of the nonequivalent (nonisomorphic) classical r-matrices for complex Lie superalgebra $\mathfrak{osp}(1|2; \mathbb{C})$ (which is a minimal supersymmetric extension of the Lie algebra $\mathfrak{sl}(2; \mathbb{C})$), and its pseudoreal $\mathfrak{osp}^*(1|\mathfrak{su}(2))$[3] and real $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2; \mathbb{R}))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1, 1))$ forms. In particular, it will be shown that $N = 1$, $D = 3$ Lorentz supersymmetry,

$$\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1)) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{sl}(2; \mathbb{R})) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{su}(1, 1)), \tag{1.14}$$

has two nonequivalent standard q-deformations and two nonequivalent (Jordanian and super-Jordanian) twist deformations. Moreover it will be shown that each real form $\mathfrak{osp}^*(1|\mathfrak{o}(3))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$ has four variants of the deformation in accordance with how the super-involutions $*$ and $\dagger$ are extended on all universal enveloping superalgebra $U(\mathfrak{osp}(1|2; \mathbb{C}))$ and its tensor square (see Sect. 4). The isomorphic Lie superalgebras (1.14) and their quantum deformations play very important role in physics as well as in mathematical considerations, so the structure of these deformations should be understood with full clarity. It should be noted also that the importance of $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$ and its deformations follows also from the unique role of the $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$ superalgebra as the lowest-dimensional rank one noncompact simple classical Lie superalgebra.

In this paper we investigate the quantum deformations of $D = 3$, $N = 1$ Lorentz supersymmetry. Firstly, following [1,10], we obtain the complete classifications of the nonequivalent (nonisomorphic) classical r-matrices for complex Lie superalgebra $\mathfrak{osp}(1|2; \mathbb{C})$ and its pseudoreal and real forms $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2; \mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1, 1))$ with the help of explicit formulas for the automorphisms of these Lie superalgebras in terms of the Cartan-Weyl bases. In the case of $\mathfrak{osp}(1|2; \mathbb{C})$ there are three nonequivalent classical r-matrices - one standard form and two Jordanian and super-Jordanian ones. For the pseudoreal superalgebra $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ there is only the standard classical

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[1] See Sect. 2 where a pseudoreality condition for the superalgebra $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ is considered.

[2] All bialgebras over the simple complex and real Lie algebras are quasitriangular, due to Whitehead lemma (see e.g. [9]). It is more likely that the Whitehead lemma is valid also for all classical Lie superalgebras.

[3] It should be noted that the compact forms for all superalgebras of the classical series $\mathfrak{osp}(1|2m; \mathbb{C})$ ($m = 1, 2, \ldots$) are pseudoreal.

r-matrix. For the real case $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ we obtained four noneqvivalent r-matrices - standard, quasi-standard, quasi-Jordanian and quasi-super-Jordanian ones. In the real case of $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ we find also four nonequivalent r-matrices - standard, quasi-standard, Jordanian and super-Jordanian ones. Then using isomorphisms $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1)) \simeq \mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1)) \simeq \mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ we express these r-matrices in terms of the Cartesian basis of the $D = 3$, $N = 1$ Lorentz superalgebra $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$ and we see that two systems with four r-matrices for $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ and $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ superalgebras coincides. Thus we obtain that the isomorphic Lie algebras $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ and $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ have the isomorphic systems of their quasi-triangular Lie bealgebras. In the case of $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$ we obtain that the $D = 3$, $N = 1$ Lorentz superalgebra has two standard q-deformations and two Jordanian and super-Jordanian ones. These Hopf deformations are presented in explicit form in terms of the quantum Cartan-Weyl generators for the quantized universal enveloping algebras of $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ and $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$. It should be noted that the full list of the nonequivalent classical r-matrices for the complex Lie superalgebra $\mathfrak{osp}(2|1;\mathbb{C})$ was obtained previously in [11] where the authors used a technique of computer algebraic computations. However, the complete list of the nonequivalent classical r-matrices and their Hopf quantizations for the real $D = 3$, $N = 1$ Lorentz superalgebra $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$ has not been presented in the literature, but some examples of such r-matrices and their quantizations were already considered (for example, see [12–15]).

The isomorphic Lie superalgebras $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$, $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$, $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ and their quantum deformations play very important role in physics as well as in mathematical considerations, so the structure of these deformations should be understood with full clarity. For example, since the real superalgebra $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ can be used as $D = 1$, $N = 1$ superconformal symmetry then the obtained results in this case allow us to interpret as deformations of $N = 1$ superconformal mechanics [15]. In mathematics and mathematical physics the importance of $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$ and its deformations follows also from the unique role of the $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$ algebra as the lowest-dimensional rank one noncompact simple Lie superalgebra, endowed only with unitary infinite-dimensional representations.

The plan of this paper is the following. In Sect. 2 we consider the complex Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ and its all real forms: $\mathfrak{osp}^{*}(1|\mathfrak{o}(3)) \simeq \mathfrak{osp}^{*}(1|\mathfrak{su}(2))$, $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1)) \simeq \mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1)) \simeq \mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$. In Sect. 3 we classify all classical r-matrices for the complex Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ and in Sect. 4 all classical r-matrices for its real forms: $\mathfrak{osp}^{*}(1|\mathfrak{su}(2))$, $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ and $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$. In Sect. 5 we provide the explicit isomorphisms between the $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$, $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ and $\mathfrak{osp}^{\dagger}(1|\mathfrak{o}(2,1))$ bialgebras. In Sect. 6 all four Hopf-algebraic quantizations (explicit quantum deformations) of the real $D = 3$ Lorentz supersymmetry are presented in detail: quantized bases, coproducts and universal R-matrices are given. In Sect. 7 we present short summary and outlook.

2 Complex Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ and its real forms

The Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ is a initial element of infinite orthosymplectic series $\mathfrak{osp}(m|2n;\mathbb{C})$ $(m, n = 1, 2, \ldots)$. Each orthosymplectic Lie superalgebra $\mathfrak{osp}(m|2n;\mathbb{C})$, as a linear space, is a direct sum of two graded (even and odd) components: $\mathfrak{osp}(m|2n;\mathbb{C}) = \mathfrak{osp}(m|2n;\mathbb{C})_0 \oplus \mathfrak{osp}(m|2n;\mathbb{C})_1$, where the even part $\mathfrak{osp}(m|2n;\mathbb{C})_0$ is a direct sum of the orthogonal and symplectic Lie algebras: $\mathfrak{osp}(m|2n;\mathbb{C})_0 = \mathfrak{o}(m;\mathbb{C}) \oplus \mathfrak{sp}(2n;\mathbb{C})$, and moreover $[\mathfrak{osp}(m|2n;\mathbb{C})_0, \mathfrak{osp}(m|2n;\mathbb{C})_1] = \mathfrak{osp}(m|2n;\mathbb{C})_1$, $\{\mathfrak{osp}(m|2n;\mathbb{C})_1, \mathfrak{osp}(m|2n;\mathbb{C})_1\} = \mathfrak{osp}(m|2n;\mathbb{C})_0$.

In the case of $\mathfrak{osp}(1|2;\mathbb{C})$ the even part has the form $\mathfrak{osp}(1|2;\mathbb{C})_0 = \mathfrak{o}(1;\mathbb{C}) \oplus \mathfrak{sp}(2;\mathbb{C})$ where $\mathfrak{o}(1;\mathbb{C})$ is null-algebra $\mathfrak{o}(1;\mathbb{C}) \simeq \{0\}$ and $\mathfrak{sp}(2;\mathbb{C}) \simeq \mathfrak{sl}(2;\mathbb{C}) \simeq \mathfrak{o}(3;\mathbb{C})$. Let $\{E_{\pm}, H\}$ be a Cartan-Weyl (CW) basis of $\mathfrak{sl}(2;\mathbb{C}) \simeq \mathfrak{o}(3;\mathbb{C})$ with the standard relations:

$$[H, E_{\pm}] = \pm E_{\pm}, \quad [E_+, E_-] = 2H, \tag{2.1}$$

then in the odd two-dimensional space $\mathfrak{osp}(1|2;\mathbb{C})_1$ one can choose the basis (v_+, v_-) satisfying the relations(see, for example, [16–18]):

$$[H, v_{\pm}] = \pm \frac{1}{2} v_{\pm}, \quad [E_{\mp}, v_{\pm}] = v_{\mp}, \quad [E_{\pm}, v_{\pm}] = 0, \tag{2.2}$$

$$\{v_{\pm}, v_{\pm}\} = \pm \frac{1}{2} E_{\pm}, \quad \{v_+, v_-\} = -\frac{1}{2} H. \tag{2.3}$$

The the CW generators H, $E_{\pm}$ of $\mathfrak{sl}(2;\mathbb{C}) \simeq \mathfrak{o}(3;\mathbb{C})$ is related with the *Cartesian* basis I_i $(i = 1, 2, 3)$ as follows:

$$H = \iota I_3, \quad E_{\pm} = \iota I_1 \mp I_2. \tag{2.4}$$

For convenience we set also

$$v_1 := v_+, \quad v_2 := v_-. \tag{2.5}$$

In terms of the generators $\{I_i, v_{\alpha} | i = 1, 2, 3; \alpha = 1, 2\}$ the defining relations (2.1)–(2.3) take the form:

$$[I_i, I_j] = \varepsilon_{ijk} I_k, \quad [I_i, v_{\alpha}] = -\frac{\iota}{2} (\sigma_i)_{\beta\alpha} v_{\beta},$$

$$\{v_1, v_1\} = \frac{1}{2} (\iota I_1 - I_2), \quad \{v_1, v_2\} = -\frac{\iota}{2} I_3, \tag{2.6}$$

$$\{v_2, v_2\} = -\frac{1}{2} (\iota I_1 + I_2),$$

where σ_i, $(i = 1, 2, 3)$ are the 2×2 Pauli matrices, and $(\alpha, \beta = 1, 2)$.

It is well known that the Lie algebra $\mathfrak{o}(3; \mathbb{C}) \simeq \mathfrak{sl}(2; \mathbb{C})$, which is a subalgebra of the superalgebra $\mathfrak{osp}(1|2; \mathbb{C})$, has two real forms: compact $\mathfrak{o}(3) \simeq \mathfrak{su}(2)$, and noncompact $\mathfrak{o}(2, 1) \simeq \mathfrak{sl}(2; \mathbb{R}) \simeq \mathfrak{su}(1, 1)$. These real forms of the subalgebra $\mathfrak{o}(3; \mathbb{C}) \simeq \mathfrak{sl}(2; \mathbb{C})$ are raised up to the odd part $\mathfrak{osp}_1(1|2; \mathbb{C})$, that is to the whole superalgebra $\mathfrak{osp}(1|2; \mathbb{C})$.

I. *The compact pseudoreal superalgebra* $\mathfrak{osp}^*(1|\mathfrak{o}(3)) \simeq \mathfrak{osp}^*(1|\mathfrak{su}(2))$.

In terms of the generators $\{I_i, v_\alpha | i = 1, 2, 3; \alpha = 1, 2\}$ this form is defined by the following conjugation:

$$I_i^* = -I_i, \quad v_1^* = \varepsilon v_2, \quad v_2^* = -\varepsilon v_1, \tag{2.7}$$

where $\varepsilon = 1$ if the conjugation (*) of the Lie superbracket is graded, i.e.

$$[\![x_a, x_b]\!]^* = (-1)^{ab} [\![x_b^*, x_a^*]\!] \quad \text{(graded)}, \tag{2.8}$$

and $\varepsilon = \iota$ if the conjugation (*) of the Lie superbracket is not graded, i.e.

$$[\![x_a, x_b]\!]^* = [\![x_b^*, x_a^*]\!] \quad \text{(ungraded)}, \tag{2.9}$$

for all homogeneous elements $x_a \in \mathfrak{g}_a$, $x_b \in \mathfrak{g}_b$ ($\mathfrak{g} := \mathfrak{osp}(1|2; \mathbb{C})$). We see that the conjugation (*) prolonged to the odd generators v_α is an antilinear antiautomorphism of four order provided that $(v_\alpha^*)^* = -v_\alpha$ which define pseudoreal condition (see [16–18]).

Therefore this form is called *pseudoreal* and in terms of the Cartesian generators (2.6)–(2.7) it is denoted by $\mathfrak{osp}^*(1|\mathfrak{o}(3))$. In terms of the CW generators $H := \iota I_3$, $E_\pm := \iota I_1 \mp I_1$, $v_+ := v_1, v_- := v_2$ this pseudoreal form $\mathfrak{osp}^*(1|\mathfrak{o}(3))$ denoted also by $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, is given as follows

$$H^* = H, \quad E_\pm^* = E_\mp, \quad v_\pm^* = \pm\varepsilon v_\mp. \tag{2.10}$$

II. *The noncompact real form* $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1)) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{sl}(2; \mathbb{R})) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{su}(1, 1))$.

In theorems of the generators $\{I_i, v_\alpha | i = 1, 2, 3; \alpha = 1, 2\}$ this form is defined by the following conjugation:

$$I_i^\dagger = (-1)^{i-1} I_i, \quad v_1^\dagger = \varepsilon v_1 \quad v_2^\dagger = \varepsilon v_2, \tag{2.11}$$

where $\varepsilon = 1$ if ($\dagger$) is the graded conjugation, and $\varepsilon = \iota$ if ($\dagger$) is the ungraded conjugation.[4] We see that the conjugation ($\dagger$) prolonged to the odd generators v_α is an antilinear antiautomorphism of second order, that is $(v_\alpha^*)^* = v_\alpha$. Therefore this form is called *real*, and it is denoted by $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$.

If we introduce the CW generators

$$H := \iota I_3, \quad E_\pm := \iota I_1 \mp I_2, \quad v_+ := v_1, \quad v_- := v_2, \tag{2.12}$$

where the Cartesian generators $\{I_i, v_\alpha | i = 1, 2, 3; \alpha = 1, 2\}$ satisfy the conjugation (2.11), then the real condition is given

as follows

$$H^\dagger = -H, \quad E_\pm^\dagger = -E_\pm, \quad v_\pm^\dagger = \varepsilon v_\pm. \tag{2.13}$$

In terms of the given CW basis the real form $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$ is also denoted by $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2; \mathbb{R}))$.

We can also introduce an alternative CW basis $H', E'_\pm, v'_\pm$ in $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$ which are expressed in terms of the Cartesian generators I_i, v_α ($i = 1, 2, 3; \alpha = 1, 2$) and the CW generators $H, E_\pm, v_\pm$ as follows:

$$H' = \iota I_2 = -\frac{\iota}{2}(E_+ - E_-),$$

$$E'_\pm = \iota I_1 \pm I_3 = \mp\iota H + \frac{1}{2}(E_+ + E_-),$$

$$v'_+ = \frac{1}{\sqrt{2}}(v_+ + \iota v_-), \quad v'_- = \frac{1}{\sqrt{2}}(\iota v_+ + v_-). \tag{2.14}$$

The CW basis $H', E'_\pm, v'_\pm$ satisfy the defining relations (2.1)–(2.3), and it has the conjugation properties:

$$(H')^\dagger = H', \quad (E'_\pm)^\dagger = -E'_\mp, \quad (v'_\pm)^\dagger = -\iota\varepsilon v'_\mp. \tag{2.15}$$

The real superalgebra $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$ in terms of the CW basis $H', E'_\pm, v'_\pm$ will be also denoted by $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1, 1))$.

It should be noted that in the case of $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1, 1))$ the Cartan generator H' is compact while for the case $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2, \mathbb{R}))$ the Cartan generator H is noncompact.

It should be noted also that the Casimir element of two order, that is a $\mathfrak{osp}(1|2; \mathbb{C})$-invariant element of the universal enveloping superalgebra $U(\mathfrak{osp}(1|2; \mathbb{C}))$:

$$C_2 := v_+v_- - v_-v_+ + \frac{1}{2}E_+E_- + \frac{1}{2}E_-E_+ + H^2$$

$$= 2v_+v_- + E_+E_- + H^2 - \frac{1}{2}H \tag{2.16}$$

satisfy the reality condition

$$C_2^\circledast = C_2 \quad (\circledast = *, \dagger) \tag{2.17}$$

with respect to all conjugations (2.10), (2.13) and (2.15) provided that these conjugations act on the product of two homogeneous elements x_a and y_b by the formulas (2.8), (2.9), where the superbracket $[\![x_a, x_b]\!]$ should be replaced on the usual product $x_a x_b$.

3 Classical *r*-matrices of the complex Lie superalgebra $\mathfrak{osp}(1|2; \mathbb{C})$

In this section we obtain complete classification bialgebras (classical *r*-matrices) for the complex Lie superalgebra $\mathfrak{osp}(1|2; \mathbb{C})$ and then in Sect. 4 we classify its real forms $\mathfrak{osp}^*(1|\mathfrak{o}(3))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1))$ using the isomorphisms: $\mathfrak{osp}(1|\mathfrak{o}(3; \mathbb{C})) \simeq \mathfrak{osp}(1|\mathfrak{sl}(2; \mathbb{C}))$, $\mathfrak{osp}^*(1|\mathfrak{o}(3)) \simeq \mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2, 1)) \simeq \mathfrak{osp}^\dagger(1|\mathfrak{sl}(2; \mathbb{R})) \simeq$

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[4] See (2.8), (2.9), where the conjugation (*) is replaced by ($\dagger$).

$\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$. In Sect. 5 we explicitly find out an isomorphism between $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ bialgebras and fix on the basis $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ bialgebra in such forms which are convenient for quantizations.

By the definition any classical r-matrix of arbitrary complex or real Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, $r \in \mathfrak{g}_{\bar{0}} \wedge \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \wedge \mathfrak{g}_{\bar{1}}$, satisfy the classical Yang–Baxter equation (CYBE):

$$[r, r]_S = \tilde{\Omega}. \tag{3.1}$$

Here $[\cdot, \cdot]_S$ is the graded Schouten bracket which for any monomial skew-symmetric even two-tensors $r_1 = x_a \wedge y_a$ and $r_2 = u_b \wedge v_b$ ($x_a, y_a \in \mathfrak{g}_a$; $u_b, v_b \in \mathfrak{g}_b$; $a, b, \in \{\bar{0}, \bar{1}\}$) is given by

$$\begin{aligned}
&[x_a \wedge y_a, u_b \wedge v_b]_S \\
&:= x_a \wedge (\llbracket y_a, u_b \rrbracket \wedge v_b + (-1)^{ab} u_b \wedge \llbracket y_a, v_b \rrbracket) \\
&\quad -(-1)^a y_a \wedge (\llbracket x_a, u_b \rrbracket \wedge v_b + (-1)^{ab} u_b \wedge \llbracket x_a, v_b \rrbracket) \\
&= [u_b \wedge v_b, x_a \wedge y_a]_S
\end{aligned} \tag{3.2}$$

and $\tilde{\Omega}$ is the $\mathfrak{g}$-invariant element, $\tilde{\Omega} \in (\overset{3}{\wedge} \mathfrak{g})_{\mathfrak{g}}$, that in the case of $\mathfrak{g} := \mathfrak{osp}(1|2;\mathbb{C})$ looks as follows:

$$\begin{aligned}
\tilde{\Omega} &= \gamma \Omega(\mathfrak{osp}(1|2;\mathbb{C})) \\
&= \gamma(4E_- \wedge H \wedge E_+ + 4v_- \wedge v_+ \wedge H \\
&\quad + 2v_- \wedge v_- \wedge E_+ - 2v_+ \wedge v_+ \wedge E_-),
\end{aligned} \tag{3.3}$$

where $\gamma \in \mathbb{C}$.

We have already mentioned that a classical r-matrix r is an even two-tensor, i.e.:

$$\begin{aligned}
r \in V_{\bar{0}} &:= \mathfrak{osp}_{\bar{0}}(1|2;\mathbb{C}) \wedge \mathfrak{osp}_{\bar{0}}(1|2;\mathbb{C}) \\
&\quad \oplus \mathfrak{osp}_{\bar{1}}(1|2;\mathbb{C}) \wedge \mathfrak{osp}_{\bar{1}}(1|2;\mathbb{C}).
\end{aligned} \tag{3.4}$$

As a basis in the linear space V_0 we can take the following two-tensors:

$$\begin{aligned}
r_0 &:= 2v_+ \wedge v_- + E_+ \wedge E_-, \\
r_\pm &:= \pm v_\pm \wedge v_\pm \pm E_\pm \wedge H,
\end{aligned} \tag{3.5}$$

$$\bar{r}_0 := E_+ \wedge E_-, \quad \bar{r}_\pm := \pm E_\pm \wedge H. \tag{3.6}$$

The following propositions are valid:

(i) Any linear combination of the elements (3.5) is a classical r-matrix, namely, if

$$r := \beta_+ r_+ + \beta_0 r_0 + \beta_- r_- \tag{3.7}$$

for $\forall \beta_+, \beta_0, \beta_- \in \mathbb{C}$, then we have

$$\begin{aligned}
[r, r]_S &= (\beta_0^2 + \beta_+ \beta_-) \Omega(\mathfrak{osp}(1|2;\mathbb{C})) \\
&\equiv \gamma \Omega(\mathfrak{osp}(1|2;\mathbb{C})).
\end{aligned} \tag{3.8}$$

(ii) Any linear combination of the elements $\bar{r}_0, \bar{r}_\pm$:

$$\bar{r} := \beta_+ \bar{r}_+ + \beta_0 \bar{r}_0 + \beta_- \bar{r}_- \tag{3.9}$$

for $\beta_0^2 + \beta_+ \beta_- = 0$ satisfies the homogeneous CYBE, i.e. $[\bar{r}, \bar{r}]_S = 0$.

(iii) Any classical r-matrix of $\mathfrak{osp}(1|2;\mathbb{C})$ is presented only in the form (3.7) or (3.9).

Firstly we prove the proposition *(i)*. Let (3.7) be an arbitrary linear combination of the elements (3.5). Because all basis elements (3.5) are classical r-matrices, moreover $[r_\pm, r_\pm]_S = 0$, as well as the Schouten brackets of the elements $r_\pm$ with r_0 are also equal to zero, $[r_\pm, r_0]_S = 0$, and we have

$$\begin{aligned}
[r, r]_S &= 2\beta_+ \beta_- [r_+, r_-]_S + \beta_0^2 [r_0, r_0]_S \\
&= (\beta_0^2 + \beta_+ \beta_-) \Omega(\mathfrak{osp}(1|2;\mathbb{C})) \\
&\equiv \gamma \Omega(\mathfrak{osp}(1|2;\mathbb{C})).
\end{aligned} \tag{3.10}$$

Thus the arbitrary element (3.7) is a classical r-matrix.

In the case of the proposition *(ii)* we have $[\bar{r}_\pm, \bar{r}_\pm]_S = 0$, $[\bar{r}_\pm, \bar{r}_0]_S = 0$, and the Schouten bracket for the arbitrary vector (3.9) looks as follows:

$$\begin{aligned}
[\bar{r}, \bar{r}]_S &= 2\beta_+ \beta_- [\bar{r}_+, \bar{r}_-]_S + \beta_0^2 [\bar{r}_0, \bar{r}_0]_S \\
&= (\beta_0^2 + \beta_+ \beta_-) \Omega(\mathfrak{sl}(2;\mathbb{C})) \\
&\equiv \gamma \Omega(\mathfrak{sl}(2;\mathbb{C})),
\end{aligned} \tag{3.11}$$

where $\Omega(\mathfrak{sl}(2;\mathbb{C})$ is the $\mathfrak{sl}(2;\mathbb{C})$-invariant element (the first term in the parenthesis on the right-hand side of (3.3)). Because $\Omega(\mathfrak{sl}(2;\mathbb{C})) \neq \tilde{\Omega}(\mathfrak{osp}(1|2;\mathbb{C}))$, therefore it should be $\gamma := \beta_0^2 + \beta_+ \beta_- = 0$ so the element $\bar{r}$ satisfies the homogeneous CYBE (3.8).

Finally we prove the proposition *(iii)*. Let us consider a general two-tensor r_g which is a sum of the two-tensors (3.7) and (3.9)

$$\begin{aligned}
r_g &= r + \bar{r} = \beta_+ r_+ + \beta_0 r_0 + \beta_- r_- \\
&\quad + \beta'_+ \bar{r}_+ + \beta'_0 \bar{r}_0 + \beta'_- \bar{r}_-
\end{aligned} \tag{3.12}$$

with arbitrary coefficients $\beta_\pm, \beta_0, \beta'_\pm, \beta'_0 \in \mathbb{C}$. It is not too difficult to see that the two-tensor (3.12) is a non-zero classical r-matrix of $\mathfrak{osp}(1|2;\mathbb{C})$ if and only if $|\beta_+| + |\beta_0| + |\beta_-| \neq 0$ and $\beta'_+ = \beta'_0 = \beta'_- = 0$ or $|\beta'_+| + |\beta'_0| + |\beta'_-| \neq 0$ and $\beta_+ = \beta_0 = \beta_- = 0$ provided that $\beta_0'^2 + \beta'_+ \beta'_- = 0$. Indeed, let us calculate the Schouten bracket of r_g;

$$
\begin{aligned}
[r_g, r_g]_S &= [r, r]_S + [\bar{r}, \bar{r}]_S + 2[r, \bar{r}]_S \\
&= (\beta_0^2 + \beta_+\beta_-)\Omega(\mathfrak{osp}(1|2; \mathbb{C})) \\
&\quad +(\beta_0'^2 + \beta_+'\beta_-')\Omega(\mathfrak{sl}(2; \mathbb{C})) + 2\beta_+\beta_+'[r_+, \bar{r}_+]_S \\
&\quad +2(\beta_+\beta_0'[r_+, \bar{r}_0]_S + \beta_0\beta_+'[r_0, \bar{r}_+]_S) \\
&\quad +2(\beta_+\beta_-'[r_+, \bar{r}_-]_S + \beta_0\beta_0'[r_0, \bar{r}_0]_S \\
&\quad +\beta_-\beta_+'[r_-, \bar{r}_+]_S) + 2(\beta_-\beta_0'[r_-, \bar{r}_0]_S \\
&\quad +\beta_0\beta_-'[r_0, \bar{r}_-]_S) + 2\beta_-\beta_-'[r_-, \bar{r}_-]_S,
\end{aligned}
\tag{3.13}
$$

where all the Schouten brackets $[r_i, \bar{r}_j]_S$, $(i, j = \pm, 0)$, are different[5] and each of them has the weight $(i1 + j1)$ with respect to the ad_H-action, $\mathrm{ad}_H([r_i, \bar{r}_j]_S) \equiv [H, [r_i, \bar{r}_j]_S]_S = (i1 + j1)[r_i, \bar{r}_j]_S$, for example, the first and the last Schouten brackets $[r_\pm, \bar{r}_\pm]_S$ have the weight ± 2. The two-tensor (3.12) will be a classical r-matrix of $\mathfrak{osp}(1|2; \mathbb{C})$ if its Schouten bracket (3.13) is $\mathfrak{osp}(1|2; \mathbb{C})$-invariant, i.e.

$$
[x, [r_g, r_g]_S]_S = 0 \quad (\forall x \in \mathfrak{osp}(1|2; \mathbb{C})).
\tag{3.14}
$$

Applying this condition to the right-hand side of (3.13) for $x = H$ first and then for $x = E_+$, and finally for $x = v_+$ we obtain the following quadratic equations:

$$
\begin{aligned}
&\beta_\pm\beta_\pm' = 0, \quad \beta_\pm\beta_0' = 0, \quad \beta_0\beta_\pm' = 0, \\
&2\beta_\pm\beta_\mp' + \beta_0\beta_0' = 0, \quad \beta_0'^2 + \beta_+'\beta_-' = 0.
\end{aligned}
\tag{3.15}
$$

It is easy to see that if $\beta_+ \neq 0$ then $\beta_+' = \beta_-' = \beta_0' = 0$. The other two cases $\beta_- \neq 0$ and $\beta_0 \neq 0$ are similar. Thus the general two-tensor (3.12), where at least one of the coefficients $\beta_\pm$, β_0 is not zero, will be a classical r-matrix if it has the form (3.7). It is evident that a similar result is obtained if we replace $\beta_i \rightleftarrows \beta_i'$ $(i = \pm, 0)$ provided that $\beta_0'^2 + \beta_+'\beta_-' = 0$, that is the general two-tensor (3.12), where at least one of the coefficients $\beta_\pm'$, β_0' is not zero, will be a classical r-matrix if it has the form (3.9).

We shall call the parameter $\gamma = \beta_0^2 + \beta_+\beta_-$ in (3.8) the γ-characteristic of the classical r-matrix (3.7). It is evident that the γ-characteristic of the classical r-matrix r is invariant under the $\mathfrak{osp}(1|2; \mathbb{C})$-automorphisms, i.e. any two r-matrices r and r', which are connected by a $\mathfrak{osp}(1|2; \mathbb{C})$-automorphism, have the same γ-characteristic, $\gamma = \gamma'$.

There are two types of explicit $\mathfrak{osp}(1|2; \mathbb{C})$-automorphisms. First type connecting the classical r-matrices with zero γ-characteristic is given by the formulas:

$$
\begin{aligned}
\varphi_0(E_+) &= \chi(\tilde{\beta}_+ E_+ - 2\tilde{\beta}_0 H + \tilde{\beta}_- E_-), \\
\varphi_0(E_-) &= \chi^{-1}(\tilde{\beta}_- E_+ - 2\kappa\tilde{\beta}_0 H + \tilde{\beta}_+ E_-), \\
\varphi_0(H) &= \tilde{\beta}_0 E_+ + (\kappa\tilde{\beta}_+ + \tilde{\beta}_-) H + \kappa\tilde{\beta}_0 E_-, \\
\varphi_0(v_+) &= \sqrt{\chi}\left(\sqrt{\tilde{\beta}_+} v_+ + \sqrt{\tilde{\beta}_-} v_-\right), \\
\varphi_0(v_-) &= \sqrt{\chi^{-1}}\left(\sqrt{\tilde{\beta}_-} v_+ + \kappa\sqrt{\tilde{\beta}_+} v_-\right),
\end{aligned}
\tag{3.16}
$$

where χ is a non-zero rescaling parameter (including $\chi = 1$), κ takes two values $+1$ or -1, and the parameters $\tilde{\beta}_i$ $(i = +, 0, -)$ satisfy the conditions:

$$
\gamma := \tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 0, \quad \kappa\tilde{\beta}_+ - \tilde{\beta}_- = 1.
\tag{3.17}
$$

Let us consider two independent pair $\{r, r'\}$ and $\{\bar{r}, \bar{r}'\}$ of the general r-matrices with zero γ-characteristics:

$$
\begin{aligned}
r &:= \beta_+ r_+ + \beta_0 r_0 + \beta_- r_-, \\
r' &:= \beta_+' r_+ + \beta_0' r_0 + \beta_-' r_-,
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\bar{r} &:= \beta_+ \bar{r}_+ + \beta_0 \bar{r}_0 + \beta_- \bar{r}_-, \\
\bar{r}' &:= \beta_+' \bar{r}_+ + \beta_0' \bar{r}_0 + \beta_-' \bar{r}_-,
\end{aligned}
\tag{3.19}
$$

where $\gamma = \beta_0^2 + \beta_+\beta_- = 0$ and $\gamma = \beta_0'^2 + \beta_+'\beta_-' = 0$. Moreover, we suppose that the parameters $\beta_\pm$ and $\beta_\pm'$ satisfy the additional relations:

$$
\kappa\beta_+ - \beta_- = \chi\beta_+' - \chi^{-1}\kappa\beta_-' \neq 0,
\tag{3.20}
$$

where the parameters κ and χ are the same as in (3.16).

One can check that the following formula are valid:

$$
r = (\varphi_0 \otimes \varphi_0)r',
\tag{3.21}
$$

$$
\bar{r} = (\varphi_0 \otimes \varphi_0)\bar{r}',
\tag{3.22}
$$

where φ_0 is the $\mathfrak{osp}(1|2; \mathbb{C})$-automorphism (3.16) with the following parameters:

$$
\begin{aligned}
\tilde{\beta}_0 &= \frac{\beta_0(\chi\beta_+' + \chi^{-1}\kappa\beta_-') - \beta_0'(\kappa\beta_+ + \beta_-)}{(\kappa\beta_+ - \beta_-)(\chi\beta_+' - \chi^{-1}\kappa\beta_-')}, \\
\tilde{\beta}_+ &= \frac{\kappa(\kappa\beta_+ + \beta_-)(\chi\beta_+' + \chi^{-1}\kappa\beta_-') + 4\beta_0\beta_0'}{2(\kappa\beta_+ - \beta_-)(\chi\beta_+' - \chi^{-1}\kappa\beta_-')} + \frac{\kappa}{2}, \\
\tilde{\beta}_- &= \frac{(\kappa\beta_+ + \beta_-)(\chi\beta_+' + \chi^{-1}\kappa\beta_-') + 4\kappa\beta_0\beta_0'}{2(\kappa\beta_+ - \beta_-)(\chi\beta_+' - \chi^{-1}\kappa\beta_-')} - \frac{1}{2}.
\end{aligned}
\tag{3.23}
$$

It is easy to check that the formulas (3.23) satisfy the condition $\tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 0$.

Let us assume in (3.23) that the parameters β_0' and β_-' are equal to zero. Then the general classical r-matrix r, satisfying the homogeneous CYBE, is reduced to usual Jordanian form by the automorphism φ_0 with the parameters:

$$
\tilde{\beta}_0 = \frac{\beta_0}{\kappa\beta_+ - \beta_-}, \quad \tilde{\beta}_\pm = \frac{\beta_\pm}{\kappa\beta_+ - \beta_-}.
\tag{3.24}
$$

Second type of $\mathfrak{osp}(1|2; \mathbb{C})$-automorphism connecting the classical r-matrices with non-zero γ-characteristic is given

as follows

$$\varphi_1(E_+) = \frac{\chi}{2}\left((\tilde{\beta}_0 + 1)E_+ + 2\tilde{\beta}_- H - \frac{\tilde{\beta}_-^2}{\tilde{\beta}_0 + 1}E_-\right),$$

$$\varphi_1(E_-) = \frac{\chi^{-1}}{2}\left(\frac{-\tilde{\beta}_+^2}{\tilde{\beta}_0 + 1}E_+ + 2\tilde{\beta}_+ H + (\tilde{\beta}_0 + 1)E_-\right),$$

$$\varphi_1(H) = \frac{1}{2}\left(-\tilde{\beta}_+ E_+ + 2\tilde{\beta}_0 H - \tilde{\beta}_- E_-\right), \qquad (3.25)$$

$$\varphi_1(v_+) = \sqrt{\frac{\chi}{2}}\left(\sqrt{\tilde{\beta}_0 + 1}v_+ + \frac{\tilde{\beta}_-}{\sqrt{\tilde{\beta}_0 + 1}}v_-\right),$$

$$\varphi_1(v_-) = \sqrt{\frac{\chi^{-1}}{2}}\left(\frac{\tilde{\beta}_+}{\sqrt{\tilde{\beta}_0 + 1}}v_+ + \sqrt{\tilde{\beta}_0 + 1}v_-\right),$$

where χ is a non-zero rescaling parameter, and $\tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 1$.

Let us consider two general r-matrices with non-zero γ-characteristics:

$$r := \beta_+ r_+ + \beta_0 r_0 + \beta_- r_-,$$
$$r' := \beta'_+ r_+ + \beta'_0 r_0 + \beta'_- r_-, \qquad (3.26)$$

where the parameters $\beta_\pm, \beta_0$ and $\beta'_\pm, \beta'_0$ can be equal to zero provided that $\gamma = \beta_0^2 + \beta_+\beta_- = \gamma' = (\beta'_0)^2 + \beta'_+\beta'_- \neq 0$, i.e. both r-matrices r and r' have the same non-zero γ-characteristic $\gamma = \gamma' \neq 0$.

One can check the following relation:

$$r = (\varphi_1 \otimes \varphi_1)r', \qquad (3.27)$$

where φ_1 is the $\mathfrak{osp}(1|2; \mathbb{C})$-automorphism (3.25) with the parameters:

$$\tilde{\beta}_0 = \frac{(\beta_0 + \beta'_0)^2 - (\beta_+ - \chi\beta'_+)(\beta_- - \chi^{-1}\beta'_-)}{(\beta_0 + \beta'_0)^2 + (\beta_+ - \chi\beta'_+)(\beta_- - \chi^{-1}\beta'_-)},$$

$$\tilde{\beta}_\pm = \frac{2(\beta_0 + \beta'_0)(\beta_\pm - \chi^{\pm 1}\beta'_\pm)}{(\beta_0 + \beta'_0)^2 + (\beta_+ - \chi\beta'_+)(\beta_- - \chi^{-1}\beta'_-)}. \qquad (3.28)$$

It is easy to check that the formulas (3.28) satisfy the condition $\tilde{\beta}_0^2 + \tilde{\beta}_+\tilde{\beta}_- = 1$.

If we assume in (3.26) that the parameters $\beta'_\pm$ are equal to zero then the general classical r-matrix r, satisfying the non-homogeneous CYBE, is reduced to the usual standard form by the automorphism φ_1, (3.25), with the following parameters:

$$\tilde{\beta}_0 = \frac{\beta_0}{\beta'_0}, \quad \tilde{\beta}_\pm = \frac{\beta_\pm}{\beta'_0}. \qquad (3.29)$$

Finally for $\mathfrak{osp}(1|2, \mathbb{C})$ we get the following result:

For the complex Lie superalgebra $\mathfrak{osp}(1|2, \mathbb{C})$ there exists up to $\mathfrak{osp}(1|2, \mathbb{C})$ automorphisms three solutions of CYBE, namely Jordanian r_J, super-Jordanian r_{sJ} and standard r_{st}:

$$r_J = \beta E_+ \wedge H, \quad [r_J, r_J]_S = 0, \qquad (3.30)$$

$$r_{sJ} = \beta_1(E_+ \wedge H + v_+ \wedge v_+),$$
$$[r_{sJ}, r_{sJ}]_S = 0, \qquad (3.31)$$

$$r_{st} = \beta_0(E_+ \wedge E_- + 2v_+ \wedge v_-),$$
$$[r_{st}, r_{st}]_S = \beta_0^2 \Omega, \qquad (3.32)$$

where the complex parameters β and β_1 can be removed by the rescaling $\mathfrak{osp}(1|2, \mathbb{C})$-automorphism: $\varphi(E_+) = \beta^{-1}E_+$, $\varphi(E_-) = \beta E_-$, $\varphi(v_+) = \sqrt{\beta^{-1}}v_+$, $\varphi(v_-) = \sqrt{\beta}v_-$, $\varphi(H) = H$; the parameter $\beta_0 = e^{\iota\phi}|\beta_0|$ for $|\phi| \leq \frac{\pi}{2}$ is effective.

So, we obtained the full classification of all nonequivalent quasitriangular Lie bisuperalgebras $(\mathfrak{osp}(1|2, \mathbb{C}), \delta_{(r)})$ in terms of all nonequivalent (nonisomorphic) classical r-matrices (3.30)–(3.32). It should be noted that the classical r-matrices for the complex Lie superalgebra $\mathfrak{osp}(1|2, \mathbb{C})$ was obtained previously in [11] where the authors used a technique of computer algebraic computations.

4 Classical r-matrices of $\mathfrak{osp}(1|2; \mathbb{C})$ real forms

The coproduct Δ_0 (1.7), which is a homomorphism $\mathfrak{g} \xrightarrow{\Delta_0} \mathfrak{g} \otimes \mathfrak{g}$ for any complex Lie superalgebra $\mathfrak{g}$ (and $\mathfrak{g} = \mathfrak{osp}(1|2; \mathbb{C})$ particularly), can be induced on all the universal enveloping superalgebra $U(\mathfrak{g}) \xrightarrow{\Delta_0} U(\mathfrak{g}) \otimes U(\mathfrak{g})$ using the linearity and multiplicativity:

$$\Delta_0(X + Y) = \Delta_0(X) + \Delta_0(Y),$$
$$\Delta_0(XY) = \Delta_0(X)\Delta_0(Y) \qquad (4.1)$$

for $\forall X, Y \in U(\mathfrak{g})$. These relations can be used to lift the $*$-conjugation, $(* = *, \dagger)$, from $U(\mathfrak{g})$ to the tensor product $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. It should be noted that the coproduct Δ_0 and the $*$-conjugation preserve the $\mathbb{Z}_2$-grading $\deg(\cdot)$: $\deg(\Delta_0(X_a)) = \deg(X_a^*) = \deg(X_a) = a \in \{\bar{0}, \bar{1}\}$ for any homogeneous element $X_a \in U(\mathfrak{g}) = U(\mathfrak{g})_{\bar{0}} \oplus U(\mathfrak{g})_{\bar{1}}$. There are two reality conditions for the coproduct Δ_0:

$$\Delta_0((X_a)^*) = (\Delta_0(X_a))^{*\otimes *}, \qquad (4.2)$$

$$\Delta_0((X_a)^*) = (\Delta_0(X_a))^{*\tilde{\otimes}*}$$
$$:= (\tau\Delta_0(X_a))^{*\otimes *} = \tau(\Delta_0(X_a))^{*\otimes *} \qquad (4.3)$$

for any homogeneous element $X_a \in U(\mathfrak{g})$, $a \in \{\bar{0}, \bar{1}\}$. Here τ is a *superpermutation* linear operator (*the superflip*) in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$:

$$\tau(X_a \otimes Y_b) = (-1)^{ab}(Y_b \otimes X_a) \qquad (4.4)$$

for all homogeneous elements $X_a, Y_b \in U(\mathfrak{g}), a, b \in \{\bar{0}, \bar{1}\}$. We note also that the definition of the Hopf superalgebra differs from that of the usual Hopf algebra by the *supermultiplication* of tensor product:

$$(X_a \otimes Y_b)(V_c \otimes W_d) = (-1)^{bc}(X_a V_c \otimes Y_b W_d) \qquad (4.5)$$

for all homogeneous elements $X_a, Y_b, V_c, W_d \in U(\mathfrak{g})$, $a, b, c, d \in \{\bar{0}, \bar{1}\}$. The conjugation $* \otimes *$ in (4.2) (cor-

respondingly $\circledast \tilde{\otimes} \circledast$ in (4.3)) will be named *direct* (correspondingly *superflipped*) one.

From the reality condition (4.2) we find that:

$$if \ \ (X_a Y_b)^{\circledast} = (-1)^{ab} Y_b^{\circledast} X_a^{\circledast} \ \ (graded) \ \ then$$
$$(X_a \otimes Y_b)^{\circledast \otimes \circledast} = (X_a^{\circledast} \otimes Y_b^{\circledast}) \ \ (ungraded), \tag{4.6}$$

and:

$$if \ \ (X_a Y_b)^{\circledast} = Y_b^{\circledast} X_a^{\circledast} \ \ (ungraded) \ \ then$$
$$(X_a \otimes Y_b)^{\circledast \otimes \circledast} = (-1)^{ab} (X_a^{\circledast} \otimes Y_b^{\circledast}) \ \ (graded). \tag{4.7}$$

From the reality condition (4.3) we find that:

$$if \ \ (X_a Y_b)^{\circledast} = (-1)^{ab} Y_b^{\circledast} X_a^{\circledast} \ \ (graded) \ \ then$$
$$(X_a \otimes Y_b)^{\circledast \tilde{\otimes} \circledast} = (-1)^{ab} (Y_b^{\circledast} \otimes X_a^{\circledast}) \ \ (graded), \tag{4.8}$$

and:

$$if \ \ (X_a Y_b)^{\circledast} = Y_b^{\circledast} X_a^{\circledast} \ \ (ungraded) \ \ then$$
$$(X_a \otimes Y_b)^{\circledast \tilde{\otimes} \circledast} = (Y_b^{\circledast} \otimes X_a^{\circledast}) \ \ (ungraded). \tag{4.9}$$

Next, we will use only the direct conjugation type (4.2), (4.6), (4.7) in the tensor product $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ since the results for the cases (4.8), (4.9) can be obtained from the cases (4.6), (4.7) using the superflip (4.4) explicitly.

The general non-reduced expression of the classical r-matrix (3.7) (and also (3.9)) is convenient for the application of reality conditions:

$$r^{\circledast \otimes \circledast} = \beta_+^* r_+^{\circledast \otimes \circledast} + \beta_0^* r_0^{\circledast \otimes \circledast} + \beta_-^* r_-^{\circledast \otimes \circledast} = -r, \tag{4.10}$$

where $\circledast$ is the conjugation associated with corresponding real form ($\circledast = *, \dagger$), and β_i^* ($i = +, 0, -$) means the complex conjugation of the number β_i. Moreover, if r-matrix is *anti-real* (anti-Hermitian), i.e. it satisfies the condition (4.10), then its γ-characteristic is real. Indeed, applying the conjugation $\circledast$ to CYBE (3.8) we have for the left-side:

$$[r, r]_S^{\circledast \otimes \circledast \otimes \circledast} = -[r^{\circledast \otimes \circledast}, r^{\circledast \otimes \circledast}]_S = -[r, r]_S = -\gamma \Omega \tag{4.11}$$

and for the right-side:

$$(\gamma \Omega)^{\circledast \otimes \circledast \otimes \circledast} = -\gamma^* \Omega \tag{4.12}$$

for all real forms $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(2;\mathbb{R}))$. It follows that the parameter γ is real, $\gamma^* = \gamma$.

I. *The compact pseudoreal real form* $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ ($H^* = H$, $E_\pm^* = E_\mp$, $v_\pm^* = \pm \varepsilon v_\mp$).

It is not difficult to see that in the case of the grade ($\varepsilon = 1$) and ungrade ($\varepsilon = \iota$) conjugations the basic two-tensors (3.5) and (3.6) have the following reality properties

$$r_0^{*\otimes*} = -r_0, \quad r_\pm^{*\otimes*} = -r_\mp, \tag{4.13}$$
$$\bar{r}_0^{*\otimes*} = -\bar{r}_0, \quad \bar{r}_\pm^{*\otimes*} = -\bar{r}_\mp, \tag{4.14}$$

In this case it follows from (4.10) that

$$\beta_0^* = \beta_0, \quad \beta_\pm^* = \beta_\mp. \tag{4.15}$$

If in (3.7) and (3.9) $\gamma = \beta_0^2 + \beta_+ \beta_- = 0$ then under conditions (4.15) we have $\beta_0 \beta_0^* + \beta_\pm \beta_\pm^* = 0$ and it follows that $\beta_0 = \beta_+ = \beta_- = 0$, i.e. *any classical r-matrix, which satisfies the homogeneous CYBE and the $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ reality condition, is equal zero.*

If in (3.7) $\gamma = \beta_0^2 + \beta_+ \beta_- \neq 0$ we have three $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ real classical r-matrices:

$$r_1 = \beta_0 r_0, \quad r_2 = \beta_+ r_+ + \beta_+^* r_-,$$
$$r_3 = \beta_+' r_+ + \beta_0' r_0 + \beta_+'^* r_-, \tag{4.16}$$

where β_0 and β_0' are real numbers and we use the conditions (4.15). The r-matrices r_i ($i = 1, 2, 3$) satisfy the non-homogeneous CYBE

$$[r_i, r_i]_S = \gamma_i \Omega, \tag{4.17}$$

where all γ_i ($i = 1, 2, 3$) are positive: $\gamma_1 = \beta_0^2 > 0$, $\gamma_2 = \beta_+ \beta_+^* > 0$, $\gamma_3 = \beta_0'^2 + \beta_+' \beta_+'^* > 0$.

Let two general classical r-matrices (3.26) with non-zero γ-characteristic be $\mathfrak{osp}^*(1|\mathfrak{su}(2))$-antireal, i.e. their parameters satisfy the reality conditions (4.15). It follows that the functions (3.28) for $\chi = e^{\iota \phi}$ have the same conjugation properties, i.e. $\tilde{\beta}_0^* = \tilde{\beta}_0$, $\tilde{\beta}_\pm^* = \tilde{\beta}_\mp$, and we obtain that the automorphism (3.24) with such parameters is $\mathfrak{osp}^*(1|\mathfrak{su}(2))$-real, i.e.:

$$\varphi_1(E_\pm)^* = \varphi_1(E_\pm^*) = \varphi_1(E_\mp),$$
$$\varphi_1(H)^* = \varphi_1(H^*) = \varphi_1(H), \tag{4.18}$$
$$\varphi_1(v_\pm)^* = \varphi_1(v_\pm^*) = \pm \varepsilon \varphi_1(v_\mp).$$

We see that the r-matrices r_2 and r_3 in (4.16) can be reduced to the standard r-matrix $r_{st} := r_1$ using the formula (3.27).

It is easy to see that the standard r-matrix $r_{st} = r_1$ in (4.16) effectively depends only on positive values of the parameter $\alpha := \beta_0$. Indeed, we see that

$$\alpha(\varphi(E_+) \wedge \varphi(E_-) + 2\varphi(v_+) \wedge \varphi(v_-))$$
$$= -\alpha(E_+ \wedge E_- + 2v_+ \wedge v_-), \tag{4.19}$$

where φ is the simple $\mathfrak{osp}^*(1|\mathfrak{su}(2))$-pseudoreal automorphism: $\varphi(E_\pm) = E_\mp$, $\varphi(H) = -H$, $\varphi(v_\pm) = \iota v_\mp$, i.e. any negative value of parameter α in r_{st} can be replaced by the positive one.

We obtain the following result:

For the compact pseudoreal form $\mathfrak{osp}^*(1|\mathfrak{su}(2))$ *with the graded ($\varepsilon = 1$) or ungraded ($\varepsilon = \iota$) conjugation ($*$), there exists up to the $\mathfrak{osp}^*(1|\mathfrak{su}(2))$-automorphisms only one solution of CYBE and this solution is the usual standard supersymmetric classical r-matrix r_{st}:*

$$r_{st} = \alpha(E_+ \wedge E_- + 2v_+ \wedge v_-),$$
$$[r_{st}, r_{st}]_S = \gamma \Omega, \tag{4.20}$$

where the effective parameter α is a positive number, and $\gamma = \alpha^2$.

II. *The non-compact real form* $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ $(H^{\dagger} = -H, E_{\pm}^{\dagger} = -E_{\pm}, v_{\pm}^{\dagger} = \varepsilon v_{\mp})$.

It is not difficult to see that in the case of the grade ($\varepsilon = 1$) and ungrade ($\varepsilon = \iota$) conjugations the basic two-tensors (3.5) and (3.6) have the following reality properties

$$r_0^{\dagger \otimes \dagger} = r_0, \quad r_{\pm}^{\dagger \otimes \dagger} = r_{\pm}, \tag{4.21}$$

$$\bar{r}_0^{\dagger \otimes \dagger} = \bar{r}_0, \quad \bar{r}_{\pm}^{\dagger \otimes \dagger} = \bar{r}_{\pm}, \tag{4.22}$$

In this case from (4.10) we obtain

$$\beta_0^* = -\beta_0, \quad \beta_{\pm}^* = -\beta_{\pm}, \tag{4.23}$$

i.e. all parameters β_i ($i = +, 0, -$) are purely imaginary.

Consider the general r-matrices (3.7) and (3.9) with the coefficients β_i and $\bar{\beta}_i$ ($i = \pm, 0$) satisfying the condition $\beta_0^2 + \beta_+ \beta_- = 0$ and (4.23) then we have six $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(2;\mathbb{R}))$ solutions of the homogeneous CYBE:

$$r_1 = \beta_+ r_+, \quad r_2 = \beta_- r_-,$$
$$r_3 = \beta'_+ r_+ + \beta'_0 r_0 + \beta'_- r_-, \tag{4.24}$$
$$\bar{r}_1 = \beta_+ \bar{r}_+, \quad \bar{r}_2 = \beta_- \bar{r}_-,$$
$$\bar{r}_3 = \beta'_+ \bar{r}_+ + \beta'_0 \bar{r}_0 + \beta'_- \bar{r}_-, \tag{4.25}$$

where all parameters β_i ($i = +, -$), β'_i ($i = +, 0, -$) are purely imaginary, and $\beta_0'^2 + \beta'_+ \beta'_- = 0$.

If the classical r-matrices (3.18) and (3.19) are $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$-antireal, i.e. their parameters satisfy the reality conditions (4.23), then for the real parameter χ all functions (3.23) are real, i.e. $\tilde{\beta}_0^* = \tilde{\beta}_0$, $\tilde{\beta}_{\pm}^* = \tilde{\beta}_{\pm}$. We obtain that the automorphism of the type (3.16) with such parameters is $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$-real, i.e.:

$$\varphi_0^{\dagger}(E_{\pm}) = \varphi_0(E_{\pm}^{\dagger}) = -\varphi_0(E_{\pm}),$$
$$\varphi_0^{\dagger}(H) = \varphi_0(H^{\dagger}) = -\varphi_0(H), \tag{4.26}$$
$$\varphi_0^{\dagger}(v_{\pm}) = \varphi_0(v_{\pm}^{\dagger}) = \varepsilon \varphi_0(v_{\pm}).$$

It allows to reduce the r-matrices r_2 and r_3 in (4.24) to the super-Jordanian r-matrix $r_{sJ} := r_1$ and the r-matrices $\bar{r}_2$ and $\bar{r}_3$ in (4.25) to the Jordanian r-matrix $r_J := \bar{r}_1$ by using the formulas (3.21) and (3.22).

In the case $\beta_0^2 + \beta_+ \beta_- \neq 0$ in (3.7) we have seven versions of $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$-real classical r-matrices. Five of them are with negative values of γ_i, ($i = 1, 2, \ldots, 5$):

$$r_1 = \beta_0 r_0, \qquad r_2 = \beta_+ r_+ + \beta_0 r_0,$$
$$r_3 = \beta_0 r_0 + \beta_- r_-, \quad r_4 = \beta'_+ r_+ + \beta'_- r_-,$$
$$r_5 = \beta''_+ r_+ + \beta''_0 r_0 + \beta''_- r_-, \tag{4.27}$$
$$[r_i, r_i]_S = \gamma_i \Omega \quad (i = 1, 2, \ldots, 5),$$

where all parameters β are purely imaginary, and $\gamma_1 = \gamma_2 = \gamma_3 = \beta_0^2 < 0$, $\gamma_4 = \beta'_+ \beta'_- < 0$, $\gamma_5 = \beta_0''^2 + \beta''_+ \beta''_- < 0$. The remaining two r-matrices r_i ($i = 6, 7$) have positive values

of γ_i:

$$r_6 = \beta'''_+ r_+ + \beta'''_- r_-,$$
$$r_7 = \beta''''_+ r_+ + \beta''''_0 r_0 + \beta''''_- r_-, \tag{4.28}$$
$$[r_i, r_i]_S = \gamma_i \Omega' \quad (i = 6, 7),$$

where $\gamma_6 = \beta'''_+ \beta'''_- > 0$ and $\gamma_7 = \beta_0''''^2 + \beta''''_+ \beta''''_- > 0$.

Let two classical r-matrices (3.26) be $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$-real, i.e. with their parameters satisfying the reality conditions (4.23). In such way the functions (3.28) for real χ are real, i.e. $\tilde{\beta}_0^* = \tilde{\beta}_0$, $\tilde{\beta}_{\pm}^* = \tilde{\beta}_{\pm}$, and we obtain that the automorphism (3.25) with such parameters is $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$-real. We can conclude that for the case of the negative γ-characteristics $\gamma_i < 0$ ($i = 1, \ldots, 5$) all r-matrices r_i ($i = 2, \ldots, 5$) in (4.27) are reduced to the standard formula $r_{st} := r_1$ and in the case of the positive γ-characteristics $\gamma_i > 0$ ($i = 6, 7$) the classical r-matrix r_7 in (4.28) is reduced to the quasi-standard r-matrix $r_{qst} := r_6$.

Let us show that the r-matrix r_{qst} effectively depend only on one positive parameter. Indeed, it is easy to see that

$$\begin{aligned} r_{qst} &= \beta'''_+ r_+ + \beta'''_- r_- = \iota \alpha (\lambda r_+ + \lambda^{-1} r_-) \\ &= \iota \alpha \big((\varphi \otimes \varphi) r_+ + (\varphi \otimes \varphi) r_- \big), \end{aligned} \tag{4.29}$$

where φ is the $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2,\mathbb{R}))$-real automorphism: $\varphi(E_{\pm}) = \lambda^{\pm 1} E_{\pm}$, $\varphi(H) = H$, $\varphi(v_{\pm}) = \lambda^{\pm \frac{1}{2}} v_{\pm}$, and $\alpha := \sqrt{\beta'''_+ \beta'''_-} \in \mathbb{R}^+$, $\lambda := -\iota \sqrt{\beta'''_+ / \beta'''_-} \in \mathbb{R}$.

Finally we obtain the following result:

For the non-compact real form $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$ *with graded* ($\varepsilon = 1$) *or ungraded* ($\varepsilon = \iota$) *conjugation* (†), *there exists up to* $\mathfrak{osp}^{\dagger}(1|\mathfrak{sl}(2;\mathbb{R}))$-*automorphisms four solutions of CYBE, namely Jordanian* r_J, *super-Jordanian* r_{sJ}, *standard* r_{st} *and quasi-standard* r_{qst}:

$$r_J = \iota \beta E_+ \wedge H, \quad [r_J, r_J]_S = 0, \tag{4.30}$$

$$r_{sJ} = \iota \beta (E_+ \wedge H + v_+ \wedge v_+),$$
$$[r_{sJ}, r_{sJ}]_S = 0, \tag{4.31}$$

$$r_{st} = \iota \alpha (E_+ \wedge E_- + 2 v_+ \wedge v_-),$$
$$[r_{st}, r_{st}]_S = -\alpha^2 \Omega, \tag{4.32}$$

$$\begin{aligned} r_{qst} &= \iota \alpha (E_+ \wedge H + v_+ \wedge v_+ \\ &\quad - E_- \wedge H - v_- \wedge v_-), \\ [r_{qst}, r_{qst}]_S &= \alpha^2 \Omega, \end{aligned} \tag{4.33}$$

where β and α are positive numbers.

III. *The non-compact real form* $\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(1,1))$ $(H'^{\dagger} = H', E_{\pm}'^{\dagger} = -E_{\mp}', v_{\pm}'^{\dagger} = -\iota \varepsilon v_{\mp}')$.

It is not difficult to see that in the case of the grade ($\varepsilon = 1$) and ungrade ($\varepsilon = \iota$) conjugations the basic two-tensors (3.5) and (3.6) have the following reality properties

$$(r_0')^{\dagger \otimes \dagger} = -r_0', \quad (r_{\pm}')^{\dagger \otimes \dagger} = r_{\mp}', \tag{4.34}$$

$$(\bar{r}_0')^{\dagger \otimes \dagger} = -\bar{r}_0', \quad (\bar{r}_{\pm}')^{\dagger \otimes \dagger} = \bar{r}_{\mp}', \tag{4.35}$$

where the primed basic two-tensors r_i' and $\bar{r}_i'$, ($i \in \{+, -, 0\}$), are given the formulas (3.5) and (3.6) in which the generators $E_\pm$, H, $v_\pm$ are replaced by the primed generators $E_\pm'$, H', $v_\pm'$. In the given case it follows from (4.10) that

$$\beta_0^* = \beta_0, \quad \beta_\pm^* = -\beta_\mp. \tag{4.36}$$

If $\beta_0^2 + \beta_+\beta_- = 0$ in (3.7) and (3.9) then $\beta_0\beta_0^* - \beta_\pm\beta_\pm^* = 0$, i.e. $\beta_\pm = \pm e^{\pm\iota\phi}|\beta_0|$, and we have the following two ϕ-families of $\mathfrak{osp}^*(1|\mathfrak{su}(1,1))$ homogeneous CYBE solutions:

$$r_\phi' := \beta_0\Big(e^{\iota\phi}\frac{|\beta_0|}{\beta_0}r_+' + r_0' - e^{-\iota\phi}\frac{|\beta_0|}{\beta_0}r_-'\Big), \tag{4.37}$$

$$\bar{r}_\phi' := \beta_0\Big(e^{\iota\phi}\frac{|\beta_0|}{\beta_0}\bar{r}_+' + \bar{r}_0' - e^{-\iota\phi}\frac{|\beta_0|}{\beta_0}\bar{r}_-'\Big), \tag{4.38}$$

where β_0 is real. By using the $\mathfrak{osp}^*(1|\mathfrak{su}(1,1))$-real rescaling automorphism $\varphi(E_\pm') = \Big(-\iota e^{\iota\phi}\frac{|\beta_0|}{\beta_0}\Big)^{\pm 1}E_\pm'$, $\varphi(H') = H'$, $\varphi(v_\pm') = \Big(-\iota e^{\iota\phi}\frac{|\beta_0|}{\beta_0}\Big)^{\pm\frac{1}{2}}v_\pm$ we can reduce the ϕ-families (4.37) and (4.38) to $r_{qsJ}' := \beta_0(\iota r_+' + r_0' + \iota r_-')$ and $r_{qJ}' := \beta_0(\iota\bar{r}_+' + \bar{r}_0' + \iota\bar{r}_-')$, respectively. Namely, we have

$$\begin{aligned} r_\phi' &= \beta_0\Big(e^{\iota\phi}\frac{|\beta_0|}{\beta_0}r_+' + r_0' - e^{-\iota\phi}\frac{|\beta_0|}{\beta_0}r_-'\Big) \\ &= \beta_0\big(\iota(\varphi\otimes\varphi)r_+' + (\varphi\otimes\varphi)r_0' + \iota(\varphi\otimes\varphi)r_-'\big). \end{aligned} \tag{4.39}$$

The same formula is also valid for the classical r-matrix $\bar{r}_\phi'$, (4.38). We shall call a $\mathfrak{osp}^*(1|\mathfrak{su}(1,1))$-antireal r-matrix of the form (4.37) (or (4.38)) as "quasi-super-Jordanian" (or "quasi-Jordanian") if it can not be reduced to super-Jordanian (or Jordanian) form by a $\mathfrak{su}(1,1)$-real automorphism, but after complexification of $\mathfrak{osp}^*(1|\mathfrak{su}(1,1))$ it can be reduced to super-Jordanian (or Jordanian) form by an appropriate complex $\mathfrak{osp}(1|\mathfrak{sl}(2,\mathbb{C}))$-automorphism. Thus all r-matrices in the ϕ-families (4.37) (or (4.38)) are quasi-super-Jordanian (or quasi-Jordanian) and they are connected with each other by the $\mathfrak{osp}^*(1|\mathfrak{su}(1,1))$-real rescaling automorphism. We take r_{qsJ}' and r_{qJ}' as representatives of the ϕ-families (4.37) and (4.38). It is easy to see that the r-matrices r_{qsJ}' and r_{qJ}' effectively depend only on positive values of the parameter β_0. Indeed, we have

$$\begin{aligned} r_{qsJ}' &= \beta_0(\iota r_+' + r_0' + \iota r_-') \\ &= -\beta_0\big(\iota(\varphi\otimes\varphi)r_+' + (\varphi\otimes\varphi)r_0' + \iota(\varphi\otimes\varphi)r_-'\big), \end{aligned} \tag{4.40}$$

where φ is the simple $\mathfrak{osp}^*(1|\mathfrak{su}(1,1))$ automorphism $\varphi(E_\pm) = E_\mp$, $\varphi(H) = -H$, $\varphi(v_\pm) = v_\mp$, i.e. any negative value of parameter β_0 in r_{qsJ} can be replaced by a positive one. The same result is also valid for r_{qJ}.

In the case $\beta_0^2 + \beta_+\beta_- \neq 0$ in (3.7) we have four versions of $\mathfrak{osp}^*(1|\mathfrak{su}(1,1))$-antireal classical r-matrices. Two of them are characterized by positive value of γ_i, ($i = 1, 2$):

$$\begin{aligned} r_1' &= \beta_0 r_0', \\ r_2' &= \beta_+' r_+' + \beta_0' r_0' - \beta_+'^* r_-', \\ [r_i', r_i']_S &= \gamma_i\Omega \quad (i = 1, 2), \end{aligned} \tag{4.41}$$

where β_0 and β_0' are real (see (4.33)), and $\gamma_1 = \beta_0^2 > 0$, $\gamma_2 = \beta_0'\beta_0'^* - \beta_+'\beta_+'^* > 0$. The remaining two are with negative values of γ_i, ($i = 3, 4$):

$$\begin{aligned} r_3' &= \beta_+'' r_+' - \beta_+''^* r_-', \\ r_4' &= \beta_+''' r_+' + \beta_0''' r_0' - \beta_+'''^* r_-', \\ [r_i', r_i']_S &= \gamma_i\Omega \quad (i = 3, 4), \end{aligned} \tag{4.42}$$

where β_0''' is real (see (4.33)), and $\gamma_3 = -\beta_+''\beta_+''^* < 0$, $\gamma_4 = \beta_0'''\beta_0'''^* - \beta_+'''\beta_+'''^* < 0$.

Let the classical r-matrices (3.26) be $\mathfrak{su}(1,1)$-antireal, i.e. their parameters satisfy the reality conditions (4.36). In such case the functions (3.28) for $\chi = e^{\iota\phi}$ have the same conjugation properties, i.e. $\tilde{\beta}_0^* = \tilde{\beta}_0$, $\tilde{\beta}_\pm^* = -\tilde{\beta}_\mp$, and we obtain that the automorphism (3.25) with these parameters is $\mathfrak{su}(1,1)$-real, i.e.

$$\begin{aligned} (\varphi_1(E_\pm'))^\dagger &= \varphi_1((E_\pm')^\dagger) = -\varphi_1(E_\mp'), \\ (\varphi_1(H'))^\dagger &= \varphi_1((H')^\dagger) = \varphi_1(H'), \\ (\varphi_1(v_\pm'))^\dagger &= \varphi_1((v_\pm')^\dagger) = -\iota\varepsilon\varphi_1(v_\mp'). \end{aligned} \tag{4.43}$$

It allows to reduce the r-matrix r_2' to the standard r-matrix $r_{st}' := r_1'$ for $\gamma_1 = \gamma_2 > 0$ and the r-matrix r_3' to the r-matrix r_4' for $\gamma_3 = \gamma_4 < 0$ by use of the formula (3.16). By analogy to the notation of quasi-Jordanian r-matrix we shall call the r-matrices r_3' and r_4' as quasi-standard ones and take $r_{qst}' = \alpha(E_+' \wedge H' + v_+' \wedge v_+' + E_-' \wedge H' + v_-' \wedge v_-')$ as their representative.[6]

Finally for $\mathfrak{su}(1,1)$ we obtain:

For the non-compact real form $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ with graded ($\varepsilon = 1$) or ungraded ($\varepsilon = \iota$) conjugation (†), there exists up to $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$-automorphisms four solutions of CYBE, namely quasi-Jordanian r_{qJ}', quasi-super-Jordanian r_{qsJ}', quasi-standard r_{qst}' and standard r_{st}':

$$\begin{aligned} r_{qJ}' &= \beta\big(\iota(E_+' - E_-')\wedge H' + E_+'\wedge E_-'\big), \\ [r_{qJ}', r_{qJ}']_S &= 0, \end{aligned} \tag{4.44}$$

$$\begin{aligned} r_{qsJ}' &= \beta\big(\iota((E_+' - E_-')\wedge H' + v_+'\wedge v_+' - v_-'\wedge v_-') \\ &\quad + E_+'\wedge E_-' + 2v_+'\wedge v_-'\big), \\ [r_{qsJ}', r_{qsJ}']_S &= 0, \end{aligned} \tag{4.45}$$

$$\begin{aligned} r_{qst}' &= \alpha(E_+'\wedge H' + v_+'\wedge v_+' \\ &\quad + E_-'\wedge H' + v_-'\wedge v_-'), \\ [r_{qst}', r_{qst}']_S &= -\alpha^2\Omega, \end{aligned} \tag{4.46}$$

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[6] The r-matrix r_{qst}' is connected with r_3' (4.42) by the following way. Substituting $\beta_+ = |\beta_+|e^{\iota\phi}$ in r_3' (4.42) and using the $\mathfrak{su}(1,1)$-real rescaling automorphism $\varphi(E_\pm') = e^{\pm\iota\phi}E_\pm'$, $\varphi(H') = H'$, $\varphi(v_\pm') = e^{\pm\frac{\iota\phi}{2}}v_\pm'$ we obtain r_{qst}' with $\alpha = |\beta_+|$.

$$r'_{st} = \alpha(E'_+ \wedge E'_- + 2v'_+ \wedge v'_-),$$
$$[r'_{st}, r'_{st}]_S = \alpha^2\Omega, \tag{4.47}$$

where β and α are positive numbers.

5 Isomorphism between $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ bisuperalgebras and its application to quantizations of $N = 1$, $D = 3$ Lorentz supersymmetry $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$

Using the formulas of connections between the CW and Cartesian bases (see (2.12), (2.14)) we can express the classical $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ r-matrices in terms of the $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ Cartesian basis. We get the following results.

For the non-compact real form $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ with graded and ungraded conjugation (†):

$$r_J = \iota\beta E_+ \wedge H$$
$$= -\beta(\iota I_1 - I_2) \wedge I_3, \tag{5.1}$$
$$[r_J, r_J]_S = 0,$$

$$r_{sJ} = \iota\beta(E_+ \wedge H + v_+ \wedge v_+)$$
$$= -\beta(\iota I_1 - I_2) \wedge I_3 + v_1 \wedge v_1) \tag{5.2}$$
$$[r_{sJ}, r_{sJ}]_S = 0,$$

$$r_{st} = \iota\alpha(E_+ \wedge E_- + 2v_+ \wedge v_-)$$
$$= -2\alpha(I_1 \wedge I_2 + 2v_1 \wedge v_2), \tag{5.3}$$
$$[r_{st}, r_{st}]_S = -\alpha^2\Omega,$$

$$r_{qst} = \iota\alpha(E_+ \wedge H + v_+ \wedge v_+$$
$$+ E_- \wedge H + v_- \wedge v_-)$$
$$= -2\iota\alpha(I_1 \wedge I_3 + v_1 \wedge v_1 - v_2 \wedge v_2), \tag{5.4}$$
$$[r_{qst}, r_{qst}]_S = \alpha^2\Omega,$$

where β and α are positive numbers.

For the noncompact real form $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ with graded and ungraded conjugation (†):

$$r'_{qJ} = \beta(\iota(E'_+ - E'_-) \wedge H' + E'_+ \wedge E'_-),$$
$$= -\beta(\iota I_1 - I_2) \wedge I_3,$$
$$[r'_{qJ}, r'_{qJ}]_S = 0,$$
$$\tag{5.5}$$

$$r'_{qsJ} = \beta(\iota((E'_+ - E'_-) \wedge H' + v'_+ \wedge v'_+ - v'_- \wedge v'_-)$$
$$+ E'_+ \wedge E'_- + 2v'_+ \wedge v'_-),$$
$$= -\beta(\iota I_1 - I_2) \wedge I_3 + v_1 \wedge v_1),$$
$$[r'_{qsJ}, r'_{qsJ}]_S = 0,$$
$$\tag{5.6}$$

$$r'_{qst} = \alpha(E'_+ \wedge H' + v'_+ \wedge v'_+$$
$$+ E'_- \wedge H' + v'_- \wedge v'_-)$$
$$= -2\alpha(I_1 \wedge I_2 + 2v_1 \wedge v_2), \tag{5.7}$$
$$[r'_{qst}, r'_{qst}]_S = -\alpha^2\Omega,$$

$$r'_{st} = \alpha(E'_+ \wedge E'_- + 2v'_+ \wedge v'_-)$$
$$= -2\iota\alpha(I_1 \wedge I_3 + v_1 \wedge v_1 - v_2 \wedge v_2),$$
$$[r'_{st}, r'_{st}]_S = \alpha^2\Omega,$$
$$\tag{5.8}$$

where β and α are positive numbers.

Comparing the r-matrix expressions (5.3)–(5.4) with (5.5)–(5.8) we obtain that

$$r_J = r'_{qJ} = -\alpha((\iota I_1 - I_2) \wedge I_3), \tag{5.9}$$
$$r_{sJ} = r'_{qsJ} = -\alpha((\iota I_1 - I_2) \wedge I_3 + v_1 \wedge v_1), \tag{5.10}$$
$$r_{st} = r'_{qst} = -2\alpha(I_1 \wedge I_2 + 2v_1 \wedge v_2), \tag{5.11}$$
$$r_{qst} = r'_{st} = -2\iota\alpha(I_1 \wedge I_3 + v_1 \wedge v_1 - v_2 \wedge v_2). \tag{5.12}$$

We see the following:

(i) The Jordanian r-matrix r_J in the $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ basis is the same as the quasi-Jordanian r-matrix r'_{qJ} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ basis.

(ii) The super-Jordanian r-matrix r_{sJ} in the $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ basis is the same as the super-quasi-Jordanian r-matrix r'_{qsJ} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ basis.

(iii) The standard r-matrix r_{st} in the $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ basis becomes the quasi-standard r-matrix r'_{qst} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1.1))$ basis.

(iv) Conversely, the quasi-standard r-matrix r_{qst} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(2,\mathbb{R}))$ basis is the same as the standard r-matrix r'_{st} in the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ basis.

The relations (5.9)–(5.12) show that the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(2,\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ superbialgebras are isomorphic. This result finally resolves the doubts about isomorphisms of these two superbialgebras [19])

Using the isomorphisms of the $\mathfrak{osp}^\dagger(1|\mathfrak{su}(2,\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ bialgebras we take as basic r-matrices for the $N = 1$, $D = 3$ Lorentz superalgebra $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ the following ones:

$$r_J = -\beta((\iota I_1 - I_2) \wedge I_3)$$
$$= \iota\beta E_+ \wedge H, \tag{5.13}$$

$$r_{sJ} = -\beta((\iota I_1 - I_2) \wedge I_3 + v_1 \wedge v_1)$$
$$= \iota\beta(E_+ \wedge H + v_+ \wedge v_+), \tag{5.14}$$

$$r_{st} = -2\alpha(I_1 \wedge I_2 + 2v_1 \wedge v_2)$$
$$= \iota\alpha(E_+ \wedge E_- + 2v_+ \wedge v_-), \tag{5.15}$$

$$r'_{st} = -2\iota\alpha(I_1 \wedge I_3 + v_1 \wedge v_1 - v_2 \wedge v_2)$$
$$= \alpha(E'_+ \wedge E'_- + 2v'_+ \wedge v'_-), \tag{5.16}$$

where β and α are positive numbers, moreover the parameter α is effective whereas the parameter β is not effective, i.e. it can be removed by a $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$-real rescal-

ing automorphism[7]: $\varphi(E_+) = \beta^{-1}E_+$, $\varphi(E_-) = \beta E_-$, $\varphi(v_+) = \sqrt{\beta^{-1}}\,v_+$, $\varphi(v_-) = \sqrt{\beta}\,v_-$, $\varphi(H) = H$.

The first two r-matrices r_J and r_{sJ} present the Jordanian and super-Jordanian twist deformations of $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$, the third and fourth r-matrices r_{st} and r'_{st} correspond to the q-analogs of $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ and $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ real algebras. In the next section we shall show how to quantize the r-matrices (5.13)–(5.16) in an explicit form.

6 Quantizations of the $N = 1$, $D = 3$ Lorentz supersymmetry $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$

Comparing the classical r-matrices of the complex Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ (3.30)–(3.32) with the classical r-matrices of its real forms: $\mathfrak{osp}^*(1|\mathfrak{o}(3))$ (4.20) and $\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1))$ (5.13)–(5.16), we see that they are given by the same formulas and differ in values of the deformation parameters and conjugation properties. There are the similar picture for the deformed structures of $\mathfrak{osp}(1|2;\mathbb{C})$ and its real forms. Therefore one can obtain the quantum deformations of all $\mathfrak{osp}(1|2;\mathbb{C})$ real forms from the quantum deformation of $\mathfrak{osp}(1|2;\mathbb{C})$ by specialization of the corresponding parameter deformation and the conjugation property.

1. q-Analogs. The quantum Hopf deformations corresponding to the standard classical r-matrices (3.32), (4.20), (5.15) and (5.16) are called the q-analogs, The q-analog of $U(\mathfrak{g})$ ($\mathfrak{g} = \mathfrak{osp}(1|2;\mathbb{C})$, $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$) is an unital associative algebra $U_q(\mathfrak{g})$ with the generators[8] $x_\pm$, $q^{\pm\frac{1}{2}X_0}$ and the defining relations:

$$\begin{aligned}
q^{\frac{1}{2}X_0}q^{-\frac{1}{2}X_0} &= q^{-\frac{1}{2}X_0}q^{\frac{1}{2}X_0} = 1, \\
q^{\frac{1}{2}X_0}x_\pm &= q^{\pm\frac{1}{4}}x_\pm q^{\frac{1}{2}X_0}, \\
\{x_+, x_-\} &= \frac{q^{-\frac{1}{2}X_0} - q^{\frac{1}{2}X_0}}{q - q^{-1}},
\end{aligned} \tag{6.1}$$

with the additional conditions:

(a) $q = e^\beta$ ($\beta \in \mathbb{C}$) for $U_q(\mathfrak{osp}(1|2;\mathbb{C}))$,
(b) $x_\pm^* = \pm\varepsilon x_\mp$, $(q^{\frac{1}{2}X_0})^* = q^{\frac{1}{2}X_0}$, $q = e^\alpha$
    for $U_q(\mathfrak{osp}^*(1|\mathfrak{su}(2)))$,
(c) $x_\pm^\dagger = -\varepsilon x_\pm$, $(q^{\frac{1}{2}X_0})^\dagger = q^{\frac{1}{2}X_0}$, $q = e^{\iota\alpha}$
    for $U_q(\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))) \simeq U_q(\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1)))$,
(c') $x_\pm^\dagger = \iota\varepsilon x_\mp$, $(q^{\frac{1}{2}X_0})^\dagger = q^{\frac{1}{2}X_0}$, $q = e^\alpha$
    for $U_q(\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))) \simeq U_q(\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1)))$, $\quad$ (6.2)

where α is real in accordance with (4.20), (5.15) and (5.16), for the graded ($\varepsilon = 1$) or ungraded ($\varepsilon = \iota$) conjugation ($*$), ($* = *, \dagger$) (see (4.6), (4.7).

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[7] Here we keep the non-effective parameter β for convenient of quantization.

[8] They are q-analogs of the Chevalley basis $v_\pm$, H with the defining relations: $\{v_+, v_-\} = -\frac{1}{2}H$, $[H, v_\pm] = \pm\frac{1}{2}v_\pm$ (see (2.2)–(2.3)).

A Hopf structure on $U_q(\mathfrak{g})$ is defined with help of three additional operations: coproduct (comultiplication) Δ_q, antipode S_q and counit ϵ_q:

$$\begin{aligned}
\Delta_q(q^{\pm\frac{1}{2}X_0}) &= q^{\pm\frac{1}{2}X_0} \otimes q^{\pm\frac{1}{2}X_0}, \\
\Delta_q(x_\pm) &= x_\pm \otimes q^{\frac{1}{4}X_0} + q^{-\frac{1}{4}X_0} \otimes x_\pm, \\
S_q(q^{\pm\frac{1}{2}X_0}) &= q^{\mp\frac{1}{2}X_0}, \quad S_q(x_\pm) = -q^{\pm\frac{1}{4}}x_\pm, \\
\epsilon_q(q^{\pm\frac{1}{2}X_0}) &= 1, \quad \epsilon_q(x_\pm) = 0,
\end{aligned} \tag{6.3}$$

with the reality conditions for the real form $U_q(\mathfrak{g}^*)$ ($* = *, \dagger$):

$$\begin{aligned}
(\Delta_q(X))^{*\otimes*} &= \Delta_q(X^*), \\
(S_q(X))^* &= S_q^{-1}(X^*), \quad (\epsilon_q(X))^* = \epsilon_q(X^*)
\end{aligned} \tag{6.4}$$

for any $X \in U_q(\mathfrak{g}^*)$ ($\mathfrak{g}^* = \mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$).

The quantum algebra $U_q(\mathfrak{g})$ is endowed also with the *opposite* Hopf structure: opposite coproduct $\tilde\Delta_q$,[9] corresponding antipode $\tilde{S}_q$ and counit $\tilde\epsilon_q$.

An invertible element $R_q := R_q(\mathfrak{g})$ which satisfies the relations:

$$\begin{aligned}
R_q\Delta_q(X) &= \tilde\Delta_q(X)R_q, \quad \forall X \in U_q(\mathfrak{g}), \\
(\Delta_q \otimes \mathrm{id})R_q &= R_q^{13}R_q^{23}, \\
(\mathrm{id} \otimes \Delta_q)R_q &= R_q^{12}R_q^{13}
\end{aligned} \tag{6.5}$$

as well as, due to (6.5), the quantum Yang–Baxter equation (QYBE)

$$R_q^{12}R_q^{13}R_q^{23} = R_q^{23}R_q^{13}R_q^{12} \tag{6.6}$$

is called the *universal R-matrix*.

Let $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ be quantum Borel subalgebras of $U_q(\mathfrak{g})$, generated by x_+, $q^{\pm\frac{1}{4}X_0}$ and x_-, $q^{\pm\frac{1}{4}X_0}$ respectively. We denote by $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ the Taylor extension of $U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-)$.[10] One can show (see [20,21]) that *there exists unique solution of equations (6.5) in the space $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ and such solution has the following form*[11]

$$R_q(\mathfrak{g}) := R_q^> = \check{R}_q^> K, \tag{6.7}$$

where

$$K = q^{X_0 \otimes X_0}, \tag{6.8}$$

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[9] The opposite coproduct $\tilde\Delta_q(\cdot)$ is a coproduct with permuted components, i.e. $\tilde\Delta_q(\cdot) = \tau \circ \Delta_q(\cdot)$ where τ is the super flip operator (see 4.4).

[10] $T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ is an associative algebra generated by formal Taylor series of the monomials $x_+^n \otimes x_-^m$ with coefficients which are rational functions of $q^{\pm\frac{1}{4}X_0}$, $q^{\pm X_0 \otimes X_0}$, provided that all values $|n - m|$ for each formal series are bounded, $|n - m| < N$.

[11] The formulas (6.7)–(6.9) are a specialization of the formulas (7.1)-(7.3) from the article [20] to the case (6.1).

$$\check{R}_q^{\succ} = \exp_{\tilde{q}}\left((q^{-1} - q)x_+ q^{-\frac{1}{4}X_0} \otimes q^{\frac{1}{4}X_0}x_-\right), \qquad (6.9)$$

Here $\tilde{q} := -q^{-\frac{1}{4}}$ and the deformation parameter q is given by the conditions (6.2). We also use the standard definition of the $\tilde{q}$-exponential:

$$\exp_{\tilde{q}}(x) := \sum_{n \geq 0} \frac{x^n}{(n)_{\tilde{q}}!}, \quad (n)_{\tilde{q}} := \frac{(1 - \tilde{q}^n)}{(1 - \tilde{q})}, \\ (n)_{\tilde{q}}! := (1)_{\tilde{q}}(2)_{\tilde{q}} \dots (n)_{\tilde{q}}. \qquad (6.10)$$

Analogously, *there exits unique solution of equations* (6.5) *in the space* $T_q(\mathfrak{b}_- \otimes \mathfrak{b}_+) = \tau \circ T_q(\mathfrak{b}_+ \otimes \mathfrak{b}_-)$ *and such solution is given by the formula*

$$R_q(\mathfrak{g}) := R_q^{\prec} = \check{R}_q^{\prec} K^{-1}, \qquad (6.11)$$

where K is given by the formula (6.8), *and*

$$\check{R}_q^{\prec} = \exp_{\tilde{q}'}\left((q^{-1} - q)x_- q^{\frac{1}{4}X_0} \otimes q^{-\frac{1}{4}X_0}x_+\right). \qquad (6.12)$$

Here $\tilde{q}' = \tilde{q}^{-1} = -q^{\frac{1}{4}}$, and q satisfies the conditions (6.2).

As formal Taylor series the solutions (6.7)–(6.9) and (6.11), (6.12) are independent and they are related by

$$R_q^{\prec} = \tau \circ R_{q^{-1}}^{\succ}. \qquad (6.13)$$

It should be noted also that

$$(R_q^{\succ})^{-1} = R_{q^{-1}}^{\succ}, \quad (R_q^{\prec})^{-1} = R_{q^{-1}}^{\prec} \qquad (6.14)$$

for all quantum superalgebras $U_q(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{osp}(1|2;\mathbb{C})$, $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$. From the explicit forms (6.7)–(6.9) and (6.11), (6.12) we also see that ($\divideontimes = *, \dagger$):

$$(R_q^{\succ})^{\divideontimes} = \tau \circ R_q^{\succ} = (R_q^{\prec})^{-1}, \\ (R_q^{\prec})^{\divideontimes} = \tau \circ R_q^{\prec} = (R_q^{\succ})^{-1} \qquad (6.15)$$

for $U_q(\mathfrak{osp}^*(1|\mathfrak{su}(2))) \simeq U_q(\mathfrak{osp}^*(1|\mathfrak{o}(3)))$ and $U_q(\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))) \simeq U_q(\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1)))$, and

$$(R_q^{\succ})^{\dagger} = (R_q^{\succ})^{-1}, \quad (R_q^{\prec})^{\dagger} = (R_q^{\prec})^{-1} \qquad (6.16)$$

for $U_q(\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))) \simeq U_q(\mathfrak{osp}^\dagger(1|\mathfrak{o}(2,1)))$. Thus, in the case $U_q(\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R})))$ both R-matrices $R_q^{\succ}$, $R_q^{\prec}$ are unitary and in the case $U_q(\mathfrak{osp}^*(1|\mathfrak{su}(2)))$ and $U_q(\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1)))$ they can be called "τ-Hermitian".

In the limit $\alpha \to 0$ ($q \to 1$) we obtain for the R-matrix (6.7)–(6.9)

$$R_q(\mathfrak{g}) = 1 + r_{BD} + O(\alpha^2). \qquad (6.17)$$

Here r_{BD} is the classical Belavin–Drinfeld r-matrix:

$$r_{BD} = \beta\left(2\tilde{x}_+ \otimes \tilde{x}_- - 16\tilde{x}_+^2 \otimes \tilde{x}_-^2 + X_0 \otimes X_0\right), \qquad (6.18)$$

where $\beta = \ln q$ (see (6.2)), and[12] $\tilde{x}_\pm = v_\pm$, $\tilde{x}_\pm^2 = \pm\frac{1}{4}E_\pm$, $X_0 = H$. The r-matrix r_{BD} is not skew-symmetric and it

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[12] In the case of $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$ we need to use the primed CW basis $\tilde{x}_\pm = E'_\pm$, $X_0 = H'$ (see (2.14), (2.15)).

satisfies the standard CYBE

$$[r_{BD}^{12}, r_{BD}^{13} + r_{BD}^{23}] + [r_{BD}^{13}, r_{BD}^{23}] = 0 \qquad (6.19)$$

which is obtained from QYBE (6.6) in the limit (6.17). The standard r-matrix (5.11) or (5.12) is the skew-symmetric part of r_{BD}, namely

$$r_{BD} = \frac{1}{2}\bar{r}_{st} + \frac{1}{2}\bar{C}_2, \qquad (6.20)$$

where $\bar{r}_{st} = r_{BD} - \tau \circ r_{BD}$ is the standard r-matrix (3.32), (4.20), (5.15) or (5.16), and $\bar{C}_2 = 2\beta C_2 = r_{BD} + \tau \circ r_{BD}$ where C_2 is the split Casimir element of $\mathfrak{g}$, ($\mathfrak{g} = \mathfrak{osp}(1|2;\mathbb{C})$, $\mathfrak{osp}^*(1|\mathfrak{su}(2))$, $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$, $\mathfrak{osp}^\dagger(1|\mathfrak{su}(1,1))$).

2. Twisted deformations. The classical r-matrices (3.30), (3.31) and (5.13), (5.14) satisfy the homogeneous CYBE (3.1) with the vanishing right side and the corresponding Hopf deformations are determined by twisting two-tensors. We remind basic properties of the twisted deformation of a Hopf (super)algebra [22]. Let $\mathcal{A} := \mathcal{A}(A; m, \Delta, S, \varepsilon)$ be a Hopf (super)algebra with multiplication m, coproduct Δ, antipode S and counit ϵ and let $F \in A \otimes A$ be an invertible two-tensor satisfies the 2-cocycle condition

$$F^{12}(\Delta \otimes \mathrm{id})(F) = F^{23}(\mathrm{id} \otimes \Delta)(F) \qquad (6.21)$$

and the "unital" normalization

$$(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = 1. \qquad (6.22)$$

Then the twisting element F defines a deformed Hopf algebra $\mathcal{A}^{(F)} := \mathcal{A}(A; m, \Delta^{(F)}, S^{(F)}, \varepsilon)$ with the new deformed coproduct and antipode are given as follows

$$\Delta^{(F)}(X) = F\Delta(X)F^{-1}, \\ S^{(F)}(X) = u^{(F)}S(X)(u^{(F)})^{-1} \qquad (6.23)$$

for any $X \in \mathcal{A}$, where $\Delta(X)$ and $S(X)$ are the coproduct and the antipode before twisting, and

$$u^{(F)} = m(\mathrm{id} \otimes S)(F) = \sum_{i,j} f_i^{(1)} S(f_j^{(2)}), \qquad (6.24)$$

if $F = \sum_{i,j} f_i^{(1)} \otimes f_j^{(2)}$.

Let $\omega \in A$ be an arbitrary invertible element, $\omega\omega^{-1} = \omega^{-1}\omega = 1$, then it is not difficult to check that the two-tensor

$$F_\omega := (\omega \otimes \omega)F\Delta(\omega^{-1}) \qquad (6.25)$$

satisfies also the cocycle Eq. (6.21). Two deformed Hopf algebras $\mathcal{A}^{(F)} = \mathcal{A}(A; m, \Delta^{(F)}, S^{(F)}, \varepsilon)$ and $\mathcal{A}^{(F_\omega)} = \mathcal{A}(A; m, \Delta^{(F_\omega)}, S^{(F_\omega)}, \varepsilon)$ with the twists F and F_ω are isomorphic by the conjugacy isomorphism: $X \to \omega X\omega^{-1}$, ($\forall X \in A$). Evidently, that $\Delta^{(F)}(X) \to \Delta^{(F_\omega)}(\omega X\omega^{-1}) = (\omega \otimes \omega)\Delta^{(F)}(X)(\omega^{-1} \otimes \omega^{-1})$. As an example of one-parameter isomorphism family $\omega(t)$ we can use the following function:

$$\omega(t) = u(t) := (u^{(F)})^t \quad (t \in \mathbb{R}), \qquad (6.26)$$

where $u^{(F)}$ is the convolution (6.24). Therefore we have one-parameter family of the twist $F_{u(t)}$, where $F_{u(0)} = F$, $F_{u(1)} = F_{u^{(F)}} = (u^{(F)} \otimes u^{(F)}) F \Delta((u^{(F)})^{-1})$ (see [23,24]).

Let R be an universal R-matrix of $\mathcal{A}$ then the operator[13]

$$R^{(F)} = F^{21} R F^{-1} \tag{6.27}$$

is also the universal R-matrix of the twisted Hopf (super) algebra $\mathcal{A}^{(F)}$, i.e. it satisfies the relations of the form (6.5), (6.6).

Let a invertible two-tensor $\Phi \in A \otimes A$ also generates a twist quantization after the quantization by the twist F, that is it satisfies the 2-cocycle condition:

$$\Phi^{12}(\Delta^{(F)} \otimes \mathrm{id})(\Phi) = \Phi^{23}(\mathrm{id} \otimes \Delta^{(F)})(\Phi), \tag{6.28}$$

and the normalization (6.22), then the new co-product $\Delta^{((F)\Phi)}$ and antipode $S^{((F)\Phi)}$ are given as follows

$$\begin{aligned}\Delta^{((F)\Phi)}(X) &= \Phi\Delta^{(F)}(X)\Phi^{-1}, \\ S^{((F)\Phi)}(X) &= u^{((F)\Phi)} S^{(F)}(X)(u^{((F)\Phi)})^{-1}\end{aligned} \tag{6.29}$$

for any $X \in \mathcal{A}$, where $\Delta^{(F)}(X)$ and $S^{(F)}(X)$ are the coproduct and the antipode after twisting with F, and

$$u^{((F)\Phi)} = m(\mathrm{id} \otimes S^{(F)})\Phi = \sum_i \phi_i^{(1)} S^{(F)}(\phi_j^{(2)}), \tag{6.30}$$

if $\Phi = \sum_{i,j} \phi_i^{(1)} \otimes \phi_j^{(2)}$. It is not hard to see that the invertible tensor $\mathcal{F} = \Phi F$ satisfies the conditions (6.21) and (6.22). Indeed, if we multiply the relation (6.30) by the relation (6.21) with the right-hand side and apply the definition (6.23) for $\Delta^{(F)}$ we obtain the 2-cocycle condition for $\mathcal{F} = \Phi F$. The normalization condition for $\mathcal{F}$ is obvious. Therefore the element $\mathcal{F}$ generates the quantization and moreover this quantization is equivalent to one in the beginning with the twist F and then with Φ, that is the following formulas are valid:

$$\begin{aligned}\Delta^{((F)\Phi)}(X) &= \Delta^{(\Phi F)}(X), \\ S^{((F)\Phi)}(X) &= S^{(\Phi F)}(X).\end{aligned} \tag{6.31}$$

The first relation in (6.31) is obvious and the second relation for the antipodes is a direct consequence of the equation for the convolutions (6.24), (6.30):

$$u^{((F)\Phi)} u^{(F)} = u^{(\Phi F)}, \tag{6.32}$$

that is proved by direct calculations. It is evident that the universal R-matrices of these quantizations also coincide

$$R^{((F)\Phi)} = \Phi^{21} R^{(F)} \Phi^{-1} = \mathcal{F}^{21} R \mathcal{F}^{-1} = R^{(\mathcal{F})}, \tag{6.33}$$

where $\mathcal{F} = \Phi F$.

Let $\mathcal{A}^{\ast}$ be a $\ast$-Hopf (super)algebra, $\mathcal{A}^{\ast} := \mathcal{A}(A; m, \Delta, S, \varepsilon, \ast)$, and the twisting element F is unitary

$$F^{\ast \otimes \ast} = F^{-1}, \tag{6.34}$$

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[13] Here and elsewhere any two tensors F^{12} and F^{21} are connected by the relation $F^{21} = \tau \circ F^{12}$, where τ is the superflip operator (see 4.4).

then the new twisting deformed $\ast$-Hopf (super)algebra $\mathcal{A}^{\ast(F)}$ is also a $\ast$-Hopf (super)algebra, i.e.

$$\begin{aligned}(\Delta^{(F)}(X))^{\ast \otimes \ast} &= \Delta^{(F)}(X^{\ast}), \\ (S^{(F)}(X))^{\ast} &= (S^{(F)})^{-1}(X^{\ast})\end{aligned} \tag{6.35}$$

for any $X \in \mathcal{A}^{\ast}$. Now we come back to our concrete Hopf superalgebras $A = U(\mathfrak{osp}(1|2;\mathbb{C}))$ and $A^{\ast} = U(\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(2,\mathbb{R})))$.

2a. Jordanian deformation. There are two well-known expressions of the twisting operator $F = F_J$ corresponding the Jordanian classical r-matrix (3.30), (5.13):

$$\begin{aligned}F_J &= (1 + 1 \otimes \beta E_+)^{H \otimes 1} \\ &= 1 + \sum_{k>0} \frac{\beta^k}{k!} H(H-1)\cdots(H-k+1) \otimes E_+^k\end{aligned} \tag{6.36}$$

$$= \exp(H \otimes 2\sigma), \tag{6.37}$$

where $2\sigma := \ln(1 + \beta E_+)$ and $\beta \in \mathbb{C}$ for $U(\mathfrak{osp}(1|2;\mathbb{C}))$ and $\beta \in \iota\mathbb{R}_+$ for $U(\mathfrak{osp}^{\dagger}(1|\mathfrak{su}(2,\mathbb{R})))$. The Jordanian twist in the form of binomial series (6.36) was first proposed in [25] (see also [26,27]) and the exponential expression (6.31) was proposed in [28] (see also [29]). Using the binomial series (6.36) we can easy obtain the explicit form of the element (6.24):

$$u^{(F)} = m(\mathrm{id} \otimes S)(F_J) = \exp(-\beta H E_+). \tag{6.38}$$

In accordance with (6.26) we put

$$u(t) = \exp(-t\beta H E_+) \quad (t \in \mathbb{R}). \tag{6.39}$$

One can calculate the following formulae for the deformed coproducts $\Delta^{(F_J)}(X) = F_J \Delta(X) F_J^{-1}$ (see [30]):

$$\begin{aligned}\Delta^{(F_J)}(e^{\pm\sigma}) &= e^{\pm\sigma} \otimes e^{\pm\sigma}, \\ \Delta^{(F_J)}(E_+) &= E_+ \otimes e^{2\sigma} + 1 \otimes E_+, \\ \Delta^{(F_J)}(H) &= H \otimes e^{-2\sigma} + 1 \otimes H, \\ \Delta^{(F_J)}(v_+) &= v_+ \otimes e^{\sigma} + 1 \otimes v_+, \\ \Delta^{(F_J)}(v_-) &= v_- \otimes e^{-\sigma} + 1 \otimes v_- + \beta H \otimes v_+ e^{-2\sigma},\end{aligned} \tag{6.40}$$

The coproducts $\Delta^{(F_J)}(E_-)$ can be calculated from the condition $\Delta^{(F_J)}(E_-) = 4\Delta^{(F_J)}(v_-)\Delta^{(F_J)}(v_-)$.

Using (6.23) and (6.38), one gets the formulas for the deformed antipode $S^{(F_J)}$:

$$\begin{aligned}S^{(F_J)}(e^{\pm\sigma}) &= e^{\mp\sigma}, \quad S^{(F_J)}(E_+) = -E_+ e^{-2\sigma}, \\ S^{(F_J)}(H) &= -He^{2\sigma}, \quad S^{(F_J)}(v_+) = -v_+ e^{-\sigma}, \\ S^{(F_J)}(v_-) &= -v_- e^{\sigma} + \beta H v_+ e^{\sigma}.\end{aligned} \tag{6.41}$$

It is easy to see the universal R-matrix $R^{(F)}$ for this twisted deformation looks as follows

$$R^{(F_J)} = F_J^{21} F_J^{-1}, \quad (R^{(F_J)})^{\ast} = (R^{(F_J)})^{-1}. \tag{6.42}$$

In the limit $\alpha \to 0$ we obtain for the R-matrix (6.23)

$$R^{(F_J)} = 1 + r_J + O(\alpha^2), \tag{6.43}$$

where r_J is the classical Jordanian r-matrix (5.13). It should be added that the coproduct $\Delta^{(F_J)}$ is real under the involution (†), i.e.

$$\Delta^{(F_J)}(a^\dagger) = (\Delta^{(F_J)}(a))^{\dagger \otimes \dagger}, \tag{6.44}$$

and the antipode $S^{(F_J)}$ satisfies the consistency

$$S^{(F_J)}((S^{(F_J)}(a^\dagger))^\dagger) = a, \tag{6.45}$$

as well as $\epsilon(a^\star) = \overline{\epsilon(a)}$ is trivially valid for $\forall a \in U_q(\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R})))$.

2b. Super-Jordanian deformation. The explicit twisting operator $F = F_{sJ}$ corresponding the super-Jordanian classical r-matrix (3.31), (5.14) was obtained in the paper [15] and it has the following factorized form:

$$F_{sJ} = \Phi F_s F_J, \tag{6.46}$$

where F_J is the Jordanian twisting two tensor (6.37), and the supersymmetric part F_s depending on the odd generator v_+ and the unitarizing factor Φ are given by the formulas:

$$F_s = 1 - 4\beta \frac{v_+}{e^\sigma + 1} \otimes \frac{v_+}{e^\sigma + 1}, \tag{6.47}$$

$$\Phi = \sqrt{\frac{(e^\sigma + 1) \otimes (e^\sigma + 1)}{2(e^\sigma \otimes e^\sigma + 1)}}, \tag{6.48}$$

$$F_s^\star = F_s^{-1} \quad \text{for } \star = \dagger \text{ or } \ddagger, \tag{6.49}$$

Such choice will modify the coproduct, $\Delta_{sJ} = \Phi \Delta_{sJ} \Phi^{-1}$, and we obtain

$$\begin{aligned}\Delta^{(F_{sJ})}(h) &= h \otimes e^{-2\sigma} + 1 \otimes h + \\ &\quad + \beta v_+ e^{-\sigma} \otimes v_+ e^{-2\sigma},\end{aligned} \tag{6.50}$$

$$\Delta^{(F_{sJ})}(v_+) = v_+ \otimes 1 + e^\sigma \otimes v_+, \tag{6.51}$$

$$\begin{aligned}\Delta^{(F_{sJ})}(v_-) &= v_- \otimes e^{-\sigma} + 1 \otimes v_- + \frac{\beta}{4}\Big\{\Big(\{h, e^\sigma\} \otimes v_+ e^{-2\sigma} \\ &\quad - \{h, v_+\} \otimes (e^\sigma - 1)e^{-2\sigma} \\ &\quad + 2v_+ \otimes h - \Big\{h, \frac{v_+ e^{-\sigma}}{e^\sigma + 1}\Big\} \otimes (e^\sigma - 1)e^{-\sigma} \\ &\quad + (e^\sigma - 1) \otimes \Big\{h, \frac{v_+}{e^\sigma + 1}\Big\}\Big) \frac{1}{e^\sigma \otimes e^\sigma + 1}\Big\}.\end{aligned} \tag{6.52}$$

The formulae for the antipode $S^{(F_{sJ})}$ look as follows:

$$S^{(F_{sJ})}(h) = -h e^{2\sigma} + \frac{1}{4}(e^{2\sigma} - 1), \tag{6.53}$$

$$S^{(F_{sJ})}(v_+) = -e^{-\sigma} v_+, \tag{6.54}$$

$$S^{(F_{sJ})}(v_-) = -v_- e^\sigma + \beta h v_+ e^\sigma - \frac{\beta}{4} v_+ e^\sigma. \tag{6.55}$$

It is easy to see that the formulae (6.50)–(6.52) satisfy the reality condition $(\Delta^{(F_{sJ})}(x))^\dagger = \Delta^{(F_{sJ})}(x^\dagger)$ for any $x \in$ $\mathfrak{osp}^\dagger(1|\mathfrak{sl}(2;\mathbb{R}))$ and the antipodes (6.53)–(6.55) satisfy the condition (6.35).

The universal R-matrix of the super-Jordanian deformation has the form

$$R^{(F_{sJ})} = \Phi F_s^{21} R^{(F_J)} F_s^{-1} \Phi^{-1}, \tag{6.56}$$

where

$$R^{(F_J)} = F_J^{21} F_J^{-1} = e^{2\sigma \otimes h} \otimes e^{-2h \otimes \sigma}. \tag{6.57}$$

7 Short summary and outlook

By the simple algebraic technique we obtain the complete classification of all basic (nonisomorphic) quantum deformations for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2;\mathbb{C})$ and its pseudoreal and real forms in terms of the classical r-matrices. In particular, we prove that compact pseudoreal form has only one quantum deformation (standard q-analog), and the $D = 3$, $N = 1$ Lorentz supersymmetry, which is the non-compact real form of $\mathfrak{osp}(1|2;\mathbb{C})$, has four different Hopf-algebraic quantum deformations: two standard q-analogs, and two (Jordanian and super-Jordanian) twist deformations. All basic Hopf-algebraic quantum deformations are presented in the explicit forms.

The q-deformation of the complex $\mathfrak{osp}^*(1|2;\mathbb{C})$ has been considered firstly in Refs. [12,13]. Further, its real forms were still studied in eighties and nineties, as quoted in the present paper (see [10–13,29,30]). We point out that in [14] there was studied the kappa-deformation of $D = 1$ superconformal mechanics as described by twisting of $osp^\star(1|o(2,1))$ superalgebra. We consider as main advantage of the present paper the first presentation in the literature of the complete classification providing all Hopf-algebraic quantum deformations of real forms of $osp(1|2;\mathbb{C})$.

The matrix realizations $o(2,1)$, $su(1,1)$ and $sl(2;R)$ of $D = 3$ Lorentz algebra and their respective superextensions can be used in different types of models, with $D = 3$ Lorentz (super)symmetries. First type of such models are described in terms of $D = 3$ Lorentz (super)vector variables, in second and third type models one employs two-dimensional (super)spinorial representations, described by supersymmetric extensions of complex $su(1,1)$ spinors and real $sl(2;R)$ spinors. The examples of models with above three types of the realizations of $D = 3$ Lorentz supersymmetries could be studied as a continuation of the present paper.

We did not consider in the paper the derivation of the supergravity models described by three different realizations of $D = 3$ Lorentz supersymmetries, but this could be a possible subject for further investigations. It is known that one can obtain the $D = 3$ gravity and supergravity actions from the knowledge of $D = 3$ Chern-Simons forms and superforms. For that purpose one can study the derivation of superforms

for the real forms of complexified $D = 3$ Lorentz superalgebra $osp(1, 2; C)$, considered in this paper.

In conclusion, we would like to note a very important point related to the fact that the results obtained here should help to solve also a more difficult problem: classification and construction of quantum deformations for the complex orthosymplectic superalgebra $\mathfrak{osp}(1|4; \mathbb{C}))$ and its real (de Sitter and anti-de Sitter) forms, which can be directly used in $D = 4$ SUGRA models. Namely, the superalgebra $\mathfrak{osp}(1|4; \mathbb{C}))$, as a linear space, has the following tensor structure

$$\mathfrak{osp}(1|4; \mathbb{C})) = \mathfrak{osp}_L(1|2; \mathbb{C})) \oplus \mathcal{P}_4 \oplus \mathfrak{osp}_R(1|2; \mathbb{C})), \quad (7.1)$$

where left $\mathfrak{osp}_L(1|2; \mathbb{C}))$ and right $\mathfrak{osp}_R(1|2; \mathbb{C}))$ superalgebras are isomorphic to the one considered here, and $\mathcal{P}_4$ is a four-dimensional linear space of a curved complexified $D = 4$ dS four-momentum generators. Since we already know the quantum deformations for the left and right superalgebras, our task is to extend them by curved four-momenta $\mathcal{P}_4$ and study the real forms of $\mathfrak{osp}(1|4; \mathbb{C}))$. If we introduce into this formula (7.1) a correct involution that characterizes the non-compact real form $osp(1|so(3, 2))$, we will get explicit expressions for this non-compact form used in ([33]).

Finally, it can be mentioned that one can introduce also quaternionic superalgebra $\mathfrak{osp}(1|2; \mathbb{H})$ [31,32] which describes $D = 5$ Lorentz or $D = 4$ de-Sitter superalgebra, with the bosonic sector $\mathfrak{o}(4, 1) \otimes \mathfrak{o}(2)$, and consider their real forms and quantum deformations.

Data Availability Statement The manuscript has associated data in a data repository. [Authors' comment: This work does not refer to data because it is a theoretical study. No comparison to observation is made.]

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