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Quantum dynamics corresponding to the chaotic BKL scenario

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Abstract We quantize the solution to the Belinski–Khalatnikov–Lifshitz (BKL) scenario using the integral quantization method. Quantization smears the gravitational singularity, preventing its localization in the configuration space. The latter is defined in terms of spatial and temporal coordinates, which are treated on the same footing that enables the respective covariance of general relativity. The relative quantum perturbations grow as the system evolves towards the gravitational singularity. The quantum randomness amplifies the deterministic classical chaos of the BKL scenario. Additionally, our results suggest that the generic singularity of general relativity can be avoided at a quantum level, giving support to the expectation that quantum gravity has a good chance of being a regular theory.

Contents

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1 Introduction

The Belinski–Khalatnikov–Lifshitz (BKL) conjecture states that general relativity includes a solution with a generic gravitational singularity [1,2]. The evolution towards the BKL singularity, the so-called BKL scenario, consists of the deterministic dynamics turning into a stochastic process near the generic singularity. There are at least two fundamental questions to be addressed in that context: What is the fate of the BKL chaos at the quantum level? Can the singularity be avoided in the corresponding quantum theory?

The evolution process presented in [1,2] is complicated and difficult to map into quantum evolution. There exists a well-defined and comparatively simple model of the BKL scenario [3–5] that can be used in the derivation of the BKL conjecture [6]. The model was obtained from the general model of the Bianchi IX spacetime for perfect fluid. The equation of state of that fluid reads $p = k\varepsilon,\ 0 \leq k < 1$, where p and ε denote the pressure and energy density of the fluid, respectively. The case $k = 1$ is excluded as it does not lead to the oscillatory dynamics specific to the Bianchi IX model. The massive BKL scenario model was obtained from that general Bianchi IX model by making the assumption (see Eq. (3)) that in the dynamics near the singularity, the anisotropy of space grows without bound so that each of the so-called directional scale factors oscillates, but never crosses the other, and evolves towards vanishing, i.e., singularity. The resulting dynamics, specified in the next section, is different from the commonly known mixmaster dynamics [7,8] in which one can divide the oscillatory evolution of the system into eras, each consisting of Kasner's epochs, evolving towards the singularity. The mixmaster model is the vacuum Bianchi IX model and can serve to derive the BKL conjecture as well [6], so we call it the vacuum model of the BKL scenario. We recently compared the dynamics of the massive and vacuum models within the dynamical systems method [9]. The dynamics of the massive BKL model depends on the matter field implicitly via the directional scale factors which are effective ones; see Eqs. (2.6), (2.7), and (2.24) in [3]. Also, due to the dependence of the general Bianchi IX model on matter components, it was possible to obtain the asymptotic form (near the singularity) of the general dynamics [3], which is mathematically simple enough to be solved analytically [10]. The dynamics of the vacuum Bianchi IX model is the same far away and near the singularity. It is so complex that the model is non-integrable [11]. Therefore, the massive BKL model is both simpler for analysis than the vacuum BKL model and better suited to describe the dynamics in the neighborhood of the cosmological singularity.

Recently, we verified numerically that the classical dynamics underlying the present paper lead to the gravitational singularity [12,13], and it is generically unstable, turning into a chaotic process near the singularity [10]. These features are consistent with the original BKL scenario [1,2,5].

As far as we are aware, there are no results available concerning the issue of the construction of quantum theory corresponding directly to the original BKL scenario [1,2]. The existing results concern the Hamiltonian framework in terms of the Ashtekar variables, which is supposed to be convenient for addressing the BKL scenario problem [14,15]. We have used the homogeneous sector of that formalism to consider the possible existence of classical and quantum spikes within that sector [16,17]. A recent article on spacelike singularities of general relativity promises to readdress the original BKL singularity problem within loop quantum gravity [18]. We do not exclude the case of joining that development in the future, but at present we rather prefer to follow our program.

Quantization of the dynamics presented in [3] can be used in examining the fate of the corresponding quantum dynamics. In fact, we have already quantized that model, with the conclusion that quantization of the dynamics leads to avoiding a gravitational singularity [19,20]. In these papers, we quantized Hamilton's dynamics derived in [21]. However, since quantization is known to be an ambiguous procedure, we have decided to examine the robustness of these results by making use of a completely different quantization method, which is one of the goals of the present paper. That method, applied recently to the quantization of the Schwarzschild spacetime [22], includes quantization of the temporal and spatial variables on the same footing. The rationale for such dealing is that the distinction between time and space variables violates the general covariance of arbitrary transformations of temporal and spatial coordinates.

The results of the present paper in the context of resolving the cosmological singularity are similar to the results of [22] addressing the issue of a singularity of an isolated object. In both cases, quantization smears the singularity, preventing its localization in configuration space. The issue of resolving the singularity within our two quite different approaches will be further discussed in the last section.

The new phenomenon we deal with in the present paper is the fate of classical chaos at the quantum level. Our analysis shows that the quantum randomness turns deterministic classical chaos into stochastic quantum chaos.

The paper is organized as follows: In Sect. 2 we recall the main results of [10] so that our paper is self-contained. Section 3 presents the main aspects of the coherent states quantization method adopted for our gravitational system. In Sect. 4 we quantize the solution to the BKL scenario. Stochastic aspects of quantum evolution are presented in Sect. 5. We conclude in the last section. The Appendix presents the essence of the coherent states quantization.

In the following, we choose $G = c = 1 = \hbar$ except where otherwise noted.

2 Solution to the BKL scenario

So that the paper will be self-contained, we recall in this section the main results of Ref. [10] to be used later.

The massive model of the BKL scenario is defined as [3,5]

$$\frac{\mathrm{d}^2 \ln a}{\mathrm{d}t^2} = \frac{b}{a} - a^2, \quad \frac{\mathrm{d}^2 \ln b}{\mathrm{d}t^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{\mathrm{d}^2 \ln c}{\mathrm{d}t^2} = a^2 - \frac{c}{b}, \tag{1}$$

subject to the constraint

$$\frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{\mathrm{d}\ln b}{\mathrm{d}t} + \frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{\mathrm{d}\ln c}{\mathrm{d}t} + \frac{\mathrm{d}\ln b}{\mathrm{d}t}\frac{\mathrm{d}\ln c}{\mathrm{d}t} = a^2 + \frac{b}{a} + \frac{c}{b}, \tag{2}$$

where $a = a(t) > 0$, $b = b(t) > 0$ and $c = c(t) > 0$ are the so-called directional scale factors, while $t \in \mathbb{R}$ is a monotonic function of proper time.

Equations (1) and (2) have been derived from the general dynamics of the Bianchi IX model under the condition that near the singularity, the following strong inequalities are satisfied [3]

$$a \gg b \gg c > 0. \tag{3}$$

It was found in [10] that the analytical solutions to Eqs. (1) and (2), for $t > t_0$, read

$$a(t) = \frac{3}{t - t_0}, \quad b(t) = \frac{30}{(t - t_0)^3}, \quad c(t) = \frac{120}{(t - t_0)^5}, \tag{4}$$

where $t - t_0 \neq 0$, and t_0 is an arbitrary real number. Thus, the solutions are parameterized by the number $t_0 \in \mathbb{R}$.

The solution (4) corresponds, for instance, in the case $t > t_0$ and $t_0 < 0$ to the following choice of the initial data

$$
\begin{aligned}
a(0) &= -3\, t_0^{-1}, & \dot{a}(0) &= -3\, t_0^{-2}, \\
b(0) &= -30\, t_0^{-3}, & \dot{b}(0) &= -90\, t_0^{-4}, \\
c(0) &= -120\, t_0^{-5}, & \dot{c}(0) &= -600\, t_0^{-6}.
\end{aligned} \tag{5}
$$

The stability analyses carried out in [10] showed that the solution (4) is unstable against small perturbations. More precisely, substituting the following functions

$$a(t) = 3(t - t_0)^{-1} + \epsilon\alpha(t), \tag{6a}$$

$$b(t) = 30(t - t_0)^{-3} + \epsilon\beta(t), \tag{6b}$$

$$c(t) = 120(t - t_0)^{-5} + \epsilon\gamma(t), \tag{6c}$$

into (1)–(2) leads, in the first order in the small parameter ϵ, to the following solution of the resulting equations

$$
\begin{aligned}
\alpha(t) =& \exp(-\theta/2)\,[K_1 \cos(\omega_1\theta + \varphi_1) \\
&+ K_2 \cos(\omega_2\theta + \varphi_2)] + K_3 \exp(-2\theta),
\end{aligned} \tag{7a}
$$

$$
\begin{aligned}
\beta(t) =& \exp(-5\theta/2)\left[\left(4 + 6\sqrt{6}\right) K_1 \cos(\omega_1\theta + \varphi_1)\right. \\
&\left. + \left(4 - 6\sqrt{6}\right) K_2 \cos(\omega_2\theta + \varphi_2)\right] \\
&+ 30K_3 \exp(-4\theta),
\end{aligned} \tag{7b}
$$

$$
\begin{aligned}
\gamma(t) =& -4\exp(-9\theta/2)\left[\left(26 + 9\sqrt{6}\right) K_1 \cos(\omega_1\theta + \varphi_1)\right. \\
&\left. + \left(26 - 9\sqrt{6}\right) K_2 \cos(\omega_2\theta + \varphi_2)\right] \\
&+ 200K_3 \exp(-6\theta),
\end{aligned} \tag{7c}
$$

where $\theta = \ln(t - t_0)$. The two frequencies read

$$\omega_1 = \frac{1}{2}\sqrt{95 - 24\sqrt{6}}, \quad \omega_2 = \frac{1}{2}\sqrt{95 + 24\sqrt{6}}, \tag{8}$$

where K_1, K_2, K_3, φ_1, and φ_2 are constants.

The manifold $\mathcal{M}$ defined by $\{K_1, K_2, K_3, \varphi_1, \varphi_2\}$ is a submanifold of $\mathbb{R}^5$. The solution defined by (6) and (7) corresponds to the choice of the set of the initial data $\mathcal{N}$ which is a small neighborhood of the initial data (5). $\mathcal{N}$ is a submanifold of $\mathbb{R}^5$, as (5) defines five independent constants due to the constraint (2). Therefore, it is clear that (7) presents a generic solution, as the measures of both $\mathcal{M}$ and $\mathcal{N}$ are nonzero. The exact solution (4) alone is of zero-measure in the space of all possible solutions to Eqs. (1) and (2).

The relative perturbations α/a, β/b, and γ/c grow proportionally as $\exp(\frac{1}{2}\theta)$. The multiplier $1/2$ plays the role of a Lyapunov exponent, describing the rate of their divergence. Since it is positive, the evolution of the system towards the gravitational singularity ($\theta \to +\infty$) is chaotic. The transition into the chaos occurs if the evolution begins with initial data which belong, for instance, to the neighborhood of the conditions (5).

The original BKL scenario [1,2] is known to enter the chaotic phase near the singularity. Its vacuum model, the mixmaster universe, has been proved to include the chaotic dynamics [23–27]. Its massive model [10] underlying the present paper has never been examined in the context of stochasticity. Finding that its dynamics is chaotic opens the door for studies of that issue at the quantum level, which is the main subject of our article. The main difference between the two BKL models (in terms of physics) is that the latter is more realistic near the singularity, as it effectively includes some contribution from the matter field.

3 Affine coherent states quantization

We propose to quantize the classical BKL scenario by using the integral quantization called affine coherent states quantization; see Appendix. We recently applied this approach in the context of cosmology [19,20] and astrophysics [22,28].

In general relativity, time and position in space are treated on the same level; however, in quantum mechanics, time is not considered to be a quantum observable, but a parameter enumerating events. In this paper, we treat time and position on the same footing in the quantum description. They are related to operators obtained by the affine coherent states quantization. This idea requires us to introduce the notion of an extended classical configuration space by including time as an additional coordinate. The correspondence between the classical time and position is determined by comparing their classical values with expectation values of their quantum counterparts.

In the following, we extend the method of quantization used in [19,20,22]. In the present description of the BKL scenario, the Hilbert space has to be extended to the carrier space of an infinite-dimensional unitary irreducible representation of the direct product of three affine groups with additional constraints determining a model of physical time. In this approach there is no distinction between kinematic and dynamical Hilbert spaces; we construct the state space of the system with time treated on the same footing as other observables. In addition, we obtain quantum states evolving similarly to classical solutions. This is achieved using the idea of the correspondence principle between quantum mechanics and its classical approximation.

We begin by introducing two configuration spaces defined as follows: the classical gravitational configuration space T_{BKL}

$$T_{BKL} := \{(t, a, b, c) : (t, a, b, c) \in \mathbb{R} \times \mathbb{R}_+^3\}, \tag{9}$$

where $\mathbb{R}_+ = (0, +\infty)$, and the affine configuration space T, defined as

$$
\begin{aligned}
T =& \{\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) : \xi \in (\mathbb{R} \times \mathbb{R}_+) \\
&\times(\mathbb{R} \times \mathbb{R}_+) \times (\mathbb{R} \times \mathbb{R}_+)\},
\end{aligned} \tag{10}
$$

where every pair (ξ_k, ξ_{k+1}), $(k = 1, 3, 5)$, parameterizes the affine group Aff($\mathbb{R}$).

The variables with even indices correspond to the scale factors $\xi_2 = a$, $\xi_4 = b$, $\xi_6 = c$. Because $a, b, c > 0$ and $\xi_1, \xi_3, \xi_5 \in \mathbb{R}$, the configuration space T parameterizes the simple product of three affine groups Aff($\mathbb{R}$) $\times$ Aff($\mathbb{R}$) $\times$ Aff($\mathbb{R}$) $=: G$ to be used in the affine quantization.

As the observational data are parameterized by a single time parameter, the variables ξ_1, ξ_3, ξ_5 should be mapped onto a single variable representing time.

The affine group Aff($\mathbb{R}$) is known to have two nontrivial unitary irreducible representations in the Hilbert space $\mathcal{H}_x := L^2(\mathbb{R}_+, \mathrm{d}\nu(x))$, where $\mathrm{d}\nu(x) := \mathrm{d}x/x$. We choose the one defined as follows (the second representation would give exactly the same results):

$$U(\xi_k, \xi_{k+1})\Psi(x) = \mathrm{e}^{i\xi_k x}\Psi(\xi_{k+1}x), \tag{11}$$

where $k = 1, 3, 5$, and $\langle x|\Psi\rangle =: \Psi(x) \in \mathcal{H}_x$. The action (11) corresponds to the standard parameterization of the affine group Aff($\mathbb{R}$) defined by the multiplication law

$$(\xi_k, \xi_{k+1}) \cdot (\xi'_k, \xi'_{k+1}) := (\xi_k + \xi_{k+1}\xi'_k, \xi_{k+1}\xi'_{k+1}) \in \mathrm{Aff}(\mathbb{R}). \tag{12}$$

The left invariant measure on the group Aff($\mathbb{R}$) reads

$$\mathrm{d}\mu(\xi_k, \xi_{k+1}) := \mathrm{d}\xi_k \, \frac{\mathrm{d}\xi_{k+1}}{\xi_{k+1}^2}, \tag{13}$$

and the corresponding invariant integration over the affine group is defined as

$$\int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_k, \xi_{k+1}) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi_k \int_0^{\infty} \mathrm{d}\xi_{k+1}/\xi_{k+1}^2. \tag{14}$$

It is clear that the direct product of three affine groups G has the unitary irreducible representation in the following Hilbert space $\mathcal{H} = \mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2} \otimes \mathcal{H}_{x_3} = L^2(\mathbb{R}_+^3, \mathrm{d}\nu(x_1, x_2, x_3))$, where $\mathrm{d}\nu(x_1, x_2, x_3) = \mathrm{d}\nu(x_1)\mathrm{d}\nu(x_2)\mathrm{d}\nu(x_3)$. This enables us to define in $\mathcal{H}$ the continuous family of affine coherent states $\langle x_1, x_2, x_3|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6\rangle := \langle x_1|\xi_1, \xi_2\rangle\langle x_2|\xi_3, \xi_4\rangle\langle x_3|\xi_5, \xi_6\rangle$, as follows:

$$\begin{aligned}&\mathcal{H} \ni \langle x_1, x_2, x_3|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6\rangle\\ &:= U(\xi)\Phi_0(x_1, x_2, x_3),\end{aligned} \tag{15}$$

where $U(\xi) := U(\xi_1, \xi_2)U(\xi_3, \xi_4)U(\xi_5, \xi_6)$, and $|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6\rangle := |\xi_1, \xi_2\rangle|\xi_3, \xi_4\rangle|\xi_5, \xi_6\rangle$, and where

$$\mathcal{H} \ni \Phi_0(x_1, x_2, x_3) = \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3). \tag{16}$$

In (16), the vectors $\Phi_k(x_k) \in L^2(\mathbb{R}_+, \mathrm{d}\nu(x_k))$, $k = 1, 2, 3$ are the so-called fiducial vectors. They are required to satisfy the two conditions

$$\int_0^{\infty} \frac{\mathrm{d}x}{x}|\Phi_k(x)|^2 = 1, \tag{17}$$

and

$$A_{\phi_l} := \int_0^{\infty} \frac{\mathrm{d}x}{x^2}|\Phi_k(x)|^2 < \infty, \tag{18}$$

where $l = 1$ for $k = 1$, $l = 3$ for $k = 2$, and $l = 5$ for $k = 3$. The fiducial vectors are the free "parameters" of this quantization scheme.

Finally, we have

$$\begin{aligned}&U(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)\Phi_0(x_1, x_2, x_3)\\ &\quad = U(\xi_1, \xi_2)U(\xi_3, \xi_4)U(\xi_5, \xi_6)\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\\ &\quad = \mathrm{e}^{i(\xi_1 x_1 + \xi_3 x_2 + \xi_5 x_3)}\Phi_1(\xi_2 x_1)\Phi_2(\xi_4 x_2)\Phi_3(\xi_6 x_3).\end{aligned} \tag{19}$$

The irreducibility of the representation leads to the resolution of the unity in $\mathcal{H}$ as follows:

$$\begin{aligned}&\frac{1}{A_\phi}\int_{\mathrm{G}} \mathrm{d}\mu(\xi)|\xi\rangle\langle\xi|\\ &:= \bigotimes_{k=1,3,5} \frac{1}{A_{\phi_k}}\int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_k, \xi_{k+1})|\xi_k, \xi_{k+1}\rangle\langle\xi_k, \xi_{k+1}|\\ &= \frac{1}{A_{\Phi_1}A_{\Phi_3}A_{\Phi_5}}\int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_1, \xi_2)|\xi_1, \xi_2\rangle\langle\xi_1, \xi_2|\\ &\quad \otimes \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_3, \xi_4)|\xi_3, \xi_4\rangle\langle\xi_3, \xi_4|\\ &\quad \otimes \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_5, \xi_6)|\xi_5, \xi_6\rangle\langle\xi_5, \xi_6| = \mathbb{1}_1\\ &\quad \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 = \mathbb{1},\end{aligned} \tag{20}$$

where $\mathrm{d}\mu(\xi) = \Pi_{k=1,3,5}\mathrm{d}\mu(\xi_k, \xi_{k+1})$.

The resolution (20) can be used for mapping a classical observable $f : T \to \mathbb{R}$ into an operator $\hat{f} : \mathcal{H} \to \mathcal{H}$ as follows [19,20,22,29]

$$\begin{aligned}\hat{f} &:= \frac{1}{A_\phi}\int_{\mathrm{G}} \mathrm{d}\mu(\xi)|\xi\rangle f(\xi)\langle\xi|\\ &= \frac{1}{A_{\Phi_1}A_{\Phi_3}A_{\Phi_5}}\int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_1, \xi_2)\int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_3, \xi_4)\\ &\quad \times \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(\xi_5, \xi_6)|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6\rangle\\ &\quad \times f(\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6)\langle\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6|,\end{aligned} \tag{21}$$

where $A_\Phi := A_{\Phi_1}A_{\Phi_3}A_{\Phi_5}$.

Here, we recall two standard characteristics of quantum observables: (i) expectation values and variances of quantum observables are the most important characteristics which allow us to compare quantum and classical worlds, and (ii) expectation values of quantum observables correspond to classical values of measured quantities, and their variances describe quantum smearing of these observables.

A general form of the expectation value of the observable $\hat{f}$ obtained from the classical function f, while the quantum system is in the state $|\Psi\rangle$, reads

$$\langle \hat{f}; \Psi \rangle := \langle \Psi | \hat{f} | \Psi \rangle = \frac{1}{A_\phi} \int_G d\mu(\xi) f(\xi) |\langle \xi | \Psi \rangle|^2. \tag{22}$$

The variance of an observable $\hat{f}$ defined as $\mathrm{var}(\hat{f}; \Psi) := \langle (\hat{f} - \langle \hat{f}; \Psi \rangle)^2, \Psi \rangle$ is more difficult for calculations because it requires $\langle \hat{f}^2; \Psi \rangle$, which involves an overlap between the coherent states, and usually depends explicitly on the fiducial vector:

$$\begin{aligned}\langle (\hat{f})^2; \Psi \rangle &:= \langle \Psi | (\hat{f})^2 | \Psi \rangle \\ &= \frac{1}{A_\phi} \int_G d\mu(\xi) \frac{1}{A_\phi} \int_G d\mu(\xi') \langle \Psi | \xi \rangle f(\xi) \\ &\quad \langle \xi | \xi' \rangle f(\xi') \langle \xi' | \Psi \rangle. \end{aligned} \tag{23}$$

The variance $\mathrm{var}(\hat{f}; \Psi)$ can be rewritten as

$$\mathrm{var}(\hat{f}; \Psi) = \langle \hat{f}^2; \Psi \rangle - \langle \hat{f}; \Psi \rangle^2. \tag{24}$$

The important quantum observables correspond to the variables of the configuration space (10). These elementary variables, ξ_k $(k = 1, 2, \dots, 6)$, can be mapped into the quantum operators as follows:

$$\hat{\xi}_k = \frac{1}{A_\Phi} \int_G d\mu(\xi) |\xi\rangle \xi_k \langle \xi|. \tag{25}$$

For every k, the above equality (25) reduces to integration over a single affine group. The other integrations give the unit operators in two remaining spaces, $\mathcal{H}_{x_l}, l \neq k$. For example,

$$\hat{\xi}_2 = \frac{1}{A_{\Phi_1}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(\xi_1, \xi_2) |\xi_1, \xi_2\rangle \xi_2 \langle \xi_1, \xi_2| \otimes \mathbb{1}_{x_1} \otimes \mathbb{1}_{x_3}. \tag{26}$$

To deal with a single time variable at the quantum level, one needs to choose a model of time in the configuration space T, defined by (10). In general, it can be introduced as either a real function or distribution, $\mathcal{T} : T \to \mathbb{R}$. Its quantization leads to the time operator $\hat{\mathcal{T}}$. However, we can impose the appropriate constraints to have the common time variable for all three operators $\hat{\xi}_1$, $\hat{\xi}_3$ and $\hat{\xi}_5$. In this paper we realize that option assuming that the only allowed quantum states Ψ of our BKL system are the states which satisfy the condition

$$\langle \Psi | \hat{\xi}_1 | \Psi \rangle = \langle \Psi | \hat{\xi}_3 | \Psi \rangle = \langle \Psi | \hat{\xi}_5 | \Psi \rangle, \tag{27}$$

which is consistent with the choice of the configuration space in the form (9). It means that we require the same expectation values for all three operators, which represent three "times" related to appropriate quantum observables $\hat{a} = \hat{\xi}_2$, $\hat{b} = \hat{\xi}_4$, and $\hat{c} = \hat{\xi}_6$.

4 Quantization of the solution to the BKL scenario

The above quantization scheme can now be applied to the solutions (7) of the BKL scenario[1]

$$\begin{aligned} &a(t) = \tilde{a}(t) + \epsilon\alpha(t), \quad b(t) = \tilde{b}(t) + \epsilon\beta(t), \\ &c(t) = \tilde{c}(t) + \epsilon\gamma(t), \end{aligned} \tag{28}$$

ascribing to them appropriate quantum states and the corresponding operators.

In quantum mechanics, contrary to classical mechanics, one needs two kinds of objects to describe the physical world. These are quantum observables represented by either appropriate operators or operator-valued measures, and quantum states being the vectors in a Hilbert space or the so-called density operators. In classical mechanics, the functions on either configuration or phase space are at the same time states and observables.

To quantize solutions of the BKL scenario, we already have the elementary observables $\hat{\xi}_k$. However, we also have to find the appropriate family of states related to the solutions (28). This family of states has to reproduce the classical solutions (28) by comparing them with expectation values of the corresponding observables.

The classical solutions are represented by three time-dependent functions. In the configuration space T, we have six variable, where three of them, ξ_1, ξ_3, ξ_5, represent the time in the state space which satisfies the condition (27). As mentioned above, the classical observables should be related to their quantum counterparts by the corresponding expectation values. This idea leads directly to the conditions for a family of states $\{\Psi_\eta(x_1, x_2, x_3) = \langle x_1, x_2, x_3 | \Psi_\eta \rangle, \eta \in \mathbb{R}^s\}$ parameterized by a set of evolution parameters $\eta = (\eta_1, \eta_2, \dots \eta_s)$ enumerating the set of trial functions.

We require the states $|\Psi_\eta\rangle$ to satisfy the following conditions [22]:

$$\langle \Psi_\eta | \hat{\xi}_k | \Psi_\eta \rangle = t, \quad k = 1, 3, 5 \tag{29}$$

$$\langle \Psi_\eta | \hat{\xi}_2 | \Psi_\eta \rangle = a(t), \tag{30}$$

$$\langle \Psi_\eta | \hat{\xi}_4 | \Psi_\eta \rangle = b(t), \tag{31}$$

$$\langle \Psi_\eta | \hat{\xi}_6 | \Psi_\eta \rangle = c(t). \tag{32}$$

Equation (29) represents the single time constraint (27). The parameter η labels the family of states to be found, and it should be a function of t, as the r.h.s. of (29)–(32) depends on t. The solution of Eqs. (29)–(32) allows us to construct the vector state dependent on classical time, $|\Psi_{\eta(t)}\rangle \in \mathcal{H}$, in our Hilbert space.

In this way, we relate the quantum dynamics to the classical dynamics. Obviously, there may exist more than one family of states satisfying the above equations of motion.

[1] In what follows, we denote by $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ the solution (4).

In what follows, we determine the states $|\Psi_{\eta(t)}\rangle$ satisfying the conditions (29)–(32). This will enable us to examine the issue of the fate of the gravitational singularity and chaos of the BKL scenario at the quantum level.

4.1 Evolving wave packets

In our paper, we consider two kinds of wave packets satisfying the conditions (29)–(32). The first kind are the affine coherent states themselves. The second type is a set of modified "exponential" wave packets, which represent a dense set of states in the Hilbert space $\mathcal{H}$.

4.1.1 Coherent states and expectation values

One can verify that, based on the results of the recent paper [22], the considered coherent states generated by a single affine group satisfy the following equations:

$$\langle\xi_k,\xi_{k+1}|\hat{\xi}_l|\xi_k,\xi_{k+1}\rangle=\xi_l,\quad \text{where} \\ l=k,k+1;\ k=1,3,5, \tag{33}$$

where the operators $\hat{\xi}_l$ are defined as

$$\hat{\xi}_l:=\frac{1}{2\pi}\frac{1}{A_{\Phi_k}}\int_{\mathbb{R}}\mathrm{d}\xi_k\int_{\mathbb{R}_+}\frac{\mathrm{d}\xi_{k+1}}{\xi_{k+1}^2}\,|\xi_k,\xi_{k+1}\rangle\xi_l\langle\xi_k,\xi_{k+1}|, \tag{34}$$

and where $l=k,k+1$, $k=1,3,5$.

The conditions (33) are the consistency conditions between the affine group parameterization and the configuration space of the quantized physical system [22].

This implies that the coherent states generated by the product of three affine groups also satisfy the consistency condition

$$\langle\xi|\hat{\xi}_l|\xi\rangle=\frac{1}{A_\phi}\int_G\mathrm{d}\mu(\xi')\langle\xi|\xi'\rangle\xi'_l\langle\xi'|\xi\rangle=\xi_l, \\ \text{where } l=1,2,\ldots,6. \tag{35}$$

The consistency conditions coincide with an idea that the coherent state $|\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6\rangle$ represents a state localized at the point $\{\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6\}$ of the affine configuration space and at the same time in the spacetime.

Therefore, the coherent states

$$|CS_\epsilon;t\rangle:=|t,\tilde{a}(t)+\epsilon\alpha(t);\ t,\tilde{b}(t) \\ +\epsilon\beta(t);\ t,\tilde{c}(t)+\epsilon\gamma(t)\rangle \tag{36}$$

satisfy the equations of motions (29)–(32). In such case, we propose using a one-dimensional parameter η and we identify it with the classical time t, i.e., the classical time is a label of the evolving family of quantum states.

Realization of (36) as a wave packet constructed in the space of square integrable functions $L^2(\mathbb{R}_+^3,\mathrm{d}\nu(x_1,x_2,x_3))$ reads

$$\Psi_{CS_\epsilon}(t,x_1,x_2,x_3)=\langle x_1,x_2,x_3|CS_\epsilon;t\rangle \\ =\mathrm{e}^{it(x_1+x_2+x_3)}\Phi_1(a(t)x_1) \\ \times\Phi_2(b(t)x_2)\Phi_3(c(t)x_3). \tag{37}$$

Using Eq. (35) and the fact that the vector (36) factorizes

$$|CS_\epsilon;t\rangle=|t,\tilde{a}(t)+\epsilon\alpha(t)\rangle|t,\tilde{b}(t) \\ +\epsilon\beta(t)\rangle|t,\tilde{c}(t)+\epsilon\gamma(t)\rangle, \tag{38}$$

we can compute the expectation value of the volume operator $\hat{V}$, where $V:=\xi_2\xi_4\xi_6$, as follows

$$\langle CS_\epsilon;t|\hat{V}|CS_\epsilon;t\rangle=(\tilde{a}(t)+\epsilon\alpha(t))\,(\tilde{b}(t) \\ +\epsilon\beta(t))\,(\tilde{c}(t)+\epsilon\gamma(t)). \tag{39}$$

4.1.2 The modified exponential packet and expectation values

Let us consider the set of Gaussian distribution wave packets (with a modified exponential part)

$$\Psi_n(x;\tau,\gamma)=Nx^n\exp\left[i\tau x-\frac{\gamma^2x^2}{2}\right],\quad N^2=\frac{2\gamma^n}{(n-1)!}, \tag{40}$$

which according to [22] is dense in $L^2(\mathbb{R}_+,\mathrm{d}\nu(x))$.

The expectation values and variances of the operators $\hat{\xi}_k$ and $\hat{\xi}_{k+1}$ have the following values:

$$\langle\Psi_n|\hat{\xi}_k|\Psi_n\rangle=\tau,\quad k=1,3,5, \tag{41}$$

$$\langle\Psi_n|\hat{\xi}_{k+1}|\Psi_n\rangle=\frac{1}{A_\Phi}\frac{\Gamma\left(n-\frac{1}{2}\right)}{(n-1)!}\gamma, \tag{42}$$

$$\mathrm{var}(\hat{\xi}_k;\Psi_n)=\frac{4n-3}{4(n-1)}\gamma^2, \tag{43}$$

$$\mathrm{var}(\hat{\xi}_{k+1};\Psi_n)=\frac{1}{A_\Phi^2}\left(\frac{1}{n-1}-\frac{\Gamma\left(n-\frac{1}{2}\right)^2}{(n-1)!^2}\right)\gamma^2. \tag{44}$$

In this case, the evolution parameter η consists of $\eta_1=\tau$ and $\eta_2=\gamma$.

In the space $L^2(\mathbb{R}_+^3,\mathrm{d}\nu(x_1,x_2,x_3))$, we take the corresponding wave packets in the form

$$\Psi_{n_1,n_3,n_5}(x_1,x_2,x_3;\ \tau_1,\tau_3,\tau_5,\gamma_1,\gamma_3,\gamma_5) \\ =\Psi_{n_1}(x_1;\tau_1,\gamma_1)\Psi_{n_3}(x_2;\tau_3,\gamma_3)\Psi_{n_5}(x_3;\tau_5,\gamma_5). \tag{45}$$

To satisfy the properties (29)–(32) for the wave packets Ψ_{n_1,n_3,n_5}, we choose the parameters τ_k and γ_k as follows:

$$\tau_1=\tau_3=\tau_5=t, \tag{46}$$

$$\gamma_k=A_{\Phi_k}\frac{(n_k-1)!}{\Gamma\left(n_k-\frac{1}{2}\right)}\cdot f_k(t),\quad k=1,3,5, \tag{47}$$

where

$$f_k(t)=\begin{cases}\tilde{a}(t)+\epsilon\alpha(t), & k=1\\ \tilde{b}(t)+\epsilon\beta(t), & k=3\\ \tilde{c}(t)+\epsilon\gamma(t), & k=5.\end{cases} \tag{48}$$

Using a similar technique as in the case of equation (39), we can calculate the expectation value of the volume operator in the state (45), and we obtain

$$\langle \Psi_{n_1,n_3,n_5} | \hat{V} | \Psi_{n_1,n_3,n_5} \rangle = f_1(t) f_3(t) f_5(t), \tag{49}$$

which coincides with the result obtained with the coherent states method (39).

It is clear that the expectation value of the volume operator converges very fast to zero as $t \to \infty$.

4.2 Variances in the Hilbert space $\mathcal{H}$

The wave packets obtained above follow the classical solutions. However, in quantum mechanics, the observables which do not commute with time and position operators fluctuate at every spacetime point. This smearing of quantum observables is determined by the Heisenberg uncertainty principle. Its most important ingredients are variances of the corresponding observables. In the following we perform an analysis of the variances of our elementary variables.

4.2.1 Using coherent states

The variances of the operators $\hat{\xi}_k, \hat{\xi}_{k+1}, k = 1, 3, 5$ in coherent states (36) read

$$\mathrm{var}(\hat{\xi}_k; |CS_\epsilon; t\rangle) = \langle \hat{\xi}_k^2 \rangle_0 f_k(t)^2, \tag{50}$$

$$\mathrm{var}(\hat{\xi}_{k+1}; |CS_\epsilon; t\rangle) = \left(\langle \hat{\xi}_{k+1}^2 \rangle_0 - 1 \right) f_k(t)^2, \tag{51}$$

where

$$\langle \hat{\xi}_i^2 \rangle_0 = \langle 0, 1 | \hat{\xi}_i^2 | 0, 1 \rangle, \quad i = 1, 2, \ldots 6, \tag{52}$$

and where $\langle \hat{\xi}_i^2 \rangle_0$ is a constant which depends on the choice of the fiducial vector.

The variance of the volume operator is found to be

$$\mathrm{var}(\hat{V}; |CS_\epsilon; t\rangle) = \left[\prod_{k=2,4,6} \langle \hat{\xi}_k^2 \rangle_0 - 1 \right] f_1(t)^2 f_3(t)^2 f_5(t)^2. \tag{53}$$

For more details concerning the r.h.s. of (52), see Appendix D of [22].

4.2.2 Using exponential wave packet

The corresponding results for the wave packets (45) under the conditions (46)–(47) read

$$\mathrm{var}(\hat{\xi}_k; \Psi_{n_1,n_3,n_5}) = \mathcal{A}_k f_k(t)^2, \tag{54}$$

$$\mathrm{var}(\hat{\xi}_{k+1}; \Psi_{n_1,n_3,n_5}) = \mathcal{B}_k f_k(t)^2, \tag{55}$$

where

$$\mathcal{A}_k = A_{\Phi_k}^2 \frac{(4n_k - 3)(n_k - 1)!(n_k - 2)!}{4\Gamma\left(n_k - \frac{1}{2}\right)^2}, \tag{56}$$

$$\mathcal{B}_k = \frac{(n_k - 1)!(n_k - 2)!}{\Gamma\left(n_k - \frac{1}{2}\right)^2} - 1. \tag{57}$$

The variance of the volume operator has the form

$$\mathrm{var}(\hat{V}; \Psi_{n_1,n_3,n_5}) = \left[\prod_{k=2,4,6} \frac{(n_k - 1)!(n_k - 2)!}{\Gamma\left(n_k - \frac{1}{2}\right)} - 1 \right] f_1(t)^2 f_3(t)^2 f_5(t)^2. \tag{58}$$

These results show that all positions of our system in time and space are smeared owing to nonzero variances. It is an important fact about the possibility of avoiding singularities in this dynamics.

5 Stochastic aspects of quantum evolution

The results of recent paper [10] give strong support to the expectation that near the generic gravitational singularity, the evolution becomes chaotic. In what follows, we examine that fundamental property of the BKL scenario at the quantum level.

To enable a direct comparison with the results of [10], we split the variances (55) into the contributions from unperturbed and perturbed states. We have

$$\begin{aligned} f_2(t)^2 &= (\tilde{a}(t) + \epsilon\alpha(t))^2 \\ &= \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t) + \epsilon^2\alpha(t)^2 \\ &\simeq \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t), \end{aligned} \tag{59}$$

$$\begin{aligned} f_4(t)^2 &= (\tilde{b}(t) + \epsilon\beta(t))^2 \\ &= \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t) + \epsilon^2\beta(t)^2 \\ &\simeq \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t), \end{aligned} \tag{60}$$

$$\begin{aligned} f_6(t)^2 &= (\tilde{c}(t) + \epsilon\gamma(t))^2 \\ &= \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t) + \epsilon^2\gamma(t)^2 \\ &\simeq \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t). \end{aligned} \tag{61}$$

The corresponding dimensionless functions describing relative quantum perturbations are defined as

$$\kappa_k := \frac{\mathrm{var}(\hat{\xi}_k; \Psi_{pert}) - \mathrm{var}(\hat{\xi}_k; \Psi_{unpert})}{\mathrm{var}(\hat{\xi}_k; \Psi_{unpert})}, \quad k = 2, 4, 6, \tag{62}$$

where Ψ_{pert} and Ψ_{unpert} denote perturbed and unperturbed wave packets, respectively.

The explicit form of (62), up to the first order in ϵ, reads

$$\kappa_a(t) := \kappa_2(t) = \frac{2\epsilon\tilde{a}(t)\alpha(t)}{\tilde{a}(t)^2} = 2\epsilon\frac{\alpha(t)}{\tilde{a}(t)}, \tag{63}$$

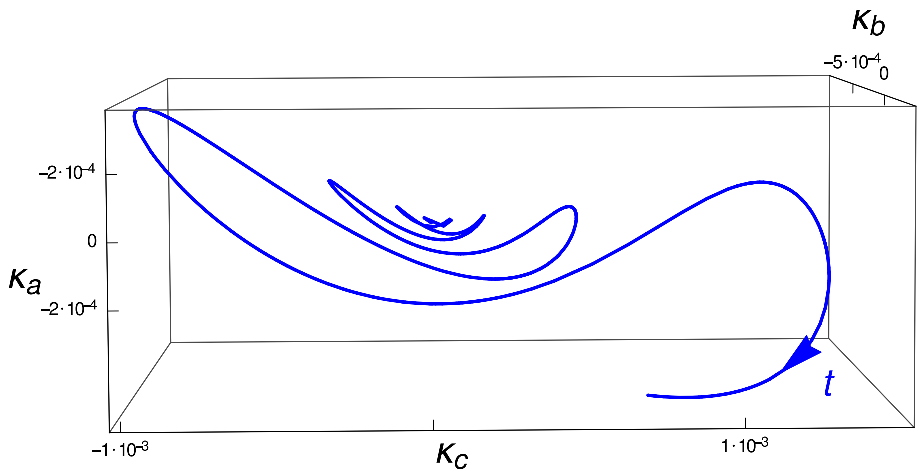

Fig. 1 The t dependence of quantum perturbation defined by (63)–(65) for $K_1 = K_2 = 0.01$, $K_3 = 0$, $\phi_1 = \phi_2 = 0$, $t_0 < 0$, $\epsilon = 0.01$. The plot presents the parametric curve $\{\kappa_a(t), \kappa_b(t), \kappa_c(t)\}$, where $t - t_0 \in (0.01, 35)$

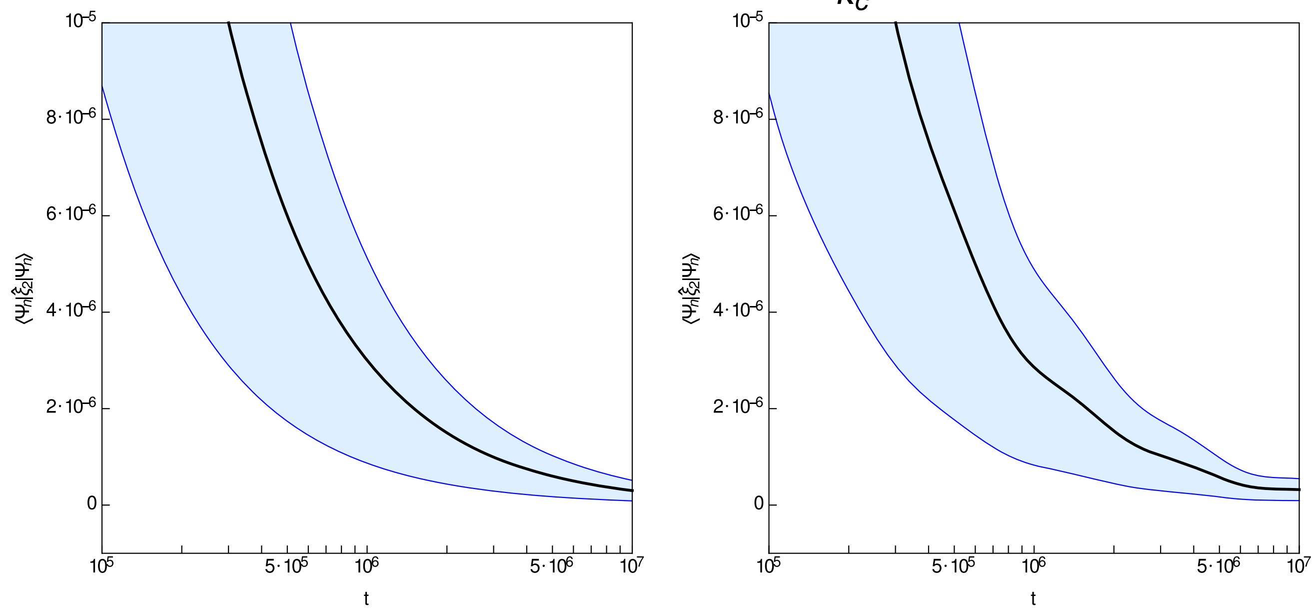

Fig. 2 The t dependence of the expectation value of the operator $\hat{\xi}_2$ defined by (42), (47) for $K_1 = K_2 = 0.01$, $K_3 = 0$, $\phi_1 = \phi_2 = 0$, any $t_0 < 0$, $n_1 = 3$. The axis of t is in logarithmic scale. The left panel corresponds to the unperturbed solution ($\epsilon = 0$), and the right panel corresponds to the perturbed solution ($\epsilon = 0.01$). The blue area defines the points for which the distance from the expected value is smaller than $\sqrt{\mathrm{var}(\xi_2; \Psi_n)}$ defined by (55) (the distance is counted along fixed t line)

$$\kappa_b(t) := \kappa_4(t) = \frac{2\epsilon\tilde{b}(t)\beta(t)}{\tilde{b}(t)^2} = 2\epsilon\frac{\beta(t)}{\tilde{b}(t)}, \tag{64}$$

$$\kappa_c(t) := \kappa_6(t) = \frac{2\epsilon\tilde{c}(t)\gamma(t)}{\tilde{c}(t)^2} = 2\epsilon\frac{\gamma(t)}{\tilde{c}(t)}. \tag{65}$$

It is clear that the relative perturbations (63)–(65) are the same for the coherent states and the exponential wave packets.

Figure 1 presents the parametric curve visualizing the relative quantum perturbations. The time dependence of the expectation values of the $\hat{\xi}_2$ operator and corresponding variances of unperturbed and perturbed solutions are presented in Fig. 2. The plots for $\hat{\xi}_4$ and $\hat{\xi}_6$ operators would look similar, so we do not present them.

6 Conclusions

It has been shown [10] that the perturbed classical solution to the dynamics of the massive model of the BKL scenario exhibits chaotic behavior. Our results present the quantum dynamics corresponding to that dynamics. Figure 1 shows that the relative quantum perturbations grow as the system evolves towards the singularity, which is consistent with the corresponding classical evolution; see Fig. 2 of [10]. Since our quantum and classical perturbations have quite similar time evolutions, we conclude that quantization does not destroy classical chaos. In fact, the quantum chaos corresponds to the classical chaos in the lowest-order approximation. Nonlinearity of classical dynamics creates deterministic chaos. Non-vanishing variances of observables of the corresponding quantum dynamics lead to stochastic chaos.

To show that behavior, we have constructed the wave packets for which expectation values of elementary observables follow the corresponding classical ones. This choice of quantum states leads to the scenario in which calculated expectation values of quantum directional scale factors evolve to vanishing similarly as their classical counterparts. However, quantum states do not represent sharp properties of a physical system. Expectation values are smeared quantities. That smearing is represented by quantum variance.

More precisely, the variance is a measure of the stochastic deviation from an expectation value of a given operator. A quantum system is in an eigenstate of an operator if and only if the variance of this operator in that state equals zero (see [22] for more details). The calculated variances depicted in Fig. 2 are always nonzero, which means that the probability of hitting the gravitational singularity is equal to zero. The nonzero variance removes the singularity from considered quantum evolution.

The quantum randomness amplifies the deterministic classical chaos. This supports the hypothesis that in the region corresponding to the neighborhood of the classical singularity, the dynamics, both classical and quantum, enter the stochastic phase. The oscillatory behavior of the expectation value of the quantum scale factor increases as $t \to \infty$, which is consistent with the classical BKL scenario [2,5].

The results of the present paper support our previous results [19,20] concerning the fate of the BKL singularity at the quantum level. One of the main differences between the results of the papers [19,20] and the present article is that in the former case, the evolution parameter (time) used at the quantum level was purely mathematical and was taken as being equal to the classical time by an assumption. In the present paper, the quantum time is established by the requirement that the temporal and spatial variables should be treated on the same footing at the quantum level, which supports the covariance of arbitrary transformations of these variables in general relativity. Another important difference is the quite different implementation of the dynamics at both the classical and quantum levels. In the case of [19,20], it was based on Hamilton's dynamics and the corresponding Schrödinger equation at the quantum level. Here, we quantize the solution to the classical dynamics ascribing to it the corresponding quantum system. Surprisingly, both approaches give physically similar results: avoidance of the classical singularity at the quantum level. Additionally, the approach of the present paper addresses the issue of the fate of the classical chaos at the quantum level, which was beyond the scope of the approach used in [19,20]. We have examined that issue by calculating the variances of expectation values of quantum observables. The variances are measures of stochasticity of considered observables at the quantum level. In fact, the calculations of variances were ignored in the recent quantizations of the vacuum BKL models [30–32].

The BKL conjecture states that general relativity includes a generic gravitational singularity. Our results strongly suggest that a generic singularity can be avoided at the quantum level so that one can expect that a theory of quantum gravity (to be constructed) has a good chance of being regular.

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Appendix A: Essence of integral quantization

If the configuration space Π is a half-plane,

$$\Pi := \{(p, q) \in \mathbb{R} \times \mathbb{R}_+\}, \quad \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\},$$

it can be identified with the affine group $\mathrm{Aff}(\mathbb{R}) =: G$.

The multiplication law can be defined as

$$(p_1, q_1) \cdot (p_2, q_2) := (p_1 + q_1 p_2, q_1 q_2). \tag{A1}$$

The unity of the group is $(0, 1)$, and the inverse reads $(p, q)^{-1} = (-p/q, 1/q)$.

This group has two nontrivial unitary irreducible representations realized in the Hilbert space $L^2(\mathbb{R}_+, \mathrm{d}\nu(x)) =: \mathcal{H}$, where $\mathrm{d}\nu(x) = \mathrm{d}x/x$. We choose the one defined as follows (the second representation would lead to exactly the same results):

$$U(p, q)\psi(x) = \mathrm{e}^{ipx}\psi(qx), \quad \psi(x) \in \mathcal{H}. \tag{A2}$$

Equation (A2) enables us to define the continuous family of affine coherent states (ACS), denoted by $\langle x|p, q\rangle \in \mathcal{H}$, as follows:

$$\langle x|p, q\rangle = U(p, q)\langle x|\phi\rangle, \tag{A3}$$

where $\langle x|\phi\rangle =: \phi(x) \in \mathcal{H}$ is the so-called fiducial vector, which is a free parameter (to some extent) of the ACS quantization scheme.

Equation (A3) can be interpreted as the correspondence

$$(p, q) \longrightarrow |p, q\rangle\langle p, q| \tag{A4}$$

between the point of the configuration space Π and the quantum projection operator acting in $\mathcal{H}$.

The irreducibility of the representation leads (due to Schur's lemma) to the resolution of the unity in $L^2(\mathbb{R}_+, \mathrm{d}\nu(x))$:

$$\frac{1}{A_\phi}\int_G \mathrm{d}\mu(p, q)|p, q\rangle\langle p, q| = \mathbb{1}, \tag{A5}$$

where $\mathrm{d}\mu(p, q) := \mathrm{d}p\ \mathrm{d}q/q^2$ is the left invariant measure on G, and where $A_\phi := \int_0^\infty |\phi(x)|^2 \frac{\mathrm{d}x}{x^2} < \infty$ is a constant.

The use of (A5) enables quantization of almost any observable $f : \Pi \to \mathbb{R}$

$$f \longrightarrow \hat{f} = \frac{1}{A_\phi}\int_G \mathrm{d}\mu(p, q)|p, q\rangle f(p, q)\langle p, q|. \tag{A6}$$

The operator $\hat{f} : \mathcal{H} \to \mathcal{H}$ is symmetric by construction. No ordering ambiguity occurs (notorious problem of canonical quantization). That operator is self-adjoint if it is bounded.

For more details concerning the integral quantization, see [29] and references therein.

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