



A correspondence between the free and interacting field

Fei Gao^{1,a}, Minghui Ding^{2,b}, Yu-xin Liu^{1,3,4,c}, Sebastian M. Schmidt^{2,5,d}

¹ Center for High Energy Physics, Peking University, Beijing 100871, China

² Helmholtz-Zentrum Dresden-Rossendorf, Bautzner Landstraße 400, 01328 Dresden, Germany

³ Department of Physics and State Key Laboratory of Nuclear Physics and Technology, Peking University, Beijing 100871, China

⁴ Collaborative Innovation Center of Quantum Matter, Beijing 100871, China

⁵ RWTH Aachen University, III. Physikalisches Institut B, 52074 Aachen, Germany

Received: 11 August 2022 / Accepted: 30 January 2023 / Published online: 14 February 2023
© The Author(s) 2023

Abstract We discover a correspondence between the free field and the interacting states. This correspondence is firstly given from the fact that the free propagator can be converted into a tower of propagators for massive states, when expanded with the Hermite function basis. The equivalence of propagators reveals that in this particular case the duality can naturally be regarded as the equivalence of one theory on the plane wave basis to the other on the Hermite function basis. More generally, the Hermite function basis provides an alternative quantization process with the creation/annihilation operators that correspond directly to the interacting fields. As an illustration, we apply this basis to the $3 + 1$ dimensional Yang–Mills theory, where the three-dimensional space being reduced through the Hermite function basis, and an auxiliary parameter ω denotes for string tension. At large ω limit, with then considering only the lowest order Hermite function (Lowest Landau Level), the equivalent action becomes the Banks–Fischler–Shenker–Susskind (BFSS) matrix model. At small ω limit, the perturbative series summed over all orders of Hermite function gives a massive gluon propagator.

1 Introduction

There have been longstanding efforts for understanding the phenomenon of duality. Generally speaking, duality is a way of showing the correspondence between two apparently different theories. The aspects covered by duality are quite numerous and include the target space duality [1–3], strong–

weak duality [4–6] and fermion–boson duality [7–11], and so on. Especially, a large class of duality has been known as particle–vortex duality, which is the dual of the Higgs model with the XY model [12, 13] for bosonic system, and the dual of the Dirac fermion with the composite one [14–18] for the fermionic case. It was later realised that this type of duality could generally be grouped into the fermion–boson duality by bosonisation [11, 19]. The idea was to attach the flux to fields that had been found to switch the statistical transmutation of particles [20–23]. After attaching the flux, the new state defined by the monopole operator which carries a transmutation different from that of the fundamental field emerges. The underlying concept of these dualities is the relation between the fundamental field and the interacting field associated with the flux.

Inspired by this, we propose a new approach based on Hermite function basis, and discover a closely related correspondence between the free field and the interacting states. This correspondence shows the equivalence of two theories, one of which can be achieved by expanding the other on the Hermite function basis. The Hermite functions are the eigenfunctions of the harmonic oscillator in quantum mechanics, forming a complete orthonormal basis. The formulae of harmonic oscillators or Hermite functions is ubiquitous and can be intuitively interpreted as the basis of interacting field. Based on these observations, one may expand the free field on account of the Hermite function basis being orthonormal, and then the resulting theory is supposedly converted into the interacting field picture. It becomes clear in the canonical quantization procedure, whereas the field holds the same canonical commutation relation, on the Hermite function basis, its respective creation/annihilation operators now directly describe the harmonic type interacting particles. By applying this, we study the correspondence for both fermion

^a e-mail: hiei@pku.edu.cn

^b e-mail: m.ding@hzdr.de (corresponding author)

^c e-mail: yxliu@pku.edu.cn

^d e-mail: s.schmidt@hzdr.de

and boson fields. We hope this will shed some light on the understanding of the phenomenon of duality.

At last, we illustrate the application of this approach on the 3 + 1 dimensional Yang–Mills theory. After reducing the three dimensional space with the Hermite function basis and constraining to the lowest order Hermite function (Lowest Landau Level), the resulting action becomes the action of the Banks–Fischler–Shenker–Susskind (BFSS) Matrix model derived through dimensional reduction [24–27]. Naturally, this approach offers an exact way of dimensional reduction without requiring compactification of the space [28–30]. Besides, the perturbative computation through the resulting theory is shown to be able to generate a dynamical mass scale of gluon, which implies the relation between the non perturbative property and the interacting states by the Hermite function basis.

The article is organised as follows. In Sect. 2 we start with the equation of motion for fermion/boson, and derive the respective propagator on the Hermite function basis. Then, in Sect. 3 we lead up to an alternative quantization process and observe a duality relation in the action of the field. In Sect. 4 we illustrate the application on the 3 + 1 dimensional Yang–Mills theory. In Sect. 5 we summarise our approach and present our conclusions.

2 Equation of motion in the Hermite function basis

We start our discussion with the equation of motion for fermions, i.e. the Dyson–Schwinger equation (DSE) for the fermion propagator. The DSE for the propagator of the fermion interacting with the gauge field is generally written as:

$$\begin{aligned}
 &(-\partial_a \gamma_z + i \not{p} + m_0)S(\bar{p}; a - b) \\
 &= \delta(a - b) + g^2 \int dk_z d^3 \bar{q} dc \\
 &\quad \times \gamma_\mu S(\bar{q}; a - c) \Gamma_\nu D^{\mu\nu}(\bar{p} - \bar{q}, k_z) \\
 &\quad e^{-ik_z(a-b)} S(\bar{p}; c - b),
 \end{aligned} \tag{1}$$

with S the fermion propagator; $D^{\mu\nu}$ the gauge boson propagator; Γ_μ the full interaction vertex with tree level as γ_μ ; g the running coupling; δ the Dirac delta function and the metric being set as $(-1, 1, 1, 1)$. Here the DSE is in a mixed representation with momentum representation for $\bar{p} = (p_t, p_x, p_y, 0)$ and coordinate representation for z -axis (a and b are on the z -axis in the coordinate space). The integrated momentum are $\bar{q} = (q_t, q_x, q_y, 0)$, and $k_z = p_z - q_z$, where p_z, q_z and k_z are on the z -axis in the momentum space. If the Fourier transform is also applied to the z -axis, the usual DSE in momentum representation can be obtained [31–40]. Here we are taking the case where the z -direction is in coordinate space as an example, and the approach is applicable

and can be extended to other cases where other directions (t, x, y) are in coordinate space.

The equation for the free fermion propagator of zero coupling ($g = 0$) is then given by:

$$(-\partial_a \gamma_z + i \not{p} + m_0)S(\bar{p}; a - b) = \delta(a - b). \tag{2}$$

With the momentum representation, the free fermion propagator can be obtained as:

$$S(\bar{p}; a - b) = \int dp_z \frac{e^{-ip_z(a-b)}}{i \not{p}_z + i \not{p} + m_0}, \tag{3}$$

which may be formally regarded as being expanded with the plane wave basis. Here instead, as we described in preceding section, the Hermite functions also form a complete orthonormal basis, and we therefore use the Hermite function basis to expand the fermion propagator on the z -axis, i.e.,

$$\begin{aligned}
 S(\bar{p}; a - b) &= \sum_n S_n(\bar{p}) f_n(a - b) \\
 &= \sum_n \left[-i\sigma_n^A(\bar{p}^2) \not{p} + \sigma_n^B(\bar{p}^2) - i\sigma_n^C(\bar{p}^2) \gamma_z \right] f_n(a - b),
 \end{aligned} \tag{4}$$

where the general form of \bar{p} -dependent scalar functions are denoted by $\sigma_n^A(\bar{p}^2)$, $\sigma_n^B(\bar{p}^2)$, and $\sigma_n^C(\bar{p}^2)$ respectively; $f_n(z)$ with $n \geq 0$ is the Hermite function (sometimes called Hermite-Gaussian function) as:

$$f_n(z) = \frac{\omega^{1/2}}{\sqrt{2^n n!} \sqrt{\pi}} H_n(\omega z) e^{-\omega^2 z^2 / 2}, \tag{5}$$

and $H_n(z)$ is the Hermite polynomial and the orthogonal normalisation condition is:

$$\int dz f_n(z) f_m(z) = \delta_{nm}. \tag{6}$$

The Hermite function, which gives rise to the wave function of the energy eigenstate of the quantum harmonic oscillator, when operated by the creation and annihilation operators ($a^\dagger = \frac{\omega^2 z - \partial_z}{\sqrt{2\omega}}$, $a = \frac{\omega^2 z + \partial_z}{\sqrt{2\omega}}$), give:

$$\begin{aligned}
 \frac{\omega^2 z - \partial_z}{\sqrt{2\omega}} f_n(z) &= \sqrt{n+1} f_{n+1}(z), \\
 \frac{\omega^2 z + \partial_z}{\sqrt{2\omega}} f_n(z) &= \sqrt{n} f_{n-1}(z).
 \end{aligned} \tag{7}$$

It should be mentioned that in Eq. (4) there exists in principle $\gamma_z \not{p}$ term, but this can be excluded by comparing it with the momentum representation of the fermion propagator in Eq. (3).

By now we have expanded the fermion propagator with the Hermite function basis on the z -axis in the coordinate space. Actually this Hermite function basis expansion method has been broadly used in the studies of continuum Schwinger method with a constant background magnetic field, namely

the Ritus formula [41–43]. The order of the Hermite function then naturally defines the Landau Level. In a constant magnetic field, the Hermite function basis can be closed to four terms, including only f_n, f_{n-1} , and f_{n+1} . Here we generalize this method by inserting an auxiliary field $A_z(a) = i\omega^2 a$ with string tension ω^2 , and decomposing the operator on the left hand side of Eq. (2) as:

$$-\partial_a \gamma_z + i \not{p} + m_0 = -\frac{1}{2}(\partial_a + \omega^2 a)\gamma_z - \frac{1}{2}(\partial_a - \omega^2 a)\gamma_z + i \not{p} + m_0. \tag{8}$$

This decomposition involves only the differential operator and does not require the background field. The Hermite function basis cannot now be closed within finite terms, thus it requires to use infinite dimensional matrices defined as linear operators for the functional analysis. To make this point clearer, one may first insert the fermion propagator expansion expression in Eq. (4) into the equation it satisfies, i.e., DSE in Eq. (2), and apply the property of Hermite function in Eq. (7) and the decomposition in Eq. (8) to obtain:

$$\sum_{m'} \{ (i \not{p} + m_0) [-i\sigma_{m'}^A(\bar{p}^2) \not{p} + \sigma_{m'}^B(\bar{p}^2) - i\sigma_{m'}^C(\bar{p}^2)\gamma_z] \times f_{m'}(a-b) - \frac{\gamma_z}{2} \sqrt{2m'} \omega [-i\sigma_{m'}^A(\bar{p}^2) \not{p} + \sigma_{m'}^B(\bar{p}^2) - i\sigma_{m'}^C(\bar{p}^2)\gamma_z] f_{m'-1}(a-b) + \frac{\gamma_z}{2} \sqrt{2(m'+1)} \omega \times [-i\sigma_{m'}^A(\bar{p}^2) \not{p} + \sigma_{m'}^B(\bar{p}^2) - i\sigma_{m'}^C(\bar{p}^2)\gamma_z] f_{m'+1}(a-b) \} = \delta(a-b). \tag{9}$$

Multiplying the resulting Eq. (9) by the integral $\int da f_m(a-b)$, and applying the orthogonal normalisation condition in Eq. (6), one get that all the Hermite functions turn to the Dirac delta functions, which may be combined with the index m' in the scalar functions $\sigma_{m'}^{A,B,C}(\bar{p}^2)$. Then comparing the corresponding Dirac terms on both sides of the equation, one can directly obtain (We drop the explicit \bar{p} index in the scalar functions $\sigma_m^{A,B,C}$ and it is included implicitly):

$$f_m(0) = \bar{p}^2 \sigma_m^A + m_0 \sigma_m^B + i \sqrt{\frac{m+1}{2}} \omega \sigma_{m+1}^C - i \sqrt{\frac{m}{2}} \omega \sigma_{m-1}^C, \tag{10a}$$

$$0 = m_0 \sigma_m^A - \sigma_m^B, \tag{10b}$$

$$0 = -i \sqrt{\frac{m+1}{2}} \omega \sigma_{m+1}^A + i \sqrt{\frac{m}{2}} \omega \sigma_{m-1}^A + \sigma_m^C. \tag{10c}$$

Let's now define an operator \hat{T} as:

$$\hat{T} = T_{mm'} = i \sqrt{\frac{m'}{2}} \omega \delta^{m,m'-1} - i \sqrt{\frac{m}{2}} \omega \delta^{m,m'+1}, \tag{11}$$

which acts on the Hermite function basis space. One can represent it in the infinite dimensional matrix form as:

$$\begin{bmatrix} 0 & \frac{i\omega}{\sqrt{2}} & 0 & 0 & 0 & 0 & \\ -\frac{i\omega}{\sqrt{2}} & 0 & i\omega & 0 & 0 & 0 & \\ 0 & -i\omega & 0 & i\sqrt{\frac{3}{2}}\omega & 0 & 0 & \\ 0 & 0 & -i\sqrt{\frac{3}{2}}\omega & 0 & i\sqrt{2}\omega & 0 & \\ 0 & 0 & 0 & -i\sqrt{2}\omega & 0 & i\sqrt{\frac{5}{2}}\omega & \\ 0 & 0 & 0 & 0 & -i\sqrt{\frac{5}{2}}\omega & 0 & \dots \\ & & & & & & \vdots \\ & & & & & & \vdots \end{bmatrix} \tag{12}$$

The free fermion propagator can then be conveniently expressed as:

$$S(\bar{p}; a-b) = \mathbf{f}(a-b) \frac{-i\hat{T}\gamma_z - i \not{p} + m_0}{\bar{p}^2 + m_0^2 + \hat{T}^2} \mathbf{f}(0), \tag{13}$$

with $\mathbf{f} = \{f_0, f_1, \dots, f_n\}$ the array of Hermite functions in Eq. (5). It follows from Eq. (12) that \hat{T} is a Hermitian operator, so that it can be diagonalised as $\hat{T} = \hat{P}^\dagger \Omega_n \hat{P}$. Interestingly, for even basis, \hat{T} has two sets of eigenvalues that differ only by the sign ± 1 , i.e., $\Omega_n = \pm |\Omega_n|$. For odd basis, there is an additional eigenvalue of zero.

In order to build a relation with the momentum representation of the free fermion propagator, as in Eq. (3), it is convenient to take the Fourier transform of $a-b$ in Eq. (13), i.e. a transformation of the z -axis in coordinate space. The Fourier transform of the Hermite function here is

$$\int da e^{-ip_z(a-b)} f_n(a-b) = \frac{\sqrt{2\pi}(-i)^n}{\omega} f_n\left(\frac{p_z}{\omega^2}\right), \tag{14}$$

and note that the odd Hermite function vanishes at the origin, and therefore for $p_z = 0$ we only need to consider the even Hermite function. We denote $\tilde{\Omega}_n = \Omega_{2n}$ for convenience and obtain:

$$S(\bar{p}, p_z = 0) = \frac{-i \not{p} + m_0}{\bar{p}^2 + m_0^2} = (-i \not{p} + m_0) \frac{\sqrt{2\pi}}{\omega} \sum_n \frac{\hat{\sigma}(\varphi^n)^*(\varphi^n)}{\bar{p}^2 + m_0^2 + \tilde{\Omega}_n^2}, \tag{15}$$

with

$$\varphi^n = \hat{P} f_{2n}\left(\frac{p_z}{\omega^2} = 0\right). \tag{16}$$

The operator $\hat{\sigma}$ acting on the $4n$ and $4n+2$ basis function will give 1 and -1 respectively. Leaving aside the Lorentz structure $-i \not{p} + m_0$, the propagator corresponds to that of the interacting theory, describing a tower of massive states of mass spectrum $\tilde{\Omega}_n$ and wave function φ^n with norm operator $\hat{\sigma} = \pm 1$. In a sense, it corresponds to two towers of massive states with positive and negative norms, that ultimately cancel and maintain a free propagation mode. Here we use the case

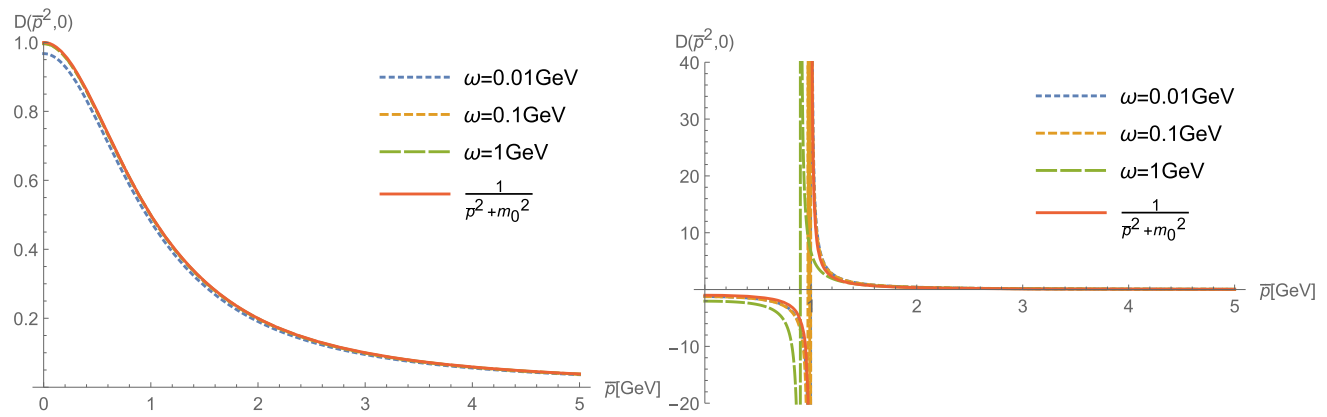


Fig. 1 The boson propagator obtained from the Hermite function basis expansion as in Eq. (20) compared to the free boson propagator $1/(\bar{p}^2 + m_0^2)$ expanded on the plane wave basis, with two different

types of masses, real mass with $m_0 = 1$ GeV (left panel) and purely imaginary mass with $m_0 = i$ GeV (right panel)

$p_z = 0$ as an example, the discussion is applicable and can be extended to other cases where p_z is non-zero.

Our work from the beginning of this section onwards has all been concerned with the fermion field. It gave a general correspondence in Eq. (15) revealing that the fermion propagator of the free field corresponds to that of the interacting field of massive states. To get a complete image we then consider also the correspondence for the bosonic propagators.

Again, we start from the equation for the free boson propagator D of zero coupling given by:

$$(-\partial_a^2 + \bar{p}^2 + m_0^2)D(\bar{p}; a - b) = \delta(a - b). \tag{17}$$

Similarly, we expand the boson propagator using the Hermite function basis as:

$$D(\bar{p}; a - b) = \sum_n D_n(\bar{p}) f_n(a - b), \tag{18}$$

and performing the same procedure as in the fermion case above, we obtain the boson propagator, expressed as:

$$D(\bar{p}; a - b) = \mathbf{f}(a - b) \frac{1}{\bar{p}^2 + m_0^2 + \hat{T}^2} \mathbf{f}(0). \tag{19}$$

Comparing it with the momentum representation of the free boson propagator, we obtain:

$$D(\bar{p}, p_z = 0) = \frac{1}{\bar{p}^2 + m_0^2} = \frac{\sqrt{2\pi}}{\omega} \sum_n \frac{\hat{\sigma}(\varphi^n)^*(\varphi^n)}{\bar{p}^2 + m_0^2 + \tilde{\Omega}_n^2}. \tag{20}$$

This gives an analogous correspondence between the boson propagator of the free field and the interacting field with a tower of massive states. Thus we note that a correspondence also exists in the boson system.

It's worth mentioning that the parameter ω in the Hermite function basis representation is in fact arbitrary, i.e., Eq. (20) can be satisfied for any ω if the infinite order of the

Hermite function is used. This is rather nontrivial, since ω in Eq. (20) is not easy to be fully cancelled. Therefore, we try to verify the correspondence in Eq. (20) numerically with two very different types of propagators, the results of which are shown in Fig. 1. In the left panel of Fig. 1, the original propagator is like a propagator with $m_0 = 1$ GeV, and its pole on the spacelike momentum axis, while in the right panel the propagator has $m_0 = i$ GeV, and its pole on the timelike momentum axis. To begin with, we find that the correspondence is indeed generally independent of the choice of ω . We considered the correspondence under three different ω , namely $\omega = 0.01, 0.1, 1$ GeV, for all of which Eq. (20) is approximately satisfied. There are some deviations due to the truncation of the basis space of Hermite functions. To be specific, for the real mass case, a smaller ω brings more deviation, while for the purely imaginary mass, a smaller ω implies a better correspondence. Additionally, it is interesting to note that the space required for the Hermite function is different in these two cases. Specifically, for $m_0 = 1$ GeV, we apply here the Hermite function $f_n(z)$ of order up to $n = 100$, while for the propagator with a pole on the momentum axis, i.e. $m_0 = i$ GeV, a good correspondence can be achieved with very few orders of the Hermite function for sufficiently small ω . For example, at $\omega = 0.01$ GeV, we apply the first six orders of the Hermite function as $f_{0,1,\dots,5}(z)$. This indicates that the Hermite function basis might be useful for nonperturbative studies towards the timelike momentum region, such as the spectrum of states or the transport properties of the system.

3 Canonical quantization on Hermite function

We further investigate the Hermite function basis more generally by the action. Clearly, an alternative typical canonical

quantization based on the Hermite function basis can be constructed. Let's begin with the free fermion field. The basic idea is to expand the $d + 1$ dimensional free fermion field ψ as:

$$\psi(t, x_1, \dots, x_p, x_{p+1}, \dots, x_d) = \sum_{n_1, \dots, n_p} \Psi_{n_1, \dots, n_p}(t, x_{p+1}, \dots, x_d) [\hat{P}f]_{n_1}(x_1) \cdots [\hat{P}f]_{n_p}(x_p). \tag{21}$$

\hat{P} , as described above, is the matrix acting on the Hermite function basis space to diagonalize the operator \hat{T} defined in Eq. (12). Ψ is a new field, the part that has not been expanded. Here we expand the fermion field in the direction n_1 to n_p , this is used as an example, in fact it can be extended to the expansion in other directions as well. The canonical anti-commutation relation of the original free fermion field ψ is:

$$\{\psi(t, x_1, \dots, x_d), \psi^\dagger(t, y_1, \dots, y_d)\} = \delta^d(x - y). \tag{22}$$

We can set the new field Ψ to meet the following constraint:

$$\{\Psi_{n_1, \dots, n_p}(t, x_{p+1}, \dots, x_d), \Psi_{m_1, \dots, m_p}^\dagger(t, y_{p+1}, \dots, y_d)\} = \delta^{n_1, m_1} \dots \delta^{n_p, m_p} \delta(x_{p+1} - y_{p+1}) \cdots \delta(x_d - y_d), \tag{23}$$

and then use the properties of the Hermite function $\sum_n f_n(x) f_n(y) = \delta(x - y)$ and the matrix used for diagonalisation $\hat{P} \hat{P}^\dagger = \mathcal{I}$, the anti-commutation relation in Eq. (22) can be reproduced. Comparing Eq. (23) with Eq. (22), we see that the new field holds a similar canonical anti-commutation relation as that of the original free fermion field, hence they must be connected in some way, at least this suggests that the new field is also a fermion field.

Let's now consider the action of the $d + 1$ dimensional free fermion field:

$$\mathcal{S} = \int dt d^d x [-\bar{\psi}(x) \not{\partial} \psi(x) + m_0 \bar{\psi}(x) \psi(x)]. \tag{24}$$

Substituting Eq. (21) in Eq. (24), the action is converted to:

$$\mathcal{S} = \int dt d^{d-p} x \sum_n [-\bar{\Psi}_n(x) \not{\partial} \Psi_n(x) - i \bar{\Psi}_n(x) \tilde{\gamma} \cdot \tilde{A}(n) \Psi_n(x) + m_0 \bar{\Psi}_n(x) \Psi_n(x)], \tag{25}$$

where n denotes for n_1, \dots, n_p ; $\not{\partial}$ and x are now used only for coordinates x_{p+1}, \dots, x_d ; $\tilde{\gamma}_\mu$ is the gamma matrix associated with coordinates x_1, \dots, x_p . In the derivation, the p dimensional $\hat{\partial}$ left acting on the Hermite function basis \mathbf{f} transforms into $i\hat{T}$. With the integral $\int d^p x$ and the orthogonal normalization condition of Hermite function, the operator becomes $\hat{P} \hat{T} \hat{P}^\dagger = \Omega_n$. The action in Eq. (25) becomes that of an interacting field theory of the fermion field Ψ in $1 + d - p$ dimensions, which interacts with a discretized gauge field in the rest p dimensions, i.e., $\tilde{A}(n) = \Omega_{n_1, \dots, n_p}$. This shows a

duality relation between a $1 + d$ dimensional free fermion field and a $1 + d - p$ dimensional interacting fermion field with interactions coming from the rest p dimensions.

One can also consider the case of $1 + d$ dimensional complex scalar field. Similarly, we expand the complex scalar field ϕ as follows:

$$\phi(t, x_1, \dots, x_p, x_{p+1}, \dots, x_d) = \sum_{n_1, \dots, n_p} \Phi_{n_1, \dots, n_p}(t, x_{p+1}, \dots, x_d) [\hat{P}f]_{n_1}(x_1) \cdots [\hat{P}f]_{n_p}(x_p). \tag{26}$$

Φ is a new field, the part that has not been expanded. Performing the same procedure as in the fermion case above, we obtain the commutation relation for the new field Φ as:

$$\begin{aligned} & [\Phi_{n_1, \dots, n_p}(t, x_{p+1}, \dots, x_d), \dot{\Phi}_{m_1, \dots, m_p}^*(t, y_{p+1}, \dots, y_d)] \\ & = i \delta^{n_1, m_1} \dots \delta^{n_p, m_p} \delta(x_{p+1} - y_{p+1}) \cdots \delta(x_d - y_d), \end{aligned} \tag{27}$$

with $\dot{\Phi}$ denoting the time derivative of Φ . It is easy to check that the original field satisfies: $[\phi(t, x), \dot{\phi}^*(t, y)] = i \delta^d(x - y)$, the commutation relation for a free complex scalar field can therefore be reproduced. Equation (27) suggests that the new field Φ is also a complex scalar field.

Let's move on to consider the action of the $d + 1$ dimensional free complex scalar field:

$$\mathcal{S} = \int dt d^d x [\partial_\mu \phi^*(x) \partial^\mu \phi(x) + m_0^2 \phi^*(x) \phi(x)]. \tag{28}$$

By inserting Eq. (26) into the Eq. (28), the action becomes $1 + d - p$ dimensional as:

$$\begin{aligned} \mathcal{S} = \int dt d^{d-p} x \sum_n & [\partial_\mu \Phi_n^*(x) \partial^\mu \Phi_n(x) \\ & + M_n^2 \Phi_n^*(x) \Phi_n(x) + m_0^2 \Phi_n^*(x) \Phi_n(x)], \end{aligned} \tag{29}$$

again, n denotes for n_1, \dots, n_p ; ∂ and x are now used only for coordinates x_{p+1}, \dots, x_d . The $1 + d$ dimensional scalar field then becomes a $1 + d - p$ dimensional scalar field interacting with a tower of massive states with $M_n^2 = \Omega_{n_1}^2 + \dots + \Omega_{n_p}^2$.

4 Yang–Mills field on Hermite function basis and matrix model

When applying the Hermite function basis to the Yang–Mills field, it eventually gives an action of the BFSS matrix model. Firstly, we write here the Yang–Mills action as [32]:

$$\mathcal{S} = -\frac{1}{4} \int d^4 x F_{\mu\nu}^a(x) F^{a, \mu\nu}(x), \tag{30}$$

where the field strength tensor $F_{\mu\nu}^a$ defined as: $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$, with g the coupling constant;

A_μ^a the bosonic gauge field; f^{abc} the structure constant of the gauge group. Now we can expand $A_\mu^a(x)$ for all three spatial coordinates as:

$$\begin{aligned} A^a(t, \mathbf{x}) &= A^{a,n}(t)[\hat{P}f]_{n_1}(x_1)[\hat{P}f]_{n_2}(x_2)[\hat{P}f]_{n_3}(x_3), \\ X_j^a(t, \mathbf{x}) &= X_j^{a,n}(t)[\hat{P}f]_{n_1}(x_1)[\hat{P}f]_{n_2}(x_2)[\hat{P}f]_{n_3}(x_3), \end{aligned} \tag{31}$$

with $n = (n_1, n_2, n_3)$, $A^a = A_0^a$ and $X_j^a = A_{j=1,2,3}^a$. Now inserting Eq. (31) into Eq. (30), the action in Eq. (30) becomes:

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{int} + \mathcal{S}_{im}, \tag{32}$$

where

$$\begin{aligned} \mathcal{S}_0 &= -\frac{1}{2} \int dt \left(|\dot{X}_j^{a,n}|^2 + \Omega_{n_j}^2 |A^{a,n}|^2 \right. \\ &\quad \left. + \frac{1}{2} |\Omega_{n_{j'}} X_j^{a,n} - \Omega_{n_j} X_{j'}^{a,n}|^2 \right), \end{aligned} \tag{33a}$$

$$\begin{aligned} \mathcal{S}_{int} &= -\frac{1}{2} \int dt \left(-2g\tau^{nml} f^{abc} \dot{X}_j^{a,n} X_j^{c,m} A^{b,l} \right. \\ &\quad \left. + g^2 f^{abc} f^{ab'c'} \lambda^{nmlk} A^{b,n} X_j^{c,m} A^{b',l} X_j^{c',k} \right. \\ &\quad \left. + \frac{1}{2} g^2 f^{abc} f^{ab'c'} \lambda^{nmlk} X_j^{b,n} X_{j'}^{c,m} X_j^{b',l} X_{j'}^{c',k} \right), \end{aligned} \tag{33b}$$

$$\begin{aligned} \mathcal{S}_{im} &= i \frac{1}{2} \int dt \left[\Omega_{n_j} A^{a,n} (\dot{X}_j^{a,n,*} - g f^{abc} A^{b,m} X_j^{c,l} \tau^{nml}) \right. \\ &\quad \left. + g f^{abc} \tau^{nml} (\Omega_{n_{j'}} X_j^{a,n} - \Omega_{n_j} X_{j'}^{a,n}) X_{j'}^{b,m} X_j^{c,l} \right], \end{aligned} \tag{33c}$$

with summation of indices a, b, c, j', j and n, m, l, k the order of Hermite function, and also

$$\begin{aligned} \tau^{nml} &= \prod_{i=1,2,3} \int d^3x [\hat{P}f]_{n_i}(x_i) [\hat{P}f]_{m_i}(x_i) [\hat{P}f]_{l_i}(x_i), \\ \lambda^{nmlk} &= \prod_{i=1,2,3} \int d^3x [\hat{P}f]_{n_i}(x_i) [\hat{P}f]_{m_i}(x_i) [\hat{P}f]_{l_i}(x_i) \\ &\quad \times [\hat{P}f]_{k_i}(x_i). \end{aligned} \tag{34}$$

One can see that the procedure defines an exact way of dimensional reduction which leads directly from the 3 + 1-dimensional Yang Mills theory to a one-dimensional theory, namely Eq. (32). Specifically, the free field with $g = 0$ becomes a tower of massive gauge fields, similar to the case in the previous section.

Now for the full theory, the arbitrary parameter ω is included in the interaction term through the coefficients τ and λ , and the theory is in principle independent of ω . However, the convergence domains of the expansion are rather different for different choices of ω . This can be viewed by treating the Lagrangian in Hermite function basis as a new Lagrangian in plane wave basis. One may forsake the Hermite function basis and consider \mathcal{S}_0 as the kinetic term of a theory with n particles of mass Ω_n . The couplings of the new

Lagrangian are then $g\tau^{nml}$ and $g^2\lambda^{nmlk}$, proportional to $\omega^{3/2}$ and ω^3 , respectively. Consequently, the convergence domains of the expansion in the large- ω limit and the small- ω limit are different. On the one hand, the large- ω limit makes the Lowest Landau Level approximation valid, which then leads to the BFSS matrix model. This shows a strong-weak duality between the Yang–Mills theory and the matrix model. On the other hand, by adjusting ω to be small, one can apply perturbative calculations even if the original coupling g is strong. It should also be mentioned that for the new Lagrangian the fundamental picture of the matter field is different, because the free field of the new Lagrangian is actually the interaction field in the old Lagrangian. In particular, the mass term in the free field can only be generated dynamically in the old Lagrangian by nonperturbatively considering the interaction terms of all orders. One may consider the Hermite function basis as a strong coupling expansion in contrast to the weak coupling perturbation with plane wave basis.

4.1 Large- ω limit

For large- ω limit, one can apply the Lowest Landau Level approximation as in a strong magnetic field, where the theory can be described by only the lowest order Hermite function $n_i, m_i, l_i, k_i = 0$, and then $\Omega_{n_j} = 0$ correspondingly. With this approximation, one can greatly reduce the action as:

$$\begin{aligned} \mathcal{S} &= -\frac{1}{2} \int dt \left((\dot{X}_j^a)^2 - \frac{2g\omega^{3/2}}{(\frac{9}{4}\pi)^{3/4}} f^{abc} \dot{X}_j^a X_j^c A^b \right. \\ &\quad \left. + \frac{g^2\omega^3}{(2\pi)^{3/2}} [A, X_j]^2 + \frac{g^2\omega^3}{2(2\pi)^{3/2}} [X_{j'}, X_j]^2 \right), \end{aligned} \tag{35}$$

with X_j^a, A^b standing for the lowest order of the gauge field, and $[X_i, X_j] = f^{abc} X_i^b X_j^c$. Now if setting $g' = g / (\frac{9}{4}\pi)^{3/4}$ with rescaling the fields as $(X_j^a, A^b) \rightarrow \omega^{-1/2}(X_j^a, A^b)$, and $DX_j^a = \dot{X}_j^a - g'\omega [A, X_j]$, one can rewrite the action as:

$$\begin{aligned} \mathcal{S} &= -\frac{1}{2} \int dt \left(\frac{(DX_j^a)^2}{\omega} + \frac{\omega}{2} (9/8)^{3/2} g'^2 [X_{j'}, X_j]^2 \right. \\ &\quad \left. + [(9/8)^{3/2} - 1] \omega g'^2 [A, X_j]^2 \right), \end{aligned} \tag{36}$$

the first line is just the bosonic part of the BFSS matrix model, and the second term is an additional term. Note that in the original BFSS model, the dimensional reduction is based on the compactification of the space, whereas here the dimensional reduction procedure in Eq. (32) is exact, and Eq. (36) is the Lowest Landau Level, which is valid in the strong limit.

4.2 Small- ω limit

Now one can also take the small ω limit and then the perturbative calculations can be applied. Note that $A^{a,n}$ has no canon-

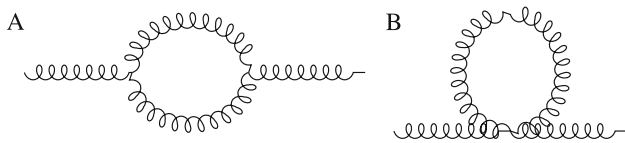


Fig. 2 One-loop self-energy Feynman diagrams of the gluon propagator, Π_A^{im} and Π_B^{im}

ical momentum and hence one may impose the Weyl gauge fixing condition, $A^{a,n} = 0$. The Weyl gauge fixing is incomplete and requires additional constraints. This incompleteness is straightforwardly depicted in the new action Eq. (32), in terms of the cross term between the two gauge fields, $X_j^{a,n}$ and $X_j^{a,n'}$. Therefore, one may simply neglect such cross terms as a gauge fixing condition. The resulting gluon propagator is then similar to that in the Feynman gauge, which reads:

$$D_{ii'}^{abnm}(p^2) = \delta^{ii'} \delta^{nm} \delta^{ab} D^{im}(p^2), \tag{37}$$

with $D^{im}(p^2) = p^2 + \sum_{j \neq i} \Omega_{n_j}^2 + \Pi^{im}(p^2)$.

Now one can compute the one-loop self-energy of the gluon propagator, which contains two contributions, i.e.

$$\begin{aligned} \Pi_A^{im}(p^2) &= \frac{g^2 N_c}{2} \sum_{n,l}^{N_{\max}} \tau^{lnm} \tau^{l'n^*m^*} \int \frac{dq}{2\pi} \sum_{j=k \neq i} \\ &\times [(\Omega_{n_j} - \Omega_{m_j})^2 D_0^{jn}(p-q) D_0^{il}(q) \\ &+ (\Omega_{m_k} - \Omega_{l_k})^2 D_0^{kn}(p-q) D_0^{il}(q) \\ &+ (\Omega_{l_k} - \Omega_{n_j})^2 D_0^{jn}(p-q) D_0^{kl}(q)], \end{aligned} \tag{38}$$

$$\begin{aligned} \Pi_B^{im}(p^2) &= g^2 N_c \sum_n^{N_{\max}} \lambda^{n^*nm^*m} \int \frac{dq}{2\pi} \sum_{j \neq i} D_0^{jn}(q^2) \\ &= \frac{g^2 N_c}{2} \sum_n^{N_{\max}} \sum_{j \neq i} \frac{\lambda^{n^*nm^*m}}{(\Omega_{n_j}^2 + \Omega_{n_i}^2)^{1/2}}, \end{aligned} \tag{39}$$

whose diagrammatic representations are given in Fig. 2. Here, $D_0^{im}(p^2) = p^2 + \sum_{j \neq i} \Omega_{n_j}^2$ is the bare gluon function. N_{\max} is the truncated order of Hermite function basis and index n^* means the complex conjugate of the respective function. Having obtained D^{im} , one may construct the gluon propagator in momentum representation with Eq. (31) and Eq. (20), and thus one gets:

$$\begin{aligned} D_{i,i'}(p^2) &= \delta^{ii'} D(p^2) \\ &= \delta^{ii'} \left(\frac{\sqrt{2\pi}}{\omega} \right)^3 \hat{\sigma}[\hat{P}f]_m^\dagger(0) D^{im}(p^2) [\hat{P}f]_m(0). \end{aligned} \tag{40}$$

Here, for simplicity, we choose only the temporal component of p^2 , which makes no difference due to Lorentz invariance.

First of all, we depict the gluon propagator with setting the coupling $g = 0$ in Fig. 3. The obtained propagator is equiva-

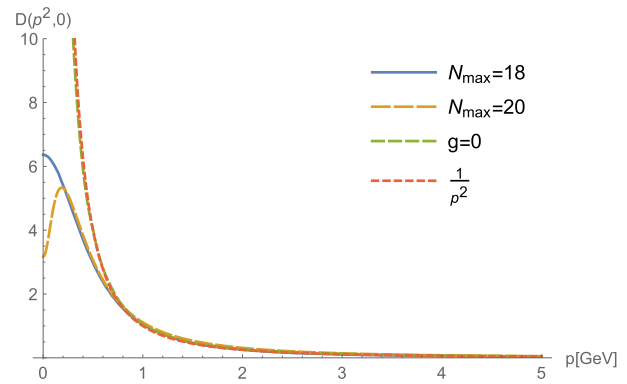


Fig. 3 Two types of gluon propagators with the Hermite functions of truncation order $N_{\max} = 18$ and $N_{\max} = 20$, together with the free field propagator. The gluon propagator with same parametrization as $N_{\max} = 20$ but with zero coupling $g = 0$ is also depicted, which coincides with the free field propagator

lent to the free field propagator, which shows the validity of the method. It is then interesting to note that the one-loop self-energy term leads directly to the dynamical mass generation in the gluon propagator, which can only be obtained non-perturbatively in the plane wave expansion scheme. Slightly surprisingly, in our results, two types of solutions exist for the gluon propagator, depending on the maximum order of the summed Hermite function, i.e. N_{\max} . As shown in Fig. 3, if the Hermite function basis is summed to the order of $4n$, the gluon propagator becomes non-monotonic in the infrared region and drops after a maximum value as it enters the deep infrared region. When summing the series to the order of $4n + 2$, we obtain a monotonic function for the gluon propagator. Now one may recall that it has been argued that there exist two types of gluon propagators, which differ in the infrared, i.e. for $p^2 \lesssim 1$ GeV ("scaling" [39,44,45], versus "decoupling" or "massive" [46–48]; for related discussions, see, e.g. [49–52]). The solution where the Hermite function basis is truncated to $4n + 2$ can immediately be regarded as the decoupling solution, whereas the non-monotonicity of the solution from truncation to $4n$ suggests a scaling solution. It should be mentioned that here we do not get a fully scaling solution, as the solution does not vanish at zero momentum. However, we did find two distinct solutions, as for each case the solution converges quickly. In Fig. 4, we can see that for solutions added to order $4n$, the solution converges at $N_{\max} \sim 20$. Similarly, as shown in Fig. 5, for solutions added to order $4n + 2$, the solution converges at approximately $N_{\max} \sim 18$. Note that the string tension parameter ω is related to the scale of the theory, and here we set the parameter ω to $\omega = 0.09$ GeV, which locates the peak of the scaling solution at $p \sim 0.3$ GeV as in the other non-perturbative studies.

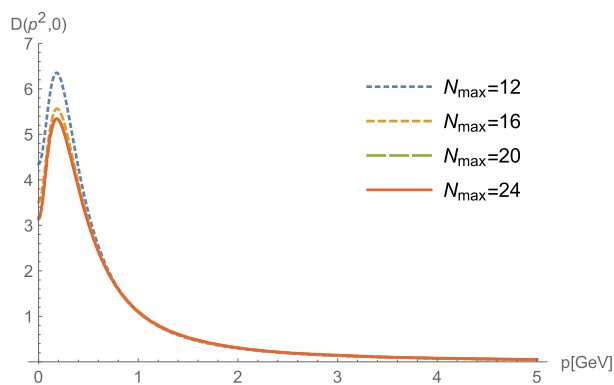


Fig. 4 The gluon propagator with the Hermite functions of truncation order $N_{\max} = 4n$

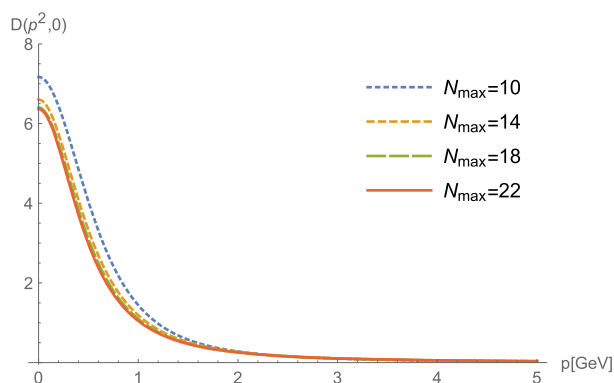


Fig. 5 The gluon propagator with the Hermite functions of truncation order $N_{\max} = 4n + 2$

5 Summary

We depict a novel duality phenomenon through the Hermite function basis. The main idea is to apply the Hermite function basis instead of the plane wave function basis for the expansion. By doing so, the free propagator becomes a tower of propagators with additional mass terms. Since the Hermite function naturally defines the basis of the interacting fields, we construct an equivalent quantization procedure on this basis. In detail, we find that the action of a $1 + d$ dimensional free fermionic field is dual to the action of a tower of $1 + d - p$ dimensional fermionic fields Ψ_n coupled to the constant gauge field $\tilde{A}(n)$ in the rest p dimensions. Similarly, for the scalar field, a $1 + d$ dimensional free scalar field is dual to a tower of $1 + d - p$ dimensional massive scalar fields with additional mass terms coming from the p dimensions. In the sense of duality, here one theory is achieved by expanding the other theory on the Hermite function basis.

This approach can be broadly applied to the study of duality phenomena. Here we apply it in particular to the $3 + 1$ -dimensional Yang–Mills theory. After applying the Hermite function to reduce the three spatial dimensions, an exact form of the corresponding one-dimensional theory can be

obtained. The action is naturally discretized and thus presumably suitable for lattice simulation. For large ω limit, the Hermite function of the lowest order (Lowest Landau Level) is applied, the resulting action is found to become the BFSS matrix model. For small ω limit, the one-loop computation of self-energy of the gluon propagator directly generates a dynamical mass scale, which can only be obtained non-perturbatively in the plane wave basis. Therefore, the Hermite function basis can be helpful in studying non-perturbative effects.

On the practical side, the Hermite function basis provides a possible way of approaching timelike momentum region non-perturbatively. Non-perturbative calculations are usually performed in spacelike momentum region, where the information in timelike region, and in particular the analytic properties of the non-perturbative propagators on plane wave basis, becomes very complicated. The Hermite function basis shows the ability to deal with different kinds of propagators, and therefore through the basis, the information in timelike region can potentially be accessed directly. The momentum representation is also readily accessible as the Hermite function is the eigenfunction of the Fourier transform.

Acknowledgements M. Ding is grateful for support by Helmholtz-Zentrum Dresden-Rossendorf High Potential Programme. Y. Liu is supported by the National Natural Science Foundation of China under Grant Nos. 11175004 and 12175007.

Data Availability Statement The manuscript has associated data in a data repository. [Authors' comment: All data generated during this study are represented in the published article.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Funded by SCOAP³. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

1. A. Giveon, M. Porrati, E. Rabinovici, Phys. Rept. **244**, 77 (1994). [https://doi.org/10.1016/0370-1573\(94\)90070-1](https://doi.org/10.1016/0370-1573(94)90070-1)
2. E. Alvarez, L. Alvarez-Gaume, Y. Lozano, Nucl. Phys. B, Proc. Suppl. **4**(1), 1 (1995). [https://doi.org/10.1016/0920-5632\(95\)00429-D](https://doi.org/10.1016/0920-5632(95)00429-D)
3. J.A. Harvey, G.W. Moore, A. Strominger, Phys. Rev. D **5**(2), 7161 (1995). <https://doi.org/10.1103/PhysRevD.52.7161>

4. C. Montonen, D.I. Olive, *Phys. Lett. B* **7**(2), 117 (1977). [https://doi.org/10.1016/0370-2693\(77\)90076-4](https://doi.org/10.1016/0370-2693(77)90076-4)
5. A. Sen, *Int. J. Mod. Phys. A* **9**, 3707 (1994). <https://doi.org/10.1142/S0217751X94001497>
6. E. Alvarez, M.A.R. Osorio, *Phys. Rev. D* **4**, 1150 (1989). <https://doi.org/10.1103/PhysRevD.40.1150>
7. S.R. Coleman, *Phys. Rev. D* **1**(1), 2088 (1975). <https://doi.org/10.1103/PhysRevD.11.2088>
8. S. Mandelstam, *Phys. Rev. D* **1**(1), 3026 (1975). <https://doi.org/10.1103/PhysRevD.11.3026>
9. E.H. Fradkin, F.A. Schaposnik, *Phys. Lett. B* **338**, 253 (1994). [https://doi.org/10.1016/0370-2693\(94\)91374-9](https://doi.org/10.1016/0370-2693(94)91374-9)
10. O. Aharony, *JHEP* (2), 093 (2016). [https://doi.org/10.1007/JHEP02\(2016\)093](https://doi.org/10.1007/JHEP02(2016)093)
11. A. Karch, D. Tong, *Phys. Rev. X* **6**(3), 031043 (2016). <https://doi.org/10.1103/PhysRevX.6.031043>
12. C. Dasgupta, B.I. Halperin, *Phys. Rev. Lett.* **4**(7), 1556 (1981). <https://doi.org/10.1103/PhysRevLett.47.1556>
13. M.E. Peskin, *Ann. Phys.* **113**, 122 (1978). [https://doi.org/10.1016/0003-4916\(78\)90252-X](https://doi.org/10.1016/0003-4916(78)90252-X)
14. D.F. Mross, J. Alicea, O.I. Motrunich, *Phys. Rev. Lett.* **117**(1), 016802 (2016). <https://doi.org/10.1103/PhysRevLett.117.016802>
15. D.T. Son, *Phys. Rev. X* **5**(3), 031027 (2015). <https://doi.org/10.1103/PhysRevX.5.031027>
16. C. Wang, T. Senthil, *Phys. Rev. X* **5**(4), 041031 (2015). <https://doi.org/10.1103/PhysRevX.5.041031>
17. M. Cheng, C. Xu, *Phys. Rev. B* **94**(21), 214415 (2016). <https://doi.org/10.1103/PhysRevB.94.214415>
18. T. Senthil, D.T. Son, C. Wang, C. Xu, *Phys. Rept.* **827**, 1 (2019). <https://doi.org/10.1016/j.physrep.2019.09.001>
19. N. Seiberg, T. Senthil, C. Wang, E. Witten, *Ann. Phys.* **374**, 395 (2016). <https://doi.org/10.1016/j.aop.2016.08.007>
20. F. Wilczek, *Phys. Rev. Lett.* **4**(8), 1144 (1982). <https://doi.org/10.1103/PhysRevLett.48.1144>
21. A.M. Polyakov, *Mod. Phys. Lett. A* **3**, 325 (1988). <https://doi.org/10.1142/S0217732388000398>
22. N. Shaji, R. Shankar, M. Sivakumar, *Mod. Phys. Lett. A* **5**, 593 (1990). <https://doi.org/10.1142/S0217732390000664>
23. S.K. Paul, R. Shankar, M. Sivakumar, *Mod. Phys. Lett. A* **6**, 553 (1991). <https://doi.org/10.1142/S0217732391000579>
24. T. Banks, W. Fischler, S.H. Shenker, L. Susskind, *Phys. Rev. D* **5**(5), 5112 (1997). <https://doi.org/10.1103/PhysRevD.55.5112>
25. A. Bilal, *Fortsch. Phys.* **4**(7), 5 (1999). [https://doi.org/10.1002/\(SICI\)1521-3978\(199901\)47:1/3<5::AID-PROP5>3.0.CO;2-B](https://doi.org/10.1002/(SICI)1521-3978(199901)47:1/3<5::AID-PROP5>3.0.CO;2-B)
26. A. Connes, M.R. Douglas, A.S. Schwarz, *JHEP* (2), 003 (1998). <https://doi.org/10.1088/1126-6708/1998/02/003>
27. G. Bergner, N. Bodendorfer, M. Hanada, E. Rinaldi, A. Schäfer, P. Vranas, *JHEP* (1), 053 (2020). [https://doi.org/10.1007/JHEP01\(2020\)053](https://doi.org/10.1007/JHEP01(2020)053)
28. O. Klein, *Z. Phys.* **3**(7), 895 (1926). <https://doi.org/10.1007/BF01397481>
29. P.G.O. Freund, M.A. Rubin, *Phys. Lett. B* **9**(7), 233 (1980). [https://doi.org/10.1016/0370-2693\(80\)90590-0](https://doi.org/10.1016/0370-2693(80)90590-0)
30. E. Witten, *Phys. Lett. B* **155**, 151 (1985). [https://doi.org/10.1016/0370-2693\(85\)90976-1](https://doi.org/10.1016/0370-2693(85)90976-1)
31. C.D. Roberts, A.G. Williams, *Prog. Part. Nucl. Phys.* **3**(3), 477 (1994). [https://doi.org/10.1016/0146-6410\(94\)90049-3](https://doi.org/10.1016/0146-6410(94)90049-3)
32. R. Alkofer, L. von Smekal, *Phys. Rept.* **353**, 281 (2001). [https://doi.org/10.1016/S0370-1573\(01\)00010-2](https://doi.org/10.1016/S0370-1573(01)00010-2)
33. G. Eichmann, H. Sanchis-Alepuz, R. Williams, R. Alkofer, C.S. Fischer, *Prog. Part. Nucl. Phys.* **9**(1), 1 (2016). <https://doi.org/10.1016/j.pnpnp.2016.07.001>
34. D. Binosi, L. Chang, J. Papavassiliou, C.D. Roberts, *Phys. Lett. B* **742**, 183 (2015). <https://doi.org/10.1016/j.physletb.2015.01.031>
35. R. Williams, C.S. Fischer, W. Heupel, *Phys. Rev. D* **93**(3), 034026 (2016). <https://doi.org/10.1103/PhysRevD.93.034026>
36. A.C. Aguilar, J.C. Cardona, M.N. Ferreira, J. Papavassiliou, *Phys. Rev. D* **98**(1), 014002 (2018). <https://doi.org/10.1103/PhysRevD.98.014002>
37. C. Tang, F. Gao, Y.X. Liu, *Phys. Rev. D* **100**(5), 056001 (2019). <https://doi.org/10.1103/PhysRevD.100.056001>
38. S.X. Qin, C.D. Roberts, *Chin. Phys. Lett.* **38**(7), 071201 (2021). <https://doi.org/10.1088/0256-307X/38/7/071201>
39. F. Gao, J. Papavassiliou, J.M. Pawłowski, *Phys. Rev. D* **103**(9), 094013 (2021). <https://doi.org/10.1103/PhysRevD.103.094013>
40. L. Chang, Y.B. Liu, K. Raya, J. Rodríguez-Quintero, Y.B. Yang, *Phys. Rev. D* **104**(9), 094509 (2021). <https://doi.org/10.1103/PhysRevD.104.094509>
41. V.I. Ritus, *Ann. Phys.* **6**(9), 555 (1972). [https://doi.org/10.1016/0003-4916\(72\)90191-1](https://doi.org/10.1016/0003-4916(72)90191-1)
42. N. Mueller, J.A. Bonnet, C.S. Fischer, *Phys. Rev. D* **8**(9), 094023 (2014). <https://doi.org/10.1103/PhysRevD.89.094023>
43. Z. Xing, J. Chao, L. Chang, Y.x. Liu, (2021)
44. A.K. Cyrol, L. Fister, M. Mitter, J.M. Pawłowski, N. Strodthoff, *Phys. Rev. D* **9**(5), 054005 (2016). <https://doi.org/10.1103/PhysRevD.94.054005>
45. C.S. Fischer, M.Q. Huber, *Phys. Rev. D* **102**(9), 094005 (2020). <https://doi.org/10.1103/PhysRevD.102.094005>
46. P. Boucaud, F. De Soto, K. Raya, J. Rodríguez-Quintero, S. Zafeiropoulos, *Phys. Rev. D* **98**(11), 114515 (2018). <https://doi.org/10.1103/PhysRevD.98.114515>
47. S. Zafeiropoulos, P. Boucaud, F. De Soto, J. Rodríguez-Quintero, J. Segovia, *Phys. Rev. Lett.* **122**(16), 162002 (2019). <https://doi.org/10.1103/PhysRevLett.122.162002>
48. A.C. Aguilar, F. De Soto, M.N. Ferreira, J. Papavassiliou, J. Rodríguez-Quintero, S. Zafeiropoulos, *Eur. Phys. J. C* **8**(2), 154 (2020). <https://doi.org/10.1140/epjc/s10052-020-7741-0>
49. A.C. Aguilar, D. Binosi, J. Papavassiliou, *Phys. Rev. D* **7**(8), 025010 (2008). <https://doi.org/10.1103/PhysRevD.78.025010>
50. M.Q. Huber, *Phys. Rept.* **879**, 1 (2020). <https://doi.org/10.1016/j.physrep.2020.04.004>
51. N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J.M. Pawłowski, M. Tissier, N. Wschebor, *Phys. Rept.* **910**, 1 (2021). <https://doi.org/10.1016/j.physrep.2021.01.001>
52. G. Eichmann, J.M. Pawłowski, J.a.M. Silva, *Phys. Rev. D* **104**(11), 114016 (2021). <https://doi.org/10.1103/PhysRevD.104.114016>