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Calculations of vacuum mean values of spinor field current and energy–momentum tensor in a constant electric background

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Abstract In the framework of strong-field QED with x-steps, we study vacuum mean values of the current density and energy–momentum tensor of the quantized spinor field placed in the so-called L-constant electric background. The latter background can be, for example, understood as the electric field confined between capacitor plates, which are separated by a sufficiently large distance L. First, we reveal peculiarities of nonperturbative calculating of mean values in strong-field QED with x-steps in general and, in the L-constant electric field, in particular. We propose a new renormalization and volume regularization procedures that are adequate for these calculations. We find necessary representations for singular spinor functions in the background under consideration. With their help, we calculate the above mentioned vacuum means. In the obtained expressions, we show how to separate global contributions due to the particle creation and local ones due to the vacuum polarization. We demonstrate how these contributions can be related to the renormalized effective Heisenberg–Euler Lagrangian.

1 Introduction

In QED with strong electric-like external fields (strong-field QED in what follows) there exists the so-called vacuum instability due to the effect of real particle creation from the vacuum caused by the external fields (the so-called Schwinger effect [1]). A number of publications, reviews and books are devoted to this effect itself and to developing different calculation methods in theories with unstable vacuum, see Refs. [2–9] for a review. In strong-field QED, nonperturbative (with respect to strong external fields) methods are well-developed for two classes of external backgrounds, namely for the so-called t-electric potential steps (t-steps) and x-electric potential steps (x-steps). t-steps represent uniform time-dependent external electric fields that are switched on and off at the initial and the final time instants, respectively whereas x-steps represent time-independent external electric fields of constant direction that are concentrated in restricted space areas. The latter fields can also create particles from the vacuum, the Klein paradox is closely related to this process [10–13]. A general nonperturbative formulation of strong-field QED with t-steps was developed many years ago in Refs. [14–18]. The study of particle creation due to the x-steps began early in the framework of relativistic quantum mechanics, see Ref. [19,20] for a review. However, until recently a consistent quantum field theory (QFT) with x-steps has not been completed. Only a short time ago a nonperturbative formulation of strong-field QED with x-steps was developed in Refs. [21,22]. In the framework of strong-field QED with x-steps calculations of particle creation effect were presented in Refs. [23–29]. In both relativistic quantum mechanics and strong-field QED the possibility of nonperturbative calculations is based on the existence of specific exact solutions (in- and out-solutions) of the Dirac equation. In strong-field QED (in explicit in the relativistic quantum mechanics as well), it is assumed that quantum processes under consideration do not affect significantly classical external fields, the back-reaction is supposed to be small. Nevertheless, it is well-understood that, in principle, the back-reaction must be calculated, at any rate, to estimate limits of the applicability of obtained results. It is also clear that the back-reaction may be strong namely for external backgrounds that can violate the vacuum stability. Here we have to say that studying the vacuum instability, one usually calculates the number den-

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sity of particles created from the vacuum. In some cases, this allows one to make phenomenological conclusions about the back-reaction; see, e.g., [30]. However, a complete study of the back-reaction is related to calculating mean values of the current density and the energy-momentum tensor (EMT) of the charge matter field. In strong-field QED with t-steps such a study was performed in Refs. [31–34]. In particular, it was demonstrated that the effect of particle-creation is precisely the main reason for the change of the energy of the matter. Making a comparison between the change of the energy density of the charged matter and the energy density of the external electric field, which is responsible for this change, restrictions on the intensity of an external field and its duration were found, see Ref. [35].

In the present article, in the framework of strong-field QED with x-steps, we study vacuum mean values of the current density and EMT of the Dirac field in a constant external electric field confined between capacitor plates, which are separated by a sufficiently large distance L. In earlier publications such a field is conditionally called L-constant electric field. In the limiting case $L \to \infty$ this field can be considered as a regularization of the constant uniform electric field. In the obtained results, we demonstrate how to separate contributions due to the global effects of particle production from the local effect due to the vacuum polarization.Some relations with the Heisenberg-Euler Lagrangian are established.

The paper is organized as follows. In Sect. 2, we describe peculiarities of calculating mean values in strong-field QED with x-steps and, in particular, in the L-constant electric field, in Sect. 3. In Sect. 4, we refine the volume regularization procedure with respect to the time-independent inner product on the t-constant hyperplane and find necessary representations of singular spinor functions in the electric field under consideration. In Sect. 5, we calculate directly the vacuum mean values of current density and EMT. In the obtained expressions, we separate contributions due to the particle creation and due to the vacuum polarization. We demonstrate how the latter contributions can be derived by the help of the Heisenberg–Euler Lagrangian. In the last Sect. 6, we summarize and discuss the main results. Some useful technical details are placed in the Appendices.

In our consideration, we use the relativistic units $\hbar = c = 1$ in which the fine structure constant is $\alpha = e^2/c\hbar = e^2$.

2 Mean values in strong-field QED with x-steps

We consider quantum and classical fields in d dimensional Minkowski space-time and use coordinates X,

$$X = \left(X^{\mu},\ \mu = 0, 1, \ldots, d-1\right) = (t, \mathbf{r}), \quad X^0 = t,$$
$$\mathbf{r} = \left(X^1, \ldots, X^{d-1}\right), \quad x = X^1.$$

for their parametrization. We assume that the basic Dirac particle is an electron with the mass m and the charge $-e$, $e > 0$, and the positron is its antiparticle. In general, the x-step is given by zero component of electromagnetic potential $A_0(x)$ that depends on the coordinate x. The corresponding electric field $E(x) = -\partial_x A_0(x) > 0$ is directed along the x axis in the positive direction and is confined in the region $S_{\text{int}} = (x_{\text{L}}, x_{\text{R}})$, where $x_{\text{L}} < 0$ and $x_{\text{R}} > 0$. The potential energy of an electron is $U(x) = -eA_0(x)$, and $\partial_x U(x) > 0$ if $x \in S_{\text{int}}$, and is constant outside the region S_{int}, $U(x) = U_{\text{L}}$ if $x < x_{\text{L}}$ and $U(x) = U_{\text{R}}$ if $x > x_{\text{R}}$. The field accelerates the electrons along the axis x in the negative direction and the positrons along the axis x in the positive direction. The x-step can create particles from the vacuum if the magnitude of its potential energy is sufficiently large, $\Delta U > 2m$. Such a x-step is called critical. In Refs. [21,22] it was developed an approach that allows one to calculate nonperturbatively effects of the vacuum instability in the presence of x-steps (the above mentioned in the Introduction strong-field QED with x-steps). It is clear that the process of pair creation is transient. Nevertheless, the condition of the smallness of backreaction shows there is a window in the parameter range of E and a time duration of its existence where the constant field approximation is consistent [35]. Physically, one can believe that the electric field of an x-step may be considered as a part of a time-dependent inhomogeneous electric field $\mathbf{E}_{\text{pristine}}(X)$ directed along the x-direction, which was switched on very fast before a time instant t_{in}, by this time it had time to spread to the whole region S_{int}. Then it was switched off very fast just after a time instant$t_{\text{out}} = t_{\text{in}} + T$. We stress that the field $\mathbf{E}_{\text{pristine}}(X)$ is equal to $E(x)$ from t_{in} to t_{out}, considered in the region S_{int}, acts as a constant field E during the sufficiently large (macroscopic) period of time T,

$$T \gg (eE)^{-1/2} \max\left\{1, m^2/eE\right\}. \tag{1}$$

We note, that there exist time-independent observables in the presence of critical x-steps. The pair-production rate and the flux of created particles are constant during the time T and main contributions to the latter quantities are independent from fast switching-on and -off effects if Eq. (1) holds true. This statement is confirmed by results obtained in considering exactly solvable cases with t-steps [36–39] and by numerical calculations[1]; see, e.g. [30]. Neglecting contributions of the fast switching-on and -off effects, one can use in calculations instead of the true initial and final vacua that existed before the time t_{in} and after the time t_{out} some time-

[1] Note that the pair-production rate per unit volume due to homogeneous fields ($x_{\text{L}} \to -\infty, x_{\text{R}} \to \infty$) of given average intensity is equal to or higher than that for the case of a finite width $x_{\text{R}} - x_{\text{L}}$; see Ref. [23].

independent vacua $|0, \text{in}\rangle$ and $|0, \text{out}\rangle$ respectively, see Refs. [21,22].

In the case of the L-constant electric field we have $E(x) = E$ and $U(x) = eEx$ in the region S_{int} and we choose that $x_{\text{L}} = -L/2$ and $x_{\text{R}} = L/2$. Its magnitude is

$$\Delta U = U_{\text{R}} - U_{\text{L}} = eEL > 0\,.$$

The L-constant field produces constant fluxes of created from the vacuum final particles during the time interval T. These particles created as electron–positron pairs and leave field area S_{int}, wherein electrons are emitted to the region S_{L} on the left of S_{int} and positrons to the region S_{R} on the right of S_{int}. In these regions the created particles have constant velocities in opposite directions, moving away from the area S_{int}. They form constant longitudinal currents and energy fluxes in the regions S_{L} and S_{R}, respectively. Since the time interval T is chosen be macroscopic, one may believe that, measuring characteristics of particles in the regions S_{L} and S_{R}, we are able to evaluate the effect of pair creation in the area S_{int} for the time interval T. As it follows from exact results [23] in the case of the L-constant field with a sufficiently large length L,

$$L \gg (eE)^{-1/2} \max\left\{1, m^2/eE\right\}\,, \tag{2}$$

one can use semiclassical description. That is, L is chosen be macroscopic finite distance. In this description, outside the area S_{int}, polarization effects are absent, therefore, the particles moving away are the final particle that will remain after the field $E_{\text{pristine}}(X)$ is turned off. They are already formed as final particles in the field area. Thus, in the case of the L-constant field with a sufficiently large length L, we are able to measure characteristics of particles in the field area S_{int} on the plane $x = \text{const}$ for the time interval T.

We consider our theory in a large space-time box that has a $d-2$ dimensional spatial volume $V_{\perp}$ of the hypersurface orthogonal to the electric field direction and the time dimension T. From the latter point of view, the vacuum mean values of the operators of physical quantities on the plane $x = \text{const}$ are defined as integrals over the area $V_{\perp}$ of the plane $x = \text{const}$ and the time interval T. Due to the translational invariance of the external field in the S_{int}, all the mean values are proportional to the spatial volume $V_{\perp}$ and the time interval T. In what follows, we consider mean values of the operators $J^{\mu}(x)$ and $T_{\mu\nu}(x)$ with respect to both initial and final vacua,

$$\begin{aligned}
&\left\langle J^{\mu}(x)\right\rangle_{\text{in/out}} = -ie\,\text{tr}\left[\gamma^{\mu} S^{c}_{\text{in/out}}(X, X')\right]|_{X=X'}\,,\\
&\left\langle J^{\mu}(x)\right\rangle^{c} = -ie\,\text{tr}\left[\gamma^{\mu} S^{c}(X, X')\right]|_{X=X'}\,;\\
&\left\langle T_{\mu\nu}(x)\right\rangle_{\text{in/out}} = i\,\text{tr}\left[A_{\mu\nu} S^{c}_{\text{in/out}}(X, X')\right]|_{X=X'}\,,\\
&\left\langle T_{\mu\nu}(x)\right\rangle^{c} = i\,\text{tr}\left[A_{\mu\nu} S^{c}(X, X')\right]|_{X=X'}\,,
\end{aligned}$$

$$A_{\mu\nu} = \frac{1}{4}\left[\gamma_{\mu}(P_{\nu} + P'^{*}_{\nu}) + \gamma_{\nu}\left(P_{\mu} + P'^{*}_{\mu}\right)\right]\,, \tag{3}$$

where γ^{μ} are γ-matrices in d dimensions,

$$\left[\gamma^{\mu}, \gamma^{\nu}\right]_{+} = 2\eta^{\mu\nu},\quad \eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1)\,.$$

In Eq. (3) there appear the generalized causal in–out propagator $S^{c}(X, X')$, the so-called in–in propagator $S^{c}_{\text{in}}(X, X')$, and out–out propagator $S^{c}_{\text{out}}(X, X')$ are used,

$$\begin{aligned}
S^{c}(X, X') &= i\,\langle 0, \text{out}|\,\hat{T}\hat{\Psi}(X)\hat{\Psi}^{\dagger}(X')\gamma^{0}\,|0, \text{in}\rangle\, c_{v}^{-1}\,,\\
S^{c}_{\text{in}}(X, X') &= i\,\langle 0, \text{in}|\,\hat{T}\hat{\Psi}(X)\hat{\Psi}^{\dagger}(X')\gamma^{0}\,|0, \text{in}\rangle\,,\\
S^{c}_{\text{out}}(X, X') &= i\,\langle 0, \text{out}|\,\hat{T}\hat{\Psi}(X)\hat{\Psi}^{\dagger}(X')\gamma^{0}\,|0, \text{out}\rangle\,,\\
c_{v} &= \langle 0, \text{out}|\,0, \text{in}\rangle\,.
\end{aligned} \tag{4}$$

Here $\hat{T}$ denotes the chronological ordering operation, $P_{\mu} = i\partial_{\mu} + eA_{\mu}(X)$, $P^{*}_{\mu} = -i\partial_{\mu} + eA_{\mu}(X)$, tr is denote the trace in the space, where γ-matrices are acting, and the Dirac Heisenberg operator $\hat{\Psi}(X)$ corresponds to the classical Dirac field $\psi(X)$. Here $\psi(X)$ is a $2^{[d/2]}$-component spinor (the brackets stand for integer part of). The Dirac Heisenberg operator satisfies the equal time canonical anticommutation relations

$$\begin{aligned}
&\left[\hat{\Psi}(X), \hat{\Psi}(X')\right]_{+}\Big|_{t=t'} = 0,\\
&\left[\hat{\Psi}(X), \hat{\Psi}^{\dagger}(X')\right]_{+}\Big|_{t=t'} = \delta\left(\mathbf{r} - \mathbf{r}'\right)\,.
\end{aligned}$$

It is clear that the vacuum polarization is a local effect, while the concept of a particle has a clear meaning only after the electric field is turned off, which in the case under consideration refers to those particles that have left the field region. Nevertheless, it is natural to assume that the created particles observed inside the field region near its boundaries x_{L} and x_{R} practically do not differ from those observed outside this region and, therefore, represent the final particles.

In this article, our main task is to establish the relationship between the matrix elements (3) and the observable quantities that describe the effects of vacuum polarization and particle production.

However, a number of important technical and principal questions still need to be answered. The point is, that in the setting of problem considered in Refs. [21,22] did not consider local effects, produced by electric field. In these works, it was assumed that the measurement of particle fluxes through some surfaces $x = x^{\text{L/R}}_{\text{meas}}$, $x^{\text{L/R}}_{\text{meas}} \in S_{\text{L/R}}$ occurs at a considerable distance from the field region S_{int} both in the region S_{L} and in the region S_{R} during the macroscopical time interval T.. This distance is assumed to be cT, which is much larger than the extent of field, $x_{\text{R}} - x_{\text{L}}$. In this case, during the time T through surfaces $x = x^{\text{L/R}}_{\text{meas}}$, only those particles pass, which at the time of the beginning of the measurement were not farther from them than at a distance cT. Then the observed fluxes consist mainly of only one type of particles, namely, electrons in the region S_L and positrons in the region S_R. Such a setting the problem allows you to neglect local characteristics of the field and calculate the vacuum-to-vacuum transition amplitude, mean differential and total numbers of created particles, mean current and EMT of created particles for the case of arbitrary x-step. In the case under consideration unlike the approach [21,22] the measurement of characteristics of particles is carried out in the field area S_{int}, where fluxes consisting of both electrons and positrons pass through any surface $x = \text{const}$. This is a new type of task in the framework of strong-field QED with x-steps, for which it is necessary to re-establish the relationship between the duration of motion of particles and a duration of observation. For this purpose, it is necessary to use a certain regularization and renormalization of the parameters used. That is why below we turn to a clarification of the physical meaning of these parameters.

3 Dirac field in the L-constant electric background

The solutions of the Dirac equation with critical x-step are known in the form of the stationary plane waves with given real longitudinal momenta p^L and p^R in the regions S_L and S_R, respectively. In this section we briefly recall some general features of these solutions established in Ref. [21] and present necessary details for the case of the L-constant electric field; see Ref. [23] for more details. We consider Dirac field in d dimensional Minkowski space-time with coordinates X. A complete set of stationary plane waves has the following form:

$$
\begin{aligned}
&\psi_n(X) = \left(\gamma^\mu P_\mu + m\right)\Phi_n(X),\\
&\Phi_n(X) = \varphi_n(t,x)\,\varphi_{\mathbf{p}_\perp}(\mathbf{r}_\perp)\,v_{\chi,\sigma},\\
&\varphi_n(t,x) = \exp(-ip_0 t)\,\varphi_n(x),\quad n = (p_0,\mathbf{p}_\perp,\sigma),\\
&\mathbf{r}_\perp = \left(X^2,\ldots,X^{d-1}\right),\ \mathbf{p}_\perp = \left(p^2,\ldots,p^{d-1}\right),
\end{aligned}
\tag{5}
$$

where $v_{\chi,\sigma}$ with $\chi = \pm 1$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{[d/2]-1})$, $\sigma_j = \pm 1$, is a set of constant orthonormalized spinors satisfying the following conditions:

$$
\gamma^0\gamma^1 v_{\chi,\sigma} = \chi v_{\chi,\sigma},\quad v^\dagger_{\chi,\sigma} v_{\chi',\sigma'} = \delta_{\chi,\chi'}\delta_{\sigma,\sigma'},
$$

In fact, functions (5) correspond to states with given momenta $\mathbf{p}_\perp$ in the perpendicular to the axis x direction. The quantum numbers χ and σ_j describe a spin polarization and provide a convenient parametrization of the solutions. Since in $(1+1)$ and $(2+1)$ dimensions ($d = 2, 3$) there are no any spin degrees of freedom, the quantum numbers σ are absent. Note that in $(2+1)$ dimensions, there are two nonequivalent representations for the γ-matrices which correspond to different fermion species parametrized by $\chi = \pm 1$ respectively. In d dimensions, for any given momenta, there exist only $J_{(d)} = 2^{[d/2]-1}$ different spin states. One can see that solutions (5), which differ only by values of χ, are linearly dependent. Without loss of generality, we set $\chi = 1$ and introduce the notation $v_\sigma = v_{1,\sigma}$. The scalar functions $\varphi_n(x)$ obey the second-order differential equation:

$$
\begin{aligned}
&\Big\{\hat{p}_x^2 - iU'(x) - [p_0 - U(x)]^2\\
&\quad + \mathbf{p}_\perp^2 + m^2\Big\}\varphi_n(x) = 0,\\
&\hat{p}_x = -i\partial_x .
\end{aligned}
\tag{6}
$$

Now we return to solving Eq. (6) in the area $x \in S_{\text{int}}$. It can be rewritten as follows:

$$
\begin{aligned}
&\left[\frac{d^2}{d\xi^2} + \xi^2 + i - \lambda\right]\varphi_n(x) = 0,\\
&\xi = \frac{eEx - p_0}{\sqrt{eE}},\quad \lambda = \frac{\pi_\perp^2}{eE},\quad \pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2}.
\end{aligned}
\tag{7}
$$

Note that $\pi_0(x) = p_0 - eEx$ is kinetic energy an electron. The general solution of Eq. (7) is completely determined by an appropriate pair of linearly independent Weber parabolic cylinder functions (WPCFs), either $D_\rho[(1-i)\xi]$ and $D_{-1-\rho}[(1+i)\xi]$ or $D_\rho[-(1-i)\xi]$ and $D_{-1-\rho}[-(1+i)\xi]$, where $\rho = -i\lambda/2 - 1$.

We assume that corresponding potential step is sufficiently large, $\Delta U = eEL \gg 2m$ (i.e., it is critical). In this case the field E and leading contributions to vacuum mean values can be considered as macroscopic physical quantities.

In the case of critical steps, and, in particular, for the step under consideration, there exist five ranges of quantum numbers n, Ω_k, $k = 1, \ldots, 5$; see section IIIB in Ref. [21]. We are interested in the Klein zone, the range Ω_3, is defined by the inequalities $U_L + \sqrt{eE\lambda} \le p_0 \le U_R - \sqrt{eE\lambda}$. Particle production from the vacuum takes place only in this range. We note that in the limit $L \to \infty$ the width of the Klein zone tends to the infinity; see section IIIB in Ref. [23] for details.

For states with quantum numbers belonging to the Klein zone the L-constant electric field can be considered as a regularization of a constant uniform electric field. That is reason why such a field with a sufficiently large length L, satisfying both condition (2) and

$$
\left[\sqrt{eE}L\left(\sqrt{eE}L - 2\sqrt{\lambda}\right)\right]^{1/2} \gg 1,
\tag{8}
$$

is of special interest. In what follows, we suppose that these conditions hold true. Besides we assume that the additional condition

$$
\sqrt{\lambda} < K_\perp,\ \sqrt{eE}L/2 \gg K_\perp^2 \gg \max\left\{1, m^2/eE\right\}
$$

takes place. Thus, in fact, we are going to consider the subrange D,

$$D \supset \Omega_3 : \sqrt{\lambda} < K_\perp,\ |p_0|/\sqrt{eE} < \sqrt{eE}L/2 - K\ ,$$
$$\sqrt{eE}L/2 \gg K \gg K_\perp^2 \gg \max\left\{1, m^2/eE\right\}\ , \quad (9)$$

where K and $K_\perp$ are any given numbers satisfying the condition (9). Namely in this subrange the pair creation is essential.

Solutions of the Dirac equation with well-defined left and right asymptotics we denote as ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$,

$$\begin{aligned}&\hat{p}_x\, {}_\zeta\psi_n(X) = p^{\mathrm{L}}\ {}_\zeta\psi_n(X),\ \ x \to x_{\mathrm{L}},\ \ \zeta = \mathrm{sgn}(p^{\mathrm{L}})\ ,\\ &\hat{p}_x\, {}^\zeta\psi_n(X) = p^{\mathrm{R}}\, {}^\zeta\psi_n(X),\ \ x \to x_{\mathrm{R}},\ \ \zeta = \mathrm{sgn}(p^{\mathrm{R}})\ ;\\ &p^{\mathrm{L}} = \zeta\sqrt{[\pi_0(\mathrm{L})]^2 - \pi_\perp^2},\ \ p^{\mathrm{R}} = \zeta\sqrt{[\pi_0(\mathrm{R})]^2 - \pi_\perp^2}\ ,\\ &\pi_0(\mathrm{L}) = p_0 - U_{\mathrm{L}},\ \ \pi_0(\mathrm{R}) = p_0 - U_{\mathrm{R}},\ \ \zeta = \pm\ , \end{aligned}\quad (10)$$

where $|\pi_0(\mathrm{L})|$ and $|\pi_0(\mathrm{R})|$ are asymptotic kinetic energies of an electron in the regions S_{L} and S_{R}, respectively.

The solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ describe particles with given momenta p^{L} as $x \to x_{\mathrm{L}}$ and p^{R} as $x \to x_{\mathrm{R}}$, respectively. One can see that the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ have the form (5) with functions $\varphi_n(x)$ denoted here as ${}_\zeta\varphi_n(x)$ or ${}^\zeta\varphi_n(x)$ respectively. The latter functions have the following asymptotics:

$$\begin{aligned}&{}_\zeta\varphi_n(x) = {}_\zeta C \exp\left[ip^{\mathrm{L}}x\right],\quad x \to x_{\mathrm{L}}\ ,\\ &{}^\zeta\varphi_n(x) = {}^\zeta C \exp\left[ip^{\mathrm{R}}x\right],\quad x \to x_{\mathrm{R}}\ .\end{aligned}$$

Here ${}_\zeta C$ and ${}^\zeta C$ are normalization constants.

We consider our theory in a large space-time box that has a spatial volume $V_\perp = \prod_{j=2}^{d-1} K_j$ and the time dimension T, where all K_j and T are macroscopically large. It is supposed that all the solutions $\psi(X)$ are periodic under transitions from one box to another. Then the integration in the inner product

$$(\psi, \psi')_x = \int_{V_\perp T} \psi^\dagger(X)\,\gamma^0\gamma^1\psi'(X)\,dt\,d\mathbf{r}_\perp\ , \quad (11)$$

over the transverse coordinates is fulfilled from $-K_j/2$ to $+K_j/2$, and over the time t from $-T/2$ to $+T/2$. Under these suppositions, the inner product (11) does not depend on x; see section IIIC1 in Ref. [21] for details. The solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ satisfy the following orthonormality relations on the $x = $ const hyperplane:

$$\begin{aligned}&\left({}_\zeta\psi_n,\ {}_{\zeta'}\psi_{n'}\right)_x = \zeta\delta_{\zeta,\zeta'}\delta_{n,n'}\ ,\\ &\left({}^\zeta\psi_n,\ {}^{\zeta'}\psi_{n'}\right)_x = -\zeta\delta_{\zeta,\zeta'}\delta_{n,n'}\ .\end{aligned}\quad (12)$$

In what follows, we will need two sets of solutions of Eq. (7) for the case $x_{\mathrm{L}} \to -\infty$ and $x_{\mathrm{R}} \to \infty$:

$$\begin{aligned}&{}_+\varphi_n(x) = Y\ {}_+C\ D_{-1-\rho}[-(1+\mathrm{i})\xi]\ ,\\ &{}_-\varphi_n(x) = Y\ {}_-C\ D_\rho[-(1-\mathrm{i})\xi]\ ,\\ &{}^+\varphi_n(x) = Y\ {}^+C\ D_\rho[(1-\mathrm{i})\xi]\ ,\\ &{}^-\varphi_n(x) = Y\ {}^-C\ D_{-1-\rho}[(1+\mathrm{i})\xi]\ ,\\ &{}^{-\zeta}C = {}_\zeta C\\ &\quad = (eE)^{-1/2}e^{\pi\lambda/8}\left[\frac{\lambda}{2}(1+\zeta) + 1 - \zeta\right]^{-1/2}\ ,\\ &Y = (V_\perp T)^{-1/2}\ .\end{aligned}\quad (13)$$

In the $V_\perp \to \infty$ and $T \to \infty$ limits one has to replace the symbol $\delta_{n,n'}$ in the normalization conditions (12) by quantity $\delta_{\sigma,\sigma'}\delta\left(p_0 - p_0'\right)\delta\left(\mathbf{p}_\perp - \mathbf{p}'_\perp\right)$ and to set $Y = (2\pi)^{-(d-1)/2}$ in Eq. (13).

In the Klein zone, in- and out-solutions are:

$$\begin{aligned}&\text{in} - \text{solutions}:\ {}_-\psi_n(X),\ {}^-\psi_n(X)\ ,\\ &\text{out} - \text{solutions}:\ {}_+\psi_n(X),\ {}^+\psi_n(X)\ .\end{aligned}\quad (14)$$

The solutions ${}^\zeta\psi_n(X)$ describe electrons, whereas the solutions ${}_\zeta\psi_n(X)$ describe positrons.

The mutual decompositions of the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ have the form:

$$\begin{aligned}&{}^\zeta\psi_n(X)\\ &\quad = {}_+\psi_n(X)g\left({}_+\middle|{}^\zeta\right) - {}_-\psi_n(X)g\left({}_-\middle|{}^\zeta\right)\ ,\\ &{}_\zeta\psi_n(X)\\ &\quad = {}^-\psi_n(X)\,g\left({}^-\middle|{}_\zeta\right) - {}^+\psi_n(X)\,g\left({}^+\middle|{}_\zeta\right)\ ,\end{aligned}\quad (15)$$

where expansion coefficients g are defined by the relations:

$$\begin{aligned}&\left({}_\zeta\psi_n,\ {}^{\zeta'}\psi_{n'}\right)_x = g\left({}_\zeta\middle|{}^{\zeta'}\right)\delta_{n,n'}\ ,\\ &\quad g\left({}^{\zeta'}\middle|{}_\zeta\right) = g\left({}_\zeta\middle|{}^{\zeta'}\right)^*\ .\end{aligned}\quad (16)$$

The coefficients g satisfy the following unitary relations:

$$\begin{aligned}&\left|g\left({}_-\middle|{}^+\right)\right|^2 = \left|g\left({}_+\middle|{}^-\right)\right|^2,\\ &\left|g\left({}_+\middle|{}^+\right)\right|^2 = \left|g\left({}_-\middle|{}^-\right)\right|^2\ ,\\ &\frac{g\left({}_+\middle|{}^-\right)}{g\left({}_-\middle|{}^-\right)} = \frac{g\left({}^+\middle|{}_-\right)}{g\left({}^+\middle|{}_+\right)},\\ &\left|g\left({}_+\middle|{}^-\right)\right|^2 - \left|g\left({}_+\middle|{}^+\right)\right|^2 = 1\ .\end{aligned}$$

The differential mean numbers of electrons and positrons from electron-positron pairs created from the vacuum are equal and present the number of created pairs,

$$N_n^{\mathrm{cr}} = \left|g\left({}_-\middle|{}^+\right)\right|^{-2}\ .$$

The total number of pairs created from the vacuum N^{cr} is the sum over the range Ω_3 of the differential mean numbers N_n^{cr}. Since the numbers N_n^{cr} do not depend on the spin polarization parameters σ_s, the sum over the spin projections produces only the factor $J_{(d)} = 2^{\left[\frac{d}{2}\right]-1}$. The sum over the momenta and the energy can be easily transformed into an integral in the following way:

$$N^{\mathrm{cr}} = \sum_{\mathbf{p}_\perp, p_0 \in \Omega_3}\sum_\sigma N_n^{\mathrm{cr}}$$

$$= \frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \int_{\Omega_3} dp_0 d\mathbf{p}_\perp N_n^{\mathrm{cr}} . \tag{17}$$

In the case of the L-constant electric field with a sufficiently large length L, satisfying Eqs. (2) and (8), functions (13) have asymptotic expansions for $|\xi| \gg \max\{1,\lambda\}$ (see, e.g. Ref. [40]) over the wide range of energies p_0 for any given λ of the subrange D given by Eq. (9). In this subrange the quantity N_n^{cr} is almost constant and coincides with the well-known result in a constant uniform electric field [2,41,42],

$$N_n^{\mathrm{cr}} \to N_n^{\mathrm{uni}} = e^{-\pi\lambda} . \tag{18}$$

One can see that the minimal length of an electric field for which these asymptotic expansions are performed is the order of the length scale,

$$\Delta l_0 = (eE)^{-1/2} \max\{1,\lambda\} . \tag{19}$$

Therefore we can call it as the formation interval over the x for the mean numbers N_n^{uni}.

Note that exact quantity N_n^{cr} has the following features (see section IIIB in Ref. [23] for details):

$$N_n^{\mathrm{cr}} \sim \left| p^{\mathrm{R}} \right| \to 0, \quad N_n^{\mathrm{cr}} \sim \left| p^{\mathrm{L}} \right| \to 0, \quad \forall\lambda \neq 0 ,$$

if n tends to the boundary with either the range Ω_2 $\left(\left|p^{\mathrm{R}}\right| \to 0\right)$ or the range Ω_4 $\left(\left|p^{\mathrm{L}}\right| \to 0\right)$) where the vacuum is stable. The contribution to the integral (17) of the entire subrange of the range Ω_3, that is not included in the subrange D, is negligibly small compared to the large contribution due to the subrange D. It means that in integral (17) N_n^{cr} plays the role of a cutoff factor. Finally, we obtain:

$$N^{\mathrm{cr}} = V_\perp T n^{\mathrm{cr}}, \quad n^{\mathrm{cr}} = r^{\mathrm{cr}} \left[L + \frac{O(K)}{\sqrt{eE}} \right] ,$$
$$r^{\mathrm{cr}} = \frac{J_{(d)} (eE)^{d/2}}{(2\pi)^{d-1}} \exp\left\{ -\pi \frac{m^2}{eE} \right\} . \tag{20}$$

Here n^{cr} is the total number density of created from the vacuum pairs per unit of time and per unit of surface orthogonal to the electric field direction.

Note that n^{cr} given by Eq. (20) is a function of the field length L. The density $r^{\mathrm{cr}} = n^{\mathrm{cr}}/L$ is known in the theory of pair creation in the constant uniform electric field as the pair-production rate (see the d dimensional case in Ref. [36]).

4 Means of currents and EMT

4.1 Regularization

Calculating some of the matrix elements considered above, one meets divergences that indicate a need of a certain regularization. Below, we consider such regularization and renormalization procedures for calculating local quantities in strong-field QED with L-constant electric field. In main, these procedures where formulated in Ref. [21], however, here they are completed by some important and the necessary refinements.

In the case of the L-constant electric field under consideration, where the distance L between capacitor plates is sufficiently large, the plane waves ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ can be identified by using one-particle mean currents and the energy fluxes in the field region S_{int}, see Ref. [23]. Thus, we can calculate the matrix elements (3) inside of the range S_{int}. However, the explicit form of the singular functions (4) depends on parameters of the volume regularization. Due to physical reasons, these parameters are significantly different from those proposed in the case when very wide regions S_{L} and S_{R} were used to measure fluxes of particles, see [22]. That is why below we turn to a clarification of the physical meaning of these parameters.

Stationary plane waves of type (10) are usually used in potential scattering theory, where they represent one-particle states with corresponding conserved longitudinal currents. Such one-particle consideration is consistent in all the ranges Ω_k, excepting the Klein zone Ω_3. The technique developed in Ref. [21] does not need any refining in these ranges. Let us consider the range Ω_3 where the strong-field QED consideration is essential. We note that for our purposes it is sufficient to consider the subrange $D \supset \Omega_3$, which gives the main contribution to the vacuum instability.

The plane waves of the type (10) are orthonormalized with respect to the inner product (11). To determine the time-independent initial $|0,\mathrm{in}\rangle$ and final $|0,\mathrm{out}\rangle$ vacua and construct the corresponding in- and out-states in an adequate Fock space, we have to use a time-independent inner product of solutions $\psi(X)$ and $\psi'(X)$ of the Dirac equation with the field $E_{\mathrm{pristine}}(X)$ on a t constant hyperplane. We recall that the periodic conditions are not imposed in the x direction. That is why, in contrast to the case of t-steps, the motion of particles in the x direction is unlimited. Unlike the approach [21,22] we assume that the large distance L is not less then cT, where T is an observation time T. In this case, one can ignore areas without the electric field and to believe that the part of the system under consideration causally related to the pair production process is situated inside the region S_{int}. The corresponding particle states are represented by solutions given by Eqs. (5) and (13). For these reasons, we refine the volume regularization procedure used in Ref. [21], defining the time-independent inner product on the t-constant hyperplane as follows:

$$(\psi,\psi') = \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(\mathrm{L})}}^{K^{(\mathrm{R})}} \psi^\dagger(X)\,\psi'(X)\,dx , \tag{21}$$

where the integral over the spatial volume $V_\perp$ is completed by the integral over the interval $\left[-K^{(\mathrm{L})}, K^{(\mathrm{R})}\right]$ in the x direction. Here $K^{(\mathrm{L/R})}$ are some arbitrary macroscopic but finite parameters of the volume regularization, which are situated in the spatial area S_{int}, $0 < K^{(\mathrm{L})} < |x_\mathrm{L}|$ and $0 < K^{(\mathrm{R})} < x_\mathrm{R}$. The length $K^{(\mathrm{R})} + K^{(\mathrm{L})} < L$ is sufficiently large,

$$K^{(\mathrm{R})} + K^{(\mathrm{L})} \gg \Delta l_0 ,$$

where Δl_0 is given by Eq. (19).

Such an inner product is time-independent if solutions $\psi(X)$ and $\psi'(X)$ obey certain boundary conditions that allow one to integrate by parts in Eq. (21) neglecting boundary terms. These boundary conditions consist of above mentioned periodicity with respect of all K_j translations and an additional condition with respect of the integral over the coordinate x; see Appendix B in Ref. [21]. The inner product (21) is conserved for such states. However, considering solutions of the type (10), which do not vanish at the spatial infinity, we must accept some additional technical assumptions to provide the time independence of the inner product (21). First of all, we note that states with different quantum numbers n are independent, therefore decompositions of the vacuum matrix elements (3) into the solutions with given n do not contain interference terms, see Appendix A for details. That is why it is enough to consider Eq. (21) only for a particular case of solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ with equal n. One can evaluate the principal value of integral (21) using relations (15) and the asymptotic behavior of functions (13) in the spatial regions where arguments of WPCF's are large, $|\xi| \gg \max\{1, \lambda\}$, see Appendix B for details. In this case the modulus of a longitudinal momentum is well defined as, $|p_x(x)| = \sqrt{[\pi_0(x)]^2 - \pi_\perp^2}$. One can see that the norms of the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ with respect to the inner product (21) are proportional to the macroscopically large parameters $\tau^{(\mathrm{L})}$ and $\tau^{(\mathrm{R})}$,

$$\tau^{(\mathrm{L})} = K^{(\mathrm{L})}/v^{\mathrm{L}},\ \tau^{(\mathrm{R})} = K^{(\mathrm{R})}/v^{\mathrm{R}} ,$$

where $v^{\mathrm{L}} = |p_x(x)/\pi_0(x)|$ at $x = -K^{(\mathrm{L})}$ and $v^{\mathrm{R}} = |p_x(x)/\pi_0(x)|$ at $x = K^{(\mathrm{R})}$ are absolute values of the longitudinal velocities of particles. In the spatial regions of interest where $|\xi|$ is large and the energy $|\pi_0(x)|$ is much bigger then $\pi_\perp$, the particles are moving almost parallel to the axis x, and the longitudinal velocities $|p_x(x)/\pi_0(x)|$ are ultrarelativistic at any x, such that $|p_x(x)/\pi_0(x)| \to c$ $(c = 1)$.

It is shown (see Appendix B in Ref. [21]) that the following couples of plane waves are orthogonal with respect to the inner product (21)

$$\left({}_\zeta\psi_n, {}^\zeta\psi_n\right) = O(1)/\tau^{(\mathrm{L/R})},\ \ n \in \Omega_3$$

if the parameters of the volume regularization $\tau^{(\mathrm{L/R})}$ satisfy the condition

$$\tau^{(\mathrm{L})} - \tau^{(\mathrm{R})} = O(1) , \tag{22}$$

where $O(1)$ are terms that are negligibly small in comparison with the macroscopic quantities $\tau^{(\mathrm{L/R})}$. In what follows we disregard the contributions of the order of $O(1)/\tau^{(\mathrm{L/R})}$. Thus, according to the physical interpretation given in latter reference, the sets (14) represent in- and out-solutions, which are linearly independent couples of complete on the t-constant hyperplane states with a given n. One can see that $\tau^{(\mathrm{L})}$ and $\tau^{(\mathrm{R})}$ are macroscopic times and they are equal,

$$\tau^{(\mathrm{L})} = \tau^{(\mathrm{R})} = \tau .$$

The L-constant field produces constant fluxes of created from the vacuum final particles during the time interval T. These particles are created with zero longitudinal kinetic momenta in a relatively small formation interval Δl_0 given by Eq. (19). After turning into real particles electrons and positrons under the action of the electric field move in opposite directions, the positrons move in the direction of the electric field to the region S_R, while the electrons in the opposite direction to the region S_L and finally leave the interval $\left[-K^{(\mathrm{L})}, K^{(\mathrm{R})}\right]$. The time which is enough to these particles to reach one of the hyperplane $x = -K^{(\mathrm{L})}$ or $x = K^{(\mathrm{R})}$ varies from zero to the maximum possible time 2τ, which is required by the ultrarelativistic particle to overcome the distance $K^{(\mathrm{R})} + K^{(\mathrm{L})}$. It is clear that the kinetic energy (and the longitudinal kinetic momentum) of a particle crossing these hyperplanes are proportional to the paths traveled by the particles. Thus, the summation over the kinetic energies when calculating fluxes of particles leaving the area between the hypersurfaces $x = -K^{(\mathrm{L})}$ and $x = K^{(\mathrm{R})}$ is equivalent to the summation over the distances that these particles traveled within the interval $\left[-K^{(\mathrm{L})}, K^{(\mathrm{R})}\right]$ in the x-direction.

Under condition (22) the norms of the solutions on the t-constant hyperplane are:

$$\begin{aligned}&\left({}_\zeta\psi_n, {}_\zeta\psi_n\right) = \left({}^\zeta\psi_n, {}^\zeta\psi_n\right) = \mathcal{M}_n ,\\ &\mathcal{M}_n = 2\frac{\tau}{T}\left|g\left({}_+|{}^-\right)\right|^2 \ \text{if}\ \ n \in \Omega_3 ,\end{aligned} \tag{23}$$

where coefficients g are defined by Eq. (16) and $\left|g\left({}_+|{}^-\right)\right|^2$ are given explicitly by Eq. (18). It is natural to assume that the observation time T (the time during which the observer registers flows of created particles leaving the area between hyperplanes $x = -K^{(\mathrm{L})}$ and $x = K^{(\mathrm{R})}$) is equal to the maximal time 2τ,

$$2\tau = T , \tag{24}$$

which is required for the created particles to leave the region with the electric field. As we see in what follows, such a relation fixes the proposed renormalization procedure. Thus, we find:

$$\mathcal{M}_n = \left|g\left({}_+|{}^-\right)\right|^2 \ \text{if}\ \ n \in \Omega_3 . \tag{25}$$

In the case $L \to \infty$, one can consider the limit $V_\perp, K^{(\mathrm{L/R})} \to \infty$ to obtain normalized solutions in the range Ω_3 as follows:

$$\begin{aligned}
\left({}_\zeta\psi_n, {}_\zeta\psi_{n'}\right) &= \left({}^\zeta\psi_n, {}^\zeta\psi_{n'}\right) \\
&= \delta_{\sigma,\sigma'}\delta\left(p_0 - p_0'\right)\delta(\mathbf{p}_\perp - \mathbf{p}_\perp')\mathcal{M}_n\,, \\
\left({}_\zeta\psi_n, {}^\zeta\psi_{n'}\right) &= 0\,,
\end{aligned}$$

where the quantity $\mathcal{M}_n$ is given by Eq. (25).

4.2 Singular functions

We recall that in the general case, in theories with unstable vacuum, the singular functions (4) do not coincide. The differences between the functions $S^c_{\mathrm{in}}(X, X')$, $S^c_{\mathrm{out}}(X, X')$ and the causal propagator $S^c(X, X')$ are denoted by $S^p(X, X')$ and $S^{\bar{p}}(X, X')$,

$$\begin{aligned}
S^p(X, X') &= S^c_{\mathrm{in}}(X, X') - S^c(X, X')\,, \\
S^{\bar{p}}(X, X') &= S^c_{\mathrm{out}}(X, X') - S^c(X, X')\,.
\end{aligned} \tag{26}$$

In the case of strong-field QED with L-constant electric field, all the functions can be expressed as sums over the solutions, given by Eqs. (5) and (13), see Ref. [21]. It can be seen that in the case under consideration with $L \to \infty$, the main contributions to the sums are due to the Klein zone. Taking this fact into account, the singular functions can be represented as:

$$\begin{aligned}
&S^c(X, X') = \theta(t - t')\, S^-\left(X, X'\right) \\
&\quad -\theta(t' - t)\, S^+\left(X, X'\right)\,, \\
&S^-(X, X') \\
&\quad = i\sum_n \mathcal{M}_n^{-1}\, {}^+\psi_n(X)\, g\left({}^+|{}_-\right) g\left({}^-|{}_-\right)^{-1}\, {}^-\bar{\psi}_n(X')\,, \qquad (27) \\
&S^+(X, X') \\
&\quad = i\sum_n \mathcal{M}_n^{-1}\, {}_-\psi_n\left(X\right) g\left({}_-|{}^+\right) g\left({}_+|{}^+\right)^{-1}\, {}_+\bar{\psi}_n\left(X'\right)\,, \\
&S^c_{\mathrm{in/out}}(X, X') = \theta(t - t')\, S^-_{\mathrm{in/out}}\left(X, X'\right) \\
&\quad -\theta(t' - t)\, S^+_{\mathrm{in/out}}\left(X, X'\right)\,, \\
&S^-_{\mathrm{in/out}}(X, X') = i\sum_n \mathcal{M}_n^{-1}\, {}^\mp\psi_n\left(X\right)\, {}^\mp\bar{\psi}_n\left(X'\right)\,, \\
&S^+_{\mathrm{in/out}}(X, X') = i\sum_n \mathcal{M}_n^{-1}\, {}_\mp\psi_n\left(X\right)\, {}_\mp\bar{\psi}_n\left(X'\right)\,, \\
&\bar{\psi} = \psi^\dagger\gamma^0\,, \qquad (28)
\end{aligned}$$

where $\mathcal{M}_n$ is given by Eqs. (25).

Using relations (15), we represent the singular functions $S^p(X, X')$ and $S^{\bar{p}}(X, X')$ given by Eq. (26) as follows:

$$\begin{aligned}
&S^p(X, X') \\
&\quad = i\sum_n \mathcal{M}_n^{-1}\, {}_-\psi_n\left(X\right) g\left({}^-|{}_-\right)^{-1}\, {}^-\bar{\psi}_n\left(X'\right)\,, \\
&S^{\bar{p}}(X, X') \\
&\quad = -i\sum_n \mathcal{M}_n^{-1}\, {}^+\psi_n\left(X\right) g\left({}_+|{}^+\right)^{-1}\, {}_+\bar{\psi}_n\left(X'\right)\,. \qquad (29)
\end{aligned}$$

We stress that both functions vanish in the absence of the vacuum instability.

5 Calculation of mean values in strong-field QED with L-constant field

5.1 Pair-creation contributions

With account taken of (26) the vacuum matrix elements, defined by Eqs. (3) and (4), can be represented as:

$$\begin{aligned}
\left\langle J^\mu(x)\right\rangle_{\mathrm{in}} &= \mathrm{Re}\left\langle J^\mu(x)\right\rangle^c + \mathrm{Re}\left\langle J^\mu(x)\right\rangle^p\,, \\
\left\langle J^\mu(x)\right\rangle_{\mathrm{out}} &= \mathrm{Re}\left\langle J^\mu(x)\right\rangle^c + \mathrm{Re}\left\langle J^\mu(x)\right\rangle^{\bar{p}}\,, \\
\left\langle J^\mu(x)\right\rangle^{p,\bar{p}} &= -ie\ \mathrm{tr}\left[\gamma^\mu S^{p,\bar{p}}(X, X')\right]\Big|_{X=X'}\,; \\
\left\langle T_{\mu\nu}(x)\right\rangle_{\mathrm{in}} &= \langle 0, \mathrm{in}|\, T_{\mu\nu}\, |0, \mathrm{in}\rangle \\
&= i\ \mathrm{tr}\left[A_{\mu\nu}S^c_{\mathrm{in}}(X, X')\right]\big|_{X=X'} \\
&= \mathrm{Re}\left\langle T_{\mu\nu}(x)\right\rangle^c + \mathrm{Re}\left\langle T_{\mu\nu}(x)\right\rangle^p\,, \\
\left\langle T_{\mu\nu}(x)\right\rangle_{\mathrm{out}} &= \langle 0, \mathrm{out}|\, T_{\mu\nu}\, |0, \mathrm{out}\rangle \\
&= i\ \mathrm{tr}\left[A_{\mu\nu}S^c_{\mathrm{out}}(X, X')\right]\big|_{X=X'} \\
&= \left\langle T_{\mu\nu}(x)\right\rangle^c + \left\langle T_{\mu\nu}(x)\right\rangle^{\bar{p}}\,, \\
\left\langle T_{\mu\nu}(x)\right\rangle^{p,\bar{p}} &= i\ \mathrm{tr}\left[A_{\mu\nu}S^{p,\bar{p}}(X, X')\right]\Big|_{X=X'}\,.
\end{aligned} \tag{30}$$

One can see with help of Eqs. (27) and (29) that all the quantities $\langle J^\mu(x)\rangle^c$ and $\left\langle T_{\mu\nu}(x)\right\rangle^c$ are finite as $L \to \infty$, whereas the current components $\left\langle J^0(x)\right\rangle^{p,\bar{p}}$, $\left\langle J^1(x)\right\rangle^{p,\bar{p}}$, $\left\langle T^{10}(x)\right\rangle^{p/\bar{p}}$, and the diagonal components $\left\langle T_{\mu\mu}(x)\right\rangle^{p,\bar{p}}$ of the EMT are growing unlimited as L. That is why here we consider the components $\langle J^\mu(x)\rangle^{p,\bar{p}}$ and $\left\langle T_{\mu\nu}(x)\right\rangle^{p,\bar{p}}$ for the case of a large but finite L.

In representation (29) the factor $\mathcal{M}_n^{-1} = N_n^{\mathrm{cr}}$ plays the role of a cutoff factor, that is why the main contribution is formed on the finite subrange D given by Eq. (9). That is why all the integrals over the momenta are finite. N_n^{cr} is given by Eq. (18) in the subrange D and does not depend on p_0. In the subrange D and for large L, the integral over p_0 is responsible for growing contributions as $L \to \infty$. That is why the main contribution to the vacuum means under consideration are formed in this subarea, such that it is enough to consider further the following expressions

$$\begin{aligned}
S^p(X, X') &= i\sum_{n\in D} N_n^{\mathrm{cr}} g\left({}^-|{}_-\right)^{-1}\, {}_-\psi_n\left(X\right)\, {}^-\bar{\psi}_n\left(X'\right) \\
&= \frac{iV_\perp T}{(2\pi)^{d-1}}\int_D dp_0 d\mathbf{p}_\perp \sum_\sigma \Big[\\
&\qquad N_n^{\mathrm{cr}} g\left({}^-|{}_-\right)^{-1}\, {}_-\psi_n\left(X\right)\, {}^-\bar{\psi}_n\left(X'\right)\Big]\,,
\end{aligned}$$

$$S^{\bar{p}}(X,X') = -i\sum_{n\in D} N_n^{\mathrm{cr}} g\left({}_+|{}^+\right)^{-1} {}^+\psi_n(X)\, {}_+\bar{\psi}_n(X')$$
$$= -\frac{iV_\perp T}{(2\pi)^{d-1}}\int_D dp_0 d\mathbf{p}_\perp \sum_\sigma \Big[N_n^{\mathrm{cr}} g\left({}_+|{}^+\right)^{-1} {}^+\psi_n(X)\, {}_+\bar{\psi}_n(X')\Big] \quad (31)$$

for the singular functions (29).

We are interested in the mean values under consideration inside the capacitor, namely for $x\in S_{\mathrm{int}}$, where $|x|<L/2$. In the range D for the given $x\sim x'$ you can select subranges $D_x^+\subset D$ and $D_x^-\subset D$,

$$D_x^+: -\left[eE\,(x-x_{\mathrm{L}}) - K\sqrt{eE}\right] < \pi_0(x) < -\sqrt{eE}K\,,$$
$$\sqrt{\lambda}<K_\perp \text{ if } \xi>0\,,$$
$$D_x^-: \sqrt{eE}K < \pi_0(x) < eE\,(x_{\mathrm{R}}-x) - K\sqrt{eE}\,,$$
$$\sqrt{\lambda}<K_\perp \text{ if } \xi<0\,, \quad (32)$$

where $|\pi_0(x)|$ is sufficiently large. In these subranges, the functions ${}_-\psi_n(X)$ and ${}^-\bar{\psi}_n(X')$ can be approximated by asymptotic forms of the WPCF's for big $|\xi|\sim|\xi'|>K$, where $\xi'=\xi|_{x\to x'}$; e.g., see Ref. [40]. Note that $|\pi_0(x)|$ is kinetic energy of a positron in D_x^+ and $\pi_0(x)$ is kinetic energy of an electron in D_x^-. Both integration domains in Eq. (32) are large enough, to provide the main contribution to the integrals (31).

Let us consider the case of $\pi_0(x)\sim\pi_0(x')\in D_x^+$. By the help of Eq. (15) we get:

$$S^p(X,X') = -\frac{iV_\perp T}{(2\pi)^{d-1}}\int_D dp_0 d\mathbf{p}_\perp \sum_\sigma N_n^{\mathrm{cr}} g({}^-|{}_-)^{-1}$$
$$\times\left[{}^+\psi_n(X)\, g\left({}^+|{}_-\right) - {}^-\psi_n(X)\, g\left({}^-|{}_-\right)\right] {}^-\bar{\psi}_n(X')\,,$$
$$S^{\bar{p}}(X,X') = -\frac{iV_\perp T}{(2\pi)^{d-1}}\int_D dp_0 d\mathbf{p}_\perp \sum_\sigma N_n^{\mathrm{cr}}\, g\left({}_+|{}^+\right)^{-1}$$
$$\times\, {}^+\psi_n(X)\left[g\left({}_+|{}^-\right) {}^-\bar{\psi}_n(X') - {}^+\bar{\psi}_n(X')\, g\left({}_+|{}^+\right)\right].$$

In the case $X\sim X'$, using asymptotics of WPCF's , given by Eq. (13) and discarding negligibly small contributions from the oscillating terms to the integral over p_0, we obtain:

$$S^p(X,X') \approx S_+^p(X,X')$$
$$= \frac{iV_\perp T}{(2\pi)^{d-1}}\int_{D_x^+} dp_0 d\mathbf{p}_\perp \sum_\sigma e^{-\pi\lambda}\, {}^-\psi_n(X)\, {}^-\bar{\psi}_n(X')$$
$$= (\gamma P+m)\Delta_+^p(X,X')\,,$$
$$S^{\bar{p}}(X,X') \approx S_+^{\bar{p}}(X,X')$$
$$= \frac{iV_\perp T}{(2\pi)^{d-1}}\int_{D_x^+} dp_0 d\mathbf{p}_\perp \sum_\sigma e^{-\pi\lambda}\left[{}^+\psi_n(X)\, {}^+\bar{\psi}_n(X')\right]$$
$$= (\gamma P+m)\Delta_+^{\bar{p}}(X,X')\,,$$

$$\Delta_+^{p/\bar{p}}(X,X') \sim \frac{-i}{2\sqrt{eE}(2\pi)^{d-1}}\int_{D_x^+} dp_0 d\mathbf{p}_\perp \xi^{-1}$$
$$\times\exp\left[-\pi\lambda - ip_0(t-t') + i\mathbf{p}_\perp(\mathbf{r}_\perp-\mathbf{r}'_\perp) \pm i\frac{\xi^2-\xi'^2}{2}\right].$$

Now we consider the integration over the transversal momenta $\mathbf{p}_\perp$. In the subrange D_x^+, given by Eq. (32), the domain of the variation of $|\mathbf{p}_\perp|$ is finite. However, taking into account that the exponential $\exp(-\pi\lambda)$ plays the role of a cutoff factor, we can extend the limits of the domain to infinity. As a result we have:

$$S_+^{p/\bar{p}}(X,X') = (\gamma P+m)\Delta_+^{p/\bar{p}}(X,X')\,,$$
$$\Delta_+^{p/\bar{p}}(X,X') = -i\, h_\perp(\mathbf{r}_\perp,\mathbf{r}'_\perp)$$
$$\times\int_{x_{\mathrm{L}}+K/\sqrt{eE}}^{x-K/\sqrt{eE}} h_\parallel^{-/+}(x,\tilde{x})d\tilde{x}\,,$$
$$p_0 = eE\,\tilde{x}\,,$$
$$h_\parallel^{-/+}(x,\tilde{x}) = \frac{1}{2\,(x-\tilde{x})}$$
$$\times\exp\left\{-ip_0(t-t') \mp \frac{i}{2}\left[\xi(x)^2 - \xi(x')^2\right]\right\}.$$
$$h_\perp(\mathbf{r}_\perp,\mathbf{r}'_\perp) = \frac{(eE)^{d/2-1}}{(2\pi)^{d-1}}$$
$$\times\exp\left(-\frac{\pi m^2}{eE} - \frac{eE\left|\mathbf{r}_\perp-\mathbf{r}'_\perp\right|^2}{4\pi}\right). \quad (33)$$

Let us consider the case of $\pi_0(x)\sim\pi_0(x')\in D_x^-$. In the same way as before, we can justify that it is enough to consider further the following expressions for the singular functions (29):

$$S_-^{p/\bar{p}}(X,X') = (\gamma P+m)\Delta_-^{p/\bar{p}}(X,X'),\quad \xi<-K\,,$$
$$\Delta_-^{p/\bar{p}}(X,X') = i\, h_\perp(\mathbf{r}_\perp,\mathbf{r}'_\perp)$$
$$\times\int_{x+K/\sqrt{eE}}^{x_{\mathrm{R}}-K/\sqrt{eE}} h_\parallel^{+/-}(x,\tilde{x})d\tilde{x}\,,$$
$$p_0 = eE\,\tilde{x}\,. \quad (34)$$

Now, we calculate the vacuum means values under consideration, which, for a given x, are formulated by contributions from both domains D_x^+ and D_x^-,

$$\langle J^\mu(x)\rangle^{p/\bar{p}} = \langle J^\mu(x)\rangle_+^{p/\bar{p}} + \langle J^\mu(x)\rangle_-^{p/\bar{p}}\,,$$
$$\langle T_{\mu\nu}(x)\rangle^{p,\bar{p}} = \langle T_{\mu\nu}(x)\rangle_+^{p,\bar{p}} + \langle T_{\mu\nu}(x)\rangle_-^{p,\bar{p}}\,,$$
$$\langle J^\mu(x)\rangle_\pm^{p/\bar{p}} = -ie\ \mathrm{tr}\left[\gamma^\mu S_\pm^{p,\bar{p}}(X,X')\right]\Big|_{X=X'}\,,$$
$$\langle T_{\mu\nu}(x)\rangle_\pm^{p,\bar{p}} = i\ \mathrm{tr}\left[A_{\mu\nu} S_\pm^{p,\bar{p}}(X,X')\right]\Big|_{X=X'}. \quad (35)$$

Using representations (33) and (34) we obtain that nonvanishing means are:

$$\left\langle J^1(x)\right\rangle_+^p = -\left\langle J^1(x)\right\rangle_+^{\bar{p}} = \left\langle J^0(x)\right\rangle_+^{p/\bar{p}}\,,$$

$$
\begin{aligned}
\left\langle J^{0}(x)\right\rangle_{+}^{p/\bar{p}} &= e\,r^{\mathrm{cr}}\left(x-x_{\mathrm{L}}-\frac{K}{\sqrt{eE}}\right)\,,\\
\left\langle T^{10}(x)\right\rangle_{+}^{p} &= -\left\langle T^{10}(x)\right\rangle_{+}^{\bar{p}}=\left\langle T^{00}(x)\right\rangle_{+}^{p/\bar{p}}\,,\\
\left\langle T^{11}(x)\right\rangle_{+}^{p/\bar{p}} &= \left\langle T^{00}((x)\right\rangle_{+}^{p/\bar{p}}\,,\\
\left\langle T^{00}(x)\right\rangle_{+}^{p/\bar{p}} &= \frac{r^{\mathrm{cr}}}{2}eE\left(x-x_{\mathrm{L}}-\frac{K}{\sqrt{eE}}\right)^{2},\\
\left\langle T^{kk}(x)\right\rangle_{+}^{p/\bar{p}} &= \frac{r^{\mathrm{cr}}}{2\pi}\log\frac{\sqrt{eE}(x-x_{\mathrm{L}})}{K}\,,\\
k &= 2,\ldots,d-1\,,
\end{aligned}
$$

if $\sqrt{eE}(x-x_{\mathrm{L}})\gg K$;

$$
\begin{aligned}
\left\langle J^{1}(x)\right\rangle_{-}^{p} &= -\left\langle J^{1}(x)\right\rangle_{-}^{\bar{p}}=-\left\langle J^{0}(x)\right\rangle^{p/\bar{p}}\,,\\
\left\langle J^{0}(x)\right\rangle_{-}^{p/\bar{p}} &= -e\,r^{\mathrm{cr}}\left(x_{\mathrm{R}}-x-\frac{K}{\sqrt{eE}}\right)\,,\\
\left\langle T^{10}(x)\right\rangle_{-}^{p} &= -\left\langle T^{10}(x)\right\rangle_{-}^{\bar{p}}=-\left\langle T^{00}(x)\right\rangle_{-}^{p/\bar{p}}\,,\\
\left\langle T^{11}(x)\right\rangle_{-}^{p/\bar{p}} &= \left\langle T^{00}((x)\right\rangle_{-}^{p/\bar{p}}\,,\\
\left\langle T^{00}(x)\right\rangle_{-}^{p/\bar{p}} &= \frac{1}{2}r^{\mathrm{cr}}eE\left(x_{\mathrm{R}}-x-\frac{K}{\sqrt{eE}}\right)^{2},\\
\left\langle T^{kk}(x)\right\rangle_{-}^{p/\bar{p}} &= \frac{r^{\mathrm{cr}}}{2\pi}\log\frac{\sqrt{eE}(x_{\mathrm{R}}-x)}{K}\,,
\end{aligned}
$$

if $\sqrt{eE}(x_{\mathrm{R}}-x)\gg K$. (36)

It entails that

$$
\begin{aligned}
&\left\langle J^{1}(x)\right\rangle^{p}=-\left\langle J^{1}(x)\right\rangle^{\bar{p}}\approx e\,r^{\mathrm{cr}}L\,,\\
&\left\langle J^{0}(x)\right\rangle^{p/\bar{p}}\approx 2e\,r^{\mathrm{cr}}x\,,\\
&\left\langle T^{10}(x)\right\rangle^{p}=-\left\langle T^{10}(x)\right\rangle^{\bar{p}}=r^{\mathrm{cr}}eELx\,,\\
&\left\langle T^{11}(x)\right\rangle^{p/\bar{p}}=\left\langle T^{00}((x)\right\rangle^{p/\bar{p}}\,,\\
&\left\langle T^{00}(x)\right\rangle_{-}^{p/\bar{p}}\approx r^{\mathrm{cr}}eE\left[\left(\frac{L}{2}\right)^{2}+x^{2}\right]\,,
\end{aligned} \tag{37}
$$

for $|x|<L/2$ and

$$
\begin{aligned}
&\left\langle T^{kk}(x)\right\rangle_{-}^{p/\bar{p}}\approx\frac{r^{\mathrm{cr}}}{2\pi}\log\left\{eE\left[\left(\frac{L}{2}\right)^{2}-x^{2}\right]\right\}\\
&\qquad\text{if } eE\left[\left(\frac{L}{2}\right)^{2}-x^{2}\right]\gg K^{2}\,,\\
&\left\langle T^{kk}(x)\right\rangle_{-}^{p/\bar{p}}\approx\frac{r^{\mathrm{cr}}}{2\pi}\log\left[\sqrt{eE}(\frac{L}{2}\pm x)\right]\\
&\text{if } \sqrt{eE}\left(\frac{L}{2}\mp x\right)\lesssim K,\ k=2,\ldots,d-1\,.
\end{aligned} \tag{38}
$$

Vacuum polarization contribution $\langle J^{\mu}(x)\rangle^{c}$ and $\left\langle T_{\mu\nu}\right\rangle^{c}$ will be calculated in the next section. Here we will show how to connect the matrix elements (37) and (38) with quantities characterizing directly pair production effect.

It should be noted that in strong-field QED with t-steps Heisenberg operators of physical quantities (for example, the kinetic energy operator of the Dirac field) are time-dependent in the general case. That is why one can determine contributions of the final particles, using in-in vacuum means, and setting $t\to\infty$ (which means considering the time instant when the external field is already switched off and all the corresponding effects of the vacuum polarization vanish). In the case under consideration we work with mean values when they already are time independent and another way of actions has to be used to determine contributions of the final particles.

We see from Eq. (36) that the charge density $\left\langle J^{0}(x)\right\rangle_{+}^{p/\bar{p}}$, formed by contributions from the domain D_{x}^{+}, is positive while the charge density $\left\langle J^{0}(x)\right\rangle_{-}^{p/\bar{p}}$, formed by contributions from the domain D_{x}^{-}, is negative. This shows that main contributors to these densities are the created positrons and electrons, respectively. The means $\left\langle J^{0}(x)\right\rangle_{+}^{p/\bar{p}}$ grow along the direction of the electric field as $x\to x_{\mathrm{R}}$, while the means $\left\langle J^{0}(x)\right\rangle_{-}^{p/\bar{p}}$ grow in the opposite direction as $x\to x_{\mathrm{L}}$. Also the energy density $\left\langle T^{00}((x)\right\rangle_{+}^{p/\bar{p}}$ and the pressure component along the direction of the electric field $\left\langle T^{11}(x)\right\rangle_{+}^{p/\bar{p}}$, increase as $x\to x_{\mathrm{R}}$, while the energy density $\left\langle T^{00}((x)\right\rangle_{-}^{p/\bar{p}}$ and the pressure $\left\langle T^{11}(x)\right\rangle_{-}^{p/\bar{p}}$ increase as $x\to x_{\mathrm{L}}$. Moving along the direction of the electric field, positrons exit the region S_{int} at $x=x_{\mathrm{R}}$, while electrons moving in the opposite direction exit the region S_{int} at $x=x_{\mathrm{L}}$. These particles maintain directions of their movements after leaving the region S_{int} at $x>x_{\mathrm{R}}$ and $x<x_{\mathrm{L}}$. Once outside the region S_{int}, these particles are not affected by the local effects of vacuum polarization, and cannot change after the field is turned off. Hence, these are final particles. Their state is described by out-solutions, given by Eq. (14), see Ref. [23]. Since the distances $x-x_{\mathrm{L}}$ and $x_{\mathrm{R}}-x$ are much larger than the formation length Δl_{0} of the created pair, one can use the semiclassical description of particle motion. From this point of view, an electron–positron pair is created with the same probability at any point inside of the region S_{int}. The particles are created with a small kinetic energy, which then increases. In the domains D_{x}^{+} and D_{x}^{-} increments of particle kinetic energy are, $|\pi_{0}(x)|-|\pi_{0}(x_{\mathrm{L}})|=eE(x-x_{\mathrm{L}})$ and $\pi_{0}(x)-\pi_{0}(x_{\mathrm{R}})=eE(x_{\mathrm{R}}-x)$ respectively. These particles are ultrarelativisic, therefore longitudinal momenta of the particles on x hyperplane are defined by their kinetic energies: $p_{x}(x)=|\pi_{0}(x)|$ for positrons with $\pi_{0}(x)\in D_{x}^{+}$ and $p_{x}(x)=-\pi_{0}(x)$ for electrons with $\pi_{0}(x)\in D_{x}^{-}$. It is natural to assume that the created particles observed inside the region S_{int} near the boundaries x_{L} and x_{R} practically do not differ from those observed outside this region and, there-

fore, represent the final particles. This situation is similar to t-step case, when final particles are those that remain after the field is switched off. Thus, we see that fluxes at $x \to x_L$ and at $x \to x_R$ hyperplanes form final particles with energies and momenta from the domains D_x^+ and D_x^-.

Thus, it is enough to know longitudinal currents and the energy fluxes through the surfaces $x = x_L$ and $x = x_R$, given by Eq. (37), that are formed in the region $S_{\rm int}$ to evaluate the contributions of the initial and final states. The normal forms of the operators J^1 and T^{10} with respect to the out -vacuum are:

$$N_{\rm out}\left(J^1\right) = J^1 - \left\langle 0, \text{out}\left|J^1\right|0, \text{out}\right\rangle ,$$
$$N_{\rm out}\left(T^{10}\right) = T^{10} - \left\langle 0, \text{out}\left|T^{10}\right|0, \text{out}\right\rangle .$$

Taking into account Eqs. (30) and (37), we calculate densities of the longitudinal current and energy flux corresponding to the final particles as means with respect to the initial vacuum state,

$$\begin{aligned}
J^1_{\rm cr}(x) &= \left\langle N_{\rm out}\left(J^1\right)\right\rangle_{\rm in} \\
&= \left\langle J^1(x)\right\rangle_{\rm in} - \left\langle J^1(x)\right\rangle_{\rm out} \\
&= \left\langle J^1(x)\right\rangle^{p} - \left\langle J^1(x)\right\rangle^{\bar p} = 2en^{\rm cr} ; \\
T^{10}_{\rm cr}(x) &= \left\langle N_{\rm out}\left[T^{10}(x)\right]\right\rangle_{\rm in} \\
&= \left\langle T^{10}(x)\right\rangle_{\rm in} - \left\langle T^{10}(x)\right\rangle_{\rm out} \\
&= \left\langle T^{10}(x)\right\rangle^{p} - \left\langle T^{10}(x)\right\rangle^{\bar p} \\
&= 2n^{\rm cr} eEx ,
\end{aligned} \tag{39}$$

where $n^{\rm cr} = r^{\rm cr} L$ is the total number density of pairs created per unit time and per unit surface orthogonal to the electric field direction, whereas $r^{\rm cr}$ is the pair-production rate, given by Eq. (20). This rate coincides with the known pair-production rate in a constant uniform electric field, see Ref. [36]. Note that $n^{\rm cr}$ is proportional to the magnitude of the potential step $\Delta U = eEL$. We stress that the longitudinal current density $J^1_{\rm cr}(x)$ is x-independent. The process of the current formation has a constant rate per unit length,

$$\frac{J^1_{\rm cr}(x)}{L} = 2e\, r^{\rm cr} . \tag{40}$$

The energy flux $T^{10}_{\rm cr}(x)$ of the final particles through the surface x, is proportional to the potential energy difference with respect of the hyperplane of the symmetry $x = 0$, $U(x) - U(0) = eEx$ and has the maximal magnitude as $x \to x_L$ and $x \to x_R$,

$$T^{10}_{\rm cr}(x_R) = -T^{10}_{\rm cr}(x_L) = n^{\rm cr}\Delta U . \tag{41}$$

We see that the fluxes of final particles are formed by the fluxes of the positrons moving along the direction of the electric field and electrons moving to the opposite direction.

Comparing two nonzero components of the d-dimensional Lorentz vectors $\left\langle J^1(x)\right\rangle^{p/\bar p}$ and $\left\langle J^0(x)\right\rangle^{p/\bar p}$, given by Eq. (37), we see the relationship of the charge density of created pairs $J^0_{\rm cr}(x)$ with the current densities $\left\langle J^0(x)\right\rangle^{p/\bar p}$. Namely,

$$J^0_{\rm cr}(x) = \left\langle J^0(x)\right\rangle^{p} + \left\langle J^0(x)\right\rangle^{\bar p} = 4e\, r^{\rm cr} x .$$

We see that there exists a charge polarization due to the electric field. In particular,

$$J^0_{\rm cr}(x_L) = -2en^{\rm cr}, \ J^0_{\rm cr}(x_R) = 2en^{\rm cr} . \tag{42}$$

A relation of the energy density $T^{00}_{\rm cr}(x)$ of created pairs to the mean value $\left\langle T^{00}(X)\right\rangle^{p/\bar p}$ can be derived in a similar manner as it was done for the current density. For this, it is suffices to note that the means $\left\langle T^{00}(X)\right\rangle^{p/\bar p}$ and $\left\langle T^{10}(X)\right\rangle^{p/\bar p}$ are two nonzero components of a d-dimensional Lorentz vector. Therefore, by rotating the coordinate system, we obtain relations between all others diagonal elements of the vacuum mean values of EMT. These relations are:

$$T^{\mu\mu}_{\rm cr}(x) = \left\langle T^{\mu\mu}(x)\right\rangle^{p} + \left\langle T^{\mu\mu}(x)\right\rangle^{\bar p} = 2\left\langle T^{\mu\mu}(x)\right\rangle^{p} , \tag{43}$$

where $\langle T^{\mu\mu}(x)\rangle^p$ are given by Eqs. (37) and (38). In particular, we have

$$\begin{aligned}
&T^{00}_{\rm cr}(x_R) = T^{00}_{\rm cr}(x_L) = n^{\rm cr}\Delta U , \\
&T^{kk}_{\rm cr}(x_R) = T^{kk}_{\rm cr}(x_L) = \frac{r^{\rm cr}}{\pi}\log\left(\sqrt{eE}L\right) , \\
&\qquad k = 2, \ldots, d-1 .
\end{aligned} \tag{44}$$

5.2 Vacuum polarization contributions

One can verify using Eqs. (27) and (29) that means $\langle J^\mu(x)\rangle^c$ and $\left\langle T_{\mu\nu}\right\rangle^c$ are finite as $L \to \infty$. That is why they can be calculated in such limit as well. In this relation, we recall that the causal propagator $S^c(X, X')$ in L-constant electric field was calculated in Ref. [43] and its limiting expression as $L \to \infty$ was found. Moreover, we have demonstrated that that expression has the form of the causal propagator $S^c(X, X')$ in T-constant electric field as in the limit $T \to \infty$. In particular, it was shown that the causal propagator given by Eq. (27) can be represented in the Schwinger's integral form,

$$\begin{aligned}
&S^c(X, X') = (\gamma P + m)\Delta^c(X, X') , \\
&\Delta^c(X, X') = \int_0^\infty f(X, X'; s)ds ,
\end{aligned} \tag{45}$$

where

$$f(X, X'; s) = \exp\left(-eE\gamma^0\gamma^1 s\right) f^{(0)}(X, X'; s) ,$$

$$f^{(0)}(X,X';s) = -\left(\frac{-i}{4\pi}\right)^{d/2} \frac{eE}{s^{(d-2)/2}\sinh(eEs)} \exp\Big[-ism^2 - \frac{i}{2}eEy_0(x+x') + \frac{i}{4s}\left|\mathbf{r}_\perp - \mathbf{r}'_\perp\right|^2 - \frac{i}{4}eE\coth(eEs)\left(y_0^2 - y_1^2\right)\Big] \quad (46)$$

is the Fock–Schwinger kernel [1]. Here $y_0 = t - t'$ and $y_1 = x' - x$. The kernel can be represented as the matrix element with respect of eigenvectors of the coordinate operators X^μ,

$$\begin{aligned} &f(X,X';s) = \langle t, \mathbf{r} \mid e^{-isM^2} \mid t', \mathbf{r}'\rangle ,\\ &M^2 = m^2 - i0 - P^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} ,\\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,\ \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] . \end{aligned} \quad (47)$$

This fact allows one to use results obtained in Ref. [34] for the renormalization of the mean values (3), see details in Appendix C.

Using this representation, we calculate $\mathrm{Re}\langle j_\mu(t)\rangle^c$ and $\mathrm{Re}\langle T_{\mu\nu}(t)\rangle^c$. It is easy to see that $\langle j_\mu(t)\rangle^c = 0$, as should be expected due to translational symmetry,[2] and $\langle T_{\mu\nu}(t)\rangle^c = 0\,,\ \mu \neq \nu$.

Let us substitute Eq. (45) into Eq. (3) with account taken of the representation (46). Then nonvanishing nonrenormalized vacuum means can be written as:

$$\begin{aligned} &\langle T_{00}\rangle^c = -\langle T_{11}\rangle^c = J_{(d)}eE\int_{\Gamma_c}\frac{f(X,X;s)}{\sinh(eEs)}ds,\\ &\langle T_{ii}\rangle^c = J_{(d)}eE\int_{\Gamma_c}\frac{\sinh(eEs)}{eEs}f(X,X;s)ds ,\\ &J_{(d)} = 2^{\lfloor d/2\rfloor - 1},\quad i = 2,\ldots,d . \end{aligned}$$

These means can be expressed via nonrenormalized one-loop Heisenberg-Euler Lagrangian $\mathcal{L}$, as

$$\begin{aligned} &\langle T_{00}\rangle^c = -\ \langle T_{11}\rangle^c = E\frac{\partial\mathcal{L}}{\partial E} - \mathcal{L},\ \langle T_{ii}\rangle^c = \mathcal{L} ,\\ &\mathcal{L} = \frac{1}{2}\int_{\Gamma_c}\frac{ds}{s}\mathrm{tr}\, f(X,X;s) ,\\ &\mathrm{tr}\, f(X,X;s) = 2J_{(d)}\cosh(eEs)f^{(0)}(X,X;s) . \end{aligned} \quad (48)$$

To renormalize the mean values (48) it is enough to renormalize the real part of the one-loop effective action $W = \int \mathcal{L}dtd\mathbf{r}$ (see Ref. [34]).

Then we may use the fact that, real parts of the renormalized finite vacuum mean values are expressed via the renormalized effective Lagrangian (C3) as:

$$\begin{aligned} \mathrm{Re}\,\langle T_{00}\rangle^c_{\mathrm{ren}} &= -\mathrm{Re}\,\langle T_{11}\rangle^c_{\mathrm{ren}}\\ &= E\frac{\partial \mathrm{Re}\mathcal{L}_{\mathrm{ren}}}{\partial E} - \mathrm{Re}\mathcal{L}_{\mathrm{ren}} ,\\ \mathrm{Re}\,\langle T_{ii}\rangle^c_{\mathrm{ren}} &= \mathrm{Re}\mathcal{L}_{\mathrm{ren}} . \end{aligned} \quad (49)$$

Thus, taking into account Eq. (C8), one can see that in the strong-field case, the quantities (49) have the following behavior:

$$\mathrm{Re}\left\langle T_{\mu\mu}\right\rangle^c_{\mathrm{ren}} \sim \begin{cases} |eE|^{d/2},\ d \neq 4n \\ |eE|^{d/2}\log\left(eE/\mu^2\right),\ d = 4n \end{cases} .$$

Finally, we have obtained nonperturbative one-loop representations for the mean current densities and renormalized EMT of a Dirac field in the L-constant electric background as:

$$\begin{aligned} &\left\langle J^\mu(x)\right\rangle_{\mathrm{in}} = \left\langle J^\mu(x)\right\rangle^p ,\\ &\left\langle J^\mu(x)\right\rangle_{\mathrm{out}} = \left\langle J^\mu(x)\right\rangle^{\bar{p}} ,\\ &\left\langle T_{\mu\nu}(x)\right\rangle^{\mathrm{ren}}_{\mathrm{in}} = \mathrm{Re}\left\langle T_{\mu\nu}\right\rangle^c_{\mathrm{ren}} + \mathrm{Re}\left\langle T_{\mu\nu}(x)\right\rangle^p ,\\ &\left\langle T_{\mu\nu}(x)\right\rangle^{\mathrm{ren}}_{\mathrm{out}} = \mathrm{Re}\left\langle T_{\mu\nu}\right\rangle^c_{\mathrm{ren}} + \mathrm{Re}\left\langle T_{\mu\nu}(x)\right\rangle^{\bar{p}} . \end{aligned} \quad (50)$$

Here $\mathrm{Re}\left\langle T_{\mu\nu}\right\rangle^c_{\mathrm{ren}}$ is given by Eq. (49), other terms are related by Eq. (37) and (38), and expressed via characteristics of pair creation as $\langle J^\mu(x)\rangle^p = J^\mu_{\mathrm{cr}}(x)/2$ and $\mathrm{Re}\langle T^{\mu\nu}(x)\rangle^p = T^{\mu\nu}_{\mathrm{cr}}(x)/2$. The components $\mathrm{Re}\left\langle T_{\mu\nu}\right\rangle^c_{\mathrm{ren}}$ describe the contribution due to vacuum polarization. These components are local. The components $J^\mu_{\mathrm{cr}}(x)$ and $T^{\mu\nu}_{\mathrm{cr}}(x)$ describe the contribution due to the creation of real particles from vacuum. They are global quantities and growing unlimited as the magnitude of potential energy tends to infinity.

6 Discussion and summary

In this work we draw the reader's attention to the fact that the technique of nonperturbative calculating of vacuum instability effects based on the original formulation of the strong-field QED with x-electric steps proposed in Refs. [21,22] must be refined and supplemented by a certain regularization procedure studying the problem of local mean values, see Sect. 2. Here we illustrate general considerations by the case of strong-field QED with L-constant field (which can be interpreted as an electric field between capacitor plates). In the same case, we propose a convenient volume regularization procedure with respect of the time-independent inner product on the t-constant hyperplane. At the same time we find adequate representations (27), (28), and (26) for all the involved singular spinor functions. Using the regularization procedure and the singular functions, we calculate the vacuum mean values of current density and EMT (3) that are local physical quantities. The new approach allows us to separate in these mean values global contributions due to

[2] Note that this current can nevertheless result in finite contributions to higher-loop diagrams; see Ref. [44].

the particle creation from local contributions due to the vacuum polarization. In Sect. 5.2 we show that real parts of the vacuum polarization contributions to EMT can be expressed via the renormalized effective Heisenberg–Euler Lagrangian. Finally, we have obtained nonperturbative one-loop representations for the mean current densities and renormalized EMT (50).

It's believed that in the limiting case $L \to \infty$ the L-constant field is a suitable regularization of the constant uniform electric field in course of describing the vacuum instability effects when the field region is considered to be small compared to the entire field region and far enough from its boundaries. In this relation, it is demonstrated that the longitudinal current density of created particles $J^{1}_{\mathrm{cr}}(x) = 2en^{\mathrm{cr}}$ is x-independent and the process of the current formation has a constant rate per unit length (40) that coincides with the known pair-production rate in a constant uniform electric field. This fact confirms the above supposition and justifies the proposed regularization procedure (24).

The new approach applied to study the vacuum instability in the L-constant field allows us to reveal details that could not be detected by calculations in the homogeneous electric field. For example, the obtained formulas show explicitly that the current density and EMT of created particles are formed separately by contributions of created positrons and created electrons. The behavior of these quantities can be described as follows. They grow with the increase the potential energy differences with respect of the symmetry hyperplane $x = 0$ and reach maximal magnitudes near the capacitor plates, namely as $x \to x_{\mathrm{L}}$ and as $x \to x_{\mathrm{R}}$. They are growing unlimited as the magnitude of the potential energy ΔU tends to infinity. Note that it explains initiation of secularly growing loop corrections to two-point correlation functions in the case of the time-independent electric field given by a linear potential step [45]. The longitudinal energy flux of final particles on both sides of the hyperplane $x = 0$ is directed in opposite directions and a charge polarization occurs due to the electric field. Continuing to move along the direction of the electric field, the positrons leave the field at the point $x = x_{\mathrm{R}}$, and the electrons moving in the opposite direction leave the field at the point $x = x_{\mathrm{L}}$. The current density and EMT calculated for these separated fluxes of electrons and positrons [22] add up to results that are consistent with the results obtained in this article.

It is useful to compare the obtained results with results on the study of the vacuum instability in the L-constant electric field presented in the work [22]. In the latter work, it was calculated the current densities and the energy flux densities of electrons and positrons, after the instant when these fluxes become completely separated and have left the region S_{int} . In the framework of the approach formulated in the present article, it is impossible to consider processes of particles leaving the region S_{int} boundaries. Nevertheless, based on physical considerations, we can expect a certain agreement between both results. In particular, we see that current density and EMT components given by Eqs. (39), (41), (42), and (44) for $x \to x_{\mathrm{L}}$ and $x \to x_{\mathrm{R}}$ are sums of the corresponding values obtained in Ref. [22] separately for electrons and positrons. This may be considered as an additional evidence that the proposed renormalization procedure (24) is consistent.

We believe also that results obtained in this work may contribute to a further development of the locally constant field approximation which is not based on the Heisenberg–Euler Lagrangian approach.

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Appendix A: Decomposition of observable into plane waves

In the framework of a field theory an observable F can be realized as an inner product of the type (21) of localizable wave packets $\psi(X)$ and $\hat{F}\psi'(X)$,

$$F\left(\psi, \psi'\right) = \left(\psi, \hat{F}\psi'\right) ,$$

where $\hat{F}$ is a differential operator and $\psi(X)$ and $\psi'(X)$ are solutions of the Dirac equation. Assuming that an observable $F\left(\psi, \psi'\right)$ is time-independent during the time T one can represent this observable in the following form of an average value over the period T:

$$\langle F\rangle = \frac{1}{T}\int_{-T/2}^{+T/2} F\left(\psi, \psi'\right) dt .$$

In general the wave packets $\psi(X)$ and $\psi'(X)$ can be decomposed into plane waves $\psi_n(X)$ and $\psi'_n(X)$ with given n,

$$\psi(X) = \sum_n \alpha_n \psi_n(X), \;\; \psi'(X) = \sum_n \alpha'_n \psi'_n(X) ,$$

where $\psi_n(X)$ and $\psi_n'(X)$ are superpositions of the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$. Taking into account the orthogonality relation (12) one finds that the decomposition of $\langle F\rangle$ into plane waves with given n does not contain interference terms,

$$\langle F\rangle = \sum_n F\left(\alpha_n\psi_n, \alpha_n'\psi_n'\right) .$$

Appendix B: Integrals on t-constant hyperplane

Integrating in (21) over the coordinates $\mathbf{r}_\perp$ and using the structure of constant spinors v_σ that enter the states $\psi_n(X)$ and $\psi_{n'}'(X)$, we obtain:

$$\left(\psi_n, \psi_{n'}'\right) = \delta_{\sigma,\sigma'}\delta_{\mathbf{p}_\perp,\mathbf{p}_\perp'}V_\perp\mathcal{R},\ \ \mathcal{R} = \int_{-K^{(\mathrm{L})}}^{K^{(\mathrm{R})}}\Theta dx ,$$
$$\Theta = e^{i(p_0-p_0')t}\varphi_n^*(x)\left[p_0+p_0'-2U(x)\right]$$
$$\times\left[p_0'-U(x)+i\partial_x\right]\varphi_{n'}'(x) .$$

Then we represent the integral $\mathcal{R}$ as follows

$$\mathcal{R} = \int_{-K^{(\mathrm{L})}}^{-k^{(\mathrm{L})}}\Theta dx + \int_{-k^{(\mathrm{L})}}^{k^{(\mathrm{R})}}\Theta dx + \int_{k^{(\mathrm{R})}}^{K^{(\mathrm{R})}}\Theta dx , \tag{B1}$$

where $0 < k^{(\mathrm{L})} \ll K^{(\mathrm{L})}$ and $0 < k^{(\mathrm{R})} \ll K^{(\mathrm{R})}$. Parameters $k^{(\mathrm{L,R})}$ are selected so that one can use the asymptotic behavior of WPCF's with large $|\xi|$. It can be seen that for a particular case of the plane waves with equal n and $\varphi_n'(x) = \varphi_n(x)$ the kernel Θ is real constant. Therefore that integrals over intervals $\left[-K^{(\mathrm{L})}, -k^{(\mathrm{L})}\right]$ and $\left[k^{(\mathrm{R})}, K^{(\mathrm{R})}\right]$ in Eq. (B1) are proportionate to lengths of these intervals. In the case of a sufficiently large length L, one can assume that both lengths $K^{(\mathrm{L})} - k^{(\mathrm{L})}$ and $K^{(\mathrm{R})} - k^{(\mathrm{R})}$ are large too, it is of order of length L and much larger than interval $k^{(\mathrm{R})} + k^{(\mathrm{L})}$. In this case the contribution to the integral (B1) from last interval can be ignored. Thus, the value of the integral (B1) is basically determined by the first and last terms. To calculate these integrals, it is enough to use the asymptotic behavior of WPCF's both in region with $\xi < 0$ and in the region with $\xi > 0$. Note that in intervals $\left[-K^{(\mathrm{L})}, -k^{(\mathrm{L})}\right]$ and $\left[k^{(\mathrm{R})}, K^{(\mathrm{R})}\right]$ the modulus of a longitudinal momentum is well defined as $|p_x(x)| = \sqrt{[\pi_0(x)]^2 - \pi_\perp^2}$. Further calculation is no different from what is given in the Appendix B in Ref. [21]. In particular, one sees that all the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ having different quantum numbers n are orthogonal with respect to the introduced inner product on the hyperplane $t = \text{const}$. One can see that the norms of the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ with respect to the inner product (21) are proportional to the macroscopically large parameters

$$\tau^{(\mathrm{L})} = K^{(\mathrm{L})}/v^{\mathrm{L}},\ \tau^{(\mathrm{R})} = K^{(\mathrm{R})}/v^{\mathrm{R}},$$

where $v^{\mathrm{L}} = |p_x(x)/\pi_0(x)| \to c$ and $v^{\mathrm{R}} = v^{\mathrm{L}} = |p_x(x)/\pi_0(x)| \to c$ $(c = 1)$ are absolute values of longitudinal velocities of particles in the spatial regions where $|\xi|$ is large.

Finally, one obtains the orthonormality relations (23).

Appendix C: Ultraviolet renormalization

The one-loop effective action $W = \int\mathcal{L}dtd\mathbf{r}$ can be represented as $W = (-i/2)\ln\det M^2$. After passing to the Euclidean metric

$$t \to -i\eta,\quad \partial_t \to i\partial_\eta,\quad eE \to -iB,\quad B > 0 ,$$

M^2 becomes the elliptic operator $\tilde{M}^2$ and W becomes the effective action $\tilde{W} = -i\left[\int\mathcal{L}d\eta d\mathbf{r}\right]_{qE\to iB}$ over the Euclidean space. To carry out the renormalization procedure, we first introduce the generalized zeta function of the operator $\tilde{M}^2$ in d-dimensional Euclidean space using the heat kernel $K(u)$:

$$\zeta^{(d)}(s) = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}du\, K(u) ,$$
$$K(u) = \int d\eta d\mathbf{r}\,\mathrm{tr}\, f_{Eucl}(X, X; u) ,$$
$$f_{Eucl}(X, X; u) = \left\langle \eta, \mathbf{r}\left| e^{-\frac{u}{\mu^2}\tilde{M}^2}\right|\eta, \mathbf{r}\right\rangle. \tag{C2}$$

Here μ is a normalization constant with the mass dimension which is necessary for the generalized zeta function to be dimensionless. Note that for on-shell renormalization one has $\mu = m$. One can write:

$$\ln\det\tilde{M}^2 = \mathrm{tr}\ln\tilde{M}^2 = -\left.\frac{d\zeta^{(d)}(s)}{ds}\right|_{s=0} .$$

The renormalized effective Lagrangian can be expressed in terms of the generalized zeta function as:

$$\mathrm{Re}\mathcal{L}_{\mathrm{ren}} = \mathrm{Re}\left.\tilde{\mathcal{L}}\right|_{B=ieE} ,$$
$$\tilde{\mathcal{L}} = -\frac{1}{2\Omega_{(d)}}\left.\frac{d\zeta^{(d)}(s)}{ds}\right|_{s=0} ,$$
$$\Omega_{(d)} = \int d\eta d\mathbf{r} . \tag{C3}$$

Next, we calculate $\mathcal{L}_{\mathrm{ren}}$. Comparing Eq. (C2) with Eq. (47) and given the expression (46) for the kernel, we calculate the trace,

$$\begin{aligned}\mathrm{tr}\, f_{Eucl}(X,X;u) &= -\,\mathrm{tr}\, f\left(X,X;-\frac{iu}{\mu^2}\right)\Big|_{eE\to -iB}\\ &= 2J_{(d)}\frac{Bu}{\mu^2}\left(\frac{\mu^2}{4\pi u}\right)^{d/2}\coth\left(\frac{Bu}{\mu^2}\right)\\ &\quad\times\exp\left[-\left(\frac{m}{\mu}\right)^2 u\right].\end{aligned}$$

Then for the zeta function in two dimensions we obtain:

$$\begin{aligned}\zeta^{(2)}(s) &= \frac{\Omega_{(2)}}{2\pi\Gamma(s)}\\ &\quad\times\int_0^\infty u^{s-1}B\coth\left(\frac{Bu}{\mu^2}\right)\exp\left[-\left(\frac{m}{\mu}\right)^2 u\right]du\,.\end{aligned}$$

The integral over u can be expressed in terms of the Hurwitz zeta function as follows:

$$\zeta_{\mathrm{H}}(s,a)=\sum_{k=0}^{\infty}(k+a)^{-s},\quad \mathrm{Re}s>1\,,$$

whose analytic continuation to the entire complex plane can be given by the integral representation

$$\zeta_{\mathrm{H}}(s,a)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\frac{e^{-ax}}{1-e^{-x}}dx\,.$$

Then for zeta function $\zeta^{(2)}(s)$ we obtain (see Refs. [46,47]):

$$\zeta^{(2)}(s)=\begin{cases}\frac{\Omega_{(2)}B}{2\pi}\left[2\left(\frac{2B}{\mu^2}\right)^{-s}\zeta_{\mathrm{H}}\left(s,1+\frac{m^2}{2B}\right)+\left(\frac{m^2}{\mu^2}\right)^{-s}\right],\\ m\neq 0\\ \Omega_{(2)}\frac{B}{\pi}\left(\frac{2B}{\mu^2}\right)^{-s}\zeta_{\mathrm{R}}(s)\,,\quad m=0\end{cases}\tag{C4}$$

For $d>2$, the zeta function $\zeta^{(d)}(s)$ can be expressed in terms of the function $\zeta^{(2)}(s)$ as follows:

$$\begin{aligned}\zeta^{(d)}(s) &= J_{(d)}\frac{\Omega_{(d)}}{\Omega_{(2)}}\left(\frac{\mu^2}{4\pi}\right)^{d/2-1}\\ &\quad\times\frac{\Gamma(s-d/2+1)}{\Gamma(s)}\zeta^{(2)}(s-d/2+1)\,.\end{aligned}\tag{C5}$$

Using Eq. (C3), we obtain the real part of $\mathcal{L}_{\mathrm{ren}}$ for an arbitrary dimension d in the following form:

$$\begin{aligned}\mathrm{Re}\mathcal{L}_{\mathrm{ren}} &= -\frac{1}{2\Omega_{(d)}}\mathrm{Re}\left.\frac{d\zeta^{(d)}(s)}{ds}\right|_{s=0,\,B=ieE}\\ &= -\frac{J_{(d)}}{2\Omega_{(2)}}\left(\frac{\mu^2}{4\pi}\right)^{d/2-1}\\ &\quad\times\mathrm{Re}\left.\frac{d}{ds}\left\{\frac{\Gamma(s-d/2+1)}{\Gamma(s)}\zeta^{(2)}(s-d/2+1)\right\}\right|_{s=0,\,B=ieE}.\end{aligned}$$

The derivative of the zeta function $\zeta^{(d)}(s)$ at the point $s=0$ reads:

$$\begin{aligned}\left.\frac{d\zeta^{(d)}(s)}{ds}\right|_{s=0} &= J_{(d)}\left(\frac{\mu^2}{4\pi}\right)^{d/2-1}\\ &\quad\times\begin{cases}\frac{(-1)^{d/2-1}}{\Gamma(d/2)}\left[\zeta^{(2)\prime}\left(1-\frac{d}{2}\right)+(\gamma+\psi(\frac{d}{2}))\zeta^{(2)}\left(1-\frac{d}{2}\right)\right],\\ for\,even\,d\\ \Gamma\left(1-\frac{d}{2}\right)\zeta^{(2)}\left(1-\frac{d}{2}\right),\ for\,odd\,d\end{cases}.\end{aligned}\tag{C6}$$

For odd d, Eq. (C6) implies that

$$\left.\frac{d\zeta^{(d)}(s)}{ds}\right|_{s=0}=\Gamma(s)\zeta^{(d)}(s)\Big|_{s=0},\quad for\,odd\,d\,.\tag{C7}$$

The corresponding final expressions for $\tilde{\mathcal{L}}$ in $d=2{,}3,4$ dimensions are treated in detail in Ref. [46]. For example, for $d=4$ and $m\neq 0$ we obtain:

$$\begin{aligned}\tilde{\mathcal{L}}_{d=4} &= \Omega_{(4)}\left(\frac{B}{\pi}\right)^2\left\{\left[\log\frac{2B}{\mu^2}-1\right]\zeta_H\left(-1,1+\frac{m^2}{2B}\right)\right.\\ &\quad\left.-\frac{\partial}{\partial s}\zeta_H\left(s-1,1+\frac{m^2}{2B}\right)\Big|_{s=0}+\frac{m^2}{2B}\left(\log\frac{m}{\mu}-\frac{1}{2}\right)\right\},\\ \mathrm{Re}\mathcal{L}^{d=4}_{\mathrm{ren}} &= \mathrm{Re}\tilde{\mathcal{L}}_{d=4}\Big|_{B=ieE}.\end{aligned}$$

In particular, in $d=3$ dimension and for $m\neq 0$ we have:

$$\begin{aligned}\mathrm{Re}\mathcal{L}^{d=3}_{\mathrm{ren}} &= \mathrm{Re}\left\{\frac{B}{2\pi}\sqrt{2B}\right.\\ &\quad\left.\times\left[\zeta_H\left(-\frac{1}{2},\frac{m^2}{2B}+1\right)+\frac{1}{2}\sqrt{\frac{m^2}{2B}}\right]_{B=ieE}\right\}.\end{aligned}$$

Using the relation (see Ref. [46], formula (4.5))

$$\begin{aligned}&\zeta_H\left(-\frac{1}{2},\frac{m^2}{2B}+1\right)=\zeta_R\left(-\frac{1}{2}\right)\\ &\quad-\sum_{l=1}^{\infty}(-1)^l\frac{(2l-3)!!}{2^l l!}\left(\frac{m^2}{2B}\right)^l\zeta_R\left(l-\frac{1}{2}\right),\end{aligned}$$

we obtain for small $m^2/(2\,|eE|)$ the following result:

$$\begin{aligned}\mathrm{Re}\mathcal{L}^{d=3}_{\mathrm{ren}} &= \frac{1}{8\pi^2}(eE)^{3/2}\zeta_R\left(\frac{3}{2}\right)+\frac{m^2}{8\pi}\sqrt{eE}\zeta_R\left(\frac{1}{2}\right)\\ &\quad-\frac{m^3}{4\pi}\sum_{l=2}^{\infty}(-1)^l\frac{\sin\frac{\pi l}{2}-\cos\frac{\pi l}{2}}{\sqrt{2}}\frac{(2l-3)!!}{2^l l!}\\ &\quad\times\left(\frac{m^2}{2eE}\right)^{l-3/2}\zeta_R\left(l-\frac{1}{2}\right).\end{aligned}$$

Note that when the field is very strong, $m^2/(eE)\ll 1$, the main contributions to $\mathcal{L}_{\mathrm{ren}}$ are determined by Eq. (C3) as $m\to 0$. Using (C4), (C5) and (C7) for these contributions, we obtain:

$$\mathrm{Re}\mathcal{L}_{\mathrm{ren}} \approx \frac{1}{2}\mathrm{Re}\begin{cases}\left[\log\left(\frac{B}{\mu^2}\right)\frac{\zeta^{(d)}(0)}{\Omega_{(d)}}\right]_{B=ieE}, & \text{for even } d\\ -\left[\Gamma(s)\frac{\zeta^{(d)}(s)}{\Omega_{(d)}}\Big|_{s=0}\right]_{B=ieE}, & \text{for odd } d\end{cases}.$$

In particular,

$$\mathrm{Re}\mathcal{L}_{\mathrm{ren}} \approx -\begin{cases}\mathrm{Re}\left[\frac{B}{4\pi}\log\left(\frac{B}{\mu^2}\right)\right]_{B=ieE} = -\frac{eE}{8}, & d=2\\ \mathrm{Re}\left[\frac{1}{2\pi^2}\left(\frac{B}{2}\right)^{3/2}\zeta_R\left(\frac{3}{2}\right)\right]_{B=ieE} = -\frac{(eE)^{3/2}}{8\pi^2}\zeta_R\left(\frac{3}{2}\right), & d=3\\ \mathrm{Re}\left[\frac{1}{2}\Omega_{(4)}\zeta'^{(4)}(0)\right]_{B=ieE} \approx \frac{(eE)^2}{24\pi^2}\log\left(\frac{eE}{\mu^2}\right), & d=4\end{cases}.$$

In the general case, for a very strong electric field, we have:

$$\mathrm{Re}\mathcal{L}_{\mathrm{ren}} \sim \begin{cases}|eE|^{d/2}, & d \neq 4n\\ |eE|^{d/2}\log\left(eE/\mu^2\right), & d = 4n\end{cases}. \tag{C8}$$

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