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Non-metric geometry as the origin of mass in gauge theories of scale invariance

To the memory of my mathematics teacher

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Abstract We discuss gauge theories of scale invariance beyond the Standard Model (SM) and Einstein gravity. A consequence of gauging this symmetry is that their underlying 4D geometry is non-metric ($\nabla_\mu g_{\alpha\beta} \neq 0$). Examples of such theories are Weyl's *original* quadratic gravity theory and its Palatini version. These theories have spontaneous breaking of the gauged scale symmetry to Einstein gravity. All mass scales have a geometric origin: the Planck scale (M_p), cosmological constant (Λ) and the mass of the Weyl gauge boson (ω_μ) of scale symmetry are proportional to a scalar field vev that has an origin in the (geometric) $\tilde{R}^2$ term in the action. With ω_μ of non-metric geometry origin, the SM Higgs field also has a similar origin, generated by Weyl boson fusion in the early Universe. This appears as a microscopic realisation of "matter creation from geometry" discussed in the thermodynamics of open systems applied to cosmology. Unlike in local scale invariant theories (with no ω_μ present) with an underlying pseudo-Riemannian geometry, in our case: (1) there are no ghosts and no additional fields beyond the SM and underlying Weyl or Palatini geometry, (2) the cosmological constant is positive and is small because gravity is weak, (3) the Weyl or Palatini connection shares the Weyl (gauge) symmetry of the action, and: (4) there exists a non-trivial, conserved Weyl current of this symmetry. An intuitive picture of non-metricity and its relation to mass generation is also provided from a solid state physics perspective where it is common and is associated with point defects (metric anomalies) of the crystalline structure.

1 Introduction

Scale symmetry may play a role in physics beyond the Standard Model (SM) and Einstein gravity. This is suggested by the fact that the SM with a vanishing Higgs mass (parameter) is scale invariant [1]. Moreover, at high energies or in the early Universe, the states of the SM are essentially massless and the theory can have a scale symmetry (global, local or gauged scale symmetry)[1]. But Einstein gravity breaks such symmetry, hence one can attempt to generate it as a spontaneously broken phase of a theory with a local or gauged scale symmetry (regarding global scale symmetry, it does not survive black hole physics [2]).

Since gravity "is" geometry, the first question is what the underlying 4D space-time geometry of a theory beyond Einstein gravity and SM is. If the action is locally scale invariant, one could expect that this should also be, for consistency, a symmetry of the underlying geometry i.e. of the connection. But the (pseudo)-Riemannian geometry and its Levi-Civita connection are not (Weyl) locally scale invariant. One may then seek an alternative geometry whose connection has the space-time symmetry of the action. A stronger motivation to do so is the gauge principle: similarly to the SM as a (quantum) gauge theory, we seek a *gauge* theory of scale invariance that recovers Einstein gravity in its broken phase.

This principle leads us to consider the Weyl conformal geometry [3–5] because Weyl connection does have a gauged scale symmetry also known as *Weyl gauge symmetry*. We also consider the Palatini approach to gravity [6] where the offshell connection, being independent of the metric, has this symmetry, too. In a gauged scale invariant theory its

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[1] In this work we make a distinction between local scale symmetry and gauged scale symmetry in that in the former case there is no gauge boson ω_μ present (or it is "pure gauge" see later).

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underlying Weyl geometry is *non-metric* i.e. $\tilde{\nabla}_\mu g_{\alpha\beta} \neq 0$; the Palatini version with such symmetry is also non-metric [7]. This differs from other theories beyond SM and Einstein gravity that assume that the underlying geometry is a metric (pseudo)Riemannian geometry ($\nabla_\mu g_{\alpha\beta} = 0$), e.g. as in conformal gravity [8–11], supergravity [12] or strings embedding [13,14]. In our case, non-metricity is present because the gauge field of scale transformations (ω_μ) is *dynamical* as expected in a true gauge theory; this is not true in conformal gravity [15] (also [16,17]) which is then a *metric* theory. Briefly, non-metricity is due to the dynamics[2] of ω_μ and is relevant for mass generation in gauge theories of scale invariance, as we discuss.

Weyl geometry was criticised for its non-metricity [3] but remained of interest [18] even though it failed to describe gravity "plus" electromagnetism as Weyl had initially intended [3–5]. Actually, the same *original* theory of Weyl, which is a *quadratic* gravity theory of action $\sqrt{g}\,(\tilde{R}^2 - F_{\mu\nu}^2)$, defined by Weyl geometry, is indeed a realistic theory of scale symmetry as first shown in [19,20]: it has a spontaneous breaking (Stueckelberg mechanism) to Einstein gravity plus a Proca action of the Weyl gauge boson of scale symmetry. This boson[3] is thus a *massive* gauge field of dilatations that decouples at a high scale (mass $\propto M_p$), below which a (metric) (pseudo)-Riemannian geometry and Einstein gravity are found [19–21]. So Weyl geometry and its gauged scale invariance give a UV completion of (pseudo)-Riemannian geometry and Einstein gravity. We thus have a gauge theory embedding of Einstein gravity – the latter is simply an effective theory and a "low energy" (spontaneously) broken phase of Weyl's original quadratic gravity theory.

In theories with a scale symmetry of the (canonical) Lagrangian, dimensionful couplings are forbidden. Then their mass scales e.g. Planck scale (M_p), cosmological constant (Λ), are generated by vacuum expectation values (vev) of some *additional* scalar field(s) beyond the SM Higgs. These extra fields are added ad-hoc as a "patch up" solution and often are *ghosts* (see later). Further, sometimes the local scale symmetry used is a "fake symmetry" since its associated current is trivial. We want to avoid all these issues that are detailed in Sect. 2. Finally, with gravity related to the underlying geometry, could M_p and Λ have a common, *geometric* origin?

In this work we review the role of non-metricity in solving these problems, based on our results in [7,19–21]. We compare (Sect. 3) the special, metric case (where ω_μ is not dynamical) to the non-metric case of Weyl *quadratic* gravity

[19–21] and of its Palatini version [7].[4] We show how (non-metric) geometry generates *all mass scales* in both Weyl and Palatini cases, *in the absence of matter,* while avoiding the aforementioned issues. These results remain valid if matter is included e.g. the SM. We shall see (Sect. 4) how non-metric geometry (in essence ω_μ) can be responsible for generating the SM Higgs field: this gives a microscopic picture of "matter creation from geometry" usually discussed in the thermodynamics of open systems applied to cosmology [27–30]. We also discuss the absence of a "second clock effect" (the initial Einstein's critique [3] of non-metricity) in a spontaneously broken gauge theory of scale invariance such as Weyl or Palatini theory. Given its important role here, we also provide a more intuitive picture of non-metricity from the solid state physics perspective where it is associated with point defects of the crystalline structure. After Conclusions (Sect. 5), Appendices A and B present technical details.

2 A "metric" example

To detail the above ideas, consider first the (pseudo-) Riemannian geometry and[5]

$$\mathcal{L}_E = -\frac{1}{2}\sqrt{g}\,(M_p^2 R + 2\,M_p^2 \Lambda) \qquad (1)$$

where M_p is the Planck scale and Λ is the cosmological constant. One can regard $\mathcal{L}_E$ as a spontaneously broken phase of a local scale invariant action. This symmetry is defined by invariance under (i) below, extended by (ii) if real scalars (ϕ) or fermions (ψ) are present

(i) $\hat{g}_{\mu\nu} = \Sigma^q\,g_{\mu\nu}, \quad \sqrt{\hat{g}} = \Sigma^{2q}\sqrt{g},$

(ii) $\hat{\phi} = \Sigma^{-q/2}\phi, \quad \hat{\psi} = \Sigma^{-3q/4}\psi, \quad (q=1),$ (2)

where $g = |\det g_{\mu\nu}|$, $\Sigma(x) > 0$ and we set the charge $q = 1$ without loss of generality.[6] Consider implementing this symmetry[7] using [32,33], so one adds 'by hand' a scalar ϕ, then

$$\mathcal{L}_E = \sqrt{g}\left\{\frac{-1}{2}\left[\frac{1}{6}\phi^2\,R + (\partial_\mu \phi)^2\right] \pm \lambda\phi^4\right\}, \qquad (3)$$

is invariant under (2). By a formal transformation (2), with $\Sigma = \phi^2/\langle\phi\rangle^2$, one fixes the gauge of this symmetry i.e. fixes $\hat{\phi}$ to a constant vev, assumed to exist, $\langle\phi^2\rangle = 6M_p^2$, so gauge fixing confirms a dynamical breaking (not vice-versa). Then

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[2] A dynamical ω_μ agrees with the view that at fundamental level gravity may be a theory of connections [22,23].

[3] Which Weyl unfortunately (and wrongly) attempted to identify with the massless, real photon.

[4] Prior to [7,19–21] realistic non-metric theories were linear-only in R and had additional scalars to generate the Planck scale e.g. [24–26].

[5] Our convention is $g_{\mu\nu} = (+, -, -, -)$, $g = |\det g_{\mu\nu}|$ while the curvature tensors are defined as in [31].

[6] The case of arbitrary q in transformations (2) and (4) is recovered by replacing $\alpha \to q\alpha$ in the results.

[7] For some interesting models beyond the SM with this symmetry see e.g. [34–40].

one generates the first term in (1) and ϕ decouples. Regarding the second term in (1), this can be obtained only if one is adding "by hand" a ϕ^4 term to (3) *and* with the right sign!

For our later comparison to "non-metric" cases, notice the following (see also [41–43]):

(a) Symmetry (2) enforces the sign of the kinetic term to be negative so ϕ is a ghost. It is common in conformal/superconformal models that ϕ, which acts as a compensator rather than a physical field, has a kinetic term of negative sign.[8] We would like to avoid a ghost in a classical action.[9] Note also that the sign of $\Lambda \sim \langle\phi\rangle^2$ is *arbitrary*, not fixed by symmetry (2) of action (3).

(b) In the absence of matter the current J^μ associated to symmetry (2) is *trivial* $J^\mu = 0$; this raised concerns on the physical meaning of symmetry (2) which was thus called "fake symmetry" [44,45]. We want to know if a non-trivial current exists in more general cases.

(c) ϕ is added "by-hand" to enforce symmetry (2), as a "compensator", so it is *not related* to the underlying geometry of Einstein gravity emergent in the broken phase. So the Planck scale generated by $\langle\phi\rangle$ is not related to the underlying geometry. Can M_p, Λ and ϕ have a geometric origin?

(d) while $\mathcal{L}_E$ is invariant under (2), the underlying geometry i.e. the Levi-Civita connection is not! Is it really consistent to have a space-time symmetry of an action while its underlying geometry (connection) does not have such symmetry? Can we avoid this?

In the following we shall see how the above issues (a), (b), (c), (d) are elegantly answered in the gauge theory of scale symmetry of Weyl, based on Weyl geometry or in its Palatini version. They ensure a geometric interpretation of this symmetry, something that is not obvious in the local scale symmetry of Eq. (2) (for a discussion on this [37,41]).

3 Non-metric geometry as the origin of mass

3.1 Weyl geometry: metricity vs non-metricity

• **Metric (integrable) case:**

Consider first the case of Weyl conformal geometry.[10] Weyl geometry is defined by classes of equivalence $(g_{\alpha\beta}, \omega_\mu)$ of the metric $(g_{\alpha\beta})$ and the Weyl gauge field (ω_μ), related by the

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[8] This has additional implications e.g. for the scalar potential in 4D N=1 supergravity [12].

[9] Alternatively, if one changed the sign of $\mathcal{L}_E$, then $\langle\phi^2\rangle < 0$, $\Sigma < 0$ but then $g_{\mu\nu}$ changes signature by (2).

[10] For a brief introduction to Weyl conformal geometry and relevant formulae see Appendix A in [21].

gauged scale symmetry (aso known as *Weyl gauge symmetry*) transformation. This symmetry is defined by transformation (2) *together with* that of ω_μ given by:

$$\hat{\omega}_\mu = \omega_\mu - \frac{1}{\alpha}\,\partial_\mu \ln \Sigma, \tag{4}$$

with α the Weyl gauge coupling. By *non-metricity* we mean the presence of dynamical ω_μ in the theory, such that

$$\tilde{\nabla}_\mu g_{\alpha\beta} = -\alpha\,\omega_\mu g_{\alpha\beta}, \tag{5}$$

$\tilde{\nabla}$ is defined by the connection $\tilde{\Gamma}$ of Weyl geometry, see Eq. (B.4). The solution is (Eq. (B.6))

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + (1/2)\,\alpha\,\left[\delta^\lambda_\mu\,\omega_\nu + \delta^\lambda_\nu\,\omega_\mu - g_{\mu\nu}\,\omega^\lambda\right]. \tag{6}$$

where Γ is the Levi-Civita connection $\Gamma^\alpha_{\mu\nu}(g) = (1/2)g^{\alpha\lambda}$ $(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$. $\tilde{\Gamma}$ is *invariant* under combined (2), (4). If ω_μ decouples ($\omega_\mu = 0$) or is "pure gauge", the theory is called Weyl integrable and is metric !. Denote by $\tilde{\Gamma}^\nu_{\mu\nu} = \tilde{\Gamma}_\mu$, $\Gamma^\nu_{\mu\nu} = \Gamma_\mu$, then

$$\omega_\mu = (1/2)\,(\tilde{\Gamma}_\mu - \Gamma_\mu) \tag{7}$$

ω_μ measures the deviation (of the trace) of the connection from the Levi-Civita connection. Since ω_μ is part of the connection $\tilde{\Gamma}$, it obviously has a (non-metric) geometric origin.

The simplest gravity action in Weyl geometry, with symmetry (2), (4) is

$$\mathcal{L}_1 = \sqrt{g}\,\frac{1}{4!\,\xi^2}\tilde{R}^2, \quad \xi < 1, \tag{8}$$

where $\tilde{R} = R(\tilde{\Gamma}, g)$ is the scalar curvature of Weyl geometry, defined by $\tilde{\Gamma}$ of (6) with the usual formulae. $\mathcal{L}_1$ is invariant under (2), (4). This is because $\tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu}(\tilde{\Gamma})$ where $\tilde{\Gamma}$ and thus $\tilde{R}_{\mu\nu}(\tilde{\Gamma})$ are invariant, hence under (2), (4), $\hat{\tilde{R}} = \tilde{R}/\Sigma$ and $\mathcal{L}_1$ is invariant. One can then show (see e.g. [21])

$$\tilde{R} = R - 3\,\alpha\,\nabla_\mu\omega^\mu - \frac{3}{2}\alpha^2\,\omega_\mu\,\omega^\mu, \tag{9}$$

where the rhs is in a Riemannian notation, so $\nabla_\mu\omega^\lambda = \partial_\mu\omega^\lambda + \Gamma^\lambda_{\mu\rho}\,\omega^\rho$. One can replace (9) in $\mathcal{L}_1$. Since $\tilde{R}^2$ contains Riemannian R^2, $\mathcal{L}_1$ is a higher derivative theory that propagates a spin-zero mode (from R^2), in addition to the graviton. It is easy to "unfold" this higher derivative theory into a second order one and extract this spin-zero mode from $\tilde{R}^2$. To this purpose, replace $\tilde{R}^2 \rightarrow -2\phi^2\tilde{R} - \phi^4$ in $\mathcal{L}_1$, where ϕ is a scalar field, to obtain

$$\mathcal{L}_1 = \sqrt{g}\,\frac{1}{4!\,\xi^2}\left[-2\phi^2\tilde{R} - \phi^4\right]. \tag{10}$$

The equation of motion of ϕ has solution $\phi^2 = -\tilde{R}$ which replaced in the action recovers Eq. (8), so Eqs. (8) and (10) are classically equivalent. Next, the equation of motion of

ω_μ is

$$\omega_\mu = (1/\alpha)\,\partial_\mu \ln \phi^2, \tag{11}$$

so ω_μ is "pure gauge" and can be integrated out. Using this back in the action, then

$$\mathcal{L}_1 = \sqrt{g}\,\frac{1}{\xi^2}\left\{ -\frac{1}{2}\left[\frac{1}{6}\phi^2 R + g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right] - \frac{1}{4!}\phi^4 \right\}. \tag{12}$$

This is called a Weyl *integrable* case since ω_μ is not dynamical: its field strength $F_{\mu\nu} = 0$ due to (11). Hence, the theory is *metric*: we apply (4) with $\Sigma = \phi^2$, therefore $\hat{\omega}_\mu = 0$, $\tilde{\Gamma} = \Gamma$, $\tilde{\nabla}_\mu g_{\alpha\beta} = 0$ so the connection is Levi-Civita and the geometry becomes (pseudo)Riemannian.

$\mathcal{L}_1$ has similarities to the case of previous section, see text after Eq. (3), but there are some good features in the case here. Firstly, unlike in (3), here ϕ was *not* added "ad-hoc" but it came from the $\tilde{R}^2$ term in the action. Secondly, the last term in (3) has a definite sign. Assuming that ϕ acquires a vev (e.g. at quantum level, etc), by applying (2) to (12) with $\Sigma = \phi/\langle\phi\rangle$, (or formally setting $\phi = \langle\phi\rangle$ in (12)), one obtains Eq. (1) of Einstein action and a cosmological constant term, with a Planck mass $M_p^2 = \langle\phi\rangle^2/(6\xi^2)$ and $\Lambda = \langle\phi\rangle^2/4$.

Briefly, Weyl quadratic gravity in the *integrable case* has certain advantages compared to the "metric" case in Sect. 2: it generates Einstein gravity and it *explains* the origin of both M_P and Λ as due to (a vev of) ϕ which was *not added* by hand but it has a *geometric* origin in the $\tilde{R}^2$ term. The case here predicts a non-zero (positive) Λ, because both Λ, $M_p \propto \langle\phi\rangle^2$. This also suggests a UV-IR connection in the physics associated to these two scales.

However, similar to Sect. 2, for action (12) the current J^μ associated to symmetry (2) is trivial $J^\mu = 0$, see [44, 45], hence their conclusion that (2) is a fake symmetry. The negative sign kinetic term in (12) may also be a concern in some cases. We want to avoid these two issues, but to retain the good features found above. This is possible in the non-metric case.

• **Non-metric case:**

The above situation improves further if ω_μ has a kinetic term. This brings us to the original Weyl gravity action [3–5] which has a gauged scale symmetry. The action is

$$\mathcal{L}_1' = \sqrt{g}\left[\frac{1}{4!}\frac{1}{\xi^2}\tilde{R}^2 - \frac{1}{4}F_{\mu\nu}^2\right]. \tag{13}$$

Here $F_{\mu\nu} = \tilde{\nabla}_\mu\omega_\nu - \tilde{\nabla}_\mu\omega_\nu$ is the field strength of ω_μ, with $\tilde{\nabla}_\mu\omega_\nu = \partial_\mu\omega_\nu - \tilde{\Gamma}^\rho_{\mu\nu}\omega_\rho$. Since $\tilde{\Gamma}^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}$ is symmetric, $F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, just like in flat space-time.

We "linearise" the $\tilde{R}^2$ term in (13) by replacing $\tilde{R}^2 \to -2\phi^2\tilde{R} - \phi^4$, where ϕ is a scalar field. In the new action, the solution of the equation of motion of ϕ is $\phi^2 = -\tilde{R}$ which

replaced in the action recovers Eq. (13), therefore we obtain a classically equivalent action. Then we replace $\tilde{R}$ in terms of Riemannian R, Eq. (9), to find [19–21]:

$$\mathcal{L}_1' = \sqrt{g}\left\{\frac{1}{4!\,\xi^2}\left[-2\phi^2\tilde{R} - \phi^4\right] - \frac{1}{4}F_{\mu\nu}^2\right\}$$

$$= \sqrt{g}\left\{\frac{-1}{2\,\xi^2}\left[\frac{\phi^2}{6}R + (\partial_\mu\phi)^2 - \frac{\alpha}{2}\nabla_\mu(\omega^\mu\phi^2)\right]\right.$$

$$\left. - \frac{\phi^4}{4!\,\xi^2} + \frac{\alpha^2}{8\,\xi^2}\phi^2\left[\omega_\mu - \frac{1}{\alpha}\partial_\mu\ln\phi^2\right]^2 - \frac{1}{4}F_{\mu\nu}^2\right\}. \tag{14}$$

From this Lagrangian one finds (see Appendix A) that

$$J^\mu = -\alpha/(4\xi^2)\,g^{\mu\nu}(\partial_\nu - \alpha\,\omega_\nu)\phi^2 \tag{15}$$

is a *conserved current*: $\nabla_\mu J^\mu = 0$ [7, 19–21]. The presence here of a non-trivial conserved J^μ is a fundamental difference from the metric/integrable cases discussed earlier and avoids the criticisms of [44, 45]. For the previous case of Weyl integrable geometry of Eq. (12), with ω_μ of (11) one easily verifies that for action (12) $J^\mu = 0$ i.e. the current is trivial.

From current conservation equation $\nabla_\mu J^\mu = 0$ or from Eq. (A.16):

$$\Box\phi^2 - \alpha\nabla^\rho(\omega_\rho\phi^2) = 0, \quad \Box = \nabla^\mu\nabla_\mu. \tag{16}$$

This shows that ϕ is a *dynamical* field, as also obvious from (12) and because $\tilde{R}^2$ contains the higher-derivative Riemannian R^2 that propagates a spin-zero mode beyond graviton. This generalises a conserved current $\nabla^\mu(\nabla_\mu\phi^2) = 0$ of global case [46–50] recovered here for $\alpha = 0$.

One would like to "fix the gauge" of Weyl gauge symmetry. As in the previous metric and integrable cases, assume that ϕ develops a vev e.g. at the quantum level or as in the global case [46–50]. For example in a Friedmann–Robertson–Walker universe, with an "isotropic" $\omega_\mu = (\omega_0(t), 0, 0, 0)$, if $\omega_0(t) \sim 1/\phi(t)^2$, then (16) gives $\Box\phi^2 \approx 0$ whose solution $\phi(t)$ evolves to a constant value (vev) at large t [46–50], so $\phi \to \langle\phi\rangle$. Then, on the ground state the current conservation $\nabla_\mu J^\mu = 0$ gives $\nabla^\rho\omega_\rho = 0$, specific to a massive Proca field ω_μ. This "fixes the gauge" of symmetry (2), (4). At the level of the Lagrangian, this gauge fixing is implemented by applying to $\mathcal{L}_1'$ transformation (2), (4) with a special $\Sigma = \phi^2/\langle\phi^2\rangle$; (or formally replace $\phi \to \langle\phi\rangle$ in Eq. (14)). In terms of the transformed fields (with a "hat"), $\mathcal{L}_1'$ becomes:

$$\mathcal{L}_1' = \sqrt{\hat{g}}\left[-\frac{1}{2}M_p^2\hat{R} + \frac{1}{2}m_\omega^2\,\hat{\omega}_\mu\hat{\omega}^\mu - \Lambda M_p^2 - \frac{1}{4}\hat{F}_{\mu\nu}^2\right], \tag{17}$$

with

$$M_p^2 = \frac{\langle \phi^2 \rangle}{6\,\xi^2}, \quad \Lambda = \frac{1}{4}\langle \phi^2 \rangle, \quad m_\omega^2 = \frac{\alpha^2 \langle \phi^2 \rangle}{4\,\xi^2}. \tag{18}$$

Equation (17) is the Einstein–Proca Lagrangian for the Weyl vector [19–21], in the Einstein gauge ("frame"). The Weyl gauge field has absorbed the derivative of the field[11] $\partial_\mu(\ln\phi)$ via a Stueckelberg mechanism [51], with the total number of degrees of freedom (dof) conserved, as expected for a spontaneous breaking: the massless ω_μ (dof = 2) and real, dynamical ϕ (dof = 1) are replaced by a massive Proca field ω_μ (dof = 3, since $\nabla_\rho \omega^\rho = 0$) [19–21]. Notice that there is no ghost in (17). The mass of ω_μ, $m_\omega \sim \alpha M_p$, is close to M_p unless one is tuning $\alpha \ll 1$; hence, any (unwanted) non-metricity effects due to ω_μ are suppressed by m_ω.

Since ω_μ is massive, it decouples to leave in the broken phase below m_ω the Einstein gravity and a *positive* cosmological constant. At the same time, since ω_μ decouples from the action, metricity *is restored* below m_ω: the connection $\tilde\Gamma$ of (6) becomes Levi-Civita (Γ) and the geometry becomes Riemannian. Hence, Weyl geometry with its gauged scale invariance acts as a non-metric ultraviolet (UV) completion (above m_ω) of Riemannian geometry.

3.2 Palatini theories: metricity vs non-metricity

● **Metric case:**
Let us now consider the Palatini approach [6] to the quadratic gravity actions with a Weyl gauge symmetry, such as Eqs. (8), (13). In this approach the connection is independent of the metric and so it is invariant under (2), (4). The connection is then determined from its equations of motion and this solution is used back in the initial action. Hence, the underlying geometry (connection) and thus its metricity or non-metricity are determined by the action and by its symmetries, as we shall see shortly: if the theory is invariant under (2) the theory is metric; while if it is Weyl gauge invariant, it is non-metric.

To begin with, consider first an action with symmetry (2) in a Palatini approach

$$\mathcal{L}_2 = \sqrt{g}\,\frac{1}{4!\,\xi^2}\,R(\tilde\Gamma, g)^2 \tag{19}$$

where

$$R(\tilde\Gamma, g) = g^{\mu\nu}\,R_{\mu\nu}(\tilde\Gamma),$$
$$R_{\mu\nu}(\tilde\Gamma) = \partial_\lambda \tilde\Gamma_{\mu\nu}^\lambda - \partial_\mu \tilde\Gamma_{\lambda\nu}^\lambda + \tilde\Gamma_{\rho\lambda}^\lambda \tilde\Gamma_{\mu\nu}^\rho - \tilde\Gamma_{\rho\mu}^\lambda \tilde\Gamma_{\nu\lambda}^\rho. \tag{20}$$

This is the Palatini version of action (8) in Weyl geometry, but now $\tilde\Gamma$ is unknown. $R_{\mu\nu}(\tilde\Gamma)$ is the metric-independent Ricci

tensor in the Palatini formalism. Since $\tilde\Gamma$ is independent of the $g_{\mu\nu}$, $\tilde\Gamma$ and $R_{\mu\nu}(\tilde\Gamma)$ are invariant under transformation (2). Therefore, $R(\tilde\Gamma, g)$ transforms like $g^{\mu\nu}$, so under (2):

$$\hat{R}(\tilde\Gamma, \hat{g}) = \frac{1}{\Sigma} R(\tilde\Gamma, g). \tag{21}$$

As a result, $\mathcal{L}_2$ is invariant under (2) and this is the simplest Palatini case that has local scale symmetry (2). $\mathcal{L}_2$ can be linearised as in the Weyl case (Eq. (10)): replace $R(\tilde\Gamma, g)^2 \to -2\phi^2 R(\tilde\Gamma, g) - \phi^4$ to obtain a classically equivalent action to (19). For this $\mathcal{L}_2$ one writes and solves the equation of motion for $\tilde\Gamma$. The solution is (see [7, Section 2]):

$$\tilde\Gamma_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha(g) + (1/2)\left(\delta_\nu^\alpha u_\mu + \delta_\mu^\alpha u_\nu - g^{\alpha\lambda} g_{\mu\nu} u_\lambda\right),$$
$$u_\mu \equiv \partial_\mu \ln \phi^2, \tag{22}$$

with Γ the Levi-Civita connection. With this $\tilde\Gamma$, one computes the scalar curvature

$$R(\tilde\Gamma, g) = R(g) - 3\nabla_\mu u^\mu - \frac{3}{2} g^{\mu\nu} u_\mu u_\nu. \tag{23}$$

$R(g)$ is the scalar curvature for $g_{\mu\nu}$ while ∇ is defined by the Levi-Civita connection (Γ). Using the last equation back in $\mathcal{L}_2$, one finds

$$\mathcal{L}_2 = \sqrt{g}\,\frac{1}{\xi^2}\left\{ -\frac{1}{2}\left[\frac{1}{6}\phi^2 R(g) + (\partial_\mu \phi)^2 \right] - \frac{1}{4!}\phi^4 \right\}. \tag{24}$$

This is the onshell version of (19) and contains a dynamical ϕ. This is because the metric part of the Palatini quadratic gravity[12] makes the action a four-derivative theory: according to (23) onshell $\tilde{R}(\tilde\Gamma, g)^2$ contains $R(g)^2$. Action (24) is similar to that seen in Eqs. (3) and (12) and its associated current vanishes again. Fixing the gauge of the symmetry which essentially means setting $\phi \to \langle\phi\rangle$ then

$$\mathcal{L}_2 = \sqrt{\hat{g}}\left\{ \frac{-1}{2} M_p^2 \hat{R}(\hat{g}) - \frac{3}{2\xi^2} M_p^4 \right\}. \tag{25}$$

And since ϕ is fixed to a constant, $\tilde\Gamma = \Gamma$, so the theory is *metric*, see e.g. discussion in [7]. This is similar to the Weyl integrable (metric) case discussed earlier.

● **Non-metric case:**
The situation changes dramatically if the theory has a gauged scale invariance. Consider

$$\mathcal{L}_2' = \sqrt{g}\left\{ \frac{1}{4!\,\xi^2} R(\tilde\Gamma, g)^2 - \frac{1}{4\alpha^2} R_{[\mu\nu]}(\tilde\Gamma)\, R^{[\mu\nu]}(\tilde\Gamma) \right\} \tag{26}$$

where $R_{[\mu\nu]} \equiv (1/2)(R_{\mu\nu} - R_{\nu\mu})$ with $R_{\mu\nu}$ as in Eq. (20). Additional scale invariant operators of dimension d = 4 can be present. One can check that the two terms in the action

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[11] See the second-last term in the last line of Eq. (14). Here $\ln\phi$ has a shift symmetry Eq. (2) and plays the role of a would-be Goldstone field of the Weyl gauge symmetry. For a detailed discussion see [7,19–21].

[12] Due to the Levi-Civita contribution to $\tilde\Gamma$, Eq. (22).

above are invariant under (2). Next, define

$$F_{\mu\nu}(\tilde{\Gamma}) = \tilde{\nabla}_\mu v_\nu - \tilde{\nabla}_\nu v_\mu, \quad \text{where}$$
$$v_\mu \equiv (1/2)(\tilde{\Gamma}_\mu - \Gamma_\mu), \quad (\tilde{\Gamma}_\mu \equiv \tilde{\Gamma}^\rho_{\mu\rho}, \ \Gamma_\mu \equiv \Gamma^\rho_{\mu\rho}),$$

(27)

so $F_{\mu\nu}$ is a function of $\tilde{\Gamma}$. Since $\tilde{\Gamma}$ is assumed symmetric in the lower indices, then $F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu = \partial_\mu \tilde{\Gamma}_\nu - \partial_\nu \tilde{\Gamma}_\mu = -R_{[\mu\nu]}$. Hence, the last term in (26) acts as a kinetic term for v_μ. Under (2), v_μ transforms like ω_μ of the Weyl case, while $F_{\mu\nu}$ is invariant. Therefore, action (26) has a bigger symmetry: it is Weyl gauge invariant, being invariant under Eqs. (2) and (4) with $\omega_\mu \to v_\mu$ i.e. $\hat{v}_\mu = v_\mu - (1/\alpha)\,\partial_\mu \ln \Sigma$. We denoted by v_μ the Weyl gauge field in the Palatini case, playing the role of ω_μ. With this, Eq. (26) is a Palatini version of the Weyl action in Eq. (13) but now $\tilde{\Gamma}$ is *unknown* – it will be determined by its equations of motion.

From (26) one proceeds as in the Weyl case to "linearise" the $R(\tilde{\Gamma}, g)^2$ term in (26) with the aid of an auxiliary scalar ϕ, so replace $R(\tilde{\Gamma}, g) \to -2\phi^2 R(\tilde{\Gamma}, g) - \phi^4$. From the resulting, equivalent Lagrangian one can then write the equations of motion of the connection $\tilde{\Gamma}^\rho_{\alpha\beta}$ which can be solved. The solution is a function of ϕ and is used to evaluate $R_{\mu\nu}(\tilde{\Gamma})$ and then $R(\tilde{\Gamma}, g)$. Using this result back in action (26) one finally finds [7] (Eq. 29):

$$\mathcal{L}'_2 = \sqrt{g} \left\{ -\frac{1}{2\xi^2} \left[\frac{1}{6} \phi^2 R + (\partial_\mu \phi)^2 \right] \right.$$
$$\left. -\frac{1}{4! \, \xi^2} \phi^4 + \frac{\alpha^2}{2\xi^2} \phi^2 \left[v_\mu - \frac{1}{\alpha^2} \partial_\mu \ln \phi^2 \right]^2 - \frac{1}{4} F^2_{\mu\nu} \right\}.$$

(28)

This result is similar to that in Weyl case Eq. (14), with $v_\mu \to \omega_\mu$. Equation (28) is the "onshell" Lagrangian, that is using the solution of $\tilde{\Gamma}$. $\mathcal{L}'_2$ is invariant under combined Eqs. (2), (4) (with $\omega_\mu \to v_\mu$). A conserved current exists similar to that in Weyl case, see Eq. (15) and Appendix A.

To fix the gauge, the same discussion as in the Weyl case applies. At the level of the Lagrangian this "gauge fixing" may formally be implemented by setting ϕ to a constant vev; this then brings us to the Einstein–Proca Lagrangian

$$\mathcal{L}'_2 = \sqrt{g} \left\{ -\frac{1}{2} M_p^2 R + 3\alpha^2 M_p^2 v_\mu v^\mu - \frac{1}{4} F^2_{\mu\nu} - \Lambda M_p^2 \right\},$$
$$\Lambda \equiv \frac{1}{4} \langle\phi\rangle^2, \quad M_p^2 \equiv \frac{\langle\phi\rangle^2}{6\xi^2}.$$

(29)

There is again a Stueckelberg mechanism, similar to the Weyl case: v_μ becomes massive after "absorbing" the dynamical ϕ which disappears from the spectrum of (29) and the number of degrees of freedom is conserved in going from (28) to (29). The Einstein–Proca action of v_μ is thus found. We see that the Stueckelberg breaking of a gauged scale symmetry is valid in a Palatini quadratic gravity model, too. This mass

mechanism may be common in gravity theories where the connection is a dynamical variable [22,23,52].

There are however two differences from the Weyl case: Firstly, there are additional quadratic operators [53] with gauged scale invariance that were not included in this analysis and that can affect the overall result. Secondly, the vectorial non-metricity obtained in the Palatini case for action (26), shown in the Appendix Eq. (B.18), is different from that in the Weyl case Eq. (5). This explains a different numerical coefficient in (28) versus (14).

The Weyl and Palatini cases above show that non-metricity that follows from gauging the scale symmetry is accompanied by a Stueckelberg breaking of this symmetry to Einstein–Proca action.[13] They have further advantages compared to their metric versions: (a) they have a non-trivial conserved current J^μ and (b) there are no ghost fields in actions (17), (29).

4 Phenomenology

4.1 Mass scales from non-metric geometry

We saw that Einstein gravity is a spontaneously broken phase of the original Weyl quadratic gravity or its Palatini version, with Weyl gauge symmetry. We have 3 mass scales: M_p, m_ω, Λ that are all proportional to the vev of the Stueckelberg field $\langle\phi\rangle$ eaten by ω_μ (v_μ), Eqs. (18), (29). Their exact values are fixed by three parameters: $\langle\phi\rangle$, ξ and α. Here ϕ is introduced by a geometric term in the action ($\tilde{R}^2$), so all masses have a *non-metric geometry* origin! ω_μ (v_μ) is also part of the underlying non-metric geometry, too. There is a difference from the Higgs mechanism, since there is no ϕ present in the final action. The cosmological constant is positive (and non-vanishing), due to a ϕ^4 term also induced by $\tilde{R}^2$. This suggests a UV - IR physics connection due to the common origin ($\propto \langle\phi\rangle$) of the scales Λ and M_p.

As in all theories with scale symmetry one can only predict ratios of scales in terms of dimensionless couplings of the theory (ξ, α). One can obtain the correct ratios

$$\Lambda/M_p^2 \sim \xi^2, \quad m_\omega^2/M_p^2 \sim \alpha^2$$

(30)

for suitable (perturbative) values of α and ξ. If the Planck scale M_p is fixed to its value, we see that Λ is small because gravity is ultraweak (coupling $\xi \ll 1$).

We also see that issues (a), (b), (c), (d) encountered in the (pseudo-)Riemannian case with Weyl symmetry of Sect. 2 are now nicely solved. To detail:

(a) There is no ghost degree of freedom in the final spectrum, since ϕ is eaten by ω_μ. Also, the Einstein gravity is

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[13] For a related scale invariant de Sitter gauge theory see [54].

recovered and the sign of Λ is predicted positive, due to the $\sqrt{g}\,\tilde{R}^2$ term;

(b) The current associated to Weyl gauge symmetry is non-trivial ($J^\mu \neq 0$) and is related to the existence of a dynamical ω_μ i.e. to non-metricity;

(c) The field ϕ was not added by hand, but was "extracted" from the $\sqrt{g}\,\tilde{R}^2$ term, hence it has an origin in (non-metric) geometry and the same is true about Λ, M_p and m_ω that $\langle\phi\rangle$ generated;

(d) Finally, the Weyl or Palatini connections are Weyl gauge invariant, hence both the action and its underlying geometry have this symmetry.

4.2 Higgs from non-metric geometry

The discussion so far was in the absence of matter, hence it was about "geometry". The next step is to see the effect of non-metricity in the presence of the SM. Consider then embedding the SM in Weyl conformal geometry – this is indeed possible, as shown in [21]. We refer the reader to this work for the technical details how this is done. This embedding is truly minimal and natural and does not require any additional degrees of freedom beyond those of the SM and of Weyl geometry (ϕ, ω_μ, $g_{\mu\nu}$). This is possible because the SM with a vanishing Higgs mass parameter is scale invariant. In fact, one can easily notice that the Lagrangian of the SM gauge bosons and fermions in the Weyl conformal geometry has a form identical to that in the (pseudo-)Riemannian geometry and has a gauged scale symmetry. Hence, the SM gauge bosons and fermions do not have any direct couplings to the Weyl gauge boson [55,56] (with one special exception for the SM fermions discussed in Section 2.3 of [21]).

However, the SM Higgs sector is modified by the Weyl gauge symmetry [21]. First, there is a non-minimal coupling of the Higgs to Weyl geometry $\sqrt{g}\,H^\dagger H \tilde{R}$ which is Weyl gauge invariant; here H is the $SU(2)_L$ Higgs doublet. The Higgs kinetic term is also modified: the SM covariant derivative $D_\mu H$ is "upgraded" to also become Weyl-covariant; hence the derivative $D_\mu H$ is replaced to include the Weyl gauge boson of scale invariance:

$$D_\mu H \rightarrow (D_\mu - \alpha/2\,\omega_\mu) H, \tag{31}$$

so that this derivative transforms like the Higgs under transformations (2), (4). Hence, the Weyl gauge invariant Higgs kinetic term becomes

$$\sqrt{g}\, g^{\mu\nu} \left[D_\mu - \alpha/2\,\omega_\mu \right) H \right]^\dagger (D_\nu - \alpha/2\,\omega_\nu)\, H. \tag{32}$$

Further, the neutral Higgs boson mixes with the field ϕ that "linearised" $\tilde{R}^2$; their "radial direction" combination is now the new Stueckelberg field eaten by ω_μ (as shown earlier), while the "angular direction" field becomes the SM neutral Higgs, hereafter called σ.

In the canonical Lagrangian, the coupling of ω_μ to σ is (see eqs.(32), (38) in [21]):

$$\mathcal{L}_H = \frac{1}{8}\,\sqrt{g}\,\alpha^2\,\omega_\mu\omega^\mu\,\sigma^2 + \mathcal{O}(\sigma^2/M_p^2). \tag{33}$$

This coupling comes from the Higgs kinetic term mentioned earlier.[14] This is the only direct coupling of the SM to the Weyl gauge boson. It has interesting consequences. In the early Universe, assuming there was no Higgs boson, this coupling can generate the Higgs from the Weyl vector boson fusion

$$\omega_\mu + \omega_\mu \rightarrow \sigma + \sigma. \tag{34}$$

Since ω_μ is part of the non-metric geometry (connection), the Higgs boson itself can have a non-metric, geometric origin! And since the Higgs generates the masses of the SM states while the Stueckelberg field ("extracted" from the $\tilde{R}^2$ term) generated M_p, Λ and m_ω, one concludes that ω_μ and non-metric geometry are the origin of all the masses of the theory. This happens without additional degrees of freedom beyond the SM or Weyl geometry!

Interestingly, the Weyl boson fusion can have an additional effect at a cosmological level, of mitigating any anisotropy that the Weyl vector would otherwise bring. This deserves careful study. A similar coupling and generation of the Higgs via Weyl boson fusion exists when considering the Higgs sector in Palatini gravity with Weyl gauge symmetry [7] (Eq. 45).

The generation of the Higgs field alone, from non-metric geometry (Weyl or Palatini), via ω_μ-ω_μ fusion in the early Universe is interesting.[15] This process appears as a possible microscopic realisation of "matter creation from geometry" discussed in a phenomenological macroscopic description in the thermodynamics of open systems applied to cosmology [27–29]. In such approach the creation of matter occurs as a process corresponding to transfer of energy from the gravitational field(s) (in our case ω_μ) or space-time curvature ($\tilde{R}^2$ that depends on ω_μ) to the matter created (in our case Higgs). The second law of thermodynamics allows space-time geometry transform into matter but the inverse transformation is forbidden. Therefore the process of matter creation from the underlying geometry is irreversible. However, this result is not valid in general but only if the specific entropy per particle (s) is $\dot{s} \leq 0$ (otherwise matter destruction takes place) [30]. If so, it would be interesting to study how irreversibility could emerge from the microscopic picture provided by our SM Lagrangian in Weyl geometry. Notice however that

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[14] The non-minimal coupling $\sqrt{g}\,H^\dagger H \tilde{R}$ impacts on the form of the Higgs potential and the mixing with initial ϕ, see Section 2.5 in [21].

[15] Matter generation from metric geometry is however familiar: e.g. MSSM states as zero modes of string compactifications [13,14], Kaluza–Klein modes of SM states in field theory orbifolds, gravitational particle production during inflation, etc.

what from the Weyl geometry viewpoint looks like matter creation from (Weyl) geometry (ω_μ), from a Riemannian picture obtained after the symmetry breaking, ω_μ looks just like another field of the theory that interacts with the Higgs! Similar considerations apply in the Palatini case.

4.3 Non-metricity and mass hierarchy

With the Higgs and Planck scale related to the underlying non-metric geometry, their hierarchy may be related to this, too. The scale of non-metricity is given by the mass of the gauge boson of scale invariance, $m_\omega \sim \alpha M_p$; below this scale metricity is restored. In general, one would expect that this scale be close to the Planck scale. But it must be mentioned that the current lower bound on the non-metricity scale is very low, of few TeV only [57,58]. Theoretically, this is realised by tuning the coupling $\alpha \ll 1$. Such small α is actually natural, because it is one of the gravity couplings of the theory (ξ, α). For detailed numerical estimates of the couplings ξ, α and Higgs non-minimal coupling, due to constraints from EW precision data and inflation, see [21].

If the non-metricity scale m_ω is low, in the TeV region, then the Higgs mass is natural, as already noticed in [21]. To see this note that quantum corrections δm_σ^2 to the Higgs mass are quadratic in the scale of "new physics" which in our case is m_ω, so

$$\delta m_\sigma^2 \sim m_\omega^2. \tag{35}$$

Above the mass of ω_μ, the gauged scale symmetry is restored together with its UV protection for the Higgs mass; indeed, this is so since no mass counterterm is then allowed and quantum corrections above m_ω could at most be of logarithmic type. This indicates a solution to the hierarchy problem that is technically natural, based on a gauged scale symmetry.[16]

4.4 Non-metricity in solid state physics

We saw that there is a link of non-metric geometry to mass generation which is an essentially geometric mechanism, valid even in the absence of matter fields; only the degrees of freedom of the Weyl geometry, like the metric, connection and curvature-squared terms were involved. The same applied to the Palatini case. One question is whether there is a more intuitive, physical interpretation of non-metricity and its link to mass generation in the condensed matter physics, as it was the case for the Higgs mechanism.

Non-metricity is common in solid state physics where it is associated with some crystalline defects. For our discussion on non-metricity in solid state physics we follow [59–

62]. For this one needs the notion of "material space" of a crystalline solid. This is a natural configuration of a body where it is described only in terms of the intrinsic structure of constituting matter. The material space is the configuration found by relaxing the solid of all internal and external stresses. If the crystalline structure has no defects, the corresponding geometry of this material space is Euclidean. If there is a (continuous) distribution of defects, that destroy the crystalline order, the associated geometry is non-Euclidean.

For a 3D crystalline structure we have defects of dimension d = 0, also known as point defects or metric anomalies, which are destroying the crystalline order; they are modifying the local notion of length, usually associated with this order. These defects can be vacancies (missing atoms), interstitials (extra atoms of same kind), substitutionals (extra atoms of different kind). Further, there are d = 1 defects such as dislocations and disclinations; d = 2 defects (phase boundary, domain walls, etc.) or d = 3 defects (inhomogeneities). With these defects distributed continuously, they give rise to effective fields of defect densities.

The material space is then described geometrically by an affine connection of a non-Riemannian space that has non-vanishing curvature, non-metricity and torsion [59–61]. Non-metricity, which we know by definition it modifies the local notion of length must therefore be related to a density of the d = 0 defects. Then the relation of mass generation to non-metricity that we found is somewhat expected, given that mass terms in the action break the local Weyl symmetry of the action, much like d = 0 defects destroy the local order of the crystalline. At low energy we saw that metricity is recovered in our case, just like the local d = 0 crystalline defects are not observable from large distances relative to the lattice size. This view gives an intuitive picture to non-metricity and its relation to mass generation.

Further, the curvature tensor is associated with a density of disclinations. In the absence of torsion (as in our Weyl and Palatini cases) the material connection is similar to Weyl connection and the geometry is then Weylian. If present, torsion is associated with dislocations and the geometry is modified. The material connection can then be expressed in terms of the torsion, non-metricity and metric. Note there is a physical distinction of these concepts, something not obvious in general gravity theories [63]. This concludes our brief description of curvature, torsion and non-metricity from a solid state physics perspective.

4.5 The multiple roles of the Stueckelberg field (ϕ)

It is worth noting the multiple role plaid by the scalar mode ϕ of $\tilde{R}^2$ in Weyl and Palatini cases: (1) It acts as a Stueckelberg field eaten by ω_μ, (2) Its vev generates M_p, Λ, m_ω, playing the role of the dilaton (note that $\ln \phi$ has a shift symmetry in (2)). (3) When the SM is embedded in Weyl

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[16] It would also be interesting to construct a supersymmetric version of Weyl quadratic action (13) – to our knowledge there is no such version in the current literature.

geometry, it mixes with the Higgs, leading to a Stueckelberg-Higgs potential which for small Higgs field values recovers the SM potential [21]. (4) The contribution of ϕ to this potential drives inflation in such $\tilde{R}^2$ models [7,21,64,65]. This explains why the inflation prediction for the tensor-to-scalar ratio ($r \sim 10^{-3}$) in Weyl case is similar to that in the Starobinsky model [66?,67] (r is larger in the Palatini case due to different coefficients in the scalar potential caused by different vectorial non-metricity) [68].

4.6 Renormalizability

In a most general case, the action in the Weyl theory (13) can include (up to a topological term) only one additional operator that also has a gauged scale invariance. This is $\tilde{C}_{\mu\nu\rho\sigma}^2$, where $\tilde{C}_{\mu\nu\rho\sigma}$ is the Weyl tensor in Weyl geometry. This is related to the usual Riemannian Weyl tensor $C_{\mu\nu\rho\sigma}$ via $\tilde{C}_{\mu\nu\rho\sigma}^2 = C_{\mu\nu\rho\sigma}^2 + (3/2)\alpha^2 F_{\mu\nu}^2$ where $F_{\mu\nu}^2$ is the kinetic term of ω_μ. The operator $\tilde{C}_{\mu\nu\rho\sigma}^2$ is essentially spectator under the mechanism of symmetry breaking presented earlier and does not affect the results shown. In a quantum analysis, it might be generated as a loop counterterm. The Riemannian version of this term ($C_{\mu\nu\rho\sigma}^2$) was extensively analysed [8–11], while the extra $F_{\mu\nu}^2$ contribution only brings a redefinition of coupling α.

Given the gauged scale invariance of the action, there are no operators of dimension larger than four that can be present in the action, since there is no scale to suppress them. In a Riemannian notation, the overall Lagrangian thus involves only $(R - 3\alpha\nabla_\mu\omega^\mu - 3/2\,\alpha^2\,\omega_\mu\omega^\mu)^2$ coming from $\tilde{R}^2$ that is linearised with the aid of ϕ, then $F_{\mu\nu}^2$, and $C_{\mu\nu\rho\sigma}^2$. The Weyl vector is massive and anomaly free [21], of mass acquired via spontaneous breaking which cannot affect renormalizability. Further, it is known that the usual quadratic gravity in the (pseudo)-Riemannian case is renormalizable [69]. For the Weyl theory, based on the symmetry of the action forbidding higher dimensional counterterms, the analysis of [69] and power-counting arguments, one expects this theory be renormalizable (but not unitary, due to spin-2 ghost of $C_{\mu\nu\rho\sigma}^2$).

In the Palatini approach a similar discussion is difficult: there are many additional Weyl gauge invariant operators that can be present in the action [7,53], then solving analytically the equations of motion for the connection and finding the non-metricity is very difficult.

4.7 Non-metricity: Weyl vs Palatini

There has been a long held view since Einstein's critique [3] that non-metricity makes a theory unphysical.[17] Firstly,

since $\tilde{\nabla}_\mu g_{\alpha\beta} \neq 0$[18] under the parallel transport of a (Weyl-covariant constant) vector, this vector changes not only the direction (as in the Riemannian geometry) but also its norm. Hence, the norm of a vector or the clock rate are path dependent (in the symmetric phase). This is detailed in Appendix B for both the Weyl and Palatini cases under the assumption of a massless Weyl boson ω_μ (Weyl gauge symmetry present). The critique is that the physical consequence of this effect is that in an experiment the distance between the spectral lines of two identical atoms of different path history will then differ, in contrast to experience (second clock effect) [3]. In the light of our result that ω_μ is actually *massive*, this claim must be reviewed.

In our view the above critique[19] is implicitly assuming a formalism which appears to break the Weyl gauge symmetry of the action; this is seen when setting the momenta on the mass shell $p^2 = m^2$ as done in [71]. This means that a mass term is actually present in the action but that means we are actually in a *broken phase*, or we already explained that in a broken phase the theory becomes *metric*, hence the formalism and critique cannot apply.

More generally, in the (non-metric) symmetric phase i.e. without masses or other dimensionful couplings present in the action, it is difficult to explain how this experiment could actually be physically realised. Indeed, if there is no mass there is no clock rate! Secondly, comparing a gauge theory (of scale invariance in our case) to the experiment first requires a "gauge fixing" of this symmetry! From the equations of motion of the Weyl field [21] and Weyl current conservation $\nabla_\mu J^\mu = 0$ [19–21] the gauge fixing follows ($\nabla_\mu\omega^\mu = 0$) *after* fixing the vev of ϕ, which in turn breaks this symmetry! Hence we are back to the broken phase of the theory which is *metric*, ω_μ is massive, decouples and the critique cannot apply! This line of reasoning implies that in Weyl's original theory the second clock effect is not there or it is suppressed by (large) m_ω as first shown for this theory in [19,20]. Hence, our results are not affected by this critique.

To summarise, we know that the gauged scale symmetry is broken, both in the Weyl quadratic gravity and in the Palatini case. When this symmetry is broken, the massive ω_μ decouples (at some high scale) from the Lagrangian and its underlying geometry: the connection becomes Levi-Civita and the theory is then metric; any non-metricity effects and implications mentioned above (if present) are then suppressed by m_ω. As long as m_ω is large enough, such effect can be ignored. As mentioned, the current lower bound on non-metricity (m_ω) derived from its effect on $e^+ e^- \to e^+ e^-$ scattering, is actually very low, 1 TeV [57,58]!

Finally, we would like to comment on the different vectorial non-metricity of Weyl versus the Palatini case, compare

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[17] This was the original critique of Einstein to Weyl's failed theory of gravity "plus" electromagnetism.

[18] For a discussion of metric versus non-metric theories see [70].

[19] This is reviewed recently in [71].

Eqs. (B.3)–(B.18). This has additional implications for the parallel transport of the vectors. As shown in the Appendix, in the Weyl case the ratio of the norms of two vectors u, v of equal (non-zero) Weyl charge is invariant under parallel transport

$$d\,\frac{|u|^2}{|v|^2} = 0. \tag{36}$$

So the *relative* length is actually invariant and this is consistent with physics being independent of the units of length. In the Palatini case, however, this ratio is not invariant, as seen from Eq. (B.20) in the Appendix

$$d\,\frac{|u|^2}{|v|^2} \neq 0. \tag{37}$$

Compared to (36), one may consider on physical grounds that the Weyl case is the only acceptable. We note, however, that the Palatini case is affected by additional operators not included in our analysis that can change (37). It may even be possible that a most general Palatini quadratic gravity with gauged scale invariance may yield onshell (after solving the equations of motion for the connection) a Weylian non-metricity and connection. This would give an interesting offshell realisation of Weyl quadratic gravity.

5 Conclusions

We discussed phenomenological aspects of non-metricity in theories beyond the SM and Einstein gravity that have a *gauged* scale symmetry. One argument in favour of this symmetry is the gauge principle: similarly to the SM as a gauge theory, we seek a gauge theory of scale invariance that recovers Einstein gravity in its broken phase.

What is the 4D underlying geometry of such theories? One can consider theories based on the Weyl geometry which has a *gauged* scale symmetry built in i.e. the Weyl connection has this symmetry. A second option is to consider the Palatini approach to gravity in which the (offshell) connection of the underlying geometry also has this symmetry, being independent of the metric and its Weyl transformation. The consequence is that the underlying geometry of these theories is non-metric i.e. $\tilde{\nabla}_\lambda g_{\mu\nu} \neq 0$; in other words, non-metricity is a result of gauging the scale symmetry (ω_μ dynamical). This situation is different from theories in which ω_μ is not dynamical, based on the (metric) pseudo-Riemannian geometry with a local scale symmetry (no ω_μ present) under which its Levi-Civita connection is *not* invariant – in such case the geometry does not share the space-time symmetry of the action – which raises concerns about their consistency.

Rather than being a problem (as it was thought in the past), non-metricity of Weyl or Palatini cases plays a crucial role in mass generation in gauge theories of scale invariance: it brings mass generation via a Stueckelberg breaking of this symmetry in Weyl or Palatini quadratic gravity *in the absence of matter*. There are additional advantages of non-metricity. Firstly, there is a non-trivial, conserved current associated with the Weyl gauge symmetry and secondly, there are no ghost degrees of freedom in the action with this symmetry. This is unlike (metric or integrable) theories with local scale symmetry only (no dynamical ω_μ) where Weyl current is trivial, as shown by Jackiw and Pi [44,45] and a ghost is present.

Our results show that in the absence of matter, *all mass scales have a geometric origin:* the Planck mass, the cosmological constant Λ and the mass of ω_μ are proportional to $\langle\phi\rangle$ which is the spin-zero mode propagated by the geometric $\tilde{R}^2$ term. Unlike in local scale invariant theories (no ω_μ present) based on the (metric) pseudo-Riemannian geometry, in our Weyl and Palatini cases, no scalar fields were added "ad-hoc" or needed to generate these mass scales. A hierarchy of scales (Λ, M_p, m_ω) is related to the smallness of the dimensionless gravitational couplings ξ, α at a classical level: with M_p fixed, the cosmological constant is small because gravity is weak ($\xi \ll 1$). There is a UV–IR physics connection associated with M_p and Λ, respectively, since these scales have a common origin ($\propto \langle\phi\rangle$). Finally, $\Lambda > 0$ because it is due to the ϕ^4 term induced again by geometric $\tilde{R}^2$.

Metricity is recovered below m_ω after the massive Weyl gauge boson of scale symmetry decouples from the spectrum. In this decoupling limit the connection becomes Riemannian (Levi-Civita) and the geometry is metric. The scale where this happens (m_ω) is naively expected to be high ($\propto \alpha M_p$), but current bounds on the non-metricity scale are actually very low (TeV scale). A low value of m_ω can be realised for a small coupling $\alpha \ll 1$. These results, obtained in the absence of matter, also apply to the Palatini case.

The above picture remains valid if matter is present, when the SM (with a massless Higgs) is embedded in Weyl geometry. This is a natural embedding, without new degrees of freedom beyond the SM and Weyl geometry. Of the SM spectrum only the Higgs field (σ) has a direct coupling to ω_μ of the form $\omega_\mu\omega^\mu\sigma^2$. This leads to the interesting possibility that the Higgs be generated by Weyl vector fusion $\omega_\mu + \omega_\mu \rightarrow \sigma + \sigma$ in the early Universe. Since ω_μ has geometric origin, this means that the Higgs itself has an origin in Weyl's non-metric geometry, too. Therefore, not only the scales of quadratic gravity are of geometric origin but this extends, in a sense, to all SM masses generated by the Higgs. This shows that Weyl geometry is more fundamental and it provides a UV completion of the (pseudo)Riemannian geometry; correspondingly, the associated Weyl quadratic gravity provides a gauge theory embedding of Einstein gravity. These results also apply to the Palatini case; however, in this case there are unknown corrections from additional operators not included in our study.

Non-metricity is common in solid state physics where it is associated with crystalline structure defects of dimension d=0 (point defects or metric anomalies). They destroy the crystalline order and modify the local notion of length associated with this order. Then the relation of mass generation to non-metricity that we found is expected, given that mass terms in the action break the Weyl (local or gauged) scale symmetry much like point defects destroy the local order/size of the lattice. At low energy we saw that metricity is recovered, much like the local crystalline defects are not observable from large distances relative to the lattice size. This gives an intuitive picture of non-metricity.

Can these ideas about non-metricity be tested experimentally? One possibility is to analyse a possible imprint on the gravitational waves due to the Weyl gauge boson of scale symmetry. The second possibility is in Higgs physics, assuming a light ω_μ near its lower bound; in this case the term $\omega_\mu \omega^\mu \sigma^2$, relating Higgs physics to non-metricity, brings corrections to the Higgs couplings (e.g. quantum corrections to the quartic coupling). In this way one may set lower bounds on m_ω which is the scale of "new physics" in this case. A third possibility is via the Stueckelberg-Higgs inflation, which predicts a low ($\sim 10^{-3}$) tensor-to-scalar ratio (r) value, testable in the near future experiments. Work to explore these interesting possibilities is in progress.

Appendix A: Weyl gauge invariant theories and conserved current

• For an arbitrary Weyl gauge invariant action we show there is a non-trivial, conserved current J^μ. We then detail this for Weyl and Palatini quadratic gravity. This information was used in the text, Sect. 3, Eq. (15) and also in the Palatini case, see text after Eq. (28).

Consider a Weyl gauge transformation:

$$\hat{g}_{\mu\nu} = \Sigma\, g_{\mu\nu}, \qquad \hat{\phi} = \Sigma^{-1/2}\phi,$$
$$\hat{\omega}_\mu = \omega_\mu - \frac{1}{\alpha}\partial_\mu \ln \Sigma \tag{A.1}$$

where ϕ is some scalar field. For an infinitesimal transformation $\delta \Sigma$

$$\delta \hat{g}_{\mu\nu} = \delta(\ln \Sigma)\,\hat{g}_{\mu\nu}, \quad \delta\hat{\phi} = -\frac{1}{2}\delta(\ln \Sigma)\,\hat{\phi},$$
$$\delta\hat{\omega}_\mu = -\frac{1}{\alpha}\delta\,\partial_\mu \ln \Sigma = -\frac{1}{\alpha}\partial_\mu \delta \ln \Sigma. \tag{A.2}$$

Consider a Weyl gauge invariant *total* action which we write as a sum $S_g + S$, where S_g is the Weyl gauge field kinetic term while S is the *remaining* part of the action that can depend on ω_μ but not on $F_{\mu\nu}$, hence:

$$S_g = -\frac{1}{4}\int \sqrt{g}\, F_{\mu\nu}^2, \qquad S = \int \sqrt{g}\, L \tag{A.3}$$

S_g and S are each Weyl gauge invariant. Under (A.2)

$$\delta S = \int \sqrt{\hat{g}}\left[-\frac{1}{2}\theta^{\mu\nu}\,\delta\hat{g}_{\mu\nu} + J^\mu\,\delta\hat{\omega}_\mu + \frac{\delta L}{\delta\hat{\phi}}\,\delta\hat{\phi}\right], \tag{A.4}$$

where we denoted

$$\theta^{\mu\nu} \equiv -\frac{2}{\sqrt{\hat{g}}}\frac{\delta L}{\delta\hat{g}_{\mu\nu}}, \qquad J^\mu \equiv \frac{1}{\sqrt{\hat{g}}}\frac{\delta L}{\delta\hat{\omega}_\mu}, \tag{A.5}$$

are the "stress-energy"-like tensor associated with S and the Weyl gauge symmetry current, respectively. The last term in δS vanishes by the equation of motion for ϕ. Since S is Weyl gauge invariant ($\delta S = 0$) and using (A.2), then (after removing the "hat" notation):

$$0 = \delta S \equiv \int \sqrt{g}\left[-\frac{1}{2}\theta^{\mu\nu}g_{\mu\nu}\,\delta \ln \Sigma - \frac{1}{\alpha}J^\mu\,\partial_\mu \delta \ln \Sigma \right]$$
$$= \int \sqrt{g}\left[-\frac{1}{2}\theta^\mu_\mu - \frac{1}{\alpha}\nabla_\mu J^\mu \right]\delta \ln \Sigma, \tag{A.6}$$

where $\sqrt{g}\nabla_\mu J^\mu = \partial_\mu(J^\mu \sqrt{g})$ was used. Therefore, for a Weyl gauge invariant action:

$$\theta^\mu_\mu = -\frac{2}{\alpha}\nabla_\mu J^\mu. \tag{A.7}$$

Finally, from the total action $S + S_g$ one can easily write the equation of motion for ω_μ:

$$J^\mu + \nabla_\sigma F^{\sigma\mu} = 0. \tag{A.8}$$

Next, multiply this equation by $\sqrt{g}$ and apply ∂_μ on it, then use the antisymmetry of $F^{\sigma\mu}$ to find that $\nabla_\mu J^\mu = 0$. Thus,

there is a conserved current, as discussed in the text. Finally, using (A.7), then onshell

$$\theta^\mu_\mu = 0, \qquad (A.9)$$

as expected for a theory with this classical symmetry.

● Let us now detail the above results for our cases of Weyl and Palatini quadratic gravities that are Weyl gauge invariant, discussed in Sect. 3. Consider our initial action in Weyl non-metric case Eq. (14) which is similar for Palatini case, Eq. (28). Therefore our S becomes

$$
\begin{aligned}
S &\equiv \int \sqrt{g}\, L \\
&= \int \sqrt{g} \left\{ \frac{-1}{12\xi^2} \phi^2 \left[R - 3\alpha \nabla_\mu \omega^\mu - \frac{3}{2} \alpha^2 \omega_\mu \omega^\mu \right] \right. \\
&\quad \left. - \frac{\phi^4}{4!\, \xi^2} \right\}.
\end{aligned}
\qquad (A.10)
$$

Using definition (A.5), we find a current

$$
J_\mu = \frac{1}{\sqrt{g}} \frac{\delta L}{\delta \omega^\mu} = \frac{1}{\sqrt{g}} \frac{\partial L}{\partial \omega^\mu} = -\frac{\alpha}{4\xi^2} (\partial_\mu - \alpha\, \omega_\mu) \phi^2. \qquad (A.11)
$$

The *total* action $S + S_g$ is identical to action (14) in the text and gives the following equation of motion for ω_μ

$$
\sqrt{g} \left\{ \frac{\alpha^2}{4\xi^2} \phi^2 \omega^\rho - \frac{\alpha}{4\xi^2} \nabla^\rho \phi^2 + \nabla_\sigma F^{\sigma\rho} \right\} = 0. \qquad (A.12)
$$

This equation is Weyl gauge invariant, as expected (since the action is invariant). Applying ∂_ρ on the last equation, using that $\sqrt{g}\, \nabla_\sigma F^{\sigma\mu} = \partial_\sigma (\sqrt{g}\, F^{\sigma\mu})$ and the antisymmetry of $F^{\sigma\rho}$, we then find

$$\nabla_\mu J^\mu = 0. \qquad (A.13)$$

Therefore, there exists a non-trivial, conserved current. This result was used in Weyl case, Eq. (15), and in the Palatini case in the text after Eq. (28).

Let us now check explicitly Eq. (A.7) for our case. From action (A.10) one has the trace:

$$
\begin{aligned}
g^{\mu\nu} \frac{\delta L}{\delta g^{\mu\nu}} &= \frac{\sqrt{g}}{12\,\xi^2} \left[\phi^2 \left(R - \frac{3}{2} \alpha^2 \omega_\mu \omega^\mu - 3\alpha \nabla_\rho \omega^\rho \right) \right. \\
&\quad \left. + \phi^4 - 3\Box \phi^2 + 3\alpha \nabla^\rho (\omega_\rho \phi^2) \right] = 0. \qquad (A.14)
\end{aligned}
$$

This vanishes by the equation of motion for $g^{\mu\nu}$. This result is actually valid for the total action $S + S_g = \int \sqrt{g} \left[L - (1/4) F_{\mu\nu}^2 \right]$ since the contribution to the trace by the (conformal) gauge kinetic term $F_{\mu\nu}^2 \sqrt{g}$ is vanishing. Finally, from the equation of motion of ϕ (which is also Weyl gauge invariant) one finds, after multiplying it by ϕ

$$
\frac{\sqrt{g}}{12\,\xi^2} \left[\phi^2 \left(R - 3\alpha \nabla_\rho \omega^\rho - \frac{3}{2} \alpha^2 \omega_\rho \omega^\rho \right) + \phi^4 \right] = 0. \qquad (A.15)
$$

Equation (A.15) is just another form of equation $\phi^2 = -\tilde{R}$ which we already know from the "linearisation" of the $\tilde{R}^2$ term, described in the text. Using Eq. (A.15) in Eq. (A.14) we find

$$
\frac{-2}{\sqrt{g}} g^{\mu\nu} \frac{\delta L}{\delta g^{\mu\nu}} = \frac{1}{2\,\xi^2} \nabla^\rho (\nabla_\rho - \alpha\omega_\rho) \phi^2 = 0. \qquad (A.16)
$$

The last equation also shows that ϕ which was "extracted" from the $\tilde{R}^2$ term, is indeed a dynamical field, as discussed in the text. This equation gives again the current conservation (also found earlier directly from (A.12)). Using notation (A.5), then from (A.16) we have

$$
\theta^\mu_\mu = \left(\frac{-2}{\alpha} \right) \left(\frac{-\alpha}{4\xi^2} \right) \nabla^\rho (\nabla_\rho - \alpha\omega_\rho) \phi^2 = 0 \qquad (A.17)
$$

in agreement with general result (A.7) and also with (A.11), (A.12). The analysis for the Palatini case Eq. (28) is very similar (with $\omega_\mu \to v_\mu$). For more details, see also [19–21] (Appendix B) and [7].

Appendix B: Parallel transport in Weyl and Palatini cases

We present here a brief review of the parallel transport of a vector in Weyl and Palatini geometries, discussed in Sect. 4.7:

● **Weyl case:** Weyl geometry[20] is represented by classes of equivalence of $(g_{\mu\nu}, \omega_\mu)$ related by (B.1). Scalars ϕ and fermions ψ transform under (B.1) as shown in (B.2) below:

$$
\hat{g}_{\mu\nu} = \Sigma^q\, g_{\mu\nu}, \quad \sqrt{\hat{g}} = \Sigma^{2q} \sqrt{g},
$$
$$
\hat{\omega}_\mu = \omega_\mu - \frac{1}{\alpha} \partial_\mu \ln \Sigma \qquad (B.1)
$$
$$
\hat{\phi} = \Sigma^{-q/2} \phi, \quad \hat{\psi} = \Sigma^{-3q/4} \psi \qquad (B.2)
$$

where, to be more general, we now allow an arbitrary Weyl charge q for the metric (one usually sets $q = 1$, as done so far in this work). The gauge covariant derivative of the scalar ϕ transforms just like the scalar itself and equals $D_\mu \phi = \left[\partial_\mu - (q/2)\, \alpha\, \omega_\mu \right] \phi$.

Weyl geometry has vectorial non-metricity

$$\tilde{\nabla}_\mu g_{\alpha\beta} = -\alpha\, q\, \omega_\mu\, g_{\alpha\beta}, \qquad (B.3)$$

where $\tilde{\nabla}$ is defined by the Weyl connection $\tilde{\Gamma}$

$$\tilde{\nabla}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \tilde{\Gamma}^\rho_{\alpha\mu} g_{\rho\beta} - \tilde{\Gamma}^\rho_{\beta\mu} g_{\rho\alpha}. \qquad (B.4)$$

Equation (B.3) may be written in a "metric" format

$$\tilde{\nabla}'_\mu g_{\alpha\beta} = 0, \quad \tilde{\nabla}' \equiv \tilde{\nabla} \Big|_{\partial_\mu \to \partial_\mu + \alpha\, q\, \omega_\mu}. \qquad (B.5)$$

---

[20] For a brief introduction to Weyl conformal geometry see Appendix A in [21].

Therefore the Weyl connection $\tilde{\Gamma}$ is found from the Levi-Civita connection (Γ) in which one makes the same substitution: $\tilde{\Gamma} = \Gamma|_{\partial_\lambda \to \partial_\lambda + \alpha\, q\, \omega_\lambda}$, or by "standard" calculation as for the Levi-Civita connection in Riemannian case. Either way, one finds

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + (q/2)\,\alpha \left[\delta^\lambda_\mu\, \omega_\nu + \delta^\lambda_\nu\, \omega_\mu - g_{\mu\nu}\, \omega^\lambda \right]. \quad \text{(B.6)}$$

Consider now a vector u^μ of some Weyl charge ($z_u/2$):

$$\hat{u}^\mu = \Sigma^{z_u/2} u^\mu. \quad \text{(B.7)}$$

The parallel transport of a constant vector (in a Weyl-covariant sense) is defined by

$$\frac{D\,u^\mu}{d\tau} = 0, \quad \text{where } D \equiv dx^\lambda\, D_\lambda,$$
$$D_\lambda\, u^\mu = \tilde{\nabla}_\lambda u^\mu \Big|_{\partial_\lambda \to \partial_\lambda + (z_u/2)\,\alpha\,\omega_\lambda}, \quad \text{(B.8)}$$

with

$$\tilde{\nabla}_\lambda u^\mu = \partial_\lambda u^\mu + \tilde{\Gamma}^\mu_{\lambda\rho}\, u^\rho, \quad \text{(B.9)}$$

and $x = x(\tau)$. Then from (B.8) the "standard" differential variation of the vector is

$$d\,u^\mu = -dx^\lambda \left[(z_u/2)\,\alpha\,\omega_\lambda\, u^\mu + \tilde{\Gamma}^\mu_{\lambda\rho}\, u^\rho \right], \quad \text{where}$$
$$d\,u^\mu \equiv dx^\lambda\, \partial_\lambda u^\mu. \quad \text{(B.10)}$$

Then under the parallel transport, the product $\langle u, v \rangle = u^\mu v^\nu g_{\mu\nu}$ of vectors u, v changes as

$$d\langle u, v \rangle = dx^\lambda \left[\tilde{\nabla}_\lambda g_{\mu\nu} - \alpha\,\omega_\lambda\, g_{\mu\nu}(z_u + z_v)/2 \right] u^\mu v^\nu. \quad \text{(B.11)}$$

This can immediately be integrated along a given path $\gamma(\tau)$.

Using non-metricity (B.3), then the norm $|u|$ of the vector varies according to

$$d|u|^2 = dx^\lambda |u|^2 \omega_\lambda\,(-\alpha)\,(q + z_u), \quad \text{(B.12)}$$

or, integrating this along a path $\gamma(\tau)$:

$$|u|^2 = |u_0|^2\, e^{-\alpha\,(q+z_u) \int_\gamma \omega_\lambda dx^\lambda}. \quad \text{(B.13)}$$

The integral and the norm are path-dependent, except when ω_μ is an exact one-form. In this case, if the path is closed the integral vanishes and the norm is invariant. This is the case of Weyl integrable geometry.

In Weyl geometry the ratio of two vectors (of same Weyl weight) is invariant under the parallel transport. This is seen by using (B.12)

$$d\,\frac{|u|^2}{|v|^2} = (-\alpha)\,\frac{|u|^2}{|v|^2}\,(z_u - z_v)\,\omega_\lambda\, dx^\lambda. \quad \text{(B.14)}$$

This vanishes if the charges of the two vectors are equal $z_u = z_v$, result used in Sect. 4.7. This is of interest since for all physical purposes it is actually the relative length that

should be invariant under parallel transport. This is consistent with physics being independent of the units of length.

● **Palatini case:** In the Palatini case the non-metricity is found from action (26) or equivalent (28) by solving the equations of motion of the connection $\tilde{\Gamma}$. This is a rather technical exercise detailed for this action in Ref. [7] (section 3.1). One finds

$$\tilde{\nabla}_\lambda(\phi^2 g_{\mu\nu}) = (-2)(g_{\mu\nu}\, V_\lambda - g_{\mu\lambda}\, V_\nu - g_{\nu\lambda}\, V_\mu)\phi^2 \quad \text{(B.15)}$$

where

$$V_\lambda = v_\lambda - \partial_\lambda \ln \phi^2 \quad \text{(B.16)}$$

and where v_λ is the Weyl gauge boson in the Palatini case. From this result, one finds the connection [7]

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu}(g) + (1/2)\,(\delta^\alpha_\nu\, \partial_\mu + \delta^\alpha_\mu\, \partial_\nu - g^{\alpha\lambda}\, g_{\mu\nu}\, \partial_\lambda)\ln \phi^2$$
$$- (3 g_{\mu\nu} V_\lambda - g_{\nu\lambda} V_\mu - g_{\lambda_\mu} V_\nu)\, g^{\lambda\alpha}. \quad \text{(B.17)}$$

After ϕ acquires a vev $\langle\phi\rangle$, (B.15) becomes

$$\tilde{\nabla}_\lambda g_{\mu\nu} = (-2)\alpha\, q\,(g_{\mu\nu}\, v_\lambda - g_{\mu\lambda}\, v_\nu - g_{\nu\lambda}\, v_\mu), \quad \text{(B.18)}$$

to be compared to the Weyl case, Eq. (B.3). With this result, the change of the norm of a vector under the parallel transport can be computed from (B.11) in which one is using the Palatini non-metricity. One finds

$$d\,|u|^2 = \alpha\, dx^\lambda \left[-\omega_\lambda(2q + z_u)\,|u|^2 + 4q\, u_\lambda(u_\beta v^\beta) \right]. \quad \text{(B.19)}$$

From this one also finds the change of the ratio of the norms of two vectors is non-zero

$$d\,\frac{|u|^2}{|v|^2} = dx^\lambda \alpha \left[(z_u - z_v)\,\omega_\lambda \right.$$
$$\left. + 4q\,\omega^\beta \left(\frac{u_\lambda u_\beta}{|u|^2} - \frac{v_\lambda v_\beta}{|v|^2} \right) \right]. \quad \text{(B.20)}$$

Unlike in Weyl geometry, the ratio of the norms of u and v changes under parallel transport even if they have the same Weyl charge $z_u = z_v$. This result was used in Sect. 4.7.

Finally, in a most general case, in the Palatini quadratic gravity there are more quadratic operators in curvature that can be present in action (26) and this makes the analysis much more difficult in such case. A list of the quadratic operators (in curvature) that can be present is found in [72], see also [52]. In such case there are additional states propagated by these (higher derivative) operators, other than ϕ (propagated by $\tilde{R}^2$); some of these may even be ghost-like; in such case it is unclear if one can still solve algebraically the second-order differential equations of motion for the Palatini connection, since these equations acquire new terms with new indices structure and new states present.

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