



Newton versus Coulomb for Kaluza–Klein modes

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Abstract We consider a set of elementary compactifications of $D + 1$ to D spacetime dimensions on a circle: first for pure general relativity, then in the presence of a scalar field, first free then with a non minimal coupling to the Ricci scalar, and finally in the presence of gauge bosons. We compute the tree-level amplitudes in order to compare some gravitational and non-gravitational amplitudes. This allows us to recover the known constraints of the $U(1)$, dilatonic and scalar weak gravity conjectures in some cases, and to show the interplay of the different interactions. We study the KK modes pair-production in different dimensions. We also discuss the contribution to some of these amplitudes of the non-minimal coupling in higher dimensions for scalar fields to the Ricci scalar.

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1 Introduction

Among the Swampland conjectures [1], one of the most popular and best tested is probably the weak gravity conjecture (WGC). Its simplest formulation [2] considers the case of a D -dimensional $U(1)$ gauge theory, with a coupling constant g , and requires the existence of at least one state of mass m and charge q which satisfies:

$$gq \geq \sqrt{\frac{D-3}{D-2}} \kappa_D m, \quad (1.1)$$

where κ_D is defined as $\kappa_D^2 = 8\pi G_D = \frac{1}{M_{P,D}^{D-2}}$ with $M_{P,D}$ the reduced Planck mass in D dimensions. This inequality implies, among others, that in the non-relativistic limit, the Newton force is not stronger than the Coulomb force. The particular states for which the equality in (1.1) is satisfied are said to saturate the WGC. In this work we will be interested in a particular case of them.

The present work is dedicated to the study of two different generalizations of the WGC: one that arises when the gauge interaction is complemented by a dilaton interaction [3,4], and another [5–7] that broadly requires the dominance of scalar interactions with respect to gravity in some scattering processes depending on the specific theory. We are interested in the modes that propagate in an extra dimension forming a tower of KK excitations [8–11]. We will explicitly show that these modes undergo gravitational and non-gravitational

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interactions of equal intensity, which allows us to use them as probes for the conjectured inequalities generalizing the one mentioned above. They will also be useful to investigate the behavior of the scalar WGC under compactification.

Obviously, the KK excitations considered here saturate the inequalities conjectured only at the classical level, to which our study will be limited, since both terms of these inequalities are in general corrected by quantum effects. However, one has in mind that extending the theory with enough supersymmetries, the KK modes can be BPS states which saturate them even at the quantum level.

The fact that KK modes saturate the inequalities of the various conjectures is a known property, but we will give a derivation of it here in a simple form that we have not found in the existing literature. Our derivation of the various inequalities will be based on amplitude calculations, not for example on the conditions for decay of extremal black holes, and some of the explicit expressions for the amplitudes needed to make the comparisons seem to be either missing or scattered and hard to find, so we hope that presenting them altogether here might be useful.

This work is organized as follows. Section 2 reviews the well-known reduction of KK from $D + 1$ to D dimensions of the Hilbert–Einstein action and a massless scalar. It allows us to introduce our notations, presents the Lagrangian expansion needed to extract the Feynmann rules for calculating amplitudes, and compute the numerical factor in the total derivative term, often misquoted in the literature, which will be useful in Sect. 5. The dilatonic WGC inequality is derived in Sect. 3, where we also calculate various KK pair production amplitudes. In Sect. 4, we consider adding a mass term for the scalar in $D + 1$ dimensions and we find our form of the scalar WGC. A non-minimal coupling to gravity is considered in Sect. 5. The interactions due to the presence of higher dimensional gauge fields are discussed in Sect. 6. Our conclusions are presented in Sect. 7. Finally, some technical details about our calculations are gathered in appendices.

2 Expansion to second order in the gravitational field

We work with the signature $(+, -, \dots, -)$. The $D + 1$ dimensional quantities will be denoted with a hat. We use Latin and Greek letters for the $D+1$ and D -dimensional coordinates, respectively. We denote by x the D non-compact and by $z \equiv z + 2\pi L$ the compact coordinates. We recall the steps of the simple dimensional reduction of a free real massless scalar field $\hat{\Phi}$ coupled to General Relativity:

$$\mathcal{S}^{(D+1)} = \mathcal{S}_{EH}^{(D+1)} + \mathcal{S}_{\Phi,0}^{(D+1)}, \tag{2.1}$$

where

$$\mathcal{S}_{EH}^{(D+1)} = \frac{1}{2\hat{\kappa}^2} \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \hat{R}, \tag{2.2}$$

and

$$\mathcal{S}_{\Phi,0}^{(D+1)} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \frac{1}{2} \hat{g}^{MN} \partial_M \hat{\Phi} \partial_N \hat{\Phi} \tag{2.3}$$

The Ricci scalar \hat{R} is computed from the metric \hat{g}_{MN} . In the simplest compactification from $D + 1$ to D dimensions it takes the form

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} - e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & -e^{2\beta\phi} \end{pmatrix} \tag{2.4}$$

with ϕ , A_μ and $g_{\mu\nu}$ D -dimensional fields independent of the z coordinate:

$$\begin{aligned} \mathcal{S}_{EH}^{(D+1)} = \frac{1}{2\hat{\kappa}^2} \int d^{D+1}x \sqrt{(-1)^{D-1} g} e^{((D-2)\alpha+\beta)\phi} \left\{ R \right. \\ \left. - [2(1-D)\alpha - 2\beta] \square\phi \right. \\ \left. - [(D-2)(1-D)\alpha^2 + 2\beta((2-D)\alpha - \beta)] (\partial\phi)^2 \right. \\ \left. - \frac{1}{4} e^{2(\beta-\alpha)\phi} F^2 \right\}. \end{aligned} \tag{2.5}$$

where g is the determinant of the D -dimensional metric.

A canonical D -dimensional Einstein–Hilbert action is obtained for

$$(D - 2)\alpha + \beta = 0. \tag{2.6}$$

and the canonical dilaton kinetic term fixes the constant α to be:

$$\alpha^2 = \frac{1}{2(D - 1)(D - 2)}. \tag{2.7}$$

Since all fields are independent of z , we can perform the integration over this coordinate to obtain, keeping only the zero modes,¹

$$\begin{aligned} \mathcal{S}_{0,0}^{(D)} = \frac{2\pi L}{2\hat{\kappa}^2} \int d^D x \sqrt{(-1)^{D-1} g} \left[R + 2\alpha \square\phi + \frac{1}{2} (\partial\phi)^2 \right. \\ \left. - \frac{1}{4} e^{2(1-D)\alpha\phi} F^2 \right]. \end{aligned} \tag{2.8}$$

We define the D -dimensional constant κ in terms of the $(D + 1)$ -dimensional $\hat{\kappa}$ as

$$\frac{1}{\kappa^2} = \frac{2\pi L}{\hat{\kappa}^2} \implies M_P^{D-2} = 2\pi L \hat{M}_P^{D-1} \tag{2.9}$$

In (2.4), the ϕ and A_μ fields are dimensionless. Dimensional fields, that we denote $\tilde{\phi}$ and \tilde{A}_μ , can be written as

$$\tilde{\phi} = \frac{\phi}{\sqrt{2\kappa}}; \quad \tilde{A}_\mu = \frac{A_\mu}{\sqrt{2\kappa}} \tag{2.10}$$

¹ The factor in front of the D’Alambertian operator, 2α , corrects the expression sometimes found in the literature, $(D - 3)\alpha$. As long as only minimal coupling to gravity is considered, the difference is harmless.

The action of the D -dimensional gauge and scalar fields, denoted as the graviphoton and the dilaton, respectively, reads:

$$S_{0,0}^{(D)} = \int d^D x \sqrt{(-1)^{D-1} g} \left[\frac{R}{2\kappa^2} + 2\alpha \square \tilde{\phi} + \frac{1}{2} (\partial \tilde{\phi})^2 - \frac{1}{4} e^{2\sqrt{2}(1-D)\alpha\kappa\tilde{\phi}} \tilde{F}^2 \right]. \tag{2.11}$$

In the following, with the exception of Sect. 5, the second term in (2.11), being a total derivative, will be discarded and, for notational simplicity, we remove the tilde in our notation.

For simplicity, we restrict to the simplest case where the field $\hat{\Phi}$ is periodic and single-valued on the compact dimension

$$\begin{aligned} \hat{\Phi}(x, z + 2\pi L) &= \hat{\Phi}(x, z), \\ \hat{\Phi}(x, z) &= \frac{1}{\sqrt{2\pi L}} \sum_{n=-\infty}^{+\infty} \varphi_n(x) e^{\frac{inz}{L}}, \end{aligned} \tag{2.12}$$

which leads to

$$\begin{aligned} S &= \int d^D x \sqrt{(-1)^{D-1} g} \left\{ \frac{R}{2\kappa^2} + \frac{1}{2} (\partial \phi)^2 \right. \\ &\quad - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}}\kappa\phi} F^2 + \frac{1}{2} \partial_\mu \varphi_0 \partial^\mu \varphi_0 \\ &\quad + \sum_{n=1}^{\infty} \left(\partial_\mu \varphi_n \partial^\mu \varphi_n^* - \frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi} \varphi_n \varphi_n^* \right) \\ &\quad + \sum_{n=1}^{\infty} \left(i\sqrt{2}\kappa \frac{n}{L} A^\mu (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) \right. \\ &\quad \left. + 2\kappa^2 \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* \right) \left. \right\}, \end{aligned} \tag{2.13}$$

where we have chosen in (2.7) the positive root for α . The complex scalars φ_n form the Kaluza–Klein (KK) tower and appear minimally coupled to the graviphoton. Around a generic background value ϕ_0 for the dilaton, the gauge coupling g is given by

$$g^2 = e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}. \tag{2.14}$$

For each KK mode, the mass and charge read

$$gq_n = \sqrt{2}\kappa \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \quad m_n = \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}. \tag{2.15}$$

This shows that they are related through

$$(gq_n)^2 = 2\kappa^2 m_n^2, \tag{2.16}$$

saturating the dilatonic WGC condition. This is expected as all the interactions unify to descend from the unique gravitational interaction of a free scalar field in higher dimensions. Useful for the rest of the manuscript is to derive this result

proceeding instead with the expansion of the metric (2.4) to second order:

$$\hat{g}_{MN} = \hat{\zeta}_{MN} + 2\hat{\kappa}\hat{h}_{MN} + 4\hat{\kappa}^2\hat{f}_{MN} + o(\hat{\kappa}^3) \tag{2.17}$$

where:

$$\hat{\zeta}_{MN} = \begin{pmatrix} e^{2\sqrt{2}\alpha\hat{\kappa}\phi_0}\eta_{\mu\nu} & 0 \\ 0 & -e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \end{pmatrix}. \tag{2.18}$$

is the background metric and $\hat{\kappa}^2\hat{f}_{MN} \ll \hat{\kappa}\hat{h}_{MN} \ll 1$, for all M, N . We write the perturbation as

$$\begin{cases} \hat{g}_{MN} = \hat{\zeta}_{MN} + 2\hat{\kappa}\hat{h}_{MN} + 4\hat{\kappa}^2\hat{f}_{MN} + \mathcal{O}(\hat{\kappa}^3) \\ \hat{g}^{MN} = \hat{\zeta}^{MN} + 2\hat{\kappa}\hat{t}^{MN} + 4\hat{\kappa}^2\hat{l}^{MN} + \mathcal{O}(\hat{\kappa}^3). \end{cases} \tag{2.19}$$

The relation $\hat{g}_{MP}\hat{g}^{PN} \equiv \delta_M^N$ reads

$$\begin{cases} \hat{t}^{MN} = -\hat{h}^{MN} \\ \hat{l}^{MN} + \hat{f}^{MN} = \hat{h}_P^M \hat{h}^{PN}, \end{cases} \tag{2.20}$$

where it is understood that the indices are raised and lowered with the background metric $\hat{\zeta}$, then

$$\begin{aligned} \sqrt{(-1)^D} \hat{g} \mathcal{L}_\Phi &= \sqrt{(-1)^D} \hat{\zeta} \left[\frac{1}{2} \partial_M \hat{\Phi} \partial^M \hat{\Phi} - \frac{\hat{\kappa}'}{2} \hat{h}^{MN} \right. \\ &\quad \times \left(\partial_M \hat{\Phi} \partial_N \hat{\Phi} - \frac{1}{2} \hat{\zeta}_{MN} \partial_P \hat{\Phi} \partial^P \hat{\Phi} \right) \\ &\quad + \frac{\hat{\kappa}'^2}{2} \left(\hat{l}^{MN} - \frac{1}{2} \hat{h}^{MN} \hat{h}_P^P \right) \partial_M \hat{\Phi} \partial_N \hat{\Phi} \\ &\quad \left. + \frac{\hat{\kappa}'^2}{4} \left(\hat{f}_P^P - \frac{1}{2} \hat{h}_{MP} \hat{h}^{PM} + \frac{1}{4} (\hat{h}_P^P)^2 \right) \partial_M \hat{\Phi} \partial^M \hat{\Phi} \right]. \end{aligned} \tag{2.21}$$

where $\hat{\kappa}' \equiv 2\hat{\kappa}$. With:

$$\hat{h}^{MN} = \frac{1}{\sqrt{2\pi L}} \begin{pmatrix} e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\sqrt{2}\alpha\phi \eta^{\mu\nu} + h^{\mu\nu} \right) & -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \frac{A^\mu}{\sqrt{2}} \\ -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \frac{A^\nu}{\sqrt{2}} & -e^{-2\sqrt{2}\beta\hat{\kappa}\phi_0} \sqrt{2}\beta\phi \end{pmatrix}, \tag{2.22}$$

and using $\sqrt{(-1)^D} \hat{\zeta} = e^{\sqrt{2}(D\alpha+\beta)\hat{\kappa}\phi_0}$, this leads to the coupling between the leading order fluctuations \hat{h}^{MN} of the metric and the stress–energy–momentum of the scalar field $\hat{T}_{MN}^{\hat{\Phi}}$:

$$\begin{aligned} \mathcal{L}_{int}^{(1)} &= -\hat{\kappa}\hat{h}^{MN} \hat{T}_{MN}^{\hat{\Phi}} = -\hat{\kappa} h^{\mu\nu} T_{\mu\nu}^{(\varphi_0, \varphi_n)} \\ &\quad - i\sqrt{2}\hat{\kappa} A^\mu \sum_{n=1}^{\infty} \frac{n}{L} (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) \\ &\quad - 2\sqrt{\frac{D-1}{D-2}} \hat{\kappa} e^{2\sqrt{\frac{D-1}{D-2}}\hat{\kappa}\phi_0} \phi \sum_{n=1}^{\infty} \frac{n^2}{L^2} \varphi_n \varphi_n^*. \end{aligned} \tag{2.23}$$

Next, we identify \hat{f}_{MN} from the metric decomposition at second order:

$$\hat{f}^{MN} = \frac{1}{2\pi L} \begin{pmatrix} e^{2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha^2\phi^2\eta_{\mu\nu} + \sqrt{2}\alpha\phi h_{\mu\nu} + f_{\mu\nu} \right) & -\frac{1}{2}e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} A_\mu A_\nu & e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta\phi A_\mu \\ e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta\phi A_\nu & & -e^{2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta^2\phi^2 \end{pmatrix}. \tag{2.24}$$

With this result, \hat{l}^{MN} in (2.21) is given by

$$\hat{l}^{MN} = \hat{h}^{MP}\hat{h}^N_P - \hat{f}^{MN} \tag{2.25} \quad \mathcal{L}_{int}^{(2)} = \frac{1}{2}\partial_\mu\varphi_0\partial_\nu\varphi_0 \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h_\rho^\rho h^{\mu\nu} \right] + \sum_{n=1}^\infty \partial_\mu\varphi_n\partial_\nu\varphi_n^* \left[\left(\frac{f_\rho^\rho}{2} \right.$$

Using (2.24) and (2.22) one obtains

$$\hat{l}^{MN} = \frac{1}{2\pi L} \begin{pmatrix} e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha^2\phi^2\eta^{\mu\nu} + \sqrt{2}\alpha\phi h^{\mu\nu} + l^{\mu\nu} \right) & -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha\phi A^\mu + \frac{1}{\sqrt{2}}h^{\mu\rho} A_\rho \right) \\ -e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \left(\alpha\phi A^\nu + \frac{1}{\sqrt{2}}h_\rho^\nu A^\rho \right) & -e^{-2\sqrt{2}\beta\hat{\kappa}\phi_0} \beta^2\phi^2 + e^{-2\sqrt{2}\alpha\hat{\kappa}\phi_0} \frac{1}{2}A_\rho A^\rho \end{pmatrix}. \tag{2.26}$$

We define $J_{\mu,n} = (\varphi_n\partial_\mu\varphi_n^* - \varphi_n^*\partial_\mu\varphi_n)$, then the second order interaction in the Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{int}^{(2)} = & \frac{1}{2}\partial_\mu\varphi_0\partial_\nu\varphi_0 \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right. \right. \\ & + \frac{1}{2}(D^2\alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2)\phi^2 \\ & + \frac{1}{2}((D-2)\alpha + \beta)\phi h_\rho^\rho \left. \right] \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h_\rho^\rho h^{\mu\nu} \\ & + \sum_{n=1}^\infty \partial_\mu\varphi_n\partial_\nu\varphi_n^* \left[\left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right. \right. \\ & + \frac{1}{2}(D^2\alpha^2 + 2D\beta\alpha + \beta^2 - 4D\alpha^2 - 4\beta\alpha + 4\alpha^2)\phi^2 \\ & + \frac{1}{2}((D-2)\alpha + \beta)\phi h_\rho^\rho \left. \right] \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h_\rho^\rho h^{\mu\nu} \\ & - \sum_{n=1}^\infty \frac{n^2}{L^2} |\varphi_n|^2 \left[-A^2 e^{\sqrt{2}(D\alpha-\beta)\kappa\phi_0} \left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} \right. \right. \\ & + \frac{(h_\rho^\rho)^2}{8} + \left. \left. \left(\frac{1}{2}(D\alpha + \beta) - \beta \right) \phi h_\rho^\rho \right. \right. \\ & + \left. \left. \frac{1}{2}(D^2\alpha^2 + 2D\alpha\beta + \beta^2 - 4D\alpha\beta - 4\beta^2 + 4\beta^2)\phi^2 \right) \right] \\ & - \sum_{n=1}^\infty i \frac{n}{L} h^{\rho\sigma} A_\rho J_{\sigma,n} + i \frac{n}{L} A^\rho J_{\rho,n} \\ & \times \left(-\frac{h_\sigma^\sigma}{2} - ((D-2)\alpha + \beta)\phi \right) \end{aligned} \tag{2.27}$$

This expression simplifies using the relation between β and α (2.6). In particular, the coefficients of ϕ^2 and ϕh_ρ^ρ vanish. One obtains

$$\begin{aligned} & -\frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \left. \right] \eta^{\mu\nu} + l^{\mu\nu} - \frac{1}{2}h_\rho^\rho h^{\mu\nu} \\ & - \sum_{n=1}^\infty \frac{n^2}{L^2} |\varphi_n|^2 e^{2\sqrt{2}(D-1)\alpha\kappa\phi_0} \left(\frac{f_\rho^\rho}{2} - \frac{h^{\rho\sigma}h_{\rho\sigma}}{4} + \frac{(h_\rho^\rho)^2}{8} \right) \\ & - \sum_{n=1}^\infty \frac{n^2}{L^2} |\varphi_n|^2 \left[-A^2 + e^{2\sqrt{2}(D-1)\alpha\kappa\phi_0} \left(2(D-1)^2\alpha^2\phi^2 \right. \right. \\ & + (D-1)\alpha\phi h_\rho^\rho \left. \right] \\ & + i \frac{n}{L} A^\rho \frac{h_\sigma^\sigma}{2} J_{\rho,n} - i \frac{n}{L} h^{\rho\sigma} A_\rho J_{\sigma,n} \end{aligned} \tag{2.28}$$

which shows how the gauge invariance of the graviphoton is recovered in this expansion at second order in $\hat{\kappa}$ and exhibits the minimal coupling of the graviphoton to the tower of scalars in $\hat{\kappa}$.

3 Scattering amplitudes and weak gravity conjectures

In this section, we will compute diverse $2 \rightarrow 2$ amplitudes in the simple model defined above and compare two sets to be identified, one denoted as *gravitational* and the other as *non-gravitational* mediated interactions.

We expand the dilaton around its background value ϕ_0 as $\phi_0 + \phi$ in the action (2.13) to obtain:

$$\begin{aligned} S_f = & \int d^Dx \sqrt{(-1)^{D-1}g} \left\{ \frac{R}{2\kappa^2} + \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2\sqrt{\frac{D-1}{D-2}\kappa}\phi} \right. \\ & \times \sum_{m=0}^\infty \left(-2\sqrt{\frac{D-1}{D-2}\kappa} \right)^m \frac{\phi^m}{m!} F^2 \\ & \left. + \frac{1}{2}\partial_\mu\varphi_0\partial^\mu\varphi_0 + \sum_{n=1}^\infty \partial_\mu\varphi_n\partial^\mu\varphi_n^* \right\} \end{aligned}$$

$$\begin{aligned}
 & - \sum_{n=1}^{\infty} \left(\frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \sum_{m=0}^{\infty} \left(2\sqrt{\frac{D-1}{D-2}}\kappa \right)^m \frac{\phi^m}{m!} \varphi_n \varphi_n^* \right) \\
 & + \sum_{n=1}^{\infty} \left(i\sqrt{2}\kappa \frac{n}{L} A^\mu (\partial_\mu \varphi_n \varphi_n^* - \varphi_n \partial_\mu \varphi_n^*) \right. \\
 & \left. + 2\kappa^2 \frac{n^2}{L^2} A_\mu A^\mu \varphi_n \varphi_n^* \right) \} \quad (3.1)
 \end{aligned}$$

where diverse interactions can be identified. For instance:

- 3 and 4-point vertices for minimally-coupled scalars to graviphotons appear in the last line. We can identify the KK electric charges

$$gq_n = \sqrt{2}\kappa \frac{n}{L} e^{\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}. \quad (3.2)$$

- In the third line, the m -th term ($m \neq 0$) in the sum gives a $(2 + m)$ -point interaction with m dilatons and two KK scalars with coupling

$$-i \left(2\sqrt{\frac{D-1}{D-2}}\kappa \right)^m \frac{n^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0}. \quad (3.3)$$

- The m -th term in the sum in front of F^2 in the first line gives a coupling of m dilatons with two gauge fields

$$-i \left(-2\sqrt{\frac{D-1}{D-2}}\kappa \right)^m (p_1 \cdot p_2 \eta_{\mu\nu} - p_{1\nu} p_{2\mu}). \quad (3.4)$$

Expansion of the metric around flat space-time $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$ gives the usual minimal couplings to gravity for both the matter fields (φ_0, φ_n) and the massless mediators (ϕ, A_μ) .

3.1 The dilatonic WGC

Consider the tree-level $2 \rightarrow 2$ scattering^{2,3} $\varphi_n(p_1)\varphi_n(p_2) \rightarrow \varphi_n(p_3)\varphi_n(p_4)$:

$$\begin{aligned}
 i\mathcal{M} = & ig^2q_n^2 \left(\frac{(p_1 + p_3) \cdot (p_2 + p_4)}{t} + \frac{(p_1 + p_4) \cdot (p_2 + p_3)}{u} \right) \\
 & - 4i \frac{D-1}{D-2} \kappa^2 m_n^4 \left(\frac{1}{t} + \frac{1}{u} \right) \\
 & - \frac{\kappa^2}{4} \left[(p_{1\mu} p_{3\nu} + p_{3\mu} p_{1\nu} - \eta_{\mu\nu} (p_1 \cdot p_3 - m_n^2)) \right. \\
 & \left. \times \frac{i\mathcal{P}^{\mu\nu\alpha\beta}}{t} (p_{2\alpha} p_{4\beta} + p_{4\alpha} p_{2\beta} - \eta_{\alpha\beta} (p_2 \cdot p_4 - m_n^2)) \right]
 \end{aligned}$$

² We adopt here this simple notation where $\varphi_n(p)$, or $|\varphi_n(p)\rangle$ should not be viewed as the field operator acting on the vacuum but to represent a one-particle state of momentum p .

³ Here and throughout, s, t and u will denote the Mandelstam variables.

$$+ (t, p_3, p_4) \leftrightarrow (u, p_4, p_3) \quad (3.5)$$

where \mathcal{P} is the usual massless spin-2 projector

$$\mathcal{P}^{\alpha\beta\rho\sigma} = \frac{\eta^{\alpha\rho}\eta^{\beta\sigma} + \eta^{\alpha\sigma}\eta^{\beta\rho}}{2} - \frac{\eta^{\alpha\beta}\eta^{\rho\sigma}}{D-2} \quad (3.6)$$

and we have separated the contributions from the exchanges of the gauge boson, the dilaton and the graviton, respectively. Taking the non-relativistic (NR) limit

$$\frac{s - 4m_n^2}{m_n^2} \rightarrow 0, \quad \frac{t}{m_n^2} \rightarrow 0, \quad \text{and} \quad \frac{u}{m_n^2} \rightarrow 0 \quad (3.7)$$

and expressing the charge in terms of the mass we obtain

$$\begin{aligned}
 i\mathcal{M} \rightarrow i\mathcal{M}_{NR} = & 4im_n^2 \left[g^2q_n^2 - \kappa^2m_n^2 \left(\frac{D-1}{D-2} + \frac{D-3}{D-2} \right) \right] \\
 & \times \left(\frac{1}{t} + \frac{1}{u} \right) = 0. \quad (3.8)
 \end{aligned}$$

The relation between the charge and the mass (2.16) ensures the cancellation between the three forces.

It is straightforward to generalize this to see that dominance of the gauge interaction requires that a state with charge q and mass m satisfying the relation

$$g^2q^2 \geq \left(\frac{\alpha^2}{2} + \frac{D-3}{D-2} \right) \kappa^2m^2, \quad (3.9)$$

where α is the dilatonic coupling of the form $e^{2\sqrt{2}\alpha\kappa\phi} F^2$, exists. We have therefore recovered in this explicit amplitude computation the dilatonic weak gravity conjecture that was derived in [3] (see also [4] for its generalization) from the study of the extremal Einstein–Maxwell-dilaton black hole solutions. In the absence of the massless dilaton field $\alpha = 0$, one trivially retrieves the original WGC condition

$$g^2q^2 \geq \frac{D-3}{D-2} \kappa^2m^2. \quad (3.10)$$

3.2 Amplitudes for pair production

Consider the production of a pair of matter states, here scalar KK states, of momenta p_3, p_4 from massless particles of momenta p_1, p_2 . We can split the production processes into two sets:

- Non-gravitational production: a pair of KK scalar modes $|\varphi_n, \varphi_n^*\rangle$ can arise from a pair of photons $\langle\gamma, \gamma|$, a pair of dilatons $\langle\phi, \phi|$, or a dilaton and a photon $\langle\phi, \gamma|$.
- Gravitational production: this includes the presence of a graviton G in initial states as $\langle G, G|, \langle G, \gamma|$ or $\langle G, \phi|$, but also gravitons as intermediate states in the production from $\langle\gamma, \gamma|$ or $\langle\phi, \phi|$. For later convenience, we further divide the gravitational production processes into purely

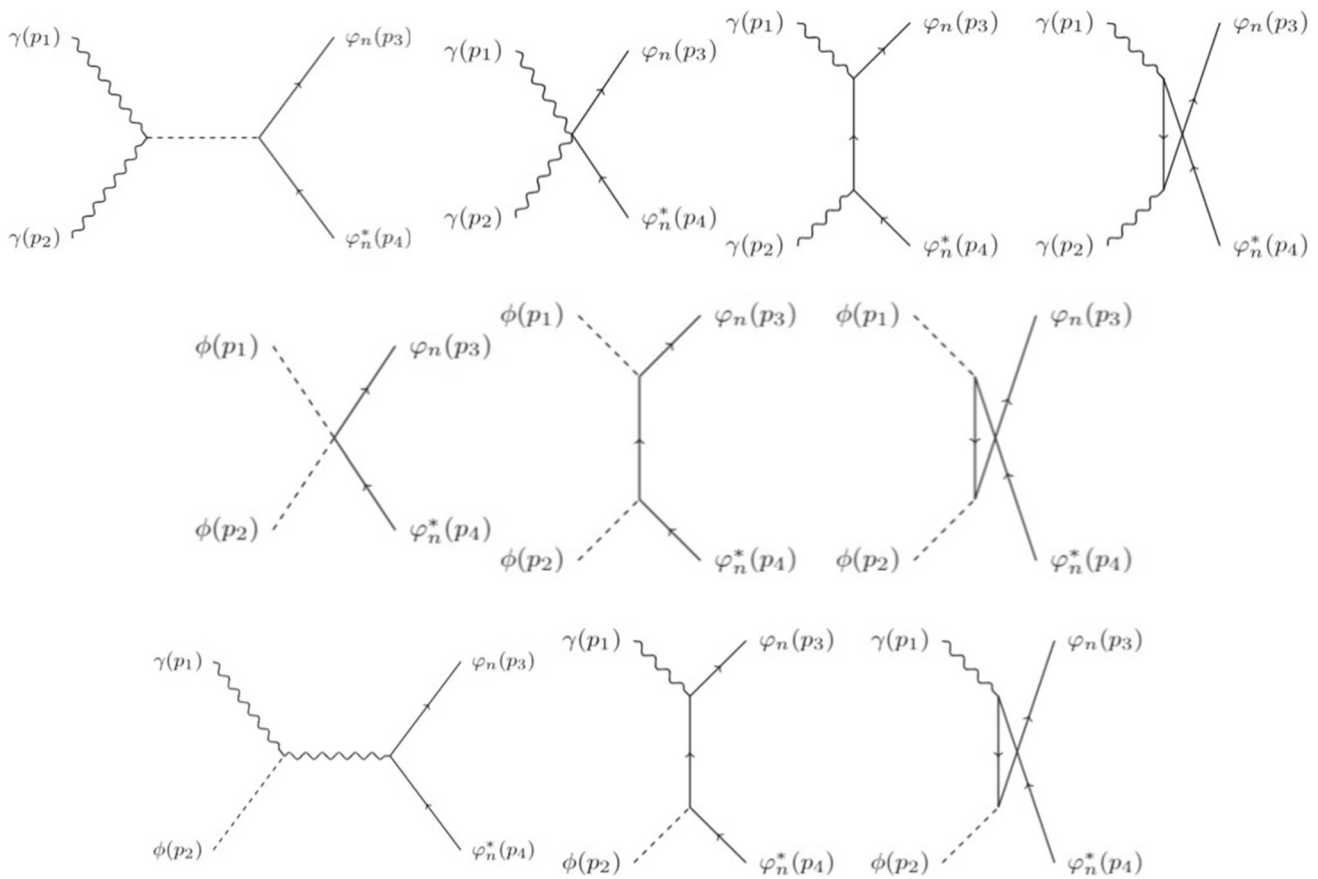


Fig. 1 Feynman diagrams for the non-gravitational production of a pair of matter states φ_n, φ_n^* from two photons (first line), two dilatons (second line) and a dilaton and a photon (third line)

gravitational (the $\langle G, G \rangle$ production) and mixed (all the others).

3.2.1 Non gravitational amplitudes

The production from photons $\gamma\gamma \rightarrow \varphi_n\varphi_n^*$ occurs through the coupling to the $U(1)$ gauge boson plus an s-channel mediated by the dilaton, as depicted in the first line of Fig. 1. These give:

$$\begin{aligned}
 i\mathcal{M}_{\gamma\gamma} = & i g^2 q_n^2 \epsilon_\mu(p_1)\epsilon_\nu(p_2) \left(\frac{(2p_3^\mu - p_1^\mu)(2p_4^\nu - p_2^\nu)}{t - m_n^2} \right. \\
 & \left. + \frac{(2p_4^\mu - p_1^\mu)(2p_3^\nu - p_2^\nu)}{u - m_n^2} + 2\eta^{\mu\nu} \right) \\
 & - 2i g^2 q_n^2 \frac{D-1}{D-2} \epsilon_\mu(p_1)\epsilon_\nu(p_2) \frac{p_1 \cdot p_2 \eta^{\mu\nu} - p_1^\nu p_2^\mu}{s}.
 \end{aligned} \tag{3.11}$$

We are interested in the threshold limit

$$\frac{s - 4m_n^2}{m_n^2} \rightarrow 0, \quad \frac{t + m_n^2}{m_n^2} \rightarrow 0, \quad \frac{u + m_n^2}{m_n^2} \rightarrow 0, \tag{3.12}$$

leading to

$$\begin{aligned}
 |\mathcal{M}_{\gamma\gamma}|^2 \xrightarrow{\text{Threshold}} & \frac{4}{(D-2)^2} \left[(D-2) - \frac{3(D-1)^2}{4(D-2)} \right. \\
 & \left. + \frac{D-1}{D-2} \right] g^4 q_n^4 = \left(\frac{D-3}{D-2} \right)^2 \frac{g^4 q_n^4}{D-2}
 \end{aligned} \tag{3.13}$$

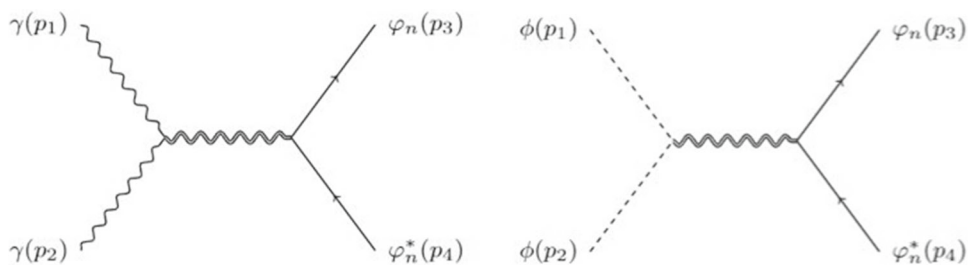
We note that, in a $U(1)$ gauge theory with no dilaton, the amplitude would be given by the first line of (3.11) only, that means in the threshold limit $4g^4 q_n^4 / (D-2)$ for a state of charge q .

The production from a dilation pair $\phi\phi \rightarrow \varphi_n\varphi_n^*$ (second line of Fig. 1) is immediately recognized to give a null result in the limit of interest:

$$\begin{aligned}
 i\mathcal{M}_{\phi\phi} = & -4i\kappa^2 \frac{D-1}{D-2} m_n^4 \left(\frac{1}{t - m_n^2} + \frac{1}{u - m_n^2} \right) \\
 & - 4i\kappa^2 \frac{D-1}{D-2} m_n^2 \xrightarrow{\text{Threshold}} 0.
 \end{aligned} \tag{3.14}$$

Finally, the production from the pair photon-dilaton $\phi\gamma \rightarrow \varphi_n\varphi_n^*$ receives contributions from the three s, t and u -

Fig. 2 Feynman diagrams for pair production, gravitationally mediated, from photons and dilatons



channels (see the third line of Fig. 1)

$$\begin{aligned}
 & i\mathcal{M}_{\gamma(p_1)\phi(p_2)} \\
 &= \epsilon_\mu(p_1) \left\{ -2\sqrt{\frac{D-1}{D-2}} \kappa g q_n (p_1 \cdot (p_1 + p_2)) g^{\mu\rho} \right. \\
 &\quad \left. - p_1^\rho (p_1 + p_2)^\mu (p_3 - p_4)_\rho \frac{i}{s} + 2i\sqrt{\frac{D-1}{D-2}} \kappa g q_n m_n^2 \right. \\
 &\quad \left. \times \left(\frac{(2p_3 - p_1)^\mu}{t - m_n^2} - \frac{(2p_4 - p_1)^\mu}{u - m_n^2} \right) \right\}, \tag{3.15}
 \end{aligned}$$

and this is easily verified to give a null contribution in the threshold limit.

3.2.2 Mixed amplitudes

We consider now the ‘‘mixed gravitational’’ processes: we start by computing the graviton s-channel mediation for $\gamma\gamma$ and $\phi\phi$ initial states, then the amplitudes with initial states γG and ϕG . We present hereafter the results for the particular case $D = 4$. When it will be of interest, we will show the results for a generic number of dimensions D .

The additional contribution to the $\gamma\gamma$ and $\phi\phi$ productions described in Fig. 2 respectively read

$$\begin{aligned}
 & i\mathcal{M}_{\gamma\gamma}^G = -\kappa^2 \left\{ (p_1 \cdot p_2)(\epsilon_{1\alpha}\epsilon_{2\beta} + \epsilon_{1\beta}\epsilon_{2\alpha}) \right. \\
 &\quad + (p_{1\alpha}p_{2\beta} + p_{1\beta}p_{2\alpha})(\epsilon_1 \cdot \epsilon_2) \\
 &\quad - (\epsilon_{1\alpha}p_{2\beta} + p_{2\alpha}\epsilon_{1\beta})(p_1 \cdot \epsilon_2) \\
 &\quad - (\epsilon_{2\alpha}p_{1\beta} + p_{1\alpha}\epsilon_{2\beta})(p_2 \cdot \epsilon_1) \\
 &\quad \left. - \eta_{\alpha\beta}(p_1 \cdot p_2 \epsilon_1 \cdot \epsilon_2 - \epsilon_1 \cdot p_2 \epsilon_2 \cdot p_1) \right\} \frac{i\mathcal{P}^{\alpha\beta\rho\sigma}}{s} \\
 &\quad \times \left\{ p_{3\rho}p_{4\sigma} + p_{3\sigma}p_{4\rho} - \eta_{\rho\sigma}(p_3 \cdot p_4 + m^2) \right\}, \tag{3.16}
 \end{aligned}$$

and

$$\begin{aligned}
 & i\mathcal{M}_{\phi\phi}^G = -\kappa^2 \left\{ p_{1\alpha}p_{2\beta} + p_{1\beta}p_{2\alpha} - \eta_{\alpha\beta}p_1 \cdot p_2 \right\} \\
 &\quad \times \frac{i\mathcal{P}^{\alpha\beta\rho\sigma}}{s} \left\{ p_{3\rho}p_{4\sigma} + p_{3\sigma}p_{4\rho} - \eta_{\rho\sigma}(p_3 \cdot p_4 + m^2) \right\}, \tag{3.17}
 \end{aligned}$$

where $\epsilon_i = \epsilon(p_i)$. For the $\gamma\gamma \rightarrow \phi_n\phi_n^*$ amplitude, a (simpler) way to compute this is through projecting onto a specific basis for the polarizations ϵ (see Appendix B).

Working in the center of mass frame for the massive particles, we obtain the different components of the graviton mediated $\gamma\gamma \rightarrow \phi_n\phi_n^*$ as follows

$$\begin{aligned}
 & i\mathcal{M}_{+,+}^G = i\mathcal{M}_{-,-}^G \\
 &= -i\frac{\kappa^2}{s} [tu - m_n^4 + (m_n^2 - u)^2 + su] = 0 \\
 & i\mathcal{M}_{+,-}^G = i\mathcal{M}_{-,+}^G = i\frac{\kappa^2}{s} [tu - m_n^4], \tag{3.18}
 \end{aligned}$$

where the \pm sign refers to the helicities of the incoming gauge bosons. In the threshold limit the graviton mediated contribution vanishes for both components.

In D dimensions, the whole $\mathcal{M}_{\gamma\gamma}$ amplitude reads

$$\left| \mathcal{M}_{\gamma\gamma} \right|^2 \xrightarrow{\text{Threshold}} \frac{(2(D-2)(gq)^2 + (D-4)\kappa^2 m^2)^2}{(D-2)^3} \tag{3.19}$$

for a generic $U(1)$ gauge theory (i.e. when the dilaton is put to zero) and

$$\left| \mathcal{M}_{\gamma\gamma} \right|^2 \xrightarrow{\text{Threshold}} \frac{((D-3)(gq_n)^2 + (D-4)\kappa^2 m_n^2)^2}{(D-2)^3} \tag{3.20}$$

in the dilatonic theory we are studying here. Both the results for the $U(1)$ and dilatonic theory ((3.13) and discussion below) are recovered in the limit $\kappa \rightarrow 0$. It is instructive to note, from these equations, that the vanishing of the graviton mediated contribution to the production from a photon pair is specific to the case of $D = 4$ dimensions, and in $D \neq 4$ dimensions mixed terms of the form $g^2 q^2 \times \kappa^2 m^2$ are generated.

For the $\phi\phi \rightarrow \phi_n\phi_n^*$ the amplitude reads

$$i\mathcal{M}_{\phi\phi}^G = -i\frac{\kappa^2}{s} [m_n^4 - ut - m_n^2 s]. \tag{3.21}$$

This results in a non vanishing contribution in the limit of interest such that

$$i\mathcal{M}_{\phi\phi}^G = i\kappa^2 m_n^2. \tag{3.22}$$

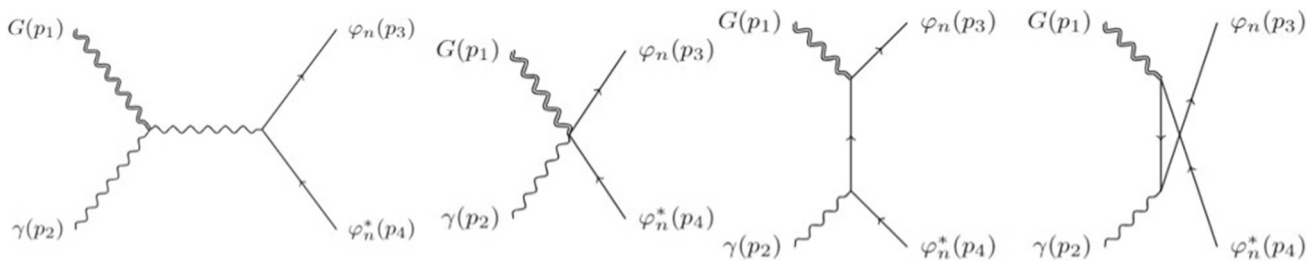


Fig. 3 Feynman diagrams for the mixed pair production from a graviton and a photon

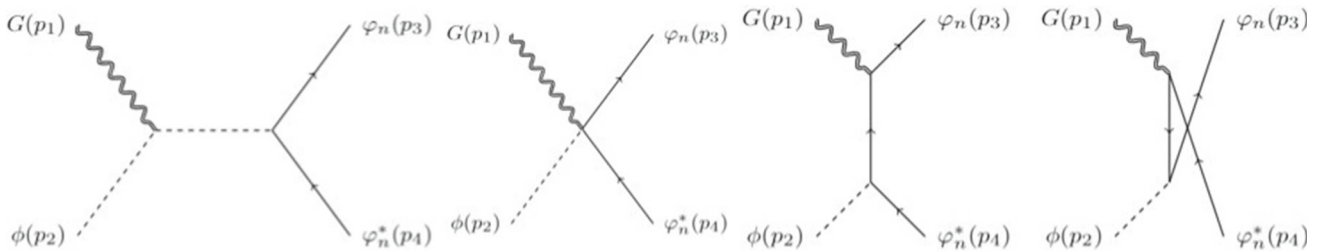


Fig. 4 Feynman diagrams for the mixed pair production from a graviton and a dilaton

Concerning the mixed initial states, we have both $\gamma G \rightarrow \varphi_n \varphi_n^*$ (see Fig. 3) and $\phi G \rightarrow \varphi_n \varphi_n^*$ (see Fig. 4). Each of these two processes receive contributions from four diagrams. Starting with the graviton-photon production, the amplitude $G(p_1)\gamma(p_2) \rightarrow \varphi_n \varphi_n^*$ takes the form

$$i\mathcal{M}_{G\gamma}^{\text{mix.}} = i\kappa g q_n \left(\frac{4(\epsilon_1 \cdot p_3)^2 \epsilon_2 \cdot p_4}{t - m_n^2} - \frac{4(\epsilon_1 \cdot p_4)^2 \epsilon_2 \cdot p_3}{u - m_n^2} + 2\epsilon_1 \cdot \epsilon_2 \epsilon_1 \cdot (p_3 - p_4) - \frac{(p_1 + p_2) \cdot p_2 (2\epsilon_1 \cdot \epsilon_2 \epsilon_1 \cdot (p_3 - p_4))}{s} \right) \quad (3.23)$$

and so for the different choices of graviton and photon helicities:

$$\begin{cases} i\mathcal{M}_{++,+}^{\text{mix.}} = -i\mathcal{M}_{--,-}^{\text{mix.}} \\ = -i\kappa g q_n \sqrt{2} \frac{tu - m_n^4}{s} \left(\frac{m_n^4 - tu}{(t - m_n^2)(u - m_n^2)} + 3 \right) \\ i\mathcal{M}_{++,-}^{\text{mix.}} = -i\mathcal{M}_{--,+}^{\text{mix.}} = i\kappa g q_n \sqrt{2} \frac{tu - m_n^4}{s} \left(\frac{m_n^4 - tu}{(t - m_n^2)(u - m_n^2)} \right). \end{cases} \quad (3.24)$$

It is immediately verified that all these contributions vanish in the threshold limit where $t \rightarrow -m_n^2$ and $u \rightarrow -m_n^2$.

The same vanishing limit at threshold holds for the mixed graviton-dilaton production, where the amplitude is

$$i\mathcal{M}_{G\phi}^{\text{mix.}} = -2i\kappa \mu_n \left(\frac{(\epsilon_1 \cdot p_3)^2}{t - m_n^2} + \frac{(\epsilon_1 \cdot p_4)^2}{u - m_n^2} \right) \quad (3.25)$$

with $\mu_n = \sqrt{6}\kappa m_n^2$ the three-point $\phi\varphi_n\varphi_n^*$ $D = 4$ coupling, and finally

$$i\mathcal{M}_{++}^{\text{mix.}} = i\mathcal{M}_{--}^{\text{mix.}} = i\kappa \mu_n \frac{tu - m_n^4}{(t - m_n^2)(u - m_n^2)}. \quad (3.26)$$

From the explicit results presented in Appendix B, it is also immediate to realize that the mixed contributions vanish at threshold for all D .

3.2.3 Gravitational production amplitudes

Finally, we discuss the purely gravitational production. The starting point for the expression of the amplitude is rather long. It receives in fact contribution from the four diagrams of Fig. 5, each one with vertices determined from a two-derivative interacting term (some details about two-derivative interactions are discussed in Appendix A). We prefer to give here a more compact expression that is obtained after some algebra:

$$i\mathcal{M}_{GG} = \frac{\kappa^2}{2} \left(-\frac{8(p_3 \cdot \epsilon_1)^2 (p_4 \cdot \epsilon_2)^2}{t - m_n^2} - \frac{8(p_3 \cdot \epsilon_2)^2 (p_4 \cdot \epsilon_1)^2}{u - m_n^2} - 2\frac{(\epsilon_1 \cdot \epsilon_2)^2 (m_n^4 - tu - sm_n^2)}{s} - 4\epsilon_1 \cdot \epsilon_2 (p_3 \cdot \epsilon_2 p_4 \cdot \epsilon_1 + p_3 \cdot \epsilon_1 p_4 \cdot \epsilon_2) \right) \quad (3.27)$$

The complete results for each one of the four diagrams contributing to the amplitude are presented in Appendix B, together with the description of the helicity method. Using now the specific basis for $D = 4$ dimensions, we find

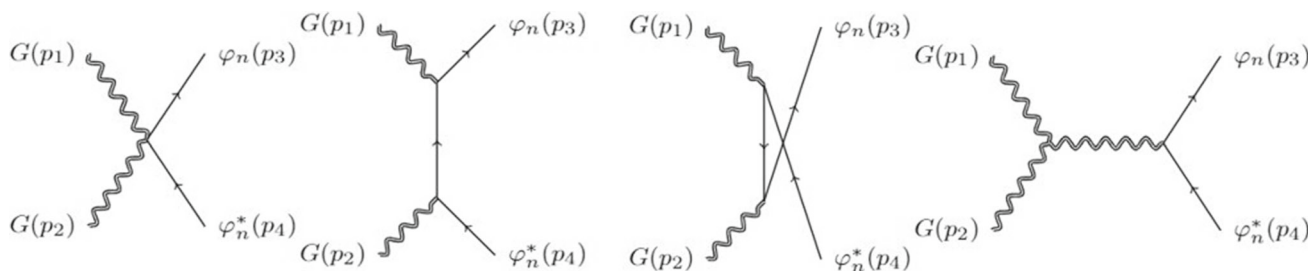


Fig. 5 Feynman diagrams for the production of a pair of matter states from two gravitons

$$\begin{aligned}
 i\mathcal{M}_{++,+} &= i\mathcal{M}_{--,-} = i\kappa^2 \left(\frac{(m_n^4 - tu)m^2}{(t - m_n^2)(u - m_n^2)} + m_n^2 \right) \\
 i\mathcal{M}_{+,-} &= i\mathcal{M}_{-,+} = i\kappa^2 \frac{(m_n^4 - tu)^2}{s(t - m_n^2)(u - m_n^2)},
 \end{aligned}
 \tag{3.28}$$

Comparing this result with the one obtained from the $\gamma\gamma$ production in the case with no dilaton, we verify the factorization

$$\begin{aligned}
 \mathcal{M}_{++,+}^{(GG)} &= \frac{\kappa^2}{4(gq)^4} \frac{(t - m_n^2)(u - m_n^2)}{s} \mathcal{M}_{++,+}^{(\gamma\gamma)} \\
 \mathcal{M}_{+,-}^{(GG)} &= \frac{\kappa^2}{4(gq)^4} \frac{(t - m_n^2)(u - m_n^2)}{s} \mathcal{M}_{+,-}^{(\gamma\gamma)}.
 \end{aligned}
 \tag{3.29}$$

The corresponding factorization for the comparison between the gravitational Compton scattering $G\varphi \rightarrow G\varphi$ (with φ a generic scalar field) and the usual Compton scattering was found in [12, 13] (see also [14]).

From the above results, in the threshold limit we have

$$\begin{aligned}
 |\mathcal{M}_{GG}|^2 &= \frac{1}{4} \left(|\mathcal{M}_{++,+}|^2 + |\mathcal{M}_{+,-}|^2 + |\mathcal{M}_{--,-}|^2 \right. \\
 &\quad \left. + |\mathcal{M}_{-,+}|^2 \right) \rightarrow \frac{\kappa^4 m_n^4}{2}.
 \end{aligned}
 \tag{3.30}$$

Note that the result $|\mathcal{M}_{GG}|^2 \rightarrow \kappa^4 m^4/2$, and more generally the “purely gravitational” pair production, is independent from the presence of the dilaton. This is easily generalized to the case of generic D (see again Appendix B for details) and leads in the threshold limit to

$$|\mathcal{M}_{GG}|^2 \rightarrow \frac{1}{D-2} \kappa^4 m_n^4.
 \tag{3.31}$$

3.2.4 Gravitational vs gauge amplitudes

When the dilaton is put to zero, the requirement

$$|\mathcal{M}_{\gamma\gamma}|^2 \Big|_{\text{Threshold}} \geq |\mathcal{M}_{GG}|^2
 \tag{3.32}$$

gives the original $U(1)$, $D = 4$ WGC bound $\sqrt{2}gq \geq \kappa m$.

Using cross-symmetry on the results of [12–14], the authors of [7] observed that (3.32) leads to the WGC relation and proposed (3.32) as a possible alternative formulation of the WGC. In [7], the graviton-mediated diagram was not taken into account in the γ amplitude. Our calculation shows that in the threshold limit, the contribution of this additional diagram disappears. Therefore, in the four-dimensional $U(1)$ gauge theory, we can safely compare, as in the (3.32), the $\gamma\gamma$ and GG productions without having to neglect any contribution.

Our calculation also shows that in $D = 4$ dimensions, the KK states saturate (3.32). In fact, we emphasize again that the gravitational amplitude \mathcal{M}_{GG} , here, does not care about the presence of the dilaton: whether the theory is a simple $U(1)$ or a dilatonic $U(1)$, the result for \mathcal{M}_{GG} is unchanged. On the other hand, the amplitude $\mathcal{M}_{\gamma\gamma}$ receives an additional contribution which changes the numerical coefficient in front of $g^4 q^4$ from 2 to 1/8. Since the $\mathcal{M}_{\gamma\phi}$ and $\mathcal{M}_{\phi\phi}$ amplitudes both vanish in the threshold limit, the comparison of the pair production processes in this KK theory leads to

$$\frac{g^4 q^4}{8} \geq \frac{\kappa^4 m_n^4}{2} \implies gq \geq \sqrt{2}\kappa m
 \tag{3.33}$$

and (2.16) shows that KK states saturate it.

However, if, in the presence of the dilaton, we consider gravitationally mediated diagrams for $\gamma\gamma$ and $\phi\phi$ amplitudes, there is a non-vanishing contribution that comes from $\mathcal{M}_{\phi\phi}^G$ in (3.22), and this would clearly spoil the saturation observed for the KK states. The inclusion of the mixed production channels $G\gamma$ (3.23) and $G\phi$ (3.25) cannot restore the saturation property, since both do not contribute in the limit of interest. The dilatonic WGC will be recovered only if the contributions from graviton exchanges in $\gamma\gamma$ and $\phi\phi$ amplitudes are not included.

Note also that the pairwise production comparison does not reproduce the constraints of WGCs in more than 4 dimensions. The $\mathcal{M}_{\gamma\gamma}$ and \mathcal{M}_{GG} amplitudes lead, for any D , to compare $\sqrt{2}gq$ and κm . For the case of a simple theory $U(1)$, setting as quoted above the dilaton to zero in our calculations, the result for the production from a photon pair in D dimen-

sions in the threshold limit is

$$|\mathcal{M}_{\gamma\gamma}|^2 = \frac{4}{D-2}(gq)^4. \tag{3.34}$$

In Appendix B we learn that the purely gravitational production of pairs gives, in the same limit of interest,

$$|\mathcal{M}_{GG}|^2 = \frac{1}{D-2}(\kappa m)^4. \tag{3.35}$$

By comparing (3.34) and (3.35), it is immediate to observe that requiring $|\mathcal{M}_\gamma|^2 \geq |\mathcal{M}_{GG}|^2$, one *does not reproduce* the WGC bound

$$gq \geq \sqrt{\frac{D-3}{D-2}}\kappa m. \tag{3.36}$$

Similarly, the comparison of purely gravitational pair production and purely non-gravitational pair production in the KK theory we consider here amounts to a comparison of the results

$$\begin{aligned} |\mathcal{M}_{\gamma\gamma}|^2 &\rightarrow \frac{(D-3)^2}{(D-2)^3}g^4q_n^4, \quad |\mathcal{M}_{\phi\phi}|^2 \rightarrow 0, \\ |\mathcal{M}_{GG}|^2 &\rightarrow \frac{1}{D-2}\kappa^4m_n^4. \end{aligned} \tag{3.37}$$

Using (2.16), it is immediate to realize that the KK states saturate the (3.32) (or an equivalent generalization of it to include the $\mathcal{M}_{\phi\phi}$ contribution which disappears here) only for $D = 4$. The results of Sect. 3.2.2 show that the addition of mixed contributions does not change this.

4 Massive and self-interacting scalars

We next consider the presence of mass and self-interacting terms in the higher dimensional scalar theory. The KK scalar modes are no more extremal states of the WGC, but this setup will allow us to retrieve scalar weak gravity conjectures which are postulated to constrain the relative strength of the additional terms.

We will consider the simple extension of (2.1)

$$\mathcal{S}_{int} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \left[-\frac{1}{2}\hat{m}^2\hat{\Phi}^2 + \frac{\hat{\mu}}{3!}\hat{\Phi}^3 - \frac{\hat{\lambda}}{4!}\hat{\Phi}^4 \right]. \tag{4.1}$$

Here, \hat{m} has mass dimension one, $\hat{\mu}$ has dimension $3 - \frac{D+1}{2}$ and $\hat{\lambda}$ has dimension $4 - (D+1)$. Using the ansatz (2.12), it is straightforward to see that the action takes the form

$$\begin{aligned} \mathcal{S} &= \mathcal{S}_f + \mathcal{S}_{int} \\ &= \int d^Dx \sqrt{(-1)^{D-1}g} \left\{ \frac{R}{2\kappa^2} + \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2\sqrt{\frac{D-1}{D-2}}\kappa\phi}F^2 \right. \\ &\quad \left. + \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}\hat{m}^2\phi_0^2} \right\} \end{aligned}$$

$$\begin{aligned} &+ \sum_{n=1}^{\infty} \partial_\mu\varphi_n\partial^\mu\varphi_n^* - \sum_{n=1}^{\infty} \left(e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi} \frac{n^2}{L^2} + e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi}\hat{m}^2} \right) \varphi_n\varphi_n^* \\ &+ \sum_{n=1}^{\infty} \left[i\sqrt{2}\kappa \frac{n}{L} A^\mu (\partial_\mu\varphi_n\varphi_n^* - \varphi_n\partial_\mu\varphi_n^*) + 2\kappa^2 \frac{n^2}{L^2} A_\mu A^\mu \varphi_n\varphi_n^* \right] \\ &+ e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0} \left[\frac{\mu}{3!}\varphi_0^3 - \frac{\lambda}{4!}\varphi_0^4 + \mu\varphi_0 \sum_{n=1}^{\infty} \varphi_n\varphi_n^* - \frac{\lambda}{2}\varphi_0^2 \sum_{n=1}^{\infty} \varphi_n\varphi_n^* \right. \\ &- \frac{\lambda}{2}\varphi_0 \sum_{m,n=1}^{\infty} (\varphi_m\varphi_m\varphi_{n+m}^* + \varphi_m^*\varphi_n^*\varphi_{n+m}) \\ &+ \frac{\mu}{2} \sum_{n,m=1}^{\infty} (\varphi_n\varphi_m\varphi_{n+m}^* + \varphi_n^*\varphi_m^*\varphi_{n+m}) \\ &- \frac{\lambda}{3!} \sum_{m,n,p=1}^{\infty} (\varphi_m\varphi_n\varphi_p\varphi_{m+n+p}^* + \varphi_m^*\varphi_n^*\varphi_p^*\varphi_{m+n+p}) \\ &\left. - \frac{\lambda}{2} \sum_{n=1}^{\infty} \varphi_n\varphi_n^* \sum_{m=1}^{\infty} \varphi_m\varphi_m^* - \frac{\lambda}{4} \sum_{\substack{m,n,p=1 \\ m \neq p, n \neq p, m+n > p}}^{\infty} \varphi_m\varphi_n\varphi_p^*\varphi_{n+m-p}^* \right] \end{aligned} \tag{4.2}$$

where we have kept the notation compact, but, in our perturbative analysis, the dilaton will again be expanded around a background value ϕ_0 as above. The couplings constants μ and λ are defined, from their higher dimensional counterpart, as

$$\mu = \frac{\hat{\mu}}{\sqrt{2\pi}L}, \quad \lambda = \frac{\hat{\lambda}}{2\pi L}. \tag{4.3}$$

The tree-level masses for the zero mode φ_0 and the KK excitations are given by:

$$\begin{aligned} m_0^2 &= e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}\hat{m}^2}, \quad m_n^2 = e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \frac{n^2}{L^2} \\ &+ e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}\hat{m}^2}. \end{aligned} \tag{4.4}$$

4.1 The scalar weak gravity conjecture

We start by computing the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ amplitude. The diagrams intervening in the scattering are presented in the Fig. 6. The non-relativistic limit of the tree-level amplitude reads

$$\begin{aligned} i\mathcal{M} &= ie^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0} \left[e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0} \frac{5}{3} \frac{\mu^2}{m_0^2} - \lambda \right] \\ &- \frac{i}{(D-1)(D-2)}\kappa^2m_0^2 \\ &- 4\frac{i}{(D-1)(D-2)}\kappa^2m_0^4 \left(\frac{1}{t} + \frac{1}{u} \right) \\ &+ i\frac{D-1}{D-2}\kappa^2m_0^2 - 4i\frac{D-3}{D-2}\kappa^2m_0^4 \left(\frac{1}{t} + \frac{1}{u} \right), \end{aligned} \tag{4.5}$$

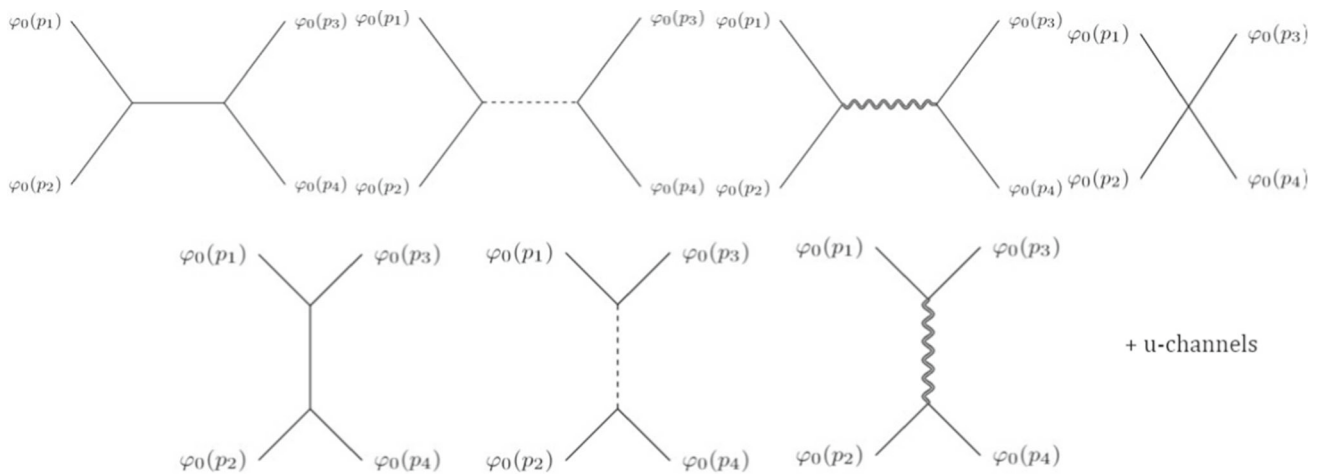


Fig. 6 Feynman diagrams for the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ scattering when a potential for the higher dimensional scalar, “parent” of φ_0 , has been turned o.n

where the different lines correspond to the contributions from the self-interaction, dilaton and graviton exchanges, respectively.

Following [6], we compare the contributions to the amplitude at the energy scale given by the (massive) external states at rest. In the non-relativistic limit, we can further split (4.5) into contributions from short and long range interactions. We can identify an effective contact interaction:

$$\begin{aligned}
 i\mathcal{M}_{CT}^{(D)} &= i e^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0} \left(\frac{5}{3} \frac{\mu^2}{\hat{m}^2} - \hat{\lambda} - \frac{1}{(D-1)(D-2)} \kappa^2 \hat{m}^2 \right. \\
 &\quad \left. + \frac{D-1}{D-2} \kappa^2 \hat{m}^2 \right) = i \frac{e^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0}}{2\pi L} \\
 &\quad \times \left(\frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \hat{\lambda} + 2\pi L \frac{D}{D-1} \kappa^2 \hat{m}^2 \right). \tag{4.6}
 \end{aligned}$$

where in the first line we can identify the contributions from the scalar interaction for the first two terms, then from the dilaton and graviton, respectively. Using (2.9) and the $(D+1)$ -gravitational coupling $\hat{\kappa} = \sqrt{2\pi L}\kappa$, the last term is recognized to be the gravitational s-channel contribution to the $\hat{\Phi}\hat{\Phi} \rightarrow \hat{\Phi}\hat{\Phi}$ scattering in $D+1$ dimensions:

$$\begin{aligned}
 i\mathcal{M}_{CT}^{(D+1)} &= i \frac{e^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0}}{2\pi L} \\
 &\quad \times \left(\frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \hat{\lambda} + \frac{(D+1)-1}{(D+1)-2} \hat{\kappa}^2 \hat{m}^2 \right). \tag{4.7}
 \end{aligned}$$

The above equation illustrates the fact that constraining the scalar interactions of the field $\hat{\Phi}$ to be dominant with respect to gravity in $D+1$ dimensions is enough to ensure that the scalar interactions of the zero mode φ_0 are dominant with respect to the combination of gravitational and dilatonic contributions in D dimensions. In other words, the effective (tree-level) non-relativistic four-point function of the zero mode φ_0 that emerges in the reduced-dimensional theory is

the same as the effective non-relativistic four-point coupling for the “parent” field $\hat{\Phi}$ in the higher-dimensional theory. Requiring that in such a contact term, the contributions of the $\hat{\Phi}$ self-interactions are the dominant ones in the $D+1$ dimensions automatically ensures that the same property holds for the φ_0 self-interactions with respect to the set of interactions that appear in the D dimensional theory.

It is interesting to observe that the higher dimensional result is recovered here thanks to a cancellation, rather than an addition, between the graviton and dilaton mediated diagrams. This is dictated by the form of the D -dependent coefficient $\gamma_s(D) \equiv (D-1)/(D-2)$ appearing in front of the graviton-mediated amplitude in the s -channel which decreases with D : $\gamma_s(D+1) < \gamma_s(D)$. The dimension-dependent factor appearing in the t and u -channels, $\gamma_{t,u}(D) \equiv (D-3)/(D-2)$ vary in the opposite direction. In other words, the peculiar feature is that, for the contact terms, the spin-2 and spin-0 bosonic mediators give opposite contributions. This feature will also appear in the amplitudes computed with the non minimal coupling to gravity. As a consequence of particular interest in the case of a massive dilaton the higher dimensional sub-dominance of gravity does not imply that gravity by itself (i.e. without the dilaton) is subdominant in the lower dimensional theory too. This violation happens in the parametric region

$$\frac{D}{D-1} \hat{\kappa}^2 \hat{m}^2 \leq \left| \frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \hat{\lambda} \right| \leq \frac{D-1}{D-2} \hat{\kappa}^2 \hat{m}^2, \tag{4.8}$$

which is an interval of length $\hat{\kappa}^2 \hat{m}^2 / (D-1)(D-2)$ inversely proportional to the dimension D .

The amplitude $\varphi_n\varphi_n \rightarrow \varphi_n\varphi_n$ provides a generalization in the presence of self-interacting terms of the computation done in Sect. 3.1. The scattering amplitude receives contributions from gauge bosons, dilatons, gravitons in the t and u -channels, φ_0 exchange, from the s -channel exchange of a

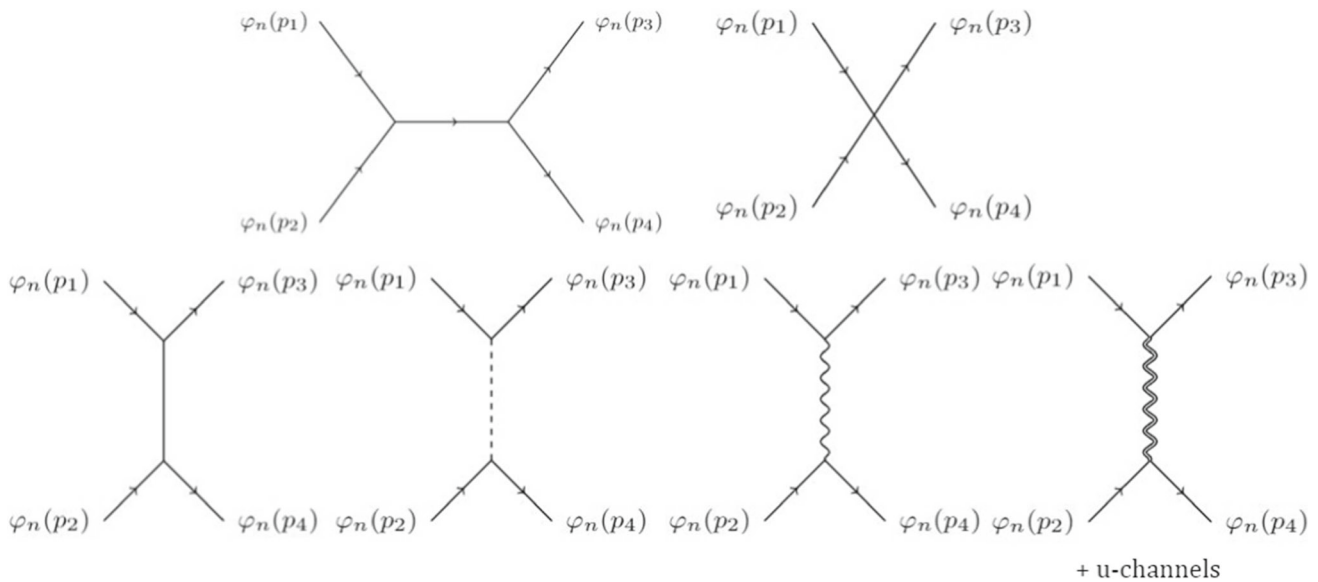


Fig. 7 Feynman diagrams for the $\varphi_n \varphi_n \rightarrow \varphi_n \varphi_n$ scattering in the t -channel

φ_{2n} particle and from a 4-point contact term. These are the diagrams that are presented in Fig. 7 and lead to

$$\begin{aligned}
 i\mathcal{M} = & -ie^{\frac{4}{\sqrt{(D-1)(D-2)}}\kappa\phi_0}\mu^2 \left(\frac{1}{s-m_{2n}^2} + \frac{1}{t-m_0^2} + \frac{1}{u-m_0^2} \right) \\
 & - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0} + i \left(\frac{1}{t} + \frac{1}{u} \right) \\
 & \times \left(4g^2q_n^2m_n^2 - 4\frac{D-3}{D-2}\frac{m_n^4}{M_P^{D-2}} - (\partial_\phi m_n^2)^2 \right) \quad (4.9)
 \end{aligned}$$

with

$$\begin{aligned}
 \partial_\phi m_n^2 = & \frac{1}{M_P^{(D-2)/2}} \left(\frac{2}{\sqrt{(D-1)(D-2)}} e^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0} \hat{m}^2 \right. \\
 & \left. + 2\sqrt{\frac{D-1}{D-2}} e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \frac{n^2}{L^2} \right). \quad (4.10)
 \end{aligned}$$

4.2 Massive dilatons

Let us consider for our illustrative discussion a simple potential for the dilaton in a polynomial expansion of the form

$$V(\phi) = \frac{1}{2}m_\phi^2\phi^2 - \frac{\mu_\phi}{3!}\phi^3 + \frac{\lambda_\phi}{4!}\phi^4. \quad (4.11)$$

In the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ scattering amplitude (4.5), the addition of a dilaton mass gives in the non-relativistic limit

$$\begin{aligned}
 i\mathcal{M}(\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0) = & ie^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0} \\
 & \times \left[e^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0} \frac{5}{3} \frac{\mu^2}{m_0^2} - \lambda \right]
 \end{aligned}$$

$$\begin{aligned}
 & - 4\frac{i}{(D-1)(D-2)}\kappa^2m_0^4\frac{1}{s-m_\phi^2} \\
 & - 4\frac{i}{(D-1)(D-2)}\kappa^2m_0^4\left(\frac{1}{t-m_\phi^2} + \frac{1}{u-m_\phi^2}\right) \\
 & + i\frac{D-1}{D-2}\kappa^2m_0^2 - 4i\frac{D-3}{D-2}\kappa^2m_0^4\left(\frac{1}{t} + \frac{1}{u}\right), \quad (4.12)
 \end{aligned}$$

where the limit still needs to be implemented in the dilaton propagators according to its mass. We can thus follow the evolution of \mathcal{M} with respect to m_ϕ to better expand it.

For the $\varphi_n\varphi_n \rightarrow \varphi_n\varphi_n$ case, the scattering amplitude with the massive dilaton reads

$$\begin{aligned}
 i\mathcal{M}(\varphi_n\varphi_n \rightarrow \varphi_n\varphi_n) = & -ie^{\frac{4}{\sqrt{(D-1)(D-2)}}\kappa\phi_0}\mu^2 \\
 & \times \left(\frac{1}{s-m_{2n}^2} + \frac{1}{t-m_0^2} + \frac{1}{u-m_0^2} \right) - i\lambda e^{\frac{2}{\sqrt{(D-1)(D-2)}}\kappa\phi_0} \\
 & - i(\partial_\phi m_n^2)^2 \left(\frac{1}{t-m_\phi^2} + \frac{1}{u-m_\phi^2} \right) \\
 & + i \left(\frac{1}{t} + \frac{1}{u} \right) \left(4g^2q_n^2m_n^2 - 4\frac{D-3}{D-2}\kappa^2m_n^4 \right). \quad (4.13)
 \end{aligned}$$

Putting all the analysis for both the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ and $\varphi_n\varphi_n \rightarrow \varphi_n\varphi_n$ scattering amplitudes together, we give a brief overview of the results here.

When the mass m_ϕ of the dilaton is less than that of the zero mode, m_0 , its mass can be neglected to first order in an expansion, in powers of m_ϕ over the exchanged momentum, and requiring that the self-interactions of a scalar field dominate in $D + 1$ dimensions is sufficient to ensure that the same property is verified by its zero mode in D dimen-

sions; a result that follows from the studies of the previous sections. As soon as the mass of the dilaton is comparable to that of the 0-mode, the massless dilaton approximation is no longer adequate and an appropriate discussion must be made for different denominators involving m_ϕ , m_0 , m_n and m_{2n} . The analysis can be done easily but it is cumbersome and not really illuminating. In short, there is no easy way to relate combinations appearing in D dimensions in this case with quantities already constrained, by assumption, in $D + 1$ dimensions.

5 $\hat{\Phi}^2 R$ interaction

Let us consider now the effect on the different D -dimensional amplitudes of the presence of a non-minimal coupling to gravity of the form

$$S_{(\xi)} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \frac{\xi}{2} \hat{\Phi}^2 \hat{R}, \tag{5.1}$$

with \hat{R} the Ricci scalar (see for example [15]). We assume here that $\langle \hat{\Phi} \rangle = 0$ as a non-vanishing vev would correspond to a redefinition of the Planck mass and a shift of the canonical fields. After compactification, one gets:

$$S_{(\xi)} = \int d^Dx \sqrt{(-1)^{D-1}g} \left[\xi \left(R - \kappa^2 (\partial\phi)^2 - \frac{2\kappa}{\sqrt{(D-1)(D-2)}} \nabla_\mu \partial^\mu \phi - \frac{1}{2} e^{-2\sqrt{\frac{D-1}{D-2}}\kappa\phi} \kappa^2 F^2 \right) \times \left(\frac{\phi_0^2}{2} + \sum_{n=1}^\infty \phi_n \phi_n^* \right) \right]. \tag{5.2}$$

This leads to new three-point couplings. First, using the linear expansion of the metric $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$, the R term gives the new coupling $\kappa (\partial_\mu \partial_\lambda h^{\mu\lambda} - \square h^\lambda_\lambda)$ ($\phi_0^2 + 2 \sum_{n=1}^\infty \phi_n \phi_n^*$) of the graviton to the scalar matter fields. Then, the $\nabla_\mu \partial^\mu \phi$ term, that we discard in previous sections as it takes the form of a total derivative, gives an additional three-point vertex between the dilaton and the matter fields and can enter, for example, in the computation of the dilatonic force in the non-relativistic limit. At first order in κ , we can write $\kappa \nabla_\mu \partial^\mu \phi = \kappa \partial_\mu \partial^\mu \phi + \mathcal{O}(\kappa^2)$, the Christoffel symbols starting themselves at order κ .

The $\phi_0 \phi_0 \rightarrow \phi_0 \phi_0$ amplitude resulting from the action (5.2), receives a contribution from the dilaton exchange (see Appendix A for some details on the Feynman rules for two-derivative vertices)

$$i\mathcal{M}_\phi = -i \frac{4}{(D-1)(D-2)} \xi^2 \kappa^2 (s+t+u) = -i \frac{16}{(D-1)(D-2)} \xi^2 \kappa^2 m_0^2. \tag{5.3}$$

and one from the graviton

$$i\mathcal{M}_G = 4i \frac{D-1}{D-2} \xi^2 \kappa^2 (s+t+u) = 16i \frac{D-1}{D-2} \xi^2 \kappa^2 m_0^2. \tag{5.4}$$

Their sum gives

$$i\mathcal{M}_{(non-minimal)} = i\mathcal{M}_\phi + i\mathcal{M}_G = 4i \frac{D}{D-1} \xi^2 \kappa^2 (s+t+u) = 16i \frac{D}{D-1} \xi^2 \kappa^2 m_0^2. \tag{5.5}$$

This matches the result one would obtain for the $\hat{\Phi} \hat{\Phi} \rightarrow \hat{\Phi} \hat{\Phi}$ scattering in $D + 1$ dimensions.

At this point, we have computed tree-level four point amplitudes where both vertices arise either from minimal or non-minimal couplings to gravity in $D+1$ dimensions. In order to compute the total $\phi_0 \phi_0 \rightarrow \phi_0 \phi_0$ amplitude we need to compute the contribution from “mixed” diagrams involving one minimal and one non-minimal vertices. This mixed gravitational diagrams give in the s -channel

$$i\mathcal{M}_{s\text{-channel}}^{G\text{-mix.}} = -2i\xi \frac{\kappa^2}{s} \left(2 p_1 \cdot (p_1 + p_2) p_2 \cdot (p_1 + p_2) - (p_1 + p_2)^2 (p_1 \cdot p_2 + m_0^2) + \frac{2}{D-2} (p_1 + p_2)^2 (p_1 \cdot p_2) - \frac{D}{D-2} (p_1 + p_2)^2 (p_1 \cdot p_2 + m_0^2) \right) \tag{5.6}$$

and in the t -channel

$$i\mathcal{M}_{t\text{-channel}}^{G\text{-mix.}} = 2i\xi \frac{\kappa^2}{t} \left(2 p_1 \cdot (p_1 - p_3) p_3 \cdot (p_1 - p_3) - (p_1 - p_3)^2 (p_1 \cdot p_3 - m_0^2) + \frac{2}{D-2} (p_1 - p_3)^2 (p_1 \cdot p_3) - \frac{D}{D-2} (p_1 - p_3)^2 (p_1 \cdot p_3 - m_0^2) \right), \tag{5.7}$$

while the u channel can be obtained through the replacements $t \leftrightarrow u$ and $p_3 \leftrightarrow p_4$. After some simple algebra, their sum reads

$$i\mathcal{M}_{s\text{-channel}}^{G\text{-mix.}} = i\xi \kappa^2 \left(s + \frac{4m_0^2}{D-2} \right); i\mathcal{M}_{t\text{-channel}}^{G\text{-mix.}} = i\xi \kappa^2 \left(t + \frac{4m_0^2}{D-2} \right); i\mathcal{M}_{u\text{-channel}}^{G\text{-mix.}} = i\xi \kappa^2 \left(u + \frac{4m_0^2}{D-2} \right) \implies i\mathcal{M}^{G\text{-mix.}} = i\xi \kappa^2 \left(s+t+u + \frac{12}{D-2} m_0^2 \right) = 4i\xi \kappa^2 \frac{D+1}{D-2} m_0^2. \tag{5.8}$$

The computation of the similar mixed diagrams with dilaton exchange gives

$$i\mathcal{M}^{\phi\text{-mix.}} = -12i\xi\kappa^2 \frac{m_0^2}{(D-1)(D-2)}, \tag{5.9}$$

where each channel contributes the same amount.

Summing up all the contributions, the final result for the amplitude is

$$i\mathcal{M}^{\text{mix.}} = 4i\xi\kappa^2 \frac{D+2}{D-1} m_0^2, \tag{5.10}$$

as it is expected from the higher dimensional Lagrangian. Again, the higher dimensional gravitational contribution is obtained after a cancellation between the effective spin-2 and spin-0 mediators. From the two results obtained above, we see that the direct non-minimal coupling to gravity (5.1) contributes with a constant term in the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ amplitude. If one takes the non-minimal coupling into account from the start and modifies the SWGC in D generic dimensions requiring

$$\left| \frac{5}{3} \frac{\hat{\mu}^2}{\hat{m}^2} - \lambda \right| \geq \left(\frac{D-1}{D-2} + 4\xi \frac{D+1}{D-2} + 16\xi^2 \frac{D-1}{D-2} \right) \hat{\kappa}^2 \hat{m}^2, \tag{5.11}$$

the same property will be respected by the zero mode φ_0 in $D-1$ dimensions with the replacement of hatted by unhatted quantities $\hat{\mu}^2, \dots \rightarrow \mu^2, \dots$

In the $\varphi_0\varphi_0 \rightarrow \varphi_0\varphi_0$ scattering, the four point amplitudes appear as a sum of the three channels s, t, u whose coefficients add-up to a factor $s + t + u = 4m_0^2$. Therefore, the total amplitude does not increase with the exchanged momentum. This is not always the case as for example in the two examples of the $\varphi_n\varphi_n \rightarrow \varphi_n\varphi_n$ or $\varphi_n\varphi_n^* \rightarrow \varphi_n\varphi_n^*$ scattering amplitudes. The computation of the available channels, t and u in the first case, s and t in the second, proceeds as in the φ_0 case described above, but these contributions with two or one non minimal vertex do not close the sum $s + t + u$, as was the case in (5.4) and (5.8).

6 Higher dimensional gauge theory

So far, we have considered gravitational and scalar interactions in the higher dimensional theory. We will discuss now the case with gauge interactions. We consider a charged scalar $\hat{\Phi}$ of charge q and mass \hat{M} minimally coupled to a $U(1)$ gauge field \hat{B}_M with gauge coupling \hat{g} in $D+1$ dimensions

$$\mathcal{S}_{EH,\Phi,H}^{(D+1)} = \int d^{D+1}x \sqrt{(-1)^D \hat{g}} \left\{ \frac{\hat{R}}{2\hat{\kappa}^2} + \hat{D}_M \hat{\Phi} \hat{D}^M \hat{\Phi}^* - \hat{M}^2 \hat{\Phi} \hat{\Phi}^* - \frac{1}{4} \hat{H}_{MN} \hat{H}^{MN} \right\}, \tag{6.1}$$

where \hat{H} is the field strength for the gauge field \hat{B} and \hat{D}_M the $D+1$ dimensional covariant derivative $\hat{D}_M \equiv \partial_M - i\hat{g}'q\hat{B}_M$, with \hat{g}' the gauge coupling. For simplicity, we choose the following periodicities for the fields

$$\begin{aligned} \hat{B}_M(x, z + 2\pi L) &= \hat{B}_M(x, z), \\ \hat{B}_M(x, z) &= \frac{1}{\sqrt{2\pi L}} \sum_{n=-\infty}^{+\infty} B_{(n)M}(x) e^{\frac{inz}{L}} \\ \hat{\Phi}(x, z + 2\pi L) &= e^{i2\pi q\Phi} \hat{\Phi}(x, z), \\ \hat{\Phi}(x, z) &= \frac{1}{\sqrt{2\pi L}} \sum_{n=-\infty}^{+\infty} \varphi_n(x) e^{i(n+q\Phi)\frac{z}{L}}, \end{aligned} \tag{6.2}$$

where $q\Phi$ is a putative charge of $\hat{\Phi}$ under an internal symmetry. The compactification of the (kinetic term of the) gauge field gives the lagrangian

$$\begin{aligned} \mathcal{L}_H^{(D)} &= -e^{-2\alpha\phi} \left(\frac{H_0^2}{4} + \sum_{n=1}^{\infty} \frac{|H_{(n)}|^2}{2} \right) \\ &+ e^{-2\beta\phi} \left(\frac{(\partial h_0)^2}{2} + \sum_{n=1}^{\infty} \left| \partial h_n - i \frac{n}{L} B_{(n)} \right|^2 \right) \\ &+ e^{-2\alpha\phi} A^\mu \left(-H_{(0)\mu\nu} \partial^\nu h_0 + \sum_{n=1}^{\infty} H_{(n)\mu\nu} \left(\partial^\nu h_n^* - i \frac{n}{L} B_{(n)}^{*\nu} \right) \right. \\ &+ H_{(n)\mu\nu}^* \left(\partial^\nu h_n - i \frac{n}{L} B_{(n)}^\nu \right) \left. \right) \\ &+ e^{-2\alpha\phi} \left[A^2 \left(\frac{(\partial h_0)^2}{2} + \sum_{n=1}^{\infty} \left| \partial h_n - i \frac{n}{L} B_{(n)} \right|^2 \right) \right. \\ &+ A^\mu A^\nu \left(\partial_\mu h_0 \partial_\nu h_0 + 2 \sum_{n=1}^{\infty} \left(\partial_\mu h_n - i \frac{n}{L} B_{(n)\mu} \right) \right. \\ &\left. \left. \times \left(\partial_\nu h_n - i \frac{n}{L} B_{(n)\nu} \right)^* \right) \right], \end{aligned} \tag{6.3}$$

where $h_0 \equiv B_{(0)z}$ is a real scalar corresponding to the zero mode of the gauge field \hat{B}_M component along the compact dimension z and $h_n \equiv B_{(n)z}$ are the complex scalars forming the KK tower of the same field. From the above action, each field h_n is seen to generate a mass for the KK excitations $B_{(n)\mu}$ of the non-compact components of the gauge field, that are then complex massive vectors, and to behave as the Goldstones in the Higgs mechanism (or in a Stueckelberg mechanism). Note that the relations $B_{(-n)\mu} = B_{(n)\mu}^*$ and $h_{-n} = h_n^*$ are valid, although the same cannot be said for the Fourier modes of the complex field $\hat{\Phi}$.

The D -dimensional lagrangian obtained from the kinetic and mass term of the scalar field $\hat{\Phi}$ reads

$$\begin{aligned} \mathcal{L}_\Phi^{(D)} &= \sum_{n=-\infty}^{+\infty} |D\varphi_n|^2 \\ &- \left(e^{2\alpha\phi} \hat{M}^2 + e^{-2(\beta-\alpha)\phi} \left[\frac{n+q\Phi}{L} - g'qh_0 \right]^2 \right) |\varphi_n|^2 \end{aligned}$$

$$\begin{aligned}
 &+ g'q \sum_{\substack{n,p=-\infty \\ n \neq 0}}^{+\infty} \left[i B_{(n)}^\mu \left(\varphi_p \partial_\mu \varphi_{n+p}^* - \partial_\mu \varphi_p \varphi_{n+p}^* \right) \right. \\
 &- 2g'q B_{(0)\mu} B_{(n)}^\mu \varphi_p \varphi_{n+p}^* - g'q \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} B_{(n)\mu} B_{(m)}^\mu \varphi_p \varphi_{n+m+p}^* \left. \right] \\
 &+ g'q A^\mu \left(\sum_{\substack{n,p=-\infty \\ n \neq 0}}^{+\infty} i h_n \left(\partial_\mu \varphi_p \varphi_{n+p}^* - \varphi_p \partial_\mu \varphi_{n+p}^* \right) \right. \\
 &- 2 \sum_{n,p=-\infty}^{+\infty} \frac{n+p+q_\Phi}{L} B_{(n)\mu} \varphi_p \varphi_{n+p}^* \\
 &+ 2g'q h_0 \sum_{n,p=-\infty}^{+\infty} B_{(n)\mu} \varphi_p \varphi_{n+p}^* \\
 &+ 2g'q \sum_{\substack{n,m,p=-\infty \\ m \neq 0}}^{+\infty} h_m B_{(n)\mu} \varphi_p \varphi_{n+p}^* \left. \right) + (A^2 + e^{-2(\beta-\alpha)\phi}) \\
 &\times \left(2g'q \sum_{\substack{n,p=-\infty \\ n \neq 0}}^{+\infty} \left[\frac{n+p+q_\Phi}{L} - g'qh_0 \right] h_n \varphi_p \varphi_{n+p}^* \right. \\
 &\left. - g'^2 q^2 \sum_{\substack{n,m,p=-\infty \\ m \neq 0}}^{+\infty} h_n h_m \varphi_p \varphi_{n+m+p}^* \right), \tag{6.4}
 \end{aligned}$$

where $g'q \equiv \hat{g}'q/\sqrt{2\pi L}$ and when acting on φ_n

$$D_\mu \equiv \partial_\mu - ig'q B_{(0)\mu} - ig \left[\left(\frac{n+q_\Phi}{L} - g'qh_0 \right) \right] A_\mu, \tag{6.5}$$

from which one can read the charge under the graviphoton. The h_0 term in this expression is a manifestation of the Aharonov–Bohm effect for the Wilson line of $B_z, \oint_z B_z$.

Here we are interested in comparing the different gravitational and non-gravitational long range classical interactions, which can be obtained from the t -channel amplitudes. The t -channel contribution to the $\varphi_n(p_1)\varphi_n(p_2) \rightarrow \varphi_n(p_3)\varphi_n(p_4)$ scattering amplitude is

$$\begin{aligned}
 i\mathcal{M}_n &= \frac{i}{t} \left(g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} + 2\kappa^2 \left(\frac{n+q_\Phi}{L} \right. \right. \\
 &- \left. \left. g'q e^{-\sqrt{\frac{D-2}{D-1}}\kappa\phi_0} \bar{h}_0 \right)^2 e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \right) (p_1 + p_3) \cdot (p_2 + p_4) \\
 &- \frac{i}{t} \left[4g'^2 q^2 \left(g'qh_0 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} - \frac{n+q_\Phi}{L} e^{\frac{D}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \right)^2 \right. \\
 &+ \left(2\sqrt{\frac{D-1}{D-2}}\kappa \left(\frac{n+q_\Phi}{L} - g'qh_0 e^{-\sqrt{\frac{D-2}{D-1}}\kappa\phi_0} \right)^2 e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \right. \\
 &\left. \left. + \frac{2}{\sqrt{(D-1)(D-2)}\kappa} \hat{M}^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \right)^2 \right], \tag{6.6}
 \end{aligned}$$

where we have omitted writing the gravitational contribution, to avoid lengthy expressions, only to reinsert it in the next step when we perform the non-relativistic limit. The mass of the n th KK state can be read from the first line of the action in (6.4)

$$\begin{aligned}
 m_n^2 &= \left(\frac{n+q_\Phi}{L} - g'q e^{-\sqrt{\frac{D-2}{D-1}}\kappa\phi_0} \bar{h}_0 \right)^2 e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \\
 &+ \hat{M}^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \tag{6.7}
 \end{aligned}$$

Let us first consider the simplest case where $q_\Phi = \bar{h}_0 = \hat{M} = 0$. In the non-relativistic limit, for $n \neq 0$, the coefficient of $\frac{1}{t}$ in the t -channel amplitude takes the form

$$\begin{aligned}
 \mathcal{M}_n^{t\text{-pole}} &= \left(g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} + 2\kappa^2 m_n^2 \right) 4m_n^2 \\
 &- 4g'^2 q^2 m_n^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \\
 &- 4\frac{D-1}{D-2}\kappa^2 m_n^4 - 4\frac{D-3}{D-2}\kappa^2 m_n^4 = 0, \tag{6.8}
 \end{aligned}$$

where m_n^2 in this case is simply $m_n^2 = e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} n^2/L^2$ and the gravitational scattering has been reinserted. The vanishing amplitude results from the (expected) two by two cancellation of interactions for the massive KK modes: namely gravitational vs dilatonic and D -dimensional gauge vs scalar from the $(D+1)$ -direction gauge field component. The $n = 0$ amplitude is different as the zero mode is massless with our specific choice. The non gravitational amplitude reads

$$\begin{aligned}
 i\mathcal{M}_0^{relativistic} &= \frac{i}{t} g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} (p_1 + p_3) \\
 &\cdot (p_2 + p_4). \tag{6.9}
 \end{aligned}$$

Let us now consider the case $q_\Phi \neq 0$. The zero mode is massive

$$m_0^2 = e^{\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \frac{q_\Phi^2}{L^2}, \tag{6.10}$$

and the corresponding four-point amplitude is given by (again, we do not include here the gravitational contribution whose expression for generic exchanger momenta is long and not very illuminating)

$$\begin{aligned}
 i\mathcal{M}_0 &= \frac{i}{t} \left(g'^2 q^2 e^{\frac{2}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} + 2\kappa^2 \frac{q_\Phi^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \right) (p_1 + p_3) \\
 &\cdot (p_2 + p_4) \\
 &- \frac{i}{t} \left[4g'^2 q^2 \frac{q_\Phi^2}{L^2} e^{2\frac{D}{\sqrt{(D-1)(D-2)}\kappa\phi_0}} \right. \\
 &\left. + 4\frac{D-1}{D-2}\kappa^2 \frac{q_\Phi^2}{L^2} e^{2\sqrt{\frac{D-1}{D-2}}\kappa\phi_0} \right]. \tag{6.11}
 \end{aligned}$$

In the non-relativistic limit, the total amplitude obtained by adding the gravitational contribution to (6.11), cancels. The non-periodicity, which makes the zero mode massive, also generates couplings at h_0 and ϕ , whose exchanges cancel, respectively, the gauge and gravitational amplitudes of the zero mode. This is to be expected since integer values of q_ϕ reshuffle the KK states; what was the zero mode becomes one of the massive modes for which we have seen that the total amplitude disappears. It is immediate to verify that the same is true for generic $n \neq 0$, \mathcal{M}_n remains null, and the same thing happens if one turns on \bar{h}_0 , as can be easily verified.

We can now study the general case. It is immediately verified that, after some algebra, in the non-relativistic limit the scattering amplitude (6.6) simplifies to

$$\begin{aligned}
 i\mathcal{M}_{NR}^{(D)} &= 4ie^{\frac{4}{\sqrt{(D-1)(D-2)}\kappa\phi_0}}\hat{M}^2\left(g'^2q^2 - \frac{D-2}{D-1}\kappa^2\hat{M}^2\right) \\
 &= 4ie^{\frac{4}{\sqrt{(D-1)(D-2)}\kappa\phi_0}}\frac{\hat{M}^2}{2\pi L}\left(\hat{g}'^2q^2 - \frac{(D+1)-3}{(D+1)-2}\hat{\kappa}^2\hat{M}^2\right) \\
 &\propto i\mathcal{M}_{NR}^{(D+1)} \tag{6.12}
 \end{aligned}$$

where one recognizes in the combination inside the parenthesis the $D + 1$ dimensional corresponding dependence. The q_ϕ and \bar{h}_0 dependences cancel out to leave this simple expression only in terms of the higher dimensional mass and charge. We conclude that the requirement that the state in $D + 1$ dimensions feels a repulsive long range force ensures that the KK modes in D dimensions also feel a repulsive long range force.

The mapping of the $D + 1$ dimensional $U(1)$ WGC into the D dimensional form of the conjecture with gauge and scalar fields was discussed in [3] from the requirement of extremal black holes and black p-branes decays, leading to the establishment of the dilatonic WGC, and in [16] for the special case of a five to four dimensional circle compactification retaining only the zero modes. The analysis presented here generalizes, from the standpoint of scattering amplitudes, the connection between these different forms of the conjecture to the case with several gauge and scalar fields with reasonings involving the whole Kaluza–Klein tower.

6.1 Effective potential for h_0

Finally, we comment on the confrontation of the effective one-loop potential for the Wilson line with the scalar WGC of [6]. The potential is generated by the integration of the KK excitations.⁴ In the case of a circle compactification from five

⁴ We use here the results of the effective potentials investigated in details for example in [17] and at the one-loop level in a type I non-supersymmetric string model in [18].

to four dimensions, the potential takes the simple form

$$\begin{aligned}
 V_{\text{eff}}(h_0) &= -\frac{3}{64\pi^6 L^4} \sum_{n=1}^{\infty} \frac{\cos(2\pi n g' q h_0 L)}{|n|^5} \\
 &= -\frac{3\left(\text{Li}_5\left(e^{-2\pi i g' q h_0 L}\right) + \text{Li}_5\left(e^{2\pi i g' q h_0 L}\right)\right)}{128\pi^6 L^4}, \tag{6.13}
 \end{aligned}$$

where the symbols Li_n denote the usual Polylogarithm functions defined as

$$\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}. \tag{6.14}$$

For the Wilson line to satisfy the Scalar WGC inequality of [6] around a generic background value \bar{h}_0 (we indicate with η the excitations around it, $h_0 = \bar{h}_0 + \eta$), one then needs

$$L^2 \geq \frac{3\kappa^2}{2\pi^2 g'^2 q^2} \left[\frac{\text{Li}_3(e^{ix}) + \text{Li}_3(e^{-ix})}{\frac{20}{9} \frac{(\text{Li}_2(e^{ix}) - \text{Li}_2(e^{-ix}))^2}{\text{Li}_3(e^{ix}) + \text{Li}_3(e^{-ix})} - \log(2 - 2\cos x)} \right]. \tag{6.15}$$

where x is defined to be $x \equiv 2\pi g' q \bar{h}_0 L$, to be respected for $m_\eta^2 > 0$, while the inequality is trivially verified for $m_\eta^2 < 0$, but this case is of no interest. In the inequality (6.15), the factor inside the square parenthesis on the right hand side is periodic and reaches a maximal value around 0.6 – 0.7 in the regions of parameters where $m_\eta^2 > 0$. Taken to be approximately an order one, the gravitational sub-dominance is then realized around any background value \bar{h}_0 if⁵

$$L^2 \geq \frac{3\hat{\kappa}^2}{2\pi^2 \hat{g}'^2 q^2} = \frac{3}{2\pi^2 g'^2 q^2} \frac{1}{M_P^2}, \tag{6.16}$$

which means that the compactification length cannot be parametrically smaller than the Planck’s one as expected.

From (6.3) and (6.4), it is immediate to observe that the self-couplings induced by radiative corrections are not the only ones that can appear in the 4-point function $\eta\eta \rightarrow \eta\eta$. A first contribution may come from the kinetic term of h_0 , coupled to the dilaton as in (6.3). This gives a two derivative vertex that would then induce contributions to the four point function proportional to the scalar product of external momenta ($p_1 \cdot p_2 \times p_3 \cdot p_4$ in the s-channel, and so on). For the effective four point non relativistic coupling, this only accounts for a shift of the gravitational contribution, the second term in (6.16). In particular, the numerical coefficient 3/2 should be changed with 5 in (6.16) and all the subsequent inequalities.

⁵ Note that $\frac{\kappa^2}{g'^2} = \frac{\hat{\kappa}^2}{\hat{g}'^2}$, so we can express the bound either in terms of five- or four-dimensional quantities in the same form.

7 Conclusions

An extra dimension for our space-time was originally introduced to unify gravity with electromagnetism: [8–11]. From the point of view of a lower dimensional observer, this unification makes the KK modes undergo attractive gravitational plus scalar interactions and repulsive electric interactions with the same intensity. This motivated the use of the KK states interactions in this work to extract the form of the inequalities that appear when one is interested in comparing gravitational interactions to other types of interactions.

Taking into account the scalar interaction due to the presence of a dilaton, the calculation of four-point amplitudes allowed us to find the inequalities of the Dilatonic WGC. Our observations go further, with the extension of the construction to include interactions in the higher dimension, and we have shown how the Scalar WGC is found as well as the behavior of these conjectures under dimensional reduction. Meanwhile, we have also computed a number of scattering amplitudes for the pair production of KK states and have been able to compare the contributions of the different channels for spacetime dimensions $D \geq 4$.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical study and no experimental data.]

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A Lagrangians with derivative interactions

One subtlety that we wish to address here is related to the nature and the use of derivative interactions in perturbation theory. The perturbative expansion is an expansion of the exponential $e^{-i \int d^D x \mathcal{H}_I}$ in powers of \mathcal{H}_I , the interaction hamiltonian in the interaction picture. When the lagrangian presents derivative interactions, one should be careful to correctly construct \mathcal{H}_I before announcing the Feynman rules. Interactions containing more than one derivative of fields can generate new genuine additional Feynman rules [19]. The analog of this result was found, in the path integral for-

malism, in [20]. We illustrate this in two simple examples closely related to the cases studied.

A.1 Interactions with derivatives of a gauge field

We first present the case of the theory defined by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-2\sqrt{\frac{D-1}{D-2}} \kappa \phi} (\partial_\mu A_\nu - \partial_\nu A_\mu) \times (\partial^\mu A^\nu - \partial^\nu A^\mu). \tag{A.1}$$

We have singled out here only the part of interest to us to highlight the interaction between the dilaton ϕ and derivatives of the graviphoton A_μ . We will work in the usual radiation gauge $A_0 = 0, \nabla \cdot \mathbf{A} = 0$. Computation of the canonical conjugate momenta give us

$$\begin{cases} \Pi_{A_0} = 0 \\ \Pi_{A_i} = - \left(1 + \sum_{m=1}^\infty \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right) F^{0i} \\ \Pi_\phi = \partial_0 \phi. \end{cases} \tag{A.2}$$

The fact that $\Pi_{A_0} = 0$ is, of course, what we should expect in a canonical formalism. The Heisenberg picture hamiltonian is obtained as

$$\begin{aligned} \mathcal{H} &= \Pi_{A_\mu} \partial_0 A_\mu + \Pi_\phi \partial_0 \phi - \mathcal{L} \\ &= -\frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \partial_0 \phi \partial_0 \phi + \frac{1}{2} \partial_i \phi \partial_i \phi \\ &\quad - \sum_{m=1}^\infty \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \left(F^{0i} F_{0i} - \frac{F_{\mu\nu} F^{\mu\nu}}{4} \right) \\ &= \frac{1}{2} \Pi_{A_i} \Pi_{A_i} + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \Pi_\phi \Pi_\phi + \frac{1}{2} \partial_i \phi \partial_i \phi \\ &\quad + \frac{1}{4} \sum_{m=1}^\infty \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} F^{\mu\nu} F_{\mu\nu} \\ &\quad + \frac{1}{2} \sum_{m=1}^\infty \left[\left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right]^2 F^{0i} F_{0i}. \end{aligned} \tag{A.3}$$

The transition to the interaction picture is done making the following replacements:

$$\begin{cases} \Pi_{A_i} \rightarrow -F^{0i} (= \Pi_{A_i, I}) \\ F_{ij} \rightarrow F_{ij} \\ F^{0i} \rightarrow F^{0i} \left(1 + \sum_{m=1}^\infty \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right)^{-1} \\ \Pi_\phi \rightarrow \partial_0 \phi \\ \phi \rightarrow \phi \\ \partial_0 \phi \rightarrow \partial_0 \phi \end{cases} \tag{A.4}$$

Some simple algebra finally get us to the interaction picture hamiltonian in the form

$$\begin{aligned} \mathcal{H} = & -\frac{1}{2} F_{0i} F^{0i} + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} \partial_0 \phi \partial_0 \phi + \frac{1}{2} \partial_i \phi \partial_i \phi \\ & + \frac{1}{4} \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} F^{\mu\nu} F_{\mu\nu} \\ & - \frac{1}{2} \frac{\sum_{m=1}^{\infty} \left[\left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right]^2}{1 + \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!}} F^{0i} F_{0i}. \end{aligned} \tag{A.5}$$

Careful construction of the interaction hamiltonian reveals the presence of an additional term to the naive expectation, to the extent that

$$\mathcal{H}_I = -\mathcal{L}_I - \frac{1}{2} \frac{\sum_{m=1}^{\infty} \left[\left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!} \right]^2}{1 + \sum_{m=1}^{\infty} \left(-2\sqrt{\frac{D-1}{D-2}} \kappa \right)^m \frac{\phi^m}{m!}} F^{0i} F_{0i}, \tag{A.6}$$

with the new term sharing the same structure with the one found in the model of [19].

Combining this result with the two derivative propagator⁶

$$\langle \partial_\mu A_\rho \partial_\nu A_\sigma \rangle(q) = i \eta_{\rho\sigma} \frac{q_\mu q_\nu}{q^2 (+i\epsilon)} - i \eta_{\rho\sigma} \eta_{\mu 0} \eta_{\nu 0} \tag{A.7}$$

we finally have the explicit form of the non standard Feynman rules we should consider in the minimally coupled (i.e. with $\xi = 0$) dimensionally reduced theory. The additional term consists in an infinite series in powers of $\kappa\phi$ starting at order 2 and defining a vertex with two gauge bosons. As such, it will not enter any of the computations we have performed, but certainly need to be considered, alongside with the propagator corrections, even at tree level, when looking at different physical processes, like $\phi\phi \rightarrow \gamma\gamma$ and $\phi\gamma \rightarrow \phi\gamma$ ones.

A.2 Toy model for the two-derivative interaction of the non-minimal coupling

The second model we present here aims to capture the main properties of the new vertices brought in by the non-minimal coupling to gravity. We explicitly show, with the simplest toy model, that the different additional pieces due to such derivatives cancel each other, allowing the use of naive perturbation theory.

Let us take, for definiteness, the following lagrangian:

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 + \frac{1}{2} (\partial\varphi)^2 + \frac{a}{2} \kappa (\partial^2\phi)\varphi^2$$

⁶ Given here in the covariant gauge, to keep a simple notation.

$$\begin{aligned} & + \frac{b}{2} \kappa^2 (\partial\phi)^2 \varphi^2 = \frac{1}{2} (\partial\phi)^2 \\ & + \frac{1}{2} (\partial\varphi)^2 - a\kappa (\partial\phi \cdot \partial\varphi)\varphi + \frac{b}{2} \kappa^2 (\partial\phi)^2 \varphi^2, \end{aligned} \tag{A.8}$$

where a and b are dimensionless constants. In keeping the parallel with the cases discussed in the text, one should think of ϕ as a massless mediator and φ the matter field. The addition of a mass term for φ does not change the computations. The conjugate momenta are

$$\begin{cases} \Pi_\phi = \partial_0\phi - a\kappa\varphi\partial_0\varphi + b\kappa^2\partial_0\phi\varphi^2 \\ \Pi_\varphi = \partial_0\varphi - a\kappa\varphi\partial_0\phi, \end{cases} \tag{A.9}$$

and, inverting the relations, we obtain

$$\begin{cases} \partial_0\phi = \frac{\Pi_\phi + a\kappa\varphi\Pi_\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \\ \partial_0\varphi = \Pi_\varphi + a\kappa\varphi \frac{\Pi_\phi + a\kappa\varphi\Pi_\varphi}{1 + (b-a^2)\kappa^2\varphi^2}. \end{cases} \tag{A.10}$$

Following the steps described above, the interaction picture hamiltonian is obtained:

$$\begin{aligned} \mathcal{H} = & \frac{\partial_0\phi(\partial_0\phi + a\kappa\varphi\partial_0\varphi)}{1 + (b-a^2)\kappa^2\varphi^2} \\ & + \partial_0\varphi \left(\partial_0\varphi + a\kappa\varphi \frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right) \\ & - \frac{1}{2} \left(\frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 \\ & - \frac{1}{2} \left(\partial_0\varphi + a\kappa\varphi \frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 + \frac{1}{2} \partial_i\phi\partial_i\phi \\ & + \frac{1}{2} \partial_i\varphi\partial_i\varphi + a\kappa\varphi\partial_0\varphi \frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \\ & + a^2\kappa^2\varphi^2 \left(\frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 \\ & - \frac{b}{2}\kappa^2\varphi^2 \left(\frac{\partial_0\phi + a\kappa\varphi\partial_0\varphi}{1 + (b-a^2)\kappa^2\varphi^2} \right)^2 \\ & - a\kappa\varphi\partial_i\phi\partial_i\varphi + \frac{b}{2}\kappa^2\varphi^2\partial_i\phi\partial_i\phi \end{aligned} \tag{A.11}$$

Expanding to second order in κ , to match the usual contributions to the $\varphi\varphi \rightarrow \varphi\varphi$ or $\phi\phi \rightarrow \varphi\varphi$ amplitudes from (A.8), we get

$$\begin{aligned} \mathcal{H} = & \frac{1}{2} (\partial_0\phi\partial_0\phi + \partial_i\phi\partial_i\phi) + \frac{1}{2} (\partial_0\varphi\partial_0\varphi + \partial_i\varphi\partial_i\varphi) \\ & + a\kappa\varphi(\partial_0\varphi\partial_0\phi - \partial_i\varphi\partial_i\phi) - \frac{b}{2}\kappa^2\varphi^2(\partial_0\phi\partial_0\phi - \partial_i\phi\partial_i\phi) \\ & + \frac{a^2}{2}\kappa^2\varphi^2(\partial_0\phi\partial_0\phi + \partial_0\varphi\partial_0\varphi) + O(\kappa^3). \end{aligned} \tag{A.12}$$

We recognize, in the first line, the sum $\mathcal{H}_{\text{free}} - \mathcal{L}_I$ that is usually found in perturbation theory with no derivative interactions. The operator in the second line, as well as all the higher orders ones that can be derived from (A.11), are due

to the derivative interactions in (A.8). Equation (A.12) shows that, at the level of the interaction picture hamiltonian, we get additional 4-point vertices with respect to the usual ones.

We now check the impact of such additional interactive terms through the explicit computation of the $\varphi(p_1)\varphi(p_2) \rightarrow \varphi(q_1)\varphi(q_2)$ scattering amplitude. Taking into account the corrections to the scalar propagator (analogous to (A.7)), the usual $(-\mathcal{L}_I)$ interactions give, in each one of the s, t and u channels

$$i\mathcal{M}_{(-\mathcal{L}_I)} = -ia^2\kappa^2 P_\mu P_\nu \left(\frac{P^\mu P^\nu}{P^2} - \eta_0^\mu \eta_0^\nu \right), \tag{A.13}$$

where P is the appropriate momentum factor in each channel ($P = p_1 + p_2, P = p_1 - p_3$ and $P = p_1 - p_4$, respectively, in s, t and u). After some algebra, the four φ contact term in (A.12) accounts for a contribution

$$i\mathcal{M}_{\text{contact}} = -2ia^2\kappa^2(p_{1,0}^2 + p_{2,0}^2 + q_{1,0}^2 - q_{1,0}p_{1,0}), \tag{A.14}$$

where the notation $p_{i,0}$ means the zero component of the momentum p_i .

Putting it all together one gets

$$i\mathcal{M} = -ia^2\kappa^2 \left\{ s + t + u - ((p_{1,0} + p_{2,0})^2 + (p_{1,0} - q_{1,0})^2 + (p_{1,0} - q_{2,0})^2) + 2(p_{1,0}^2 + p_{2,0}^2 + q_{1,0}^2 - q_{1,0}p_{1,0}) \right\}. \tag{A.15}$$

Using momentum conservation one can show that, again after some algebra, the non covariant pieces cancel leaving the same result one would have guessed using the naive Feynman rules from the lagrangian (A.8) associating the appropriate momentum factor to each derivative:

$$i\mathcal{M} = -ia^2\kappa^2 (s + t + u). \tag{A.16}$$

The type of vertices being the same, this same cancellation happens in the ‘‘pair production’’-like amplitude $\phi\phi \rightarrow \varphi\varphi$.

This toy model explicitly shows the cancellation between different non covariant pieces arising in the computation of amplitudes with two derivative vertices and justifies, a posteriori, the use of naive perturbation theory we made in Sect. 5.

B Helicity basis and Mandelstam variables

In the computation of the pair production diagrams, we need to deal with external states polarizations for massless helicity-1 and helicity-2 particles. This is of no concern when we compute the squared amplitude, as it is usually treated by means of the replacements $\sum_{\text{pol}} \epsilon_\mu(p)\epsilon_\nu^*(p) \rightarrow -g_{\mu\nu}$ for photon amplitudes and $\sum_{\text{pol}} \epsilon_{\mu\nu}(p)\epsilon_{\rho\sigma}^*(p) = \sum_{\text{pol}} \epsilon_\mu(p)\epsilon_\nu(p)\epsilon_\rho^*(p)\epsilon_\sigma^*(p) \rightarrow \mathcal{P}_{\mu\nu\rho\sigma}$ for graviton ones. If, on the other hand, we want to consider the amplitude

more directly and not its square, we need to choose a basis for the polarizations and the momentum, and perform the calculations within this basis.

For the case of the pair production, the in-going states relevant here are either photons or gravitons, while the outgoing ones are massive particles. We perform here the computations in the center of momentum frame.

Starting from the $D = 4$ case, we write the momenta

$$\begin{aligned} p_1 &= E_p(1, 0, 0, 1), & p_2 &= E_p(1, 0, 0, -1), \\ p_3 &= (E_p, p \sin \theta, 0, p \cos \theta), \\ p_4 &= (E_p, -p \sin \theta, 0, -p \cos \theta) \end{aligned} \tag{B.1}$$

and the polarizations

$$\begin{aligned} \epsilon_1^\pm &\equiv \epsilon(p_1)^\pm = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0), \\ \epsilon_2^\pm &\equiv \epsilon(p_2)^\pm = \frac{1}{\sqrt{2}}(0, \pm 1, -i, 0). \end{aligned} \tag{B.2}$$

The scalar products appearing in the amplitudes can now be explicitly performed in this particular basis and the results can then be rewritten in terms of the Mandelstam variables using the following relations:

$$\begin{aligned} p^2 &= \frac{s - 4m^2}{4}, & \sin^2 \theta &= \frac{(t - u)^2}{s(s - 4m^2)}, \\ \cos^2 \theta &= \frac{4tu - 4m^2}{s(s - 4m^2)} \end{aligned} \tag{B.3}$$

At this point, we need to separate the contributions coming from different helicities. For definiteness, we refer now to the amplitude in (3.11), that we report here for the reader’s convenience

$$\begin{aligned} i\mathcal{M}_{\gamma\gamma} &= ig^2 q_n^2 \epsilon_\mu(p_1)\epsilon_\nu(p_2) \left(\frac{(2p_3^\mu - p_1^\mu)(2p_4^\nu - p_2^\nu)}{t - m_n^2} \right. \\ &\quad \left. + \frac{(2p_4^\mu - p_1^\mu)(2p_3^\nu - p_2^\nu)}{u - m_n^2} + 2\eta^{\mu\nu} \right) \\ &\quad - 2ig^2 q_n^2 \frac{D - 1}{D - 2} \epsilon_\mu(p_1)\epsilon_\nu(p_2) \frac{p_1 \cdot p_2 \eta^{\mu\nu} - p_1^\nu p_2^\mu}{s}. \end{aligned}$$

A great simplification comes when we deal more directly with the amplitudes components. We can in fact use the property⁷ $\epsilon(p) \cdot p = 0$. With our choice of basis, we also have $\epsilon(p_1) \cdot p_2 = \epsilon(p_2) \cdot p_1 = 0$, so that, for the purposes of the calculation with the helicity method, we can use the following expression for the amplitude

⁷ When using the usual shortcut $\sum_{\text{pol}} \epsilon_\mu(p)\epsilon_\nu(p) = -g_{\mu\nu}$ this simplification cannot be used.

$$\mathcal{M}_{\gamma\gamma} = 4g^2q_n^2 \left\{ \frac{\epsilon(p_1) \cdot p_3 \epsilon(p_2) \cdot p_4}{t - m_n^2} + \frac{\epsilon(p_1) \cdot p_4 \epsilon(p_2) \cdot p_3}{u - m_n^2} + \frac{\epsilon(p_1) \cdot \epsilon(p_2)}{2} \left(1 - \frac{D-1}{D-2} \frac{p_1 \cdot p_2}{s} \right) \right\}. \tag{B.4}$$

We denote with $\mathcal{M}_{\pm\pm}$ the different contributions, with the \pm referring to the helicities of the polarization. We have then

$$i\mathcal{M}_{++} = 2i(gq_n)^2 \left(\frac{m_n^2 s}{(t - m_n^2)(u - m_n^2)} - \gamma_d \frac{3}{4} \right),$$

$$i\mathcal{M}_{+-} = -2i(gq_n)^2 \frac{(m_n^4 - ut)}{(t - m_n^2)(u - m_n^2)}, \tag{B.5}$$

where we have introduced a factor γ_d in front of the term arising from the dilaton such that we retrieve the result for our KK theory when $\gamma_d = 1$ and the usual result for a $U(1)$ gauge theory when $\gamma_d = 0$. To compute the total amplitude, we average over the in-going polarizations and obtain in the threshold limit

$$|\mathcal{M}_{\gamma\gamma}|^2 = \frac{1}{4} (2|\mathcal{M}_{++}|^2 + 2|\mathcal{M}_{+-}|^2) \rightarrow 2 \left(1 - \gamma_d \frac{3}{4} \right)^2 (gq_n)^4. \tag{B.6}$$

When $\gamma_d = 0$, the overall numerical factor is 2, while for $\gamma_d = 1$, it is 1/8, matching the results obtained in Sect. 3.2 for $D = 4$. It is immediate to realize that, in the threshold limit, only the $\epsilon(p_1) \cdot \epsilon(p_2)$ term contributes.

The same method outlined above can be used for any other number of dimensions D , where the gauge bosons have $D-2$ independent helicity states. For instance, in the $D = 5$ case, the helicity basis can be taken as

$$\begin{aligned} \epsilon_1^1 &= \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0) & \epsilon_2^1 &= \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0) \\ \epsilon_1^2 &= \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0) & \epsilon_2^2 &= \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0) \\ \epsilon_1^3 &= (0, 0, 0, 1, 0) & \epsilon_2^3 &= (0, 0, 0, -1, 0). \end{aligned} \tag{B.7}$$

For any $D > 4$, the polarization basis can be chosen such that, for both p_1 and p_2 , the first two polarizations are the same as in $D = 4$, while the other polarizations are $\epsilon_1^i = (0, \dots, \underbrace{1}_{i+1}, \dots, 0)$ and $\epsilon_2^i = (0, \dots, \underbrace{-1}_{i+1}, \dots, 0)$.

For an even number of dimensions, one may chose the basis in an equivalent way as an ensemble of two by two circular polarizations. In $D = 6$ dimensions, for instance, this would give

$$\epsilon_1^1 = \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0, 0) \quad \epsilon_2^1 = \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0, 0)$$

$$\begin{aligned} \epsilon_1^2 &= \frac{1}{\sqrt{2}}(0, 1, -i, 0, 0, 0) & \epsilon_2^2 &= \frac{1}{\sqrt{2}}(0, -1, -i, 0, 0, 0) \\ \epsilon_1^3 &= \frac{1}{\sqrt{2}}(0, 0, 0, -1, -i, 0) & \epsilon_2^3 &= \frac{1}{\sqrt{2}}(0, 0, 0, 1, -i, 0) \\ \epsilon_1^4 &= \frac{1}{\sqrt{2}}(0, 0, 0, 1, -i, 0) & \epsilon_2^4 &= \frac{1}{\sqrt{2}}(0, 0, 0, -1, -i, 0). \end{aligned} \tag{B.8}$$

Of course, the results are independent of the particular choice.

Whatever specific basis one choses, from (B.4) it follows that in the threshold limit, as already observed for the specific case $D = 4$, only the diagonal terms \mathcal{M}_{ii} are non zero, and they all give the same contribution

$$\mathcal{M}_{ii} \rightarrow 2(gq)^4 \left(1 - \frac{1}{2} \frac{D-1}{D-2} \right). \tag{B.9}$$

It is then straightforward to extract the value of the amplitude in the threshold limit for D generic dimensions as

$$\begin{aligned} |\mathcal{M}|^2 &\rightarrow \frac{1}{(D-2)^2} (D-2) |\mathcal{M}_{ii}|^2 \\ &= \frac{4}{D-2} (gq)^4 \left(1 - \frac{1}{2} \frac{D-1}{D-2} \right)^2 = \left(\frac{D-3}{D-2} \right)^2 \frac{(gq_n)^4}{D-2}. \end{aligned} \tag{B.10}$$

This result of course matches that shown in (3.13), that was obtained by means of the usual trick $\sum_{\text{pol}} \epsilon_\mu(p) \epsilon_\nu(p) = -g_{\mu\nu}$. Note also that when the dilaton is put to zero (i.e. when the second contribution in the parenthesis (B.10) is put to zero) we re-obtain the result

$$|\mathcal{M}_{\gamma\gamma}|^2 \rightarrow \frac{4}{D-2} (gq)^4. \tag{B.11}$$

The same procedure can now be used to extract the different components of the purely gravitational amplitude of Sect. 3.2.3. The four diagrams contribute in the amount

$$\begin{aligned} \mathcal{M}_{t\text{-pole}} &= -\frac{4\kappa^2(\epsilon_1 \cdot p_3)^2(\epsilon_2 \cdot p_4)^2}{t - m_n^2} \\ \mathcal{M}_{u\text{-pole}} &= -\frac{4\kappa^2(\epsilon_2 \cdot p_3)^2(\epsilon_1 \cdot p_4)^2}{u - m_n^2} \\ \mathcal{M}_{\text{seagull}} &= 2\kappa^2 \epsilon_1 \cdot \epsilon_2 \left(\epsilon_1 \cdot \epsilon_2 (p_3 \cdot p_4 + m_n^2) - 2\epsilon_2 \cdot p_3 \epsilon_1 \cdot p_4 - 2\epsilon_1 \cdot p_3 \epsilon_2 \cdot p_4 \right) \end{aligned} \tag{B.12}$$

and

$$\begin{aligned} \mathcal{M}_{g\text{-pole}} &= \frac{2\epsilon_1 \cdot \epsilon_2}{D-2} \left\{ 2p_1 \cdot p_2 \left[(D-2)(\epsilon_{2,\lambda} \epsilon_{1,\tau} + \epsilon_{1,\lambda} \epsilon_{2,\tau}) \right. \right. \\ &\quad \left. \left. + \epsilon_{1,\lambda} \epsilon_{2,\tau} \right] - \epsilon_1 \cdot \epsilon_2 \eta_{\lambda\tau} \right\} \\ &\quad + p_1 \cdot p_2 \left[4\epsilon_1 \cdot \epsilon_2 \eta_{\lambda\tau} - 2(D-2)(\epsilon_{2,\lambda} \epsilon_{1,\tau} + \epsilon_{1,\lambda} \epsilon_{2,\tau}) \right] \\ &\quad + D\epsilon_1 \cdot \epsilon_2 (p_{1,\lambda} p_{1,\tau} + p_{2,\lambda} p_{2,\tau} + p_{1,\lambda} (p_1 + p_2)_\tau) \end{aligned}$$

$$\begin{aligned}
 &+ p_{2,\lambda}(p_1 + p_2)_\tau + 2D p_1 \cdot p_2 \epsilon_{2,\lambda\epsilon_1,\tau} \\
 &+ 2(D - 2) p_1 \cdot p_2 \epsilon_{1,\lambda\epsilon_2,\tau} - 2 p_1 \cdot p_2 \epsilon_1 \cdot \epsilon_2 \eta_{\lambda\tau} \\
 &+ 2 \epsilon_1 \cdot \epsilon_2 p_{2,\lambda} p_{1,\tau} + 2 \epsilon_1 \cdot \epsilon_2 p_{1,\lambda} p_{2,\tau} \\
 &- 2 \epsilon_1 \cdot \epsilon_2 (p_1 + p_2)_\lambda p_{1,\tau} \\
 &- 2 \epsilon_1 \cdot \epsilon_2 (p_1 + p_2)_\lambda p_{2,\tau} - 2 \epsilon_1 \cdot \epsilon_2 p_{1,\lambda} (p_1 + p_2)_\tau \\
 &- 2 \epsilon_1 \cdot \epsilon_2 p_{2,\lambda} (p_1 + p_2)_\tau - 4 p_2 \cdot (p_1 + p_2) \epsilon_{2,\lambda\epsilon_1,\tau} \Big\} \\
 &\times \left(p^{3,\lambda} p^{4,\tau} + p^{4,\lambda} p^{3,\tau} - g^{\lambda\tau} \left(p_3 \cdot p_4 + m_n^2 \right) \right)
 \end{aligned} \tag{B.13}$$

to give (3.27), reported here for simplicity

$$\begin{aligned}
 i\mathcal{M}_{GG} = &\frac{\kappa^2}{2} \left(-\frac{8(p_3 \cdot \epsilon_1)^2 (p_4 \cdot \epsilon_2)^2}{t - m_n^2} - \frac{8(p_3 \cdot \epsilon_2)^2 (p_4 \cdot \epsilon_1)^2}{u - m_n^2} \right. \\
 &- 2 \frac{(\epsilon_1 \cdot \epsilon_2)^2 (m_n^4 - tu - sm_n^2)}{s} \\
 &\left. - 4 \epsilon_1 \cdot \epsilon_2 (p_3 \cdot \epsilon_2 p_4 \cdot \epsilon_1 + p_3 \cdot \epsilon_1 p_4 \cdot \epsilon_2) \right)
 \end{aligned}$$

As in the previous case, it is again easily verified that in the threshold limit only the diagonal \mathcal{M}_{ii} terms are non-vanishing and that they all give the same result. In terms of the above amplitude, such non-vanishing contribution is given by the $(\epsilon_1 \cdot \epsilon_2)^2$ term that results in

$$\mathcal{M}_{ii} \rightarrow \kappa^2 m_n^2. \tag{B.14}$$

It is now straightforward to obtain, from these considerations, the result for the squared amplitude in D generic dimensions:

$$|\mathcal{M}|^2 \rightarrow \frac{1}{(D - 2)^2} (D - 2) |\mathcal{M}_{ii}|^2 = \frac{\kappa^4 m_n^4}{D - 2}, \tag{B.15}$$

which is the result quoted in the text (3.35).

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