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The non-singular Trautman–Kopczyński big-bang model and a torsional spinor description of dark matter

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Abstract A view is taken up whereby the non-singular Trautman–Kopczyński big-bang creation of the Universe produced a highly torsional hot state at early stages of the cosmic evolution which particularly brought about the formation of a dark matter cloud. It is thus assumed that the combination of Einstein–Cartan's theory with the torsionful version of the two-component ε-formalism of Infeld and van der Waerden supplies a natural local description of dark matter in terms of uncharged spin-one massive fields. In the case of either handedness, the pertinent spinor field equation arises directly from a suitable form of the world Bianchi identity. It appears that each such field equation bears a term that is thought of as carrying part of the information on the mass of the fields. The whole information turns out to be extracted by well prescribed derivatives of certain couplings involving the fields and spinor torsion pieces in such a way that the mass of dark matter is really thought of as arising from the interaction between the fields and spinor torsion.

1 Introduction

It is well known that the theory of general relativity, as formulated in terms of Einstein's field equations, gives rise to isotropic and homogeneous cosmological models which predict the occurrence of singular gravitational colapses in a way that does not depend upon both the physical contents of a certain class of energy–momentum tensors and the symmetries of the models [1–7]. According to this framework, there was at least one moment during the evolution of the Universe at which the density of matter and energy was infinite. Several attempts to circumvent this singularity situation were made by just introducing a cosmological constant into the geometric side of the generally relativistic field equations, but such a procedure nevertheless has not been fully satisfactory from the theoretical viewpoint [8,9].

In the work of Ref. [10], it was suggested for the first time that Einstein–Cartan's theory [11,12] could be utilized for drawing up alternative cosmological models which impede the production of singular gravitational and cosmological collapses. Soon after the publication of this work, it was shown [13] that Einstein–Cartan's equations admit a two-parameter family of spherically symmetric solutions of the Friedmann type, which supply a lower bound for the final radius of a gravitationally collapsing cloud of dust, thereby providing a construction of what may be surely designated as *Trautman–Kopczyński cosmological models*. More recently, the work of Ref. [14] has notably given a clear explanation on how spacetime torsion at a microscopic level may generate a gravitational repulsion that prevents the cosmological singularity to occur, while presenting an explicit solution to the cosmological spatial flatness and horizon problems by supplying physically plausible torsional mechanisms without having to call upon any cosmic inflationary scenario.

An intrinsic property of Einstein–Cartan's theory relies upon the fact that the characteristic asymmetry borne by the Ricci tensor for any torsionful world affine connexion always entails the presence of asymmetric energy–momentum tensors on the right-hand sides of the field equations. The skew parts of such tensors were originally identified [15–17] with sources for densities of intrinsic angular momentum of matter that presumably generate spacetime torsion locally. Thus, as remarked in Ref. [10], spin density of matter plays a physical role in Einstein–Cartan's theory which is analogous to that played by mass in general relativity.

Amongst the most elementary spacetime properties of Einstein–Cartan's theory, there is a striking feature associated to this theory which bears a deeper mathematical character, namely, the fact that any spacetime endowed with a torsionful affinity admits a local spinor structure in much the

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same way as in the case of generally relativistic spacetimes [18,19]. This admissibility has supported the construction of an essentially unique torsional extension [20] of the famous two-component spinor $\gamma\varepsilon$-formalisms of Infeld and van der Waerden for general relativity [21,22]. Such as traditionally posed, the torsionless ε-formalism supplied the first description of gravitons [23] wherein the relevant degrees of freedom are all carried by the totally symmetric part of one of the Witten curvature spinors for a Riemann tensor [24], with a systematic adaptation of this description to the torsionless γ-formalism having been exhibited later on in Ref. [22]. The work of Ref. [22] also suggested a geometric definition of wave functions for the cosmic microwave background, which show up in both torsionless formalisms as spinors that occur in irreducible decompositions of Maxwell bivectors produced by suitably contracted spin affinities. Within this torsionless spinor framework, a complete set of wave equations that control the spacetime propagation of the cosmic microwave background and gravitons, was derived on the basis of the implementation of some formal algebraic expansions and symbolic valence-reduction devices built up by Penrose (see Ref. [12]).

The torsional extension referred to above of the classical Infeld-van der Waerden formalisms, was primarily aimed at proposing a local description whereby dark energy should be supposedly looked upon as a torsional cosmic background [25]. Accordingly, dark energy fields were broadly defined as uncharged spin-one massive fields which come from spin-affine pieces that amount to gauge-invariant Proca potentials, and thence enter into irreducible decompositions of torsional bivectors. Roughly speaking, these invariant spin-affine contributions account for the underlying spacetime torsionfulness, and likewise enter additively together with contracted spin-affine pieces for the cosmic microwave background into overall contracted spin affinities. It became evident that, in contradistinction to the cosmic microwave background, dark energy can *not* propagate alone in spacetime. It was realized, in effect, that any torsional affine potential must be accompanied by an adequate torsionless spin-affine contribution such that the propagation of the dark energy background has to be united together with that of the cosmic microwave background. However, one background propagates in spacetime as if the other were absent whence they do not interact with one another. Yet, the propagation of the cosmic microwave background in regions of the Universe where the values of torsional bivectors are negligible, may be described alone within the torsionless framework. Moreover, the calculational procedures involved in the derivation of the wave equations for the dark energy background, were based upon the property that the algebraic expansions and valence-reduction devices which had been utilized in the torsionless framework, still apply formally to the torsionful framework.

In the present paper, we take up the view that the nonsingular big-bang creation of the Universe, as suggested by Trautman–Kopczyński cosmological models [10,13], produced a highly torsional hot state at early stages of the cosmic evolution which brought about the formation of the dark energy background as well as the appearance of dark matter and gravitons. As the theoretical framework we have been considering geometrically imposes that the propagation of dark energy must be accompanied by the propagation of the cosmic microwave background, it may be asserted that such physical backgrounds were produced together. Hence, it can be said that we are presumptively dealing here with a *physical four-feature picture* of the Universe which arises directly from the combination of Einstein–Cartan's theory with a torsionful two-component spinor framework. It is our belief that this combination should afford a transparent description of some of the physical features of the Universe. By this point, we shall work on such situation by proposing a description of dark matter within the context of the torsionful ε-formalism. Dark matter is thus physically characterized as uncharged spin-one massive fields which were produced together with gravitons. This characterization stems only from the result that wave functions for dark matter arise together with ones for gravitons in the expansion for a curvature spinor of a torsionful spin affinity. We stress that our attitude whereby dark matter should be theoretically described by a curvature spinor, goes hand-in-hand with one of the basic assumptions made in the original spinor description of gravitons [23]. We will see that a spinor version of Einstein–Cartan's equations, which naturally emerges in our framework, exhibits a physical identification as dark matter of the densities of spinning matter that produce spacetime torsion. So, as the world form of Einstein–Cartan's theory stands, it will become manifest that it is only dark matter which constitutes the physical source for spacetime torsion. We assume from the beginning that the Hubble expansion of the Universe at accelerated stages of the cosmic evolution, is partially due to the gravitational action on inner matter-energy contents of a dense dark matter cloud. Our spinor field equations arise from a suitable form of the world Bianchi identity, and suggest that the cosmic distribution of dark matter generally did not possess spatial uniformity during the evolutive eras of the Universe. As we believe, this may provide an easier macroscopic explanation of the anisotropy of the presently observable acceleration of the Universe (see, for instance, Refs. [26–28]). The corresponding wave equations are derived out of utilizing the same calculational techniques as those implemented for describing the dark energy background. It will be seen by means of these wave equations that dark matter interacts with gravitons and torsion, but the description of dark energy as proposed in Ref. [25] will turn out to show that dark matter interacts also with both of the cosmic backgrounds according to typical coupling patterns. This insight seemingly identifies the main responsible for the ocurrence of the observed angular anisotropy of the cosmic microwave background [29]. It will appear that the dark energy background should contribute significantly to the Hubble acceleration when it propagates inside the dark matter cloud. Upon carrying out the pertinent derivation procedures for the case of either handedness, we shall have to claim that one of the terms borne by the respective field equation encodes part of the information on the mass of dark matter. In the case of either handedness, the whole information then turns out to be extracted by well prescribed derivatives of couplings involving dark matter fields and spinor torsion pieces in such a way that the mass of dark matter is really thought of as arising from the interaction between the fields and spinor torsion.

Throughout the work, we will use the natural system of units in which $c = \hbar = 1$. All the world and spinor conventions adhered to in Ref. [25] will be adopted in what follows. In particular, symmetrizations and skew-symmetrizations over index blocks will be, respectively, denoted by round and square brackets surrounding the indices singled out by the symmetry operations, whilst vertical bars surrounding an index block will mean that the indices sorted out are not to partake of a symmetry operation.

2 Einstein–Cartan's theory

In the realm of Einstein–Cartan's theory, any spacetime carries a symmetric metric tensor $g_{\mu\nu}$ along with a torsionful covariant derivative operator ∇_μ that satisfies the metric[1] compatibility condition $\nabla_\mu g_{\lambda\sigma} = 0$. The Riemann tensor $R_{\mu\nu\lambda}{}^{\sigma}$ of ∇_μ enters either of the configurations

$$D_{\mu\nu} u^{\alpha\ldots\beta} = R_{\mu\nu\tau}{}^{\alpha} u^{\tau\ldots\beta} + \cdots + R_{\mu\nu\tau}{}^{\beta} u^{\alpha\ldots\tau} \tag{1}$$

and

$$D_{\mu\nu} w_{\lambda\ldots\sigma} = -R_{\mu\nu\lambda}{}^{\tau} w_{\tau\ldots\sigma} - \cdots - R_{\mu\nu\sigma}{}^{\tau} w_{\lambda\ldots\tau}, \tag{2}$$

with $u^{\alpha\ldots\beta}$ and $w_{\lambda\ldots\sigma}$ being some world tensors, and

$$D_{\mu\nu} = 2(\nabla_{[\mu}\nabla_{\nu]} + T_{\mu\nu}{}^{\lambda}\nabla_\lambda). \tag{3}$$

The operator $D_{\mu\nu}$ thus possesses the Leibniz rule property. When acting on a world-spin scalar h, it gives $D_{\mu\nu}h = 0$ by definition. The object $T_{\mu\nu}{}^{\lambda}$ is the torsion tensor of ∇_μ. It amounts to the skew part $\Gamma_{[\mu\nu]}{}^{\lambda}$ of the world affinity $\Gamma_{\mu\nu}{}^{\lambda}$ of ∇_μ, and thereby satisfies the defining property

$$T_{\mu\nu}{}^{\lambda} = T_{[\mu\nu]}{}^{\lambda}, \tag{4}$$

whence $D_{\mu\nu} = D_{[\mu\nu]}$. For the relevant Ricci tensor and scalar, one has

$$R_{\mu\nu} = R_{\mu\lambda\nu}{}^{\lambda}, \;\; R = R^{\lambda\sigma} g_{\lambda\sigma}. \tag{5}$$

The tensor $R_{\mu\nu\lambda\sigma}$ possesses skewness in the indices of the pairs $\mu\nu$ and $\lambda\sigma$, but the traditional index-pair symmetry now fails to hold because of the applicability of the cyclic identity[2]

$$R_{[\mu\nu\lambda]}{}^{\sigma} - 2\nabla_{[\mu} T_{\nu\lambda]}{}^{\sigma} + 4T_{[\mu\nu}{}^{\tau} T_{\lambda]\tau}{}^{\sigma} = 0. \tag{6}$$

Thus, the Ricci tensor of ∇_μ bears asymmetry. Additionally, the Bianchi identity should now read

$$\nabla_{[\mu} R_{\nu\lambda]\sigma}{}^{\rho} - 2T_{[\mu\nu}{}^{\tau} R_{\lambda]\tau\sigma}{}^{\rho} = 0. \tag{7}$$

By invoking the dualization schemes given in Ref. [12] and making some index manipulations thereafter, we may rewrite (6) and (7) in terms of first-left duals as

$${}^{*}R^{\lambda}{}_{\mu\nu\lambda} + 2\nabla^{\lambda*} T_{\lambda\mu\nu} + 4{}^{*}T_{\mu}{}^{\lambda\tau} T_{\lambda\tau\nu} = 0 \tag{8}$$

and

$$\nabla^{\rho*} R_{\rho\mu\lambda\sigma} + 2{}^{*}T_{\mu}{}^{\rho\tau} R_{\rho\tau\lambda\sigma} = 0. \tag{9}$$

It should be clear from Eqs. (8) and (9) that both ${}^{*}R^{\lambda}{}_{\mu\nu\lambda}$ and $\nabla^{\rho*} R_{\rho\mu\lambda\sigma}$ do not vanish here in contraposition to the torsionless framework.

The tensor $g_{\mu\nu}$ comes into play as a solution to Einstein–Cartan's equations[3]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa E_{\mu\nu}, \tag{10}$$

whereas $T_{\mu\nu}{}^{\lambda}$ is locally related [12] to the spin density of matter $S_{\mu\nu}{}^{\lambda}$ present in spacetime through

$$T_{\mu\nu}{}^{\lambda} = -\kappa(S_{\mu\nu}{}^{\lambda} - S_{[\mu} g_{\nu]}{}^{\lambda}), \tag{11}$$

with $S_\mu \doteqdot S_{\mu\lambda}{}^{\lambda}$. Equation (11) right away yields the trace relation

$$T_\mu = \frac{\kappa}{2} S_\mu, \tag{12}$$

along with the definition $T_\mu \doteqdot T_{\mu\lambda}{}^{\lambda}$. Consequently, we can write down the supplementary relationship

$$-\kappa S_{\mu\nu}{}^{\lambda} = T_{\mu\nu}{}^{\lambda} - 2T_{[\mu} g_{\nu]}{}^{\lambda}. \tag{13}$$

Hence, working out the relation (6) leads to the equation

$$\nabla_\lambda T_{\mu\nu}{}^{\lambda} + 2(\nabla_{[\mu} T_{\nu]} + T_{\mu\nu}{}^{\lambda} T_\lambda) = \kappa E_{[\mu\nu]}, \tag{14}$$

which amounts to the same thing as

$$(\nabla_\lambda + \kappa S_\lambda) S_{\mu\nu}{}^{\lambda} = -E_{[\mu\nu]}. \tag{15}$$

We can thus say that the skew part of $E_{\mu\nu}$ is a source for $S_{\mu\nu}{}^{\lambda}$, and thence also for $T_{\mu\nu}{}^{\lambda}$.

[1] It is convenient to attribute to $g_{\mu\nu}$ the local signature $(+ - - -)$.

[2] As posed in Ref. [12], the tensor $T_{\mu\lambda\sigma}$ conventionally equals (-2) times ours.

[3] The quantity κ is identified with Einstein's gravitational constant of general relativity.

In fact, if the trace pattern

$$T_\mu = \nabla_\mu \Phi \tag{16}$$

is taken for granted, with Φ standing for a world-spin invariant, then Eqs. (13) and (15) may be fitted together so as to yield the statement

$$\nabla_\lambda T_{\mu\nu}{}^{\lambda} = \kappa E_{[\mu\nu]}, \tag{17}$$

which would also come straightaway from (14). Whence, $T_{\mu\nu}{}^{\lambda}$ appears to play a dynamical role similar to that played by $g_{\mu\nu}$, whilst[4]

$$\nabla_{[\mu} T_{\nu]} + T_{\mu\nu}{}^{\lambda} T_\lambda = 0 \tag{18}$$

and

$$\nabla_\lambda S_{\mu\nu}{}^{\lambda} + \nabla_{[\mu} S_{\nu]} = -E_{[\mu\nu]}. \tag{19}$$

We shall see in Sect. 6 that the gradient model for T_μ as given by (16), imparts a somewhat simpler form to our expression for the mass of dark matter.

3 Curvature spinors and derivatives

It is shown in Refs. [20,25] that, in either formalism, the curvature spinors for ∇_μ occur in the prescriptions

$$\check{D}_{AB}\zeta^C = \varpi_{ABM}{}^{C}\zeta^M, \ \check{D}_{A'B'}\zeta^C = \varpi_{A'B'M}{}^{C}\zeta^M, \tag{20}$$

where ζ^A is a spin vector. In the ε-formalism, $\check{D}_{AB}$ and $\check{D}_{A'B'}$ enter the operator bivector decomposition

$$\Sigma^\mu_{AA'}\Sigma^\nu_{BB'}D_{\mu\nu} = \varepsilon_{A'B'}\check{D}_{AB} + \varepsilon_{AB}\check{D}_{A'B'}, \tag{21}$$

with $D_{\mu\nu}$ being given by Eq. (3) and the Σ-objects amounting to appropriate connecting objects. The ε-objects are the only covariant metric spinors for the formalism being allowed for since they bear invariance under the action of the generalized Weyl gauge group [20]. Both of them and any Σ-connecting object are usually taken as covariantly constant entities such that, for instance,

$$\nabla_\mu \varepsilon_{AB} = 0. \tag{22}$$

Indeed, any curvature spinors obey a general symmetry relation like

$$\varpi_{ABCD} = \varpi_{(AB)CD}, \ \varpi_{A'B'CD} = \varpi_{(A'B')CD}. \tag{23}$$

It may be established [20] that the spinor pair $(\varpi_{AB(CD)}, \varpi_{A'B'(CD)})$ constitutes the irreducible decomposition of the Riemann tensor for ∇_μ. We have, in effect, the expression

$$R_{AA'BB'CC'DD'} = (\varepsilon_{A'B'}\varepsilon_{C'D'}\varpi_{AB(CD)} + \varepsilon_{AB}\varepsilon_{C'D'}\varpi_{A'B'(CD)}) + \text{c.c.}, \tag{24}$$

together with its first-left dual

$${}^*R_{AA'BB'CC'DD'} = [-i(\varepsilon_{A'B'}\varepsilon_{C'D'}\varpi_{AB(CD)} - \varepsilon_{AB}\varepsilon_{C'D'}\varpi_{A'B'(CD)})] + \text{c.c.}, \tag{25}$$

where the symbol "c.c." denotes here as elsewhere an overall complex conjugate piece. The unprimed curvature spinor is expandable as[5]

$$\mathrm{X}_{ABCD} = \Psi_{ABCD} - \varepsilon_{(A|(C}\xi_{D)|B)} - \frac{1}{3}\varkappa\varepsilon_{A(C}\varepsilon_{D)B}, \tag{26}$$

with

$$\Psi_{ABCD} = \mathrm{X}_{(ABCD)}, \ \xi_{AB} = \mathrm{X}^M{}_{(AB)M}, \ \varkappa = \mathrm{X}_{LM}{}^{LM} \tag{27}$$

and Ψ_{ABCD} defining a formal prototype of a wave function for gravitons, which vanishes identically in case the underlying spacetime bears conformal flatness. The quantity $\varkappa$ thus appears as a complex-valued world-spin invariant that satisfies

$$R = 4\mathrm{Re}\varkappa, \ {}^*R_{\mu\nu}{}^{\mu\nu} = 4\,\mathrm{Im}\varkappa. \tag{28}$$

We emphasize that the occurrence of the ξ-spinor in the expansion (26), is just associated to the torsional property $R_{\mu\nu\lambda\sigma} \neq R_{\lambda\sigma\mu\nu}$. Then, the only symmetries borne by the X-spinor are exhibited by

$$\mathrm{X}_{ABCD} = \mathrm{X}_{(AB)(CD)}. \tag{29}$$

The contracted pieces $(\varpi_{ABM}{}^{M}, \varpi_{A'B'M}{}^{M})$ fulfill additivity relations that carry wave functions for the cosmic microwave and dark energy backgrounds of both handednesses, namely (for further details, see Ref. [25])

$$\varpi_{ABM}{}^{M} = -2i(\phi_{AB} + \psi_{AB}) \tag{30}$$

and

$$\varpi_{A'B'M}{}^{M} = -2i(\psi_{A'B'} + \phi_{A'B'}). \tag{31}$$

We can therefore recast the prescriptions (20) into the form

$$\check{D}_{AB}\zeta^C = \mathrm{X}_{ABM}{}^{C}\zeta^M - i(\phi_{AB} + \psi_{AB})\zeta^C \tag{32}$$

and

$$\check{D}_{A'B'}\zeta^C = \Xi_{A'B'M}{}^{C}\zeta^M - i(\phi_{A'B'} + \psi_{A'B'})\zeta^C. \tag{33}$$

[4] Making the choice (16) also implies that $\nabla_{[\mu}S_{\nu]} = \kappa S_{\mu\nu}{}^{\lambda}S_\lambda$.

[5] From now onwards, we will for convenience employ the definitions $\varpi_{AB(CD)} \doteqdot \mathrm{X}_{ABCD}$ and $\varpi_{A'B'(CD)} \doteqdot \Xi_{A'B'CD}$. We should also stress that $\varepsilon_{A(C}\varepsilon_{D)B} = \varepsilon_{(A|(C}\varepsilon_{D)|B)}$.

The prescriptions for computing $\check{D}$-derivatives of a covariant spin vector η_A can be immediately obtained from (32) and (33) by requiring that

$$\check{D}_{AB}(\zeta^C\eta_C) = 0,\ \check{D}_{A'B'}(\zeta^C\eta_C) = 0, \tag{34}$$

and carrying out Leibniz expansions thereof.[6] We thus obtain

$$\check{D}_{AB}\eta_C = -[\mathrm{X}_{ABC}{}^M\eta_M - i(\phi_{AB} + \psi_{AB})\eta_C] \tag{35}$$

and

$$\check{D}_{A'B'}\eta_C = -[\Xi_{A'B'C}{}^M\eta_M - i(\phi_{A'B'} + \psi_{A'B'})\eta_C], \tag{36}$$

along with the complex conjugates of Eqs. (32) through (36). Hence, whenever $\check{D}$-derivatives of Hermitian spin tensors are actually computed, all occurrent $\phi\psi$-contributions get cancelled. Obviously, such a cancellation likewise happens when we allow the $\check{D}$ -operators to act freely upon spin tensors having the same numbers of covariant and contravariant indices of the same kind. In any such case, the relevant expansion thus carries only gravitational contributions.

The development of Sect. 6 formally demands the use of derivatives of geometric spinor densities [22,25], which will be very briefly touched upon now. The $\check{D}$-derivatives of a complex spin-scalar density α of weight w are written as[7]

$$\begin{aligned}&\check{D}_{AB}\alpha = 2iw\alpha(\phi_{AB} + \psi_{AB}),\\ &\check{D}_{A'B'}\alpha = 2iw\alpha(\phi_{A'B'} + \psi_{A'B'}),\end{aligned} \tag{37}$$

whence the patterns for $\check{D}$-derivatives of some spin-tensor density can always be specified by symbolic expansions like

$$\check{D}_{AB}(\alpha\Upsilon_{C\ldots D}) = (\check{D}_{AB}\alpha)\Upsilon_{C\ldots D} + \alpha\check{D}_{AB}\Upsilon_{C\ldots D}, \tag{38}$$

with $\Upsilon_{C\ldots D}$ being a spin tensor. When $w < 0$, a cancellation of the $\phi\psi$-pieces will occur in the expansion (38) as well if $\Upsilon_{C\ldots D}$ is taken to carry $-2w$ indices and $\operatorname{Im}\alpha \neq 0$ everywhere. A similar property also holds for situations that involve outer products between contravariant spin tensors and complex spin-scalar densities having suitable positive weights. In the ε-formalism, each of the objects ϕ_{AB}, ψ_{AB} and ξ_{AB} neatly fits in with the situations just described, inasmuch as each of them is effectively identified thereabout with a spin-tensor density of weight -1.

Equations (3) and (21) give rise to the derivative operators

$$\check{D}_{AB} = \Delta_{AB} + 2\tau_{AB}{}^\mu\nabla_\mu,\ \Delta_{AB} \doteq -\nabla^{C'}_{(A}\nabla_{B)C'}, \tag{39}$$

where $\tau_{AB}{}^\mu$ is borne by the bivector expansion for $T_{\lambda\sigma}{}^\mu$, that is to say,

$$T_{AA'BB'}{}^\mu = \varepsilon_{A'B'}\tau_{AB}{}^\mu + \text{c.c.}. \tag{40}$$

[6] It should be obvious that the Leibniz-rule property of $D_{\mu\nu}$ is set forth to both $\check{D}_{AB}$ and $\check{D}_{A'B'}$.

[7] The complex conjugate of the spin density α is said to possess an *antiweight* w such that $|\alpha|^2$ possesses an *absolute weight* $2w$.

In conjunction with (32) and (33), we thus have the differential relations

$$\Delta_{AB}\zeta^C = \check{D}_{AB}\zeta^C - 2\tau_{AB}{}^\mu\nabla_\mu\zeta^C \tag{41}$$

and

$$\Delta_{A'B'}\zeta^C = \check{D}_{A'B'}\zeta^C - 2\tau_{A'B'}{}^\mu\nabla_\mu\zeta^C, \tag{42}$$

along with the ones for η_C. The defining property $D_{\mu\nu}h = 0$ can then be recovered as

$$\Delta_{AB}h = -2\tau_{AB}{}^\mu\nabla_\mu h. \tag{43}$$

Hence, we can spell out the expansion

$$\nabla_{AC'}\nabla^{C'}_B = \Delta_{AB} + \frac{1}{2}\varepsilon_{AB}\Box, \tag{44}$$

as well as its contravariant version, with $\Box = \nabla_\mu\nabla^\mu$.

4 Einstein–Cartan's theory in spinor form

Equations (24) and (26) supply us with the following expression for the spinor version of $R_{\mu\nu}$

$$\begin{aligned}&R_{AA'BB'} = \varepsilon_{AB}\varepsilon_{A'B'}\operatorname{Re}\varkappa\\ &-[(\varepsilon_{A'B'}\xi_{AB} + \Xi_{A'B'AB}) + \text{c.c.}].\end{aligned} \tag{45}$$

Under certain affine circumstances, the symmetric part of Einstein–Cartan's equations leads to the limiting case of general relativity, but this is not of a primary interest at this stage. Instead, we should now allow for the skew part

$$R_{[\mu\nu]} = -\kappa E_{[\mu\nu]}, \tag{46}$$

whose spinor version is, then, constituted by

$$\varepsilon_{A'B'}\xi_{AB} + \text{c.c.} = \kappa(\varepsilon_{A'B'}\check{E}_{AB} + \text{c.c.}). \tag{47}$$

Hence, if we implement a decomposition for the spin density of matter $S_{\mu\nu}{}^\lambda$ like the one exhibited by (40), namely,

$$S_{AA'BB'}{}^\mu = \varepsilon_{A'B'}\check{S}_{AB}{}^\mu + \text{c.c.}, \tag{48}$$

after calling for Eqs. (11) and (15), we will get the relation

$$\tau_{AB}{}^{CC'} = -\kappa(\check{S}_{AB}{}^{CC'} + \frac{1}{2}S^{C'}_{(A}\varepsilon_{B)}{}^C), \tag{49}$$

together with

$$(\nabla_\mu + \kappa S_\mu)\check{S}_{AB}{}^\mu = -\frac{1}{\kappa}\xi_{AB}. \tag{50}$$

Equation (13) thus gets translated into

$$-\kappa\check{S}_{AB}{}^{CC'} = \tau_{AB}{}^{CC'} + T^{C'}_{(A}\varepsilon_{B)}{}^C. \tag{51}$$

The dynamical role played by $T_{\mu\nu}{}^\lambda$ becomes considerably enhanced when the spinor version of (17) is set up. We have, in effect,

$$\nabla_\mu\tau_{AB}{}^\mu = \xi_{AB}, \tag{52}$$

while the choice (19) should be transcribed as

$$\nabla_\mu \check{S}_{AB}{}^{\mu} + \frac{1}{2}\nabla_{C'(A} S^{C'}_{B)} = -\frac{1}{\kappa}\xi_{AB}. \tag{53}$$

A glance at Eqs. (14), (47) and (50) tells us that, as far as the physical inner structure of Einstein–Cartan's theory is concerned, a ξ-curvature spinor must be taken as the only source for spacetime torsion and densities of intrinsic angular momentum of matter. [8] We will elaborate a little further upon this point in the concluding Section.

5 Field equations

The attitude we have been taking herein involves identifying the world field equation for dark matter as the statement

$$\nabla^{\rho}\, {}^{*}R^{\lambda}{}_{[\rho\sigma]\lambda} = -2{}^{*}T^{\lambda\rho\tau} R_{\tau[\rho\sigma]\lambda}, \tag{54}$$

which is nothing else but a skew contracted version of the Bianchi identity (9). The spinor version of ${}^{*}R^{\lambda}{}_{[\rho\sigma]\lambda}$ yields the expansion

$$\Sigma^{\rho}_{BB'}\Sigma^{\sigma}_{CC'}\, {}^{*}R^{\lambda}{}_{[\rho\sigma]\lambda} = {}^{*}R^{DD'}{}_{(BC)[B'C']DD'} + \text{c.c.}, \tag{55}$$

which is written out explicitly as

$${}^{*}R^{DD'}{}_{(BC)[B'C']DD'} + \text{ c.c.} = -i(\varepsilon_{B'C'}\xi_{BC} - \text{c.c.}). \tag{56}$$

Thus, the left-hand side of Eq. (54) takes the form

$$\nabla^{BB'}[-i(\varepsilon_{B'C'}\xi_{BC} - \text{c.c.})] = -i(\nabla^{B}_{C'}\xi_{BC} - \text{c.c.}), \tag{57}$$

whence the entries of the pair $(\xi_{AB}, \xi_{A'B'})$ assumably constitute wave functions of opposite handednesses for dark matter, with each of which thus possessing six real independent components. For the right-hand side of Eq. (54), we have the correspondence[9]

$$-2{}^{*}T^{\lambda\rho\tau} R_{\tau[\rho\sigma]\lambda} \longmapsto -i(\varepsilon^{DB}\tau^{D'B'AA'} - \text{c.c.}) \times (\varepsilon_{B'C'} R_{AA'(BC)L'}{}^{L'}{}_{DD'} + \text{c.c.}), \tag{58}$$

together with the expression

$$R_{AA'(BC)L'}{}^{L'}{}_{DD'} = -\varepsilon_{A'D'}\mathrm{X}_{A(BC)D} + \Xi_{A'D'D(C}\varepsilon_{B)A} + \varepsilon_{A(B}\varepsilon_{C)D}\mathrm{X}_{L'A'D'}{}^{L'} + \varepsilon_{D(C}\Xi_{B)AA'D'}. \tag{59}$$

Towards carrying through the derivation of our field equations, it is convenient to require the unprimed and primed wave functions to bear algebraic independence throughout spacetime. This requirement enables us to rearrange some of the complex conjugate pieces of Eqs. (58) and (59) to the extent that the left-right handednesses of the fields become transparently separated. We should then expand the whole Riemann term of (58) in agreement with

$$2R_{\tau[\rho\sigma]\lambda} \longmapsto [\varepsilon_{B'C'}(-\varepsilon_{A'D'}\mathrm{X}_{A(BC)D} + \varepsilon_{D(C}\Xi_{B)AA'D'}) + \varepsilon_{BC}(\Xi_{ADD'(C'}\varepsilon_{B')A'} + \varepsilon_{A'(B'}\varepsilon_{C')D'}\mathrm{X}_{LAD}{}^{L})] + \text{c.c.}, \tag{60}$$

whose unprimed X-contributions, by virtue of Eq. (26), are expressed as

$$\mathrm{X}_{A(BC)D} = \Psi_{ABCD} - \frac{1}{2}\varepsilon_{AD}\xi_{BC} + \frac{1}{6}\varkappa\varepsilon_{A(B}\varepsilon_{C)D} \tag{61}$$

and

$$\mathrm{X}_{LAD}{}^{L} = -\left(\xi_{AD} - \frac{1}{2}\varkappa\varepsilon_{AD}\right). \tag{62}$$

Therefore, putting into effect the rearrangement given by (60) allows us to work out naively the τR-couplings that occur in Eq. (58).

For the kernel $\tau^{D'B'AA'}$, we thus utilize trivial index-displacement rules for obtaining the X-contribution[10]

$$\varepsilon^{DB}\tau^{D'B'AA'}(-\varepsilon_{B'C'}\varepsilon_{A'D'}\mathrm{X}_{A(BC)D} + \varepsilon_{BC}\varepsilon_{A'(B'}\varepsilon_{C')D'}\mathrm{X}_{LAD}{}^{L}) = \tau_{A'C'}{}^{AA'}\xi_{AC} - \frac{1}{2}\varkappa\tau_{A'C'C}{}^{A'}, \tag{63}$$

along with the Ξ-one

$$\varepsilon^{DB}\tau^{D'B'AA'}(\varepsilon_{B'C'}\varepsilon_{D(C}\Xi_{B)AA'D'} + \varepsilon_{BC}\Xi_{ADD'(C'}\varepsilon_{B')A'}) = 2\tau^{B'D'B}{}_{C'}\Xi_{BCB'D'} - \tau^{B'D'B}{}_{D'}\Xi_{BCB'C'}. \tag{64}$$

In turn, for the kernel $\tau^{DBAA'}$, we have

$$\varepsilon^{D'B'}\tau^{DBAA'}(-\varepsilon_{B'C'}\varepsilon_{A'D'}\mathrm{X}_{A(BC)D} + \varepsilon_{BC}\varepsilon_{A'(B'}\varepsilon_{C')D'}\mathrm{X}_{LAD}{}^{L}) = \tau^{ABD}{}_{C'}\Psi_{ABCD} + \frac{3}{2}\tau^{AB}{}_{CC'}\xi_{AB} - 2\tau^{AB}{}_{BC'}\xi_{AC} - \frac{5}{6}\varkappa\tau^{A}{}_{CAC'} \tag{65}$$

and

$$\varepsilon^{D'B'}\tau^{DBAA'}(\varepsilon_{B'C'}\varepsilon_{D(C}\Xi_{B)AA'D'} + \varepsilon_{BC}\Xi_{ADD'(C'}\varepsilon_{B')A'}) = \tau^{AB}{}_{B}{}^{A'}\Xi_{ACA'C'} - \tau^{AB}{}_{C}{}^{A'}\Xi_{ABA'C'}. \tag{66}$$

It follows that, if we take into consideration the relation

$$\tau^{AB}{}_{BB'} - \tau_{A'B'}{}^{AA'} = T^{A}_{B'} = \Sigma^{\mu A}_{B'}T_{\mu}, \tag{67}$$

after performing some further index manipulations and implementing the required algebraic independence between the conjugate wave functions, we will arrive at the field equation

$$\nabla^{B}_{C'}\xi_{BC} + m_{CC'} = \sigma^{(\mathrm{X})}_{CC'} + \sigma^{(\Xi)}_{CC'}, \tag{68}$$

[8] This statement remains valid even when the pattern (16) is brought forth along with Eq. (52).

[9] For ${}^{*}T^{\lambda\rho\tau}$, we have the expansion $i(\varepsilon^{DB}\tau^{D'B'AA'} - \text{c.c.})$.

[10] We should notice that one of the couplings which occur on the left-hand side of (63) annihilates the Ψ-term of (61)

together with

$$m_{CC'} = (T^A_{C'} - 3\tau^{AB}{}_{BC'})\xi_{AC} + \frac{3}{2}\tau^{AB}{}_{CC'}\xi_{AB} \tag{69}$$

and the complex conjugates of (68) and (69). The σ-pieces of Eq. (68) are complex sources for ξ_{BC}, which are irreducibly given by

$$\sigma^{(X)}_{CC'} = -\tau^{ABD}{}_{C'}\Psi_{ABCD} + \varkappa(\frac{1}{2}T_{CC'} + \frac{1}{3}\tau^A{}_{CAC'}) \tag{70}$$

and

$$\sigma^{(\Xi)}_{CC'} = \tau^{AB}{}_C{}^{A'}\Xi_{ABA'C'} + 2\tau^{A'B'A}{}_{C'}\Xi_{ACA'B'} - T^{AA'}\Xi_{CAA'C'}. \tag{71}$$

We can see that the m-terms borne by the field equations for ξ_{AB} and $\xi_{A'B'}$, arise strictly from the gravitational interaction between torsion kernels and the fields themselves, while the corresponding sources carry only couplings of torsion kernels with $\varkappa\Psi\Xi$-curvatures. In Sect. 7, a world interpretation of this feature will be made.

6 Wave equations

To derive the wave equations that govern the propagation in spacetime of the ξ-fields, we follow up the procedure which amounts to implementing the expansions and derivatives exhibited previously to work out the ξ-field equation for either handedness. For ξ_{BC}, say, we thus let the operator $\nabla^{C'}_A$ act on both sides of Eq. (68). By making use of the contravariant form of (44), we then obtain [11]

$$\frac{1}{2}\Box\xi_{AC} + \check{D}^B_A\xi_{BC} - (2\tau^{B\mu}_A\nabla_\mu\xi_{BC} + \nabla^{C'}_A m_{CC'}) = -(\nabla^{C'}_A\sigma^{(X)}_{CC'} + \nabla^{C'}_A\sigma^{(\Xi)}_{CC'}), \tag{72}$$

with Eq. (39) having been employed. As ξ_{BC} stands for a two-index spin-tensor density of weight -1, the derivative $\check{D}^B_A\xi_{BC}$ consists of a purely gravitational expansion, which may be made up of

$$\check{D}^B_{(A}\xi_{C)B} = \frac{2}{3}\varkappa\xi_{AC} - \Psi_{AC}{}^{BD}\xi_{BD} \tag{73}$$

and

$$\check{D}^B_{[A}\xi_{C]B} = \varepsilon_{AC}\xi^{BD}\xi_{BD}. \tag{74}$$

It is obvious that the symmetry of ξ_{AC} brings about the occurrence of the relation

$$\nabla_\mu m^\mu - \Delta^{BD}\xi_{BD} = \nabla^\mu\sigma^{(X)}_\mu + \nabla^\mu\sigma^{(\Xi)}_\mu, \tag{75}$$

which just amounts to the skew part in A and C of the statement (72). Hence, accounting for (74) and (75), we get the subsidiary condition

$$2(\xi^{BD}\xi_{BD} + \tau^{BD\mu}\nabla_\mu\xi_{BD}) + \nabla_\mu m^\mu - (\nabla^\mu\sigma^{(X)}_\mu + \nabla^\mu\sigma^{(\Xi)}_\mu) = 0. \tag{76}$$

The overall symmetric piece of (72) thus reads

$$\frac{1}{2}\Box\xi_{AC} + \check{D}^B_{(A}\xi_{C)B} - 2(\nabla_\mu\xi_{B(A})\tau^{B\mu}_{C)} - \nabla^{C'}_{(A}m_{C)C'} = -(\nabla^{C'}_{(A}\sigma^{(X)}_{C)C'} + \nabla^{C'}_{(A}\sigma^{(\Xi)}_{C)C'}). \tag{77}$$

With the help of Eq. (52), we can still reexpress the explicit $\nabla\xi\tau$-term of (77) as

$$(\nabla_\mu\xi_{B(A})\tau^{B\mu}_{C)} = \nabla_\mu(\xi_{B(A}\tau^{B\mu}_{C)}), \tag{78}$$

provided that $\xi^B_{(A}\xi_{C)B} \equiv 0$.

We saw that Einstein–Cartan's theory assigns a dynamical meaning to torsion kernels. Based upon this fact and the formal commonness between the couplings borne by $\nabla^{C'}_{(A}m_{C)C'}$ and that shown up by Eq. (78), we claim that the mass μ_{DM} of dark matter is produced by the interaction between the ξ-fields and torsion kernels, in accordance with the Klein–Gordon prescription[12]

$$\frac{1}{2}\mu^2_{DM}\xi_{AC} = -2\nabla_\mu(\xi_{B(A}\tau^{B\mu}_{C)}) - \nabla^{C'}_{(A}m_{C)C'} \tag{79}$$

and the reality requirement $\mu^2_{DM} > 0$. The wave equation for ξ_{AC} comes about when (73) and (79) are inserted into (77). We thus end up with

$$\left(\Box + \mu^2_{DM} + \frac{4}{3}\varkappa\right)\xi_{AB} - 2\Psi_{AB}{}^{CD}\xi_{CD} = -2(\nabla^{C'}_{(A}\sigma^{(X)}_{B)C'} + \nabla^{C'}_{(A}\sigma^{(\Xi)}_{B)C'}), \tag{80}$$

along with the complex conjugate of (80).

7 World dynamics, mass terms and energy density

Within the geometric context of Sect. 2, we can deduce the equivalent relations [20]

$$\widetilde{\nabla}_\mu g_{\lambda\sigma} - 2T_{\mu(\lambda\sigma)} = 0 \Leftrightarrow \Gamma_\mu = \widetilde{\Gamma}_\mu + T_\mu = \partial_\mu \log(-\mathfrak{g})^{1/2}, \tag{81}$$

where

$$\widetilde{\Gamma}_\mu = \widetilde{\Gamma}_{\mu\lambda}{}^\lambda,\ \Gamma_{(\mu\lambda)\sigma} \doteq \widetilde{\Gamma}_{\mu\lambda\sigma}, \tag{82}$$

with $\widetilde{\nabla}_\mu$ amounting to the covariant derivative operator for $\widetilde{\Gamma}_{\mu\lambda\sigma}$, and $\mathfrak{g}$ denoting the determinant of $g_{\mu\nu}$. Allowing for

[11] It has been unnecessary here to stagger the indices of any symmetric two-index configuration.

[12] We recall that the natural system of units has been adopted by us from the beginning.

the secondary metric condition

$$\widetilde{\nabla}_{\lambda} g_{\mu\nu} = 0, \tag{83}$$

thus implies that $T_{\mu} = 0$ everywhere in spacetime. Noticeably, this implication yields the property

$$T_{\mu\nu\lambda} = T_{[\mu\nu\lambda]}, \tag{84}$$

such that $T_{\mu\nu\lambda}$ would possess only four real independent components if the condition (83) were actually implemented. Therefore, once we are given a solution to Eq. (10), with the components of $E_{[\mu\nu]}$ being accordingly prescribed at the outset, we may promptly make use of the contortion tensor for $T_{\mu\nu\lambda}$ and identify $\widetilde{\Gamma}_{\mu\lambda\sigma}$ with a Christoffel connexion, not only to determine all the components of $\Gamma_{(\mu\lambda)\sigma}$, but also to select out a set of outer products that obey the relationship

$$2(\Sigma^{00'}_{(\mu}\Sigma^{11'}_{\nu)} - \Sigma^{01'}_{(\mu}\overline{\Sigma^{01'}_{\nu)}}) = g_{\mu\nu}. \tag{85}$$

Consequently, if the expressions (45) and (46) were accounted for afterwards, we would become able to set the corresponding components of $R_{[\mu\nu]}$, ξ_{AC} and $\xi_{A'C'}$. Hence, in view of (83), we could utilize the simplified expansion

$$\frac{1}{\sqrt{-\mathfrak{g}}}\partial_{\lambda}(\sqrt{-\mathfrak{g}}T_{\mu\nu}{}^{\lambda}) + 2\widetilde{\Gamma}_{\lambda[\mu}{}^{\sigma}T_{\nu]\sigma}{}^{\lambda} = \kappa E_{[\mu\nu]}, \tag{86}$$

to integrate out Eq. (17) towards getting the components of $\tau_{AB}{}^{\mu}$ and making up the information on those of $\Gamma_{\mu\nu\lambda}$ which would, in turn, provide us with the ones of

$$R_{\mu\nu\lambda}{}^{\rho} \doteqdot 2(\partial_{[\mu}\Gamma_{\nu]\lambda}{}^{\rho} + \Gamma_{[\mu|\tau|}{}^{\rho}\Gamma_{\nu]\lambda}{}^{\tau}) \tag{87}$$

and

$$^{*}R_{\mu\nu\lambda\sigma} = \frac{1}{2}\sqrt{-\mathfrak{g}}e_{\mu\nu\rho\tau}R^{\rho\tau}{}_{\lambda\sigma}, \tag{88}$$

with $e_{\lambda\sigma\rho\tau}$ being one of the invariant Levi-Civitta world densities. In this way, all the components that enter the prescriptions (69) and (79) might be determined consistently, and an estimation of μ^2_{DM} could then be attained. It should be emphasized at this point that the transcription (53) guarantees the legitimacy of the strong assumption on spin alignment, which is rooted in the obtainment of Kopczyński solutions [13] (see also Ref. [11]).

In order to formulate the world dynamics of dark matter in a more explicit manner, it may be expedient to define[13]

$$f_{\mu\nu} = 2\nabla_{[\mu}B_{\nu]}, \tag{89}$$

where B_{λ} is a dark matter potential, and

$$f_{\mu\nu} \doteqdot {}^{*}R^{\lambda}{}_{[\mu\nu]\lambda}. \tag{90}$$

[13] There is no relationship between B_{μ} and any contracted spin affinity, contrarily to the case of the cosmic backgrounds.

The action for dark matter is settled as

$$S_{DM} = \int_{\omega}\sqrt{-\mathfrak{g}}\left(-\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \frac{1}{2}M^{\mu}B_{\mu} + j^{\mu}B_{\mu}\right)d^4x, \tag{91}$$

where ω denotes a spacetime volume whose closure is compact, and

$$d^4x = \frac{1}{4!}e_{\lambda\sigma\rho\tau}dx^{\lambda}\wedge dx^{\sigma}\wedge dx^{\rho}\wedge dx^{\tau}. \tag{92}$$

In setting up Eq. (91), we have split the right-hand side of the statement (54) as follows

$$2^{*}T^{\lambda\rho\tau}R_{\tau[\rho\mu]\lambda} = M_{\mu} + j_{\mu}. \tag{93}$$

The only reason for this splitting procedure is, indeed, that the derivation of the field equations of Sect. 5 formally brings forward a disjunction between sources and torsion-field couplings, to which S_{DM} must be subject. We will see in a moment that, whenever $T_{\mu} = 0$, the contributions M_{μ} and j_{μ} must carry the information on the remaining pieces borne by the m-pattern and σ-sources of Eq. (68), respectively (see Eqs. (101) and (103) below).

The least-action principle involving S_{DM} reads off

$$\delta S_{DM} = 0. \tag{94}$$

By definition, the δ-variation bears linearity and enjoys the Leibniz-rule property, in addition to being defined so as to commute with partial derivatives and integrations. The quantity δB_{μ} takes arbitrary values in ω and vanishes on the boundary $\partial\omega$ of ω, while $\delta j_{\mu} = 0$ on the closure of ω. We suppose that δM_{μ} should be expressed as $\mu^2_m\delta B_{\mu}$, with μ^2_m being identified with the contribution to μ^2_{DM} that comes from the m-term exhibited as Eq. (69), whence

$$\delta(M^{\mu}B_{\mu}) = 2\mu^2_m B^{\mu}\delta B_{\mu} = 2M^{\mu}\delta B_{\mu}. \tag{95}$$

Since

$$\frac{1}{4}\delta(f^{\mu\nu}f_{\mu\nu}) = \frac{1}{2}f^{\mu\nu}\delta f_{\mu\nu} = f^{\mu\nu}(\partial_{\mu}\delta B_{\nu} - T_{\mu\nu}{}^{\lambda}\delta B_{\lambda}), \tag{96}$$

after performing the integration by parts

$$\int_{\omega}\sqrt{-\mathfrak{g}}f^{\mu\nu}\partial_{\mu}\delta B_{\nu}d^4x = \int_{\partial\omega}\sqrt{-\mathfrak{g}}f^{\mu\nu}\delta B_{\nu}d^3x_{\mu} - \int_{\omega}[\frac{1}{\sqrt{-\mathfrak{g}}}\partial_{\mu}(\sqrt{-\mathfrak{g}}f^{\mu\nu})]\delta B_{\nu}\sqrt{-\mathfrak{g}}d^4x, \tag{97}$$

with

$$d^3x_{\mu} = \frac{1}{3!}e_{\mu\lambda\sigma\rho}dx^{\lambda}\wedge dx^{\sigma}\wedge dx^{\rho}, \tag{98}$$

and fitting pieces together, we thus obtain the equations of motion

$$\frac{1}{\sqrt{-\mathfrak{g}}}\partial_\mu(\sqrt{-\mathfrak{g}}f^{\mu\lambda}) + f^{\mu\nu}T_{\mu\nu}{}^{\lambda} + M^\lambda = -j^\lambda. \quad (99)$$

Now, if we take into account the relations (81), (83) and (93) together with the expansion

$$\begin{aligned}\nabla_\mu f^{\mu\nu} &= \partial_\mu f^{\mu\nu} + \Gamma_{\mu\lambda}{}^{\mu} f^{\lambda\nu} + \Gamma_{\mu\lambda}{}^{\nu} f^{\mu\lambda}\\ &= \partial_\mu f^{\mu\nu} + (\widetilde{\Gamma}_\mu - T_\mu) f^{\mu\nu} + T_{\mu\lambda}{}^{\nu} f^{\mu\lambda}\\ &= \frac{1}{\sqrt{-\mathfrak{g}}}\partial_\mu(\sqrt{-\mathfrak{g}}f^{\mu\nu}) - 2T_\mu f^{\mu\nu} + T_{\mu\lambda}{}^{\nu} f^{\mu\lambda},\end{aligned} \quad (100)$$

effectively setting $T_\mu = 0$ in it, we will recover the field equation (54) as

$$\nabla_\mu f^{\mu\lambda} + M^\lambda = -j^\lambda, \quad (101)$$

while obtaining the wave equation

$$\begin{aligned}&\Box B_\mu + R_\mu{}^{\lambda} B_\lambda - \nabla_\mu(\nabla^\lambda B_\lambda) - 2T_\mu{}^{\lambda\sigma}\nabla_\sigma B_\lambda\\ &= -2{}^*T^{\lambda\rho\tau} R_{\tau[\rho\mu]\lambda}.\end{aligned} \quad (102)$$

It becomes clear that, if $T_\mu = 0$ then the constituents of the splitting (93) should occur in the correspondences

$$2\mathrm{Im}\,\Sigma_\mu^{CC'} m_{CC'} = M_\mu,\ \ 2\mathrm{Im}\,\Sigma_\mu^{CC'}(\sigma^{(\mathrm{X})}_{CC'} + \sigma^{(\Xi)}_{CC'}) = -j_\mu. \quad (103)$$

From Eq. (91), it follows that the overall Lagrangian density for dark matter may be symbolically written as

$$\mathfrak{L}_{DM} = \mathfrak{L}_M + \mathfrak{L}_\sigma, \quad (104)$$

with the individual pieces

$$\mathfrak{L}_M = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu} + \frac{1}{2}M^\mu B_\mu,\ \ \mathfrak{L}_\sigma = j^\mu B_\mu. \quad (105)$$

Hence, by employing the following definition for the respective energy–momentum tensor

$$T^{DM}_{\mu\nu} = g_{\mu\nu}\mathfrak{L}_M - f_{\mu\lambda}g_{\rho\nu}\frac{\partial\mathfrak{L}_M}{\partial\nabla_\rho B_\lambda}, \quad (106)$$

we get the explicit expression[14]

$$T^{DM}_{\mu\nu} = g_{\mu\nu}\left(-\frac{1}{4}f^{\lambda\sigma}f_{\lambda\sigma} + \frac{1}{2}M^\lambda B_\lambda\right) + f_{\mu\lambda}f_\nu{}^{\lambda}. \quad (107)$$

Evidently, rather than allowing for Eq. (102) to obtain the components of B_μ, we may absorb the property (84) to integrate the equation

$$\partial_{[\mu}B_{\nu]} - T_{\mu\nu}{}^{\lambda}B_\lambda = \frac{1}{2}f_{\mu\nu}. \quad (108)$$

[14] We should note that $f_{[\mu|\lambda|}f_{\nu]}{}^{\lambda} = 0$ such that $T^{DM}_{\mu\nu}$ still bears symmetry.

With the components of M_μ at hand, such as given by Eq. (103), we can say that the above procedure would easily produce the value

$$\mu_m = \sqrt{\frac{M^\lambda B_\lambda}{B^\tau B_\tau}}. \quad (109)$$

Equation (101) could, then, be reinstated as

$$\nabla_\mu f^{\mu\lambda} + \mu_m^2 B^\lambda = -j^\lambda, \quad (110)$$

whilst the energy density carried by $f_{\mu\nu}$ would be expressed by

$$\begin{aligned}T^{DM}_{00} &= g_{00}\left(-\frac{1}{4}f^{\lambda\sigma}f_{\lambda\sigma} + \frac{1}{2}\mu_m^2 B^\lambda B_\lambda\right)\\ &\quad + f_{01}f_0{}^1 + f_{02}f_0{}^2 + f_{03}f_0{}^3.\end{aligned} \quad (111)$$

8 Concluding remarks and outlook

The expansion (26) is what suggests that gravitons and dark matter were produced together by the big-bang creation of the Universe, whence we can say that the earliest states of very high density of spin matter must have occurred in the absence of conformal flatness. Nevertheless, the spinor form of Einstein–Cartan's theory shows us that it is solely dark matter which produces spacetime torsion, but the theory of dark matter we have proposed particularly asserts that the sources for dark matter must be prescribed in terms of well defined couplings between curvatures and torsion. Hence, our theory appears to assign a double physical character to dark matter. According to the formulation of Sects. 5 and 6, the propagation of dark matter in spacetime is controlled by the wave equation (80) and its complex conjugate. A systematic treatment of these differential equations can be carried out by utilizing a torsional extension of the methods provided by Ref. [30]. Such a treatment would thus supply a description of the effects of curvature and torsion on the wave functions that are presumably ascribed to dark matter, while possibly shedding some light on the assumption that the massiveness of dark matter arises strictly from the interaction between dark matter fields and torsion kernels just as simply prescribed by Eq. (79).

It was seen that the implementation of the gradient pattern (16) ensures the formal commonness of $\nabla^{C'}_{(A}m_{C)C'}$ with the right-hand side of Eq. (78), whence we may think of the mass of dark matter as being expressed as

$$\mu_{DM} = \sqrt{\frac{[-4\nabla_\mu(\xi_{B(A}\tau_{C)}^{B\mu}) - 2\nabla^{C'}_{(A}m_{C)C'}]\xi^{AC}}{\xi_{AC}\xi^{AC}}}. \quad (112)$$

The significance of the derivation procedures exibited in Sect. 6 partially originates from the fact that Eq. (80) brings out the overall information on the mass of dark matter, while the field equation (68) can supply only the part of this information that is carried by the prescription (69). Remarkably enough, this observation still applies to the description of the cosmic microwave and dark energy backgrounds we had referred to in Sect. 1. Whenever the cosmic backgrounds propagate inside dark matter, the microwave background thus acquires a certain amount of mass μ_{CMB} while the dark energy background acquires an effective mass μ_{eff}. The corresponding relations are formally the same as the one given by (79), namely,

$$\mu^2_{CMB}\phi_{AC} = -4\nabla_\mu(\phi_{B(A}\tau_{C)}^{B\mu}) - 2\nabla^{C'}_{(A}M_{C)C'} \tag{113}$$

and

$$\mu^2_{DE}\psi_{AC} = -4\nabla_\mu(\psi_{B(A}\tau_{C)}^{B\mu}) - 2\nabla^{C'}_{(A}\mathcal{M}_{C)C'}, \tag{114}$$

with the characteristic expressions

$$\begin{aligned} M_{CC'} &= 2(T^A_{C'}\phi_{AC} - \tau^{AB}{}_{CC'}\phi_{AB}), \\ \mathcal{M}_{CC'} &= 2(T^A_{C'}\psi_{AC} - \tau^{AB}{}_{CC'}\psi_{AB}) \end{aligned} \tag{115}$$

and the requirements

$$\mu^2_{CMB} > 0,\ \mu^2_{DE} > 0. \tag{116}$$

We may conclude that any wave functions for dark energy and dark matter can not be taken as null fields as they propagate in spacetime, whereas any one for the cosmic microwave background must not be taken as a null field when it propagates in torsional regions of the Universe. In the ε -formalism, the terms of the wave equations that keep track of the propagation of the cosmic backgrounds can therefore be rearranged as

$$\begin{aligned} &(\Box + \mu^2_{CMB} + \frac{4}{3}\varkappa)\phi_{AB} \\ &\quad -2\Psi_{AB}{}^{CD}\phi_{CD} + 2\xi^C_{(A}\phi_{B)C} = 0 \end{aligned} \tag{117}$$

and

$$\begin{aligned} &(\Box + \mu^2_{eff} + \frac{4}{3}\varkappa)\psi_{AB} - 2\Psi_{AB}{}^{CD}\psi_{CD} + 2\xi^C_{(A}\psi_{B)C} \\ &\quad = 2m^2\tau_{AB}{}^{\mu}A_\mu, \end{aligned} \tag{118}$$

where A_μ is a gauge-invariant affine potential, m amounts to the Proca mass of ψ_{AB} and

$$\mu_{eff} = \sqrt{m^2 + \mu^2_{DE}}. \tag{119}$$

It is worth pointing out that the procedures involved in the derivation of Eq. (80), yield the occurrence of a ξ^2-term at the intermediate steps of the pertinent calculations, which is eliminated from the basis of the theory because of Eq. (76) and the symmetry of ξ_{AB}. Our wave equations for dark matter of both handednesses thus bear linearity in contrast to the ones for gravitons that occur in general relativity, which carry couplings of the type $\Psi_{MN(AB}\Psi_{CD)}{}^{MN}$. It is an observational fact that curvature fields possess darkness, but dark matter nonetheless interacts with spin 1/2 fermions through couplings which look like $\xi^B_A\psi_B$ and $\xi^{A'}_{B'}\chi^{B'}$ while gravitons do not. A noteworthy feature of our framework is related to its prediction that both the cosmic backgrounds interact with dark matter, but the relevant couplings are stringently borne by the wave equations for the backgrounds. Such a peculiar occurrence had taken place in connection with the derivation of the wave equations for the microwave background and gravitons in torsionless spacetime environments. It could take place once again if a torsional description of gravitons were carried out within the framework we have been allowing for, since any ε-formalism wave functions for gravitons (Ψ^{ABCD}, Ψ_{ABCD}) are four-index spin-tensor densities of weight ± 2. Hence, the dark energy background must contribute significantly to the Hubble acceleration when it propagates inside the dark matter cloud. The cosmic distribution of dark matter as coming from Eqs. (68) and (80) is far from bearing uniformity, whence we can say that the couplings $\Psi_{AB}{}^{CD}\phi_{CD}$ and $\xi^C_{(A}\phi_{B)C}$ may supply an explanation of the observed angular anisotropy of the cosmic microwave background in non-conformally flat regions of the Universe. Of course, a similar explanation can also be accomplished from the couplings that occur in the wave equation for ψ_{AB} such that we could expect that the dark energy background bears anisotropy. Then, notwithstanding the result that the attainment of solutions to the wave equations (117) and (118) would likewise lead to theoretical values for the angular anisotropies of the cosmic microwave and dark energy backgrounds, both of these anisotropies could have been predicted beforehand in a qualitative way.

Of the utmost importance as regards the physical characters of the work just presented is, in fact, the achievement of a geometric estimation of the mass of dark matter along with an observational confirmation that the cosmic microwave background should behave as a massive field as it propagates inside dark matter. The work of Sect. 7 describes a world situation which enables one to estimate the contribution to the mass of dark matter that comes from the pattern (69) along with the contribution owing to the derivative (78). It turns out that if the condition (83) is used up in the case of a conformally flat Kopczyński solution, then all wave functions for gravitons should be set equal to zero together with the torsion trace T_μ, and we might settle the world version of Eq. (80) as the dark matter statement

$$\Box f_{\mu\nu} + M^*_{\mu\nu} + F^*_{\mu\nu} = s^*_{\mu\nu}, \tag{120}$$

with the bivector formulae

$$\begin{aligned} M^*_{\mu\nu} &= 2\,\mathrm{Im}(\mu^2_{DM}\Sigma^A_{\mu B'}\Sigma^{BB'}_\nu\xi_{AB}), \\ F^*_{\mu\nu} &= \frac{8}{3}\,\mathrm{Im}(\varkappa\Sigma^A_{\mu B'}\Sigma^{BB'}_\nu\xi_{AB}) \end{aligned} \tag{121}$$

and

$$s^{*}_{\mu\nu} = -4\,\mathrm{Im}[\Sigma^{A}_{\mu B'}\Sigma^{BB'}_{\nu}(\nabla^{C'}_{(A}\hat{s}^{(\mathrm{X})}_{B)C'} + \nabla^{C'}_{(A}\hat{s}^{(\Xi)}_{B)C'})], \qquad (122)$$

whence we could equally well estimate the Klein–Gordon masses μ_{CMB} and μ_{DE} too, in conformity with the Trautman–Kopczyński cosmological models. Having at our disposal an estimated value of μ_{DE} obtained in this way, would permit us to draw a comparison of it with those which come from the negative-pressure interpretation of dark energy as provided in detail by the work of Ref. [31]. This might eventually yield an interesting estimation of the Proca mass of ψ_{AB}. It would be worthwhile, also, to compare the values of acquired masses coming from covariant derivatives of couplings between fields and torsion kernels with the ones that arise from the dynamical description of the very early stages of the evolution of the Universe as formulated by the theory of cosmological pertubations [32,33]. This latter comparison may perhaps make it possible to investigate whether dark matter and dark energy should play some important role in the establishment of a definitive inflationary solution to the cosmological spatial flatness and horizon problems. Thus, one could compare the results that come from the popular scalar-field scenario with those supplied by the dark dynamics of

$$\mathfrak{L}_D = \mathfrak{L}_{EC} + \mathfrak{L}_M + \mathfrak{L}_E, \qquad (123)$$

where $\mathfrak{L}_{EC}$ stands for the Einstein–Cartan Lagrangian density whilst $\mathfrak{L}_M$ is given by Eq. (105). The contribution $\mathfrak{L}_E$ amounts to the Lagrangian density for dark energy, which is expressed as

$$\mathfrak{L}_E = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}E^{\mu}A_{\mu} + \frac{1}{2}m^2 A_{\mu}A^{\mu}, \qquad (124)$$

with E^{μ} being defined in a way similar to M^{μ}, and

$$F_{\mu\nu} \doteqdot 2(\nabla_{[\mu}A_{\nu]} + T_{\mu\nu}{}^{\lambda}A_{\lambda}) \qquad (125)$$

amounting to a bivector that arises from a suitably contracted torsional spin-affinity, as utilized in Ref. [25].

Cosmological descriptions in spacetimes equipped with torsionful affinities, have supplied many very striking physical features. Among these, the most important ones are associated to the occurrence of geometric wave functions in spinor descriptions, which could not emerge within any purely world framework. The explanations concerning the cosmological singularity prevention and gravitational repulsion, which come straightforwardly from the world form of Einstein–Cartan's theory, do not require at all the use of the $\gamma\varepsilon$-formalisms. However, as we said before, it is our belief that either torsional spinor formalism recovers in terms of predictions a bosonic four-feature picture of the Universe which involves the cosmic backgrounds, gravitons and dark matter, that is to say,

$$\{\phi_{AB}, \psi_{AB}, \Psi_{ABCD}, \xi_{AB}\}.$$

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