



# Maxwell extension of $f(R)$ gravity

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**Abstract** Inspired by the Maxwell symmetry generalization of general relativity (Maxwell gravity), we have constructed the Maxwell extension of  $f(R)$  gravity. We found that the semi-simple extension of the Poincare symmetry allows us to introduce geometrically a cosmological constant term in four-dimensional  $f(R)$  gravity. This symmetry also allows the introduction of a non-vanishing torsion to the Maxwell  $f(R)$  theory. It is found that the antisymmetric gauge field  $B^{ab}$  associated with Maxwell extension is considered as a source of the torsion. It is also found that the gravitational equation of motion acquires a new term in the form of an energy–momentum tensor for the background field. The importance of these new equations is briefly discussed.

## 1 Introduction

Although general relativity (GR) is widely accepted as a fundamental theory to describe the gravitational phenomena on an astrophysical scale, it does not explain for the rotational curves of galaxies that have been measured do not fit the predictions of GR with baryonic matter and predict the accelerated expansion of the universe that was observed at the end of the last century [1]. The explanation in the case of rotational curves is to introduce a new type of non-baryonic matter (dark matter) [2,3]. The accelerated expansion of the universe is usually explained by invoking a mysterious substance called dark energy. The simplest candidate for dark energy is the cosmological constant [4]. Needless to say, the cosmological constant problem is one of the major challenges in theoretical physics. Introducing mysterious substances to match experimentally observed values with the theoretical

predictions of GR is one of the approaches to overcome the problem. In this approach, one modifies the matter part of the Einstein field equations. Another approach is to modify the left-hand side (geometric part) of the Einstein field equation, called as a modified gravitational theory, in which the standard Einstein–Hilbert (E–H) action is replaced by an arbitrary function of the Ricci scalar  $R$ . Such a modification first was put forward by Buchdahl in 1970 [5]. This theory is called today  $f(R)$  gravity and became an established field of theoretical gravity and cosmology after the influential work by Starobinsky [6]. The current acceleration of the universe can be explained by  $f(R)$  gravity [7–13]. Viable models of dark energy satisfying the Solar system and cosmological observational data based on  $f(R)$  gravity where  $f(R)$  is finite at  $R = 0$  were first independently constructed in [11–13] and previous models where  $f(R)$  diverges at  $R = 0$  were shown to be not viable in [14]. For more information as well as recent developments and their applications to the physically relevant models of  $f(R)$  theories, see one of the excellent reviews [15–23] and references therein.

There exists another interesting class of modified gravity theory which may easily produce the cosmological constant by gauging the Maxwell algebra, so-called Maxwell-gravity [24]. Starting with the work of Bacry et al. [25,26], the idea of Maxwell symmetry has been systematically studied by Schrader [27]. Such a symmetry describes a charged particle moving in a four-dimensional Minkowski background in the presence of a constant electromagnetic field. The Maxwell algebra is an extension of the Poincare algebra by six additional tensorial abelian symmetry generators that make the four-momenta non-commutative  $[P_a, P_b] = i\lambda Z_{ab}$  [28]. In 2012, the semi-simple tensor extension of the Poincare group was given by Soroka with a new non-abelian tensorial generator [29]. In this study, another alternative approach to the cosmological term problem is proposed. After the work of Azcarraga and Soroka, there has been a renewed interest in

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the cosmological constant problem due to Maxwell symmetry. Various studies on the gauge theory of the (super) Maxwell symmetry algebras carried out and different aspects has been studied in [30–41].

As is well known, the simplest candidate for describing dark energy is the cosmological constant. Then, it becomes interesting to study the ways in which cosmological constant terms can be introduced in the  $f(R)$  theories. In particular, the  $f(R)$  theory of gravity with a cosmological term can be developed in a geometric formulation, where the theory is constructed from the curvatures of the semi-simple extended Poincare algebra. The main purpose of the present paper is to generalize the metric  $f(R)$  theory to a situation with extra degrees coming from Maxwell symmetry extension. In this way, we give an alternative way of introducing the cosmological term to the  $f(R)$  theory.

The paper is organized as follows: In Sect. 2, we briefly recall the construction of  $f(R)$  theory and summarize its basic equations in both metric and Palatini formalism. In Sect. 3, we present the construction of a four-dimensional gravity model containing a cosmological constant term only from the (linear in) curvatures of semi-simple Poincare algebra, the action we reached corresponds to an Einstein–Hilbert like action. In Sect. 4, we propose our simple model of the Maxwell generalized  $f(R)$  gravity action involving a cosmological term and comment on the obtaining of the equations of motion. Finally, in Sect. 5, we discuss the obtained results and possible future developments.

## 2 $f(R)$ theory in terms of differential forms

There are three versions of  $f(R)$  gravity: Metric- $f(R)$  theory (second order formalism) is fully described by the metric field alone, Palatini- $f(R)$  theory (first order formalism) in which metric and connection are handled as independent fields, similarly the “metric-affine  $f(R)$  theory” in which matter Lagrangian also includes connection [15]. In this section, we very briefly summarize the main ingredients of  $f(R)$  gravity in both metric and Palatini approaches. The starting point for the metric- $f(R)$  gravity is the Einstein–Hilbert action,

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \tag{1}$$

in which  $\kappa = 8\pi G/c^4$  with  $G$  being Newton’s gravitational constant and the Ricci scalar  $R$  is constructed from the Riemann curvature tensor. One of the simplest modifications to general relativity is the  $f(R)$  gravity. It generalizes the Lagrangian density of the E–H action. Specifically, it replaces the Ricci scalar  $R$  in action Eq. (1), with some function  $f(R)$

of the scalar curvature:

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R). \tag{2}$$

The source-free vacuum field equations that are obtained taking the variations of action with respect to the metric  $g_{\mu\nu}$  are

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + (g_{\mu\nu} \Delta - \nabla_\mu \nabla_\nu) f' = 0, \tag{3}$$

where  $\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu$  is the d’Alembertian operator. Note that the cosmological constant term does not appear in this field equation. These equations can be re-arranged in the Einstein-like form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}^{eff}, \tag{4}$$

where

$$T_{\mu\nu}^{eff} = \frac{1}{f'} \left\{ \nabla_\mu \nabla_\nu f' - g_{\mu\nu} \Delta f' + \frac{1}{2} g_{\mu\nu} (f - f' R) \right\}, \tag{5}$$

is an effective energy–momentum tensor which can be interpreted as an extra gravitational energy–momentum tensor due to higher order curvature effects. Including the function  $f(R)$  gives extra freedom in defining the behavior of gravity. The detailed structure of  $f(R)$  gravity theories arising from the action Eq. (2), in  $4D$  space-time has been discussed in ref. [15, 16, 18–20, 22].

In the context of the first order (Palatini) formalism, the entities basis 1-forms  $e^a$  and the connection 1-forms  $\omega^{ab}$  are independent from each other. That is, the connection is not previously fixed to be given by Christoffel’s symbols, but must be determined dynamically. Consequently, the general form of the field equations does not forbid the presence of torsion [20]. In the language of exterior differential forms, E–H action takes the following form

$$S_{EH} = \frac{1}{2\kappa} \int \mathcal{L}_{EH}, \tag{6}$$

with the gravitational Lagrangian 4-form is given by

$$\mathcal{L}_{EH} = \frac{1}{2} \varepsilon_{abcd} R^{ab} \wedge e^c \wedge e^d = R^{ab} \wedge *e_{ab} = R^* 1, \tag{7}$$

where we denote the exterior products of basis 1-forms  $e^a$  as  $e^{ab} = e^a \wedge e^b$ ,  $e^{abc} = e^a \wedge e^b \wedge e^c$  etc.,  $R^{ab}$  is the Ricci 2-form,  $*$  denotes the Hodge dual operator acting on the basis forms and  $*1 = \frac{1}{4!} \varepsilon_{abcd} e^{abcd}$  is the invariant volume element.

We consider the modified action

$$S_{f(R)} = \frac{1}{2\kappa} \int \mathcal{L}_{f(R)}, \tag{8}$$

and take the following Lagrangian 4-form in differential form language:

$$\mathcal{L}_{f(R)} = f(R)^* 1. \tag{9}$$

The full variation of the 4-form  $\mathcal{L}_{f(R)}$  can be calculated as follows. Using the product rule for the variation gives

$$\begin{aligned} \delta \mathcal{L}_{f(R)} &= \delta f(R)^* 1 + f(R) (\delta^* 1) \\ &= f' \delta(R^* 1) + (f - f' R) \delta^* 1, \end{aligned} \tag{10}$$

where we have been used  $\delta f(R) = f' \delta R$  with  $f' = \frac{df}{dR}$ . Consequently, the variation Eq. (10) takes the following form:

$$\delta \mathcal{L}_{f(R)} = f' \delta \mathcal{L}_{EH} + (f - f' R) \delta^* 1. \tag{11}$$

Inserting the total variation of E–H Lagrangian Eq. (7) into Eq. (11), one obtains

$$\begin{aligned} \delta \mathcal{L}_{f(R)} &= f' \left[ D\delta\omega^{ab} \wedge^* e_{ab} + \delta e^c \wedge (R^{ab} \wedge^* e_{abc}) \right] \\ &\quad + (f - f' R) \delta e^c \wedge^* e_c \\ &= f' D\delta\omega^{ab} \wedge^* e_{ab} + \delta e^c \wedge \\ &\quad \times \left[ f' R^{ab} \wedge^* e_{abc} + (f - f' R)^* e_c \right] \\ &= \delta\omega^{ab} \wedge (df' \wedge^* e_{ab}) + \delta e^c \wedge \\ &\quad \times \left[ f' R^{ab} \wedge^* e_{abc} + (f - f' R)^* e_c \right]. \end{aligned} \tag{12}$$

In the last line we have used the fact that torsion is zero, i.e.,  $D(*e_{ab}) = T^c \wedge^* e_{abc} = 0$  and discharged the boundary term and took  $Df' = df'$ . Since the connection is a non-propagating field, one can solve variation of the connection  $\delta\omega^{ab}$  and express it in terms of the frame-like field  $\delta e^a$ :

$$\delta\omega^{ab} = \frac{1}{2} i^a i^b (D\delta e^c \wedge e_c) - i^a (D\delta e^b) + i^b (D\delta e^a). \tag{13}$$

Inserting this back in Eq. (12), one computes the vierbein equation of motion in second-order formalism as

$$f' R^{ab} \wedge^* e_{abc} + (f - f' R)^* e_c + 2D \wedge^* (df' \wedge e_c) = 0. \tag{14}$$

This is the field equation for “metric- $f(R)$  theory”. In terms of Einstein’s tensor Eq. (14) takes the following form

$$f' G_c^a * e_a + \frac{1}{2} (f' R - f)^* e_c - D^* (df' \wedge e_c) = 0, \tag{15}$$

where Einstein tensor in flat space-time coordinates is given by  $G_c^a = R_c^a - \frac{1}{2} \delta_c^a R$ . It is clearly seen here that the equation gives the well-known Einstein field equations at the limit of  $f(R) \rightarrow R$ . In terms of curved space-time coordinates Eq. (15) can be expressed as Eq. (3). A detailed calculation on  $f(R)$  theory using differential forms can be found in [42, 43].

### 3 Maxwell extension of Einstein gravity

In [24], the authors presented the construction of a local four-dimensional gauge theory based on the Maxwell algebra and applied it to generalize Einstein’s gravity (Maxwell Gravity). Their construction of the action involves bilinear invariant curvature 2-forms and their action is not invariant under local Maxwell transformations but only under local Lorentz transformations. The action respecting the local Maxwell symmetry invariance is given in [44]. They also considered bilinear invariant curvature 2-forms associated with  $AdSL_4$ -valued 1-form gauge connection, and then constructed a four-dimensional action that generalizes the Einstein gravity. They showed that Maxwell Gravity can be obtained from  $AdSL_4$ -gravity by means of the Inönü–Wigner contraction method [44, 45]. Both of these works make use of the bilinear invariant combination of curvature 2-forms in their actions. On the contrary, we prefer to make use of the linear combination of the gauge curvature 2-forms that will be useful in the next section on the construction of the Maxwell generalization of  $f(R)$  theory.

We begin by giving an overview of the semi-simple extension of the Poincare group [29]. The algebra of this group was also re-derived in [30] through the deformation of Maxwell algebra. Nowadays, it is also known as the AdS-Lorentz algebras [46]. The  $AdSL_4$  algebra, on the other hand, is constructed from the  $AdS$  algebra by means of the S-expansion method introduced in [47, 48]. The commutators of the algebra read

$$\begin{aligned} [M_{ab}, M_{cd}] &= i (\eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}), \\ [M_{ab}, Z_{cd}] &= i (\eta_{ad} Z_{bc} + \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}), \\ [Z_{ab}, Z_{cd}] &= i \mu (\eta_{ad} Z_{bc} + \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}), \\ [P_a, P_b] &= i \lambda Z_{ab}, \\ [M_{ab}, P_c] &= i (\eta_{bc} P_a - \eta_{ac} P_b), \\ [Z_{ab}, P_c] &= i \mu (\eta_{bc} P_a - \eta_{ac} P_b), \end{aligned} \tag{16}$$

where the generators  $X_A = \{P_a, M_{ab}, Z_{ab}\}$  correspond to the translations, the Lorentz transformations and the Maxwell symmetry. Here, for dimensional reasons the constant  $\lambda$  is related to the cosmological constant  $\Lambda$  and is given by  $\Lambda = \mu \lambda$  and the tangent space metric defined as  $\eta_{ab} = \text{diag}(+, -, -, -)$ . On the contrary to Maxwell algebra, here  $Z_{ab}$  are tensorial but non-abelian generators. An interesting

feature of the semi-simple Poincare algebra is that it reproduces the Maxwell algebra through the Inönü–Wigner contraction method.

In the following, we work along the lines of [24] by using differential forms. To construct an action based on the semi-simple Poincare algebra, we start from the following 1-form connection  $A(x) = A^A X_A$ :

$$A(x) = e^a P_a - \frac{1}{2} \omega^{ab} M_{ab} + \frac{1}{2} B^{ab} Z_{ab}, \tag{17}$$

where  $A^A(x) = \{e^a, \omega^{ab}, B^{ab}\}$  are the gauge fields corresponding to the 16 generators of symmetry transformations.

The curvature 2-form  $F = dA + \frac{i}{2} [A, A]$  associated with the 1-form connection (17) reads

$$F(x) = F^a P_a - \frac{1}{2} R^{ab} M_{ab} + \frac{1}{2} F^{ab} Z_{ab}, \tag{18}$$

where  $F^a$ ,  $R^{ab}$  and  $F^{ab}$  are generalized (Maxwell) torsion, Ricci and Maxwell curvature 2-forms, respectively. Explicit expressions for the curvature 2-forms are given by

$$\begin{aligned} F^a &= de^a + \omega^a_c \wedge e^c - \mu B^a_c \wedge e^c \\ &= T^a - \mu B^a_c \wedge e^c \\ &= \tilde{D}e^a, \end{aligned} \tag{19}$$

$$R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} = D\omega^{ab}, \tag{20}$$

$$\begin{aligned} F^{ab} &= dB^{ab} + \omega^{[a}_c \wedge B^{c|b]} - \mu B^a_c \wedge B^{cb} - \lambda e^a \wedge e^b \\ &= DB^{ab} - \mu B^a_c \wedge B^{cb} - \lambda e^a \wedge e^b \\ &= \tilde{D}B^{ab} - \lambda e^a \wedge e^b, \end{aligned} \tag{21}$$

where  $D = d + \omega$  is the Lorentz exterior covariant derivative and  $\tilde{D} = d + \tilde{\omega}$  is the covariant exterior derivative with respect to the shifted connection  $\tilde{\omega}^{ab} = \omega^{ab} - \mu B^{ab}$ .

Within the scope of the study, we will make use of the shifted curvature obtained by taking the difference between Ricci and Maxwell curvature 2-forms:

$$J^{ab} = \tilde{R}^{ab}(\omega) - \mu F^{ab} \equiv \tilde{R}^{ab}(\tilde{\omega}) + \mu \lambda e^a \wedge e^b \tag{22}$$

where  $\tilde{R}^{ab}(\tilde{\omega}) = \tilde{D}\tilde{\omega}^{ab}$  is the Ricci 2-form for the shifted connection 1-form  $\tilde{\omega}^{ab} = \omega^{ab} - \mu B^{ab}$ .

Interestingly, the shifted connection  $\tilde{\omega}^{ab}$  can be interpreted as an extension of the Riemannian connection  $\omega^{ab}$  to a non-Riemannian one with torsion. In this sense, our result may also be considered as the specific extension to a non-Riemannian framework determined by the structure of the semi-simple Poincare algebra. Indeed, in this context, the antisymmetry  $B^{ab} = -B^{ba}$  implies that we are dealing with an Einstein–Cartan geometry with non-metricity tensor equal to zero because there is no symmetric part of the shifted connection [24].

In close analogy with the Einstein–Hilbert action, we start from the following action of pure gravity:

$$S_{EHJ} = \frac{1}{2\kappa} \int \mathcal{L}_{EHJ} \tag{23}$$

If we promote  $R^{ab} \rightarrow J^{ab}$ , the Lagrangian 4-form in differential form language will be given as

$$\mathcal{L}_{EHJ} = \frac{1}{2} \varepsilon_{abcd} J^{ab} \wedge e^c \wedge e^d = J^{ab} \wedge *e_{ab} = J^*1. \tag{24}$$

Clearly, the Lagrangian contains the Einstein–Hilbert, the cosmological constant and a gravitational Maxwell term for the shifted connection.

At this point, there are two ways to proceed. First, one can either vary the Lagrangian with respect to 1-form basis  $e^a$  and the connection 1-forms  $\omega^{ab}$  independently (Palatini’s method) and then solve for the Maxwell torsion. Second, one can set the Maxwell torsion equal to zero at the beginning and consider the zero Maxwell torsion constrained variations of the Lagrangian. For the Einstein–Hilbert-like Lagrangians (linear in curvatures) without any coupling to matter, both of these cases imply the same set of field equations, namely the source-free Einstein field equations with the cosmological term [21]. We derive the field equations by a first-order formalism (Palatini’s method). The dynamics of Maxwell gravity in terms of vierbein  $e^a$  and the shifted connection  $\tilde{\omega}^{ab}$  is derived by varying the Lagrangian:

$$\begin{aligned} \delta \mathcal{L}_{EHJ} &= \delta J^{ab} \wedge *e_{ab} + J^{ab} \wedge \delta *e_{ab} \\ &= \left( \tilde{D}\delta\tilde{\omega}^{ab} + \mu\lambda\delta e^{[a} \wedge e^{b]} \right) \wedge *e_{ab} + J^{ab} \wedge \delta e^c \wedge *e_{abc} \\ &= \tilde{D}\delta\tilde{\omega}^{ab} \wedge *e_{ab} + \delta e^c \wedge \left[ J^{ab} \wedge *e_{abc} + 6\mu\lambda *e_c \right]. \end{aligned} \tag{25}$$

This full variation will be used for the Maxwell extension of  $f(R)$  theory in the next section. One can write the Eq. (25) up to a total exterior derivative term and set the variation to zero:

$$\begin{aligned} \delta \mathcal{L}_{EHJ} &= \delta\tilde{\omega}^{ab} \wedge F^c \wedge *e_{abc} \\ &\quad + \delta e^c \wedge \left[ J^{ab} \wedge *e_{abc} + 6\mu\lambda *e_c \right] = 0, \end{aligned} \tag{26}$$

gives two equations of motion. One is the torsion equation,

$$F^c = 0 \Rightarrow T^c = B^c_d \wedge e^d. \tag{27}$$

It is worth stressing that torsion comes quite naturally since it is introduced by gauge fields  $B^{ab}$ . Although the torsion associated with the gauge field  $\omega^{ab}$  need not be zero, the torsion of the re-defined fields  $\tilde{\omega}^{ab}$  should be zero. In the general case, from Eq. (27) it is seen that  $B$ -field is the source of torsion, and hence the connection is no longer on-shell torsion-free even in the source-free region.

The other is the generalized Einstein’s equation:

$$J^{ab} \wedge *e_{abc} + 6\mu\lambda *e_c = 0 \Rightarrow R^{ab} \wedge *e_{abc} + 6\mu\lambda *e_c = \mu F^{ab} \wedge *e_{abc}. \tag{28}$$

and it can be decomposed as

$$R^{ab} \wedge *e_{abc} + 6\mu\lambda *e_c = \mu F^{ab} \wedge *e_{abc}. \tag{29}$$

To recast it in a more familiar form one can use the relations:  $e_a^\mu e_b^\nu R^{ab} = \frac{1}{2} R^{\mu\nu}_{\rho\sigma} dx^\rho \wedge dx^\sigma$ ,  $e_a^\mu e_b^\nu F^{ab} = \frac{1}{2} F^{\mu\nu}_{\rho\sigma} dx^\rho \wedge dx^\sigma$

$$R^v_\sigma - \frac{1}{2} R \delta^v_\sigma - 6\mu\lambda \delta^v_\sigma = \mu T^v_\sigma(B), \tag{30}$$

where

$$T^v_\sigma(B) = e_a^\mu e_b^\nu \left( D_{[\mu} B_{\sigma]}^{ab} - \mu B_{[\mu c}^a \wedge B_{\sigma]}^{cb} \right) - \frac{1}{2} \delta^v_\sigma e_a^\gamma e_b^\kappa \left( D_{[\gamma} B_{\kappa]}^{ab} - \mu B_{[\gamma c}^a \wedge B_{\kappa]}^{cb} \right), \tag{31}$$

is the energy–momentum tensor for the background field and the square brackets around the indices imply antisymmetrization. It is important to note that the cosmological constant term is naturally contained here without the need to introduce it by hand.

#### 4 Maxwell extension of $f(R)$ gravity

We propose a new version of  $f(R)$  theory by combining two existing ideas, i.e., by invoking the gauge theory of gravity as the proper description of gravitational effects and by assuming semi-simple extended Poincare symmetry as the underlying gauge group of the universe. Now, to construct a Maxwell generalized  $f(R)$  action from the curvature of the semi-simple Poincare algebra, we consider the shifted curvature 2-form  $J^{ab}$  in Eq. (22). If we promote  $f(R) \rightarrow f(J)$ , the Lagrangian 4-form for the Maxwell generalized  $f(R)$  theory can be written in differential form language as

$$\mathcal{L}_{f(J)} = f(J) *1. \tag{32}$$

Following the same procedure presented in Sect. 2, the total variation of the 4-form  $\mathcal{L}_{f(J)}$  can be calculated as follows. Using again the product rule, the variation takes the following form

$$\delta \mathcal{L}_{f(J)} = f' \delta \mathcal{L}_{EHJ} + (f - f' J) \delta *1, \tag{33}$$

where this time  $\delta f(J) = f' \delta J$  with  $f' = \frac{df}{dJ}$ .

Inserting the variation of  $\delta \mathcal{L}_{EHJ}$  from Eq. (25) into Eq. (33), one gets

$$\delta \mathcal{L}_{f(J)} = f' \left[ \tilde{D} \delta \tilde{\omega}^{ab} \wedge *e_{ab} + \delta e^c \wedge (J^{ab} \wedge *e_{abc} + 6\mu\lambda *e_c) \right] + (f - f' J) \delta e^c \wedge *e_c. \tag{34}$$

Rearranging gives:

$$\begin{aligned} \delta \mathcal{L}_{f(J)} &= f' \tilde{D} \delta \tilde{\omega}^{ab} \wedge *e_{ab} + \delta e^c \wedge \\ &\times \left[ f' J^{ab} \wedge *e_{abc} + (f - f' J + 6\mu\lambda f') *e_c \right] \\ &= \delta \tilde{\omega}^{ab} \wedge \tilde{D} (f' *e_{ab}) + \delta e^c \wedge \\ &\times \left[ f' J^{ab} \wedge *e_{abc} + (f - f' J + 6\mu\lambda f') *e_c \right] \\ &= \delta \tilde{\omega}^{ab} \wedge (df' \wedge *e_{ab}) + \delta e^c \wedge \\ &\times \left[ f' J^{ab} \wedge *e_{abc} + (f - f' J + 6\mu\lambda f') *e_c \right]. \end{aligned} \tag{35}$$

In the last line we have used the fact that Maxwell-torsion is zero, i.e.,  $\tilde{D} (*e_{ab}) = F^c \wedge *e_{abc} = 0$  and discharged the boundary term and took  $\tilde{D} f' = df'$ . It is helpful to remark here that the  $f(R)$ -theories of gravity with torsion is also considered in [49] where torsion is geometrically inserted in f(R) gravity and in [50] where  $f(R)$  gravity with torsion is formulated in the  $J$ -bundle approach. In [49], the authors start with the metric-affine formulation in which the affine connection is assumed to be metric compatible. As is well known for a given metric tensor  $g_{\mu\nu}$ , every metric connection is expressible as the sum of Levi-Civita connection and contorsion tensor, i.e.,  $\Gamma^\mu_{\alpha\beta} = \overset{\circ}{\Gamma}^\mu_{\alpha\beta} + K^\mu_{\alpha\beta}$ . The antisymmetry property of the contorsion tensor  $K^\mu_{\alpha\beta} = -K^\mu_{\beta\alpha}$  ensures the metric compatibility of the connection  $\Gamma^\mu_{\alpha\beta}$ . In this way, one can identify the actual degrees of freedom of the theory with the independent components of the metric  $g_{\mu\nu}$  and the contorsion tensor  $K^\mu_{\alpha\beta}$ . In [50], the authors used the  $J$ -bundle approach to construct the  $f(R)$ -theories of gravity with torsion. In the  $J$ -bundles framework, one starts from several Lagrangian densities which depend on the fields only through their antisymmetric combinations. This is the case of the Einstein–Hilbert like Lagrangian which, in the tetrad-affine formulation, depends on the antisymmetric derivatives of the spin-connection through the curvature. In view of this fact, they defined a suitable quotient space of the first jet bundle, made equivalent two sections which have a first order contact with respect to the exterior differentiation, instead of the whole set of derivatives. The resulting fiber coordinates of the so defined new spaces are exactly the antisymmetric combinations appearing in the Lagrangian densities. For general relativity ( $GR$ ) as well as  $f(R)$  gravity, they also showed that the fiber coordinates of the quotient space can

be identified with the components of the torsion and curvature tensors. For further information, we refer the reader to [49,50] and the references therein.

The very fact that both approaches, being formally different, give essentially equivalent results and in both approaches a torsion arises from the non-linearity of the gravitational Lagrangian function even in the absence of spin and its existence does not affect the metric field equations. Our approach, on the other hand, starts from the antisymmetric shifted connection  $\tilde{\omega}^{ab} = \omega^{ab} - \mu B^{ab}$  in the tetrad-affine formulation, and therefore the antisymmetric background field  $B^{ab}$  is the source of the torsion.

After these remarks were given, we proceed as in the previous section. The connection is again a non-propagating field, then we can define a relation between vierbein  $\delta e^a$  and shifted connection  $\delta \tilde{\omega}^{ab}$  variations for the zero-Maxwell torsion case as follows:

$$\delta \tilde{\omega}^{ab} = \frac{1}{2} i^a i^b (\tilde{D} \delta e^c \wedge e_c) - i^a (\tilde{D} \delta e^b) + i^b (\tilde{D} \delta e^a). \tag{36}$$

Inserting this in Eq. (35), one can obtain the field equations of motion in second-order formalism as

$$f' J^{ab} \wedge * e_{abc} + (f - f' J + 6\mu \lambda f') * e_c + 2\tilde{D} \wedge *(df' \wedge e_c) = 0. \tag{37}$$

The Maxwell modified  $f(R)$ -metric field equation for the presence of the background field  $B^{ab}$  that follows from Eq. (37) takes the form

$$f' (G_c^a - 3\mu \lambda \delta_c^a) * e_a - \frac{1}{2} (f - f' R) * e_c - \tilde{D} \wedge *(df' \wedge e_c) = \mu f' F_c^a \wedge * e_a, \tag{38}$$

where  $G_c^a = R_c^a - \frac{1}{2} \delta_c^a R$  is the Einstein tensor in flat space-time coordinates. Maxwell extended  $f(R)$  gravity with cosmological constant are naturally contained here without the need to introduce a cosmological constant by hand. To recast it in a more familiar form one can pass from the tangent indices to world indices by using the relations:  $e_a^\mu e_b^\nu R^{ab} = \frac{1}{2} R^{\mu\nu}_{\rho\sigma} dx^\rho \wedge dx^\sigma$ ,  $e_a^\mu e_b^\nu F^{ab} = \frac{1}{2} F^{\mu\nu}_{\rho\sigma} dx^\rho \wedge dx^\sigma$ , one gets the following field equation,

$$R^\nu_\sigma - \frac{1}{2} R \delta^\nu_\sigma - 6\mu \lambda \delta^\nu_\sigma = T^\nu_\sigma(eff) + \mu T^\nu_\sigma(B), \tag{39}$$

where

$$T^\nu_\sigma(eff) = \frac{1}{f'} \left\{ \nabla^\nu \nabla_\sigma f' - \delta^\nu_\sigma \Delta f' + \frac{1}{2} \delta^\nu_\sigma (f - f' R) \right\}, \tag{40}$$

and

$$T^\nu_\sigma(B) = e_a^\mu e_b^\nu \left( D_{[\mu} B_{\sigma]}^{ab} - \mu B_{[\mu c}^a \wedge B_{\sigma]}^{cb} \right) - \frac{1}{2} \delta^\nu_\sigma e_a^\gamma e_b^\kappa \left( D_{[\gamma} B_{\kappa]}^{ab} - \mu B_{[\gamma c}^a \wedge B_{\kappa]}^{cb} \right), \tag{41}$$

here  $T^\nu_\sigma(eff)$  is an extra gravitational energy–momentum tensor due to higher order curvature effects and  $T^\mu_\rho(B)$  is energy–momentum tensor for the background gauge field  $B^{ab}(x)$  due to Maxwell extension. As is expected, in the limit  $\mu \rightarrow 0$ , Eqs. (39), (40) and (41) turn to the well known equations of motion for metric- $f(R)$  gravity. Result obtained in (39) is actually  $AdSL_4$  extension of  $f(R)$  gravity. If one performs a Inönü–Wigner contraction of  $AdSL_4$  down to Maxwell group by re-scaling the generators  $P_a \rightarrow \xi P_a$ ,  $Z_{ab} \rightarrow \xi^2 Z_{ab}$  as well as the gauge fields  $e^a \rightarrow \xi^{-1} e^a$ ,  $B^{ab} \rightarrow \xi^{-2} B^{ab}$  and then taking the limit  $\xi \rightarrow \infty$ , we obtain the equations of motion for the Maxwell extension of  $f(R)$  gravity with the cosmological term. In doing so, the derivative in Eq. (40) with respect to  $J$  becomes derivative with respect to  $R$  and the last energy–momentum tensor changes to

$$T^\nu_\sigma(B) = e_a^\mu e_b^\nu D_{[\mu} B_{\sigma]}^{ab} - \frac{1}{2} \delta^\nu_\sigma e_a^\gamma e_b^\kappa D_{[\gamma} B_{\kappa]}^{ab}. \tag{42}$$

### 5 Conclusion

In this paper, we have presented the Maxwell extension of  $f(R)$  theory of gravity as the gravitational part of the action is a function of the shifted curvature scalar  $J$ , i.e.,  $f(J)$ . This could be a linear function, or non-linear. We have considered the curvature 2-forms associated with the semi-simple extension of Poincare algebra ( $AdSL_4$ )-valued one-form gauge connection, and then we constructed a four-dimensional action that generalizes the  $f(R)$  gravity. It is shown that the Maxwell extension of  $f(R)$  gravity can be obtained from  $AdSL_4$ - $f(R)$  gravity making use of the Inönü–Wigner contraction method. It is found that the Maxwell extension modifies the results of the metric- $f(R)$  gravity not only by introducing the cosmological constant term but also the new gauge fields  $B_\mu^{ab}(x)$  terms. These could play the role of inflaton vector fields which drive accelerated expansion from Maxwell  $f(R)$  gravity [24,51]. It would be interesting to apply our formalism to Gauss-Bonnet type generalization of  $f(R)$  gravity respecting the local Maxwell symmetry. In such a theory, the shifted curvature, Maxwell torsion and the vector gauge fields may give rise to a modified  $f(R)$  gravity theory which is capable, in principle, to address the problem of the dark side of the universe in a very general geometric scheme.

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