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Closed timelike curves and energy conditions in regular spacetimes

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Abstract In this article, we explore the relationship between the existence of closed timelike curves and energy conditions that occur in the Kerr–Newman spacetime. To quantify the dependence, we define a correlation index between energy conditions and closed timelike curves. We investigate the validity of Hawking's chronology protection conjecture for the closed time like curves that occur in the noncommutative version of the Kerr–Newman spacetime that has the right ingredients to examine the conjecture . We report the results outlining the possible role played by violations of energy conditions in eliminating the closed timelike curves.

1 Introduction

The formulation of the fundamental laws of physics intrinsically assumes the preservation of causality (the causal ordering of events in spacetime). A Closed Timelike Curve (CTC) is a closed curve whose tangent is everywhere timelike. Therefore, the existence of CTCs in spacetime implies the violation of cause-effect relations that are sacrosanct to predictability in physics [1]. From a classical general relativity perspective, many spacetimes have the pathology of the CTCs; these are bona-fide solutions to Einstein's field equations e.g Gött, Gödel, Kerr and Kerr–Newman solutions [2–4] (see also [5] and references therein). To deal with the pathology, Hawking came up with the Chronology Protection Conjecture, which states: *The laws of physics do not allow the appearance of closed timelike curves*.

The (curvature) singularity of spacetime is dangerous pathology that the theory of relativity is plagued with. There exists an extensive study on reconciling singularities with predictability in physics. The cosmic censorship conjecture by Penrose [6] states that the singularities are always hidden behind a horizon, therefore protecting large portions of spacetime from the unpredictability that arises due to causal communication with the singularity. Though there is no formal proof yet of the conjecture, there are many models proposed with the intention of proving/disproving the same. A review of the status of the problem can be found in [7–12]. The models that present the possibilities of naked singularities (where the causal rays from the singularity can reach non-singular points in spacetime) are not generic enough to conclude against the conjecture [9]. It is well accepted that once we have a quantum theory of gravity, the singularities are expected to be smeared away into a high curvature region, suggesting that the singularities are the artefacts of classical general relativity [13–16].

The pathologies related to the existence of closed timelike curves (CTCs) are different in nature from that of curvature singularities, in the sense that the singularity causes the breakdown of laws of physics. In contrast, the existence of CTCs signals the breakdown of predictability. Of particular interest are the CTCs present in the Kerr–Newman spacetime. When horizons exist, the CTCs are present in the interior of the inner horizon in the Kerr–Newman spacetime. They are also sometimes present in the Kerr–Newman spacetime that has a globally naked singularity.

As mentioned before, the existence of CTCs manifests in the violation of causality relations. From the point of view of semi-classical quantum gravity theory, this reflects the fact that the two-point correlation function of quantum field does not take Hadamard form, leading to non-renormalizable stress tensor [17]. Some interesting aspects arise in the spacetimes endowed with CTCs that appear when one studies quantized fields on such a background, and have been discussed extensively in [17–23]. Although substantial efforts

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have been made (e.g. see [24,25]) towards understanding the issues related to CTCs, it still lacks the rigorous status as the singularity theorems associated with curvature singularities. In short, the existence, formation and avoidance of CTCs in spacetime are poorly understood.

A major step towards understanding the nature and existence of CTCs was taken by Hawking. In a seminal paper, [26] he stated that (from now onwards, we will refer to this as 'Hawking's statement'): *In spacetimes exhibiting closed timelike curves, either there exist singularities* (as in Kerr–Newman black holes)*, or energy–momentum tensor shows pathological behaviour at infinity* (as in the Gödel and Gött spacetimes)*, or the weak energy condition is violated*. This statement is a criterion for the existence of CTCs in a given spacetime. There are some interesting observations that one could make about Hawking's statement. Consider a spacetime devoid of curvature singularities and energy–momentum tensor that is regular in the asymptotic regions. Then the CTCs can exist if the weak energy condition gets violated. Now let us divide such a spacetime into two sets – a first that contains a patch (or patches) wherein CTCs exist and a second set that contains a patch (or patches) where CTCs do not exist. According to Hawking's statement, the weak energy condition is violated either in the first set or in the second set or both. In this sense the Hawking statement is global in its nature, i.e. the statement does not include enough information to state the relative location in the spacetime where CTCs exist and WEC is violated. Turning this observation into questions, one could ask - is there a correlation between the causality violation and violation of any of the energy conditions? Is the region containing CTCs a subset of the region where WEC is violated? Can the WEC violating region and CTCs region be mutually exclusive? Are energy condition violations needed if the final aim is to eliminate CTCs present in the solutions for classical general relativity? Can we preserve energy conditions and eliminate CTCs? This article is aimed at an open investigation to address some of the above questions.

One factor that is detrimental to addressing some of the above questions is the lack of an adequate number of models that present the pathology of CTCs. Much work has been carried out for regular black hole geometries (e.g. the models in which the singularities are smeared out) in [27–29] where the elimination of CTCs is usually not addressed.

Our recent work [30] is the first attempt to address the issue of eliminating CTCs using variants of gravity. In the above work [30] we analyzed the noncommutative version of the Kerr–Newman spacetime presented in [31]. It is shown that there exists a region in the parameter space of (M, Q, a, Θ) of the Kerr–Newman spacetime for which the pathology due to the presence of CTCs exists, and a region in the parameter space in which the CTCs are eliminated. We showed that for a given set of parameters (M, Q, a), we could choose the noncommutative parameter $\Theta = \Theta_0$ such that CTCs are eliminated. The paper also describes the analysis of the Kerr–Newman like solutions in $f(R)$ gravity. It discusses the possibility of eliminating the CTCs using the parameters present in the model. From this study, it is clear that variants of Kerr–Newman spacetimes present us with an excellent laboratory to test out the dependence of CTCs on the various energy conditions and may provide answers to the above questions. A priori, it can be expected that the prevention or elimination of CTCs might be possible, provided we compromise one (or more) energy conditions.

In this work, we study the correlation between energy conditions and the occurrence of CTCs. As mentioned before, Hawking's statement connects the violation of the energy conditions and the presence of CTCs at the global level. We would like to know how the violation or preservation of weak energy conditions in a given sufficiently small region of spacetime could affect the presence or absence of CTCs, thus providing the local analogue of Hawking's statement. We explore the possibility of refining Hawking's statement in the following way. Suppose we consider a point in spacetime through which a CTC passes. Some of the energy conditions may or may not be violated at the location of CTCs. It is, therefore, interesting to study the interplay between the energy conditions and CTCs from the local view.

The understanding of this correlation can be beneficial in many contexts. Firstly, when one considers models for modified gravity, the selection of one model over the other is usually based on consistency with cosmological observations. One more parameter for this selection criteria could be the CTC elimination. Suppose one constructs equivalent models of certain solutions, e.g. the Kerr–Newman spacetime, using phenomenological quantum gravity models. In that case, the status of causality preservation due to the elimination of CTCs could be yet another independent criterion for choosing one model over the other.

The noncommutative counterpart of the Kerr–Newman solutions is regular, and the curvature singularity is smeared out. So this presents a model with CTCs but no curvature singularity and therefore works as an excellent theoretical testing ground to examine the dependence of CTCs on the various energy conditions and may provide answers to some of the above questions.

The paper is organized as follows. In Sect. 2, we discuss in detail the NCKN spacetime, the diagonalization of its energy momentum tensor and its classification according to Hawking and Ellis. We then compute the energy conditions and discuss their general behavior. Then, we investigate their relationship with CTCs for the NCKN spaceitme, examining the local validity of Hawking's statement. In Sect. 3, we propose a general formalism for quantifying the association of energy conditions with the existence of CTCs by defining a *correlation index*. In Sect. 3.1, we apply this general procedure in to the NCKN spacetime and discuss the possible correlation between the various energy conditions and CTCs. In Sect. 4 we make concluding remarks.

Throughout, we use geometric units in which $G = c = 1$. We also set the ADM mass of the spacetime to be 1, i.e. $M = 1$. For algebraic computations, we use *Mathematica* [32] and RG tensor (RGTC) package. Although we deal with solutions that are used to describe the exterior geometries of black holes, we will refer to them as spacetimes as, for some of the parameters, the spacetime may or may not contain horizons and black hole regions.

2 CTCs and energy conditions in NCKN spacetime

2.1 The NCKN spacetime

We consider charged rotating black hole spacetimes inspired by noncommutative geometry. The motivation behind introducing the noncommutativity in the usual black holes setup is to cure point-like curvature singularities (e.g. the Dirac-delta like the singularity of Schwarzschild BH, the ring singularity of the Kerr BH) by effectively replacing it by appropriate mass distribution [33–35]. Within the framework of coordinates coherent state approach together with expected first order quantum gravitational correction to classical general relativity, it can be shown that the Dirac-delta-like singularities will be replaced by Gaussian distributions [31]. For the point particle at the origin, the noncommutative nature of coordinates reflects into a modification to usual delta function distribution $\delta(\mathbf{x})$ as

$$\rho_0(\mathbf{x}) = \frac{1}{2\pi\Theta} e^{-\frac{\mathbf{x}^2}{2\Theta}} \tag{1}$$

The width of the Gaussian distribution is now characterized by the noncommutative (NC) parameter Θ. It has been shown that [36] the effective corrections to the Einstein's-field equations due to the above replacement can be modelled by replacing the point-like sources by a suitable Gaussian distribution while keeping the differential operators unchanged. The noncommutative solution for the Kerr–Newman (KN) spacetime can be written as [31]

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\rho^2} dt^2 - 2a \\ &\times \sin^2\theta\left(1 - \frac{\Delta - a^2\sin^2\theta}{\rho^2}\right) dt d\phi \\ &+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta d\phi^2 \end{aligned} \tag{2}$$

with

$$\Delta = r^2 - 2m(r)r + a^2 + q^2(r)$$

$$\begin{aligned} \Sigma^2 &= (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \\ \rho^2 &= r^2 + a^2\cos^2\theta \end{aligned} \tag{3}$$

where the mass and charge functions take the form

$$m(r) = M\frac{\gamma(3/2, r^2/(4\Theta))}{\Gamma(3/2)} \tag{4}$$

$$\begin{aligned} q^2(r) = \frac{Q^2}{\pi}\Big[&\gamma^2(1/2, r^2/(4\Theta)) - \frac{r}{\sqrt{2\Theta}}\gamma(1/2, r^2/(2\Theta)) \\ &+ r\sqrt{\frac{2}{\Theta}}\gamma(3/2, r^2/(4\Theta))\Big] \end{aligned} \tag{5}$$

Here $\gamma(x, y)$ is lower incomplete gamma function defined as

$$\gamma(x, y) = \int_0^y dt\, t^{x-1} e^{-t}$$

With this definition the functions $m(r)$ and $q(r)$ vanishes as r approaches to zero and in the asymptotic limit $(r \to \infty)$ $m(r)$ and $q(r)$ approaches to the total mass M and total charge Q, respectively.

Using the Einstein equations for the above metric, the expression for components of energy–momentum tensor read as

$$T^\mu_\nu = \begin{bmatrix} T^t_t & 0 & 0 & T^t_\phi \\ 0 & T^r_r & 0 & 0 \\ 0 & 0 & T^\theta_\theta & 0 \\ T^\phi_t & 0 & 0 & T^\phi_\phi \end{bmatrix} \tag{6}$$

At this point, we would like to make the following observations. The energy–momentum tensor is off-diagonal in the $t-\phi$ sector. To extract the information about the energy conditions, we shall seek a tetrad frame in which $t-\phi$ sector is diagonalized while keeping the other components intact since other components are already in diagonal form. It is worth mentioning that our energy–momentum tensor Eq. (6) falls under the type-I of Hawking-Ellis classification [5] which is based on the extent to which the orthonormal components of the energy–momentum tensor can be expressed in a diagonal form [37]. General expressions for the components of energy–momentum tensor are complicated and are given in the appendix. Since the region near the equator ($\theta = \pi/2$) is densely populated with CTCs, we shall restrict ourselves to the near-equatorial region.[1] After carrying out the diagonalization procedure for the $t-\phi$ components of energy–momentum tensor, we get[2]

$$8\pi\tilde{T}^t_t = -\frac{1}{r^4}\Big[q(r,\Theta,Q) + 2r^2 m'(r,\Theta) - r q'(r,\Theta,Q)\Big]$$

[1] This point is explained with plots at the beginning of Sect. 2.2.

[2] See Appendix A for the details of the diagonalization procedure.

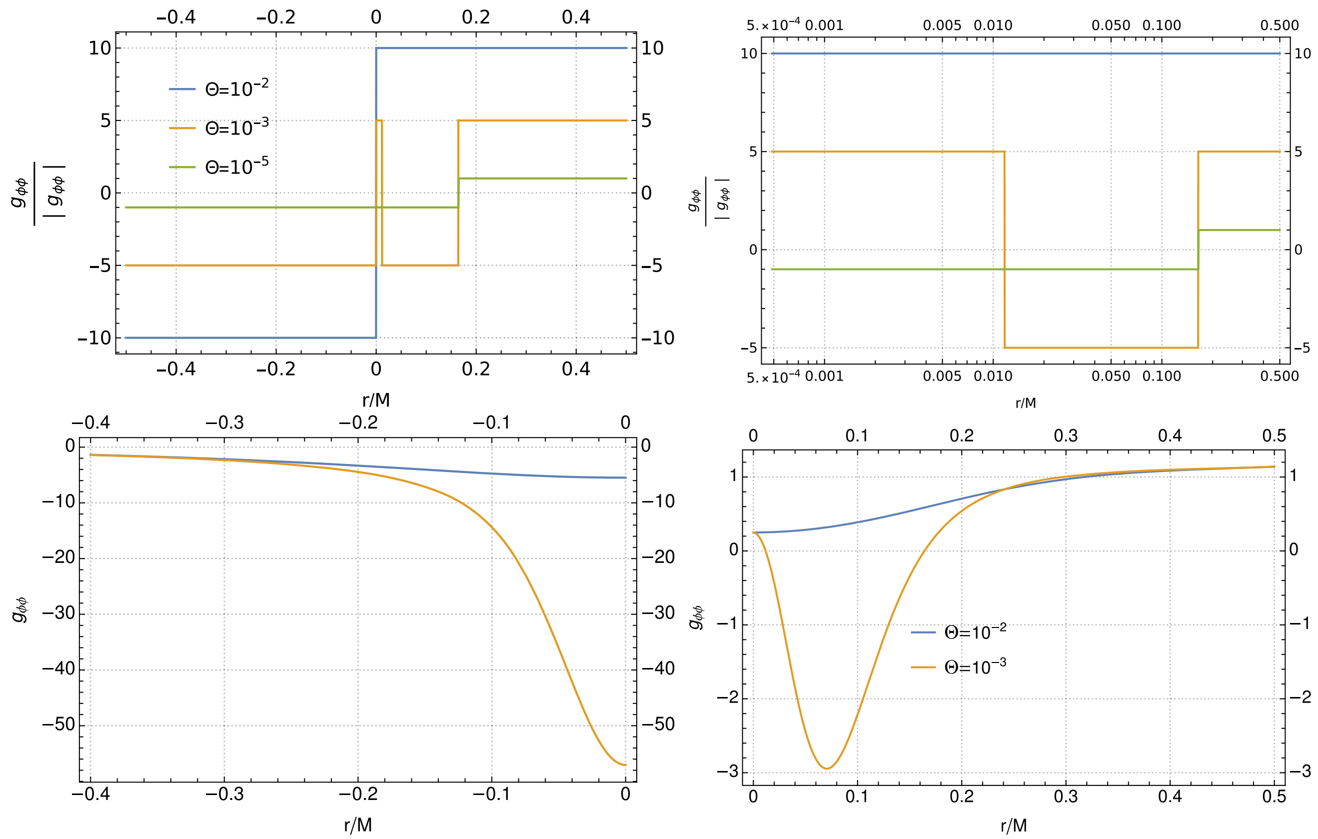

Fig. 1 Top: Behaviour of the *sign* of $g_{\phi\phi}$ in the equatorial plane ($\theta = \pi/2$) for the fixed value of charge $Q = 0.6$ and dimensionless spin $a = 0.5$ with $\Theta = 10^{-2}$ (blue), $\Theta = 10^{-3}$ (orange) and $\Theta = 10^{-5}$ (green). Region (in r space) containing the CTCs exists for orange and blue curves. The top right panel shows the curves in the top left panel zoomed into the region $r \to 0^+$. Note that to distinguish the green, orange and blue curves we have scaled the sign of $g_{\phi\phi}$ by suitable multiplicative factor for orange and blue curves. Bottom: In the bottom panels, we show the functional behaviour of the metric component $g_{\phi\phi}$ in the negative-r domain (left) and positive r domain (right). Blue curve is for $\Theta = 10^{-2}$ while the orange one is for $\Theta = 10^{-3}$

$$
\begin{aligned}
8\pi \tilde{T}^{\phi}_{\phi} &= \frac{1}{r^4}\Big[q(r,\Theta,Q) - r^3 m''(r,\Theta) - r q'(r,\Theta,Q) \\
&\quad + 2r^2 q''(r,\Theta,Q)\Big] \\
8\pi \tilde{T}^{t}_{\phi} &= 8\pi \tilde{T}^{\phi}_{t} = 0 \qquad (7)
\end{aligned}
$$

2.2 The energy conditions and CTCs

As demonstrated in [30], for a given charge, angular momentum and mass of a black hole, we can choose the noncommutative parameter ($\Theta > \Theta_{critical}$) such that the CTCs are eliminated. In contrast, the usual Kerr–Newman spacetime has CTCs in its spacetime for the same set of parameters (this can be achieved by taking the limit $\Theta = 0$).

In Fig. 1 (left panel) we have plotted sign of $g_{\phi\phi}$ against positive as well as negative re-scaled radial coordinate r/M, for three different values of noncommutative parameter $\Theta = 10^{-2}$, 10^{-3} and 10^{-5} (blue, orange and green curves, respectively) while keeping Q and a fixed. We observe that in all the cases, CTCs always exist for $r < 0$. For $r > 0$, $\Theta = 10^{-3}$ (orange curve) and 10^{-5} (green curve) admit CTC regions with different radial extent. In the region very close to $r = 0$, CTCs are absent. This is shown in right panel of Fig. 1 wherein we restrict the radial extent till $r/M = 10^{-2}$. We note that

1. for the orange curve CTCs are absent in the region $0 < r < 10^{-2}$, they re-appear and exists between $10^{-2} < r < 0.15$ and then they vanish for all $r > 0.15$.
2. For green curve CTCs are absent in the region $0 < r < 10^{-4}$, they re-appear and exists between $10^{-4} < r < 0.15$ and then they vanish for all $r > 0.15$.

This pattern continues for all the reasonable values of Θ. We recall from (4) that for a given value of Θ, the effect of noncommutativity is more prominent near the origin.

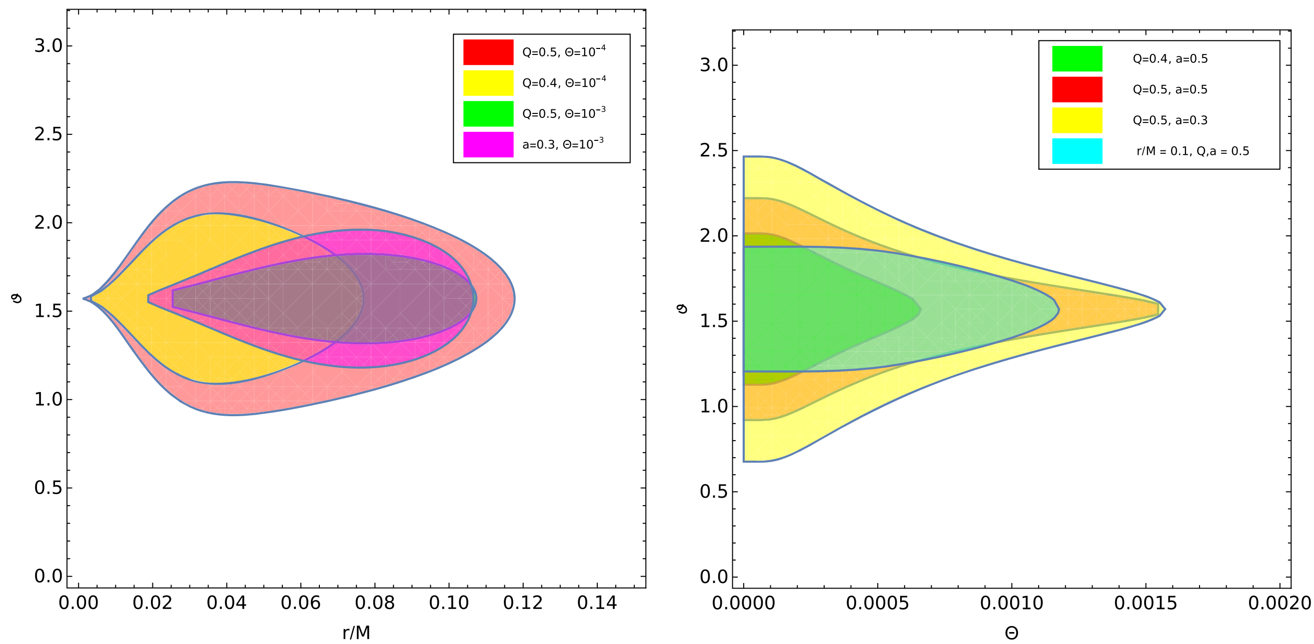

Fig. 2 The shape of the CTC regions. The left panel shows the CTC region in the $r-\vartheta$ plane of co-ordinate space for different BH parameter values. The parameter values that are not mentioned in the legend are defaulted to $a=0.5$, $Q=0.5$, $\vartheta=\pi/2$. Also, the region in green for which $Q=0.5, a=0.5, \Theta=10^{-3}$ is almost a subset of the region in magenta for which $a=0.3$, $Q=0.5$, $\Theta=10^{-3}$. It is to be noted that the extent of the CTC region in the longitudinal direction *decreases* and is oblated as the spin increases, i.e. BH spins faster. The CTC region increases in radial and longitudinal extent as the BH charge Q increases. One can see that the extent of the CTC region is most sensitive to the BH charge parameter Q and is less sensitive to the BH spin a. In the right panel, the shape of the CTC region in the $\Theta-\vartheta$ parameter space is shown. Firstly it can be seen that the region in green for which $Q=0.4, a=0.5, r=0.05$ is a subset of the region in red for which $Q=0.5, a=0.5, r=0.05$, which is in turn is a subset of the region in yellow for which $Q=0.5, a=0.3, r=0.05$. These plots agree with the previous plot in the sense that the extent of the CTC region also increases with increasing Q and decreasing a in the $\Theta-\vartheta$ plane. The region in cyan has a different shape than the other regions and is for $Q=0.5, a=0.5, r=0.1$. This shows that the extent of the region in this plane decreases as we move away from the origin $r=0$

The CTCs are absent for a larger radius r as the metric components approach the usual Kerr–Newman spacetime asymptotically [30,31].

In Fig. 2, we show that the CTCs are more prominent on the equatorial plane. On the top panel of Fig. 2, the x-axis is radial coordinate r and y-axis is the angle θ in radians. The shaded portion is the region where CTCs are present. At the bottom panel, we plot Θ on x-axis and θ on y-axis for fixed radius r. This plot illustrates that CTCs are concentrated more near $\theta=\pi/2$.

For correlating this information with the energy conditions, we similarly obtain the violation details of the energy conditions as a function of r. We then superimposed the CTC region and the energy condition violating region on the same plot. The correlation obtained is, therefore, a local one, where we address the question of whether, for a given point, the presence of CTCs is correlated with the violation of a particular energy condition or not.

We present here the observations regarding the possible correlation of the CTCs and energy conditions. For the type I matter field the various energy conditions are given by

$$\rho \geq 0\,,\ \rho+p_i \geq 0 \qquad \text{WEC}$$

$$\rho+\sum_{i=1}^{3} p_i \geq 0 \qquad \text{SEC}$$

$$\rho+p_i \geq 0 \qquad \text{NEC}$$

$$\rho-|p_i| \geq 0 \qquad \text{DEC}$$

We now illustrate the interplay between the weak energy condition and CTCs. To achieve that we first define the function $f(r,\Theta,a,Q)$ as

$$f = Min[\rho, \rho+p_1, \rho+p_2, \rho+p_3]. \tag{8}$$

Figure 3 illustrates the behaviour of sign of f (yellow) and sign of $g_{\phi\phi}$ (blue) against radial coordinate r/M for $Q=0.6$, $a=0.5$ and for various values for noncommutative parameter $\Theta=10^{-9}$, 10^{-5}, 10^{-3} and 10^{-1}. We can see two disconnected CTC regions for each case, one in $r>0$ and the other in $r<0$. No CTCs are present in the vicinity of $r\rightarrow 0^{+}$ as $g_{\phi\phi}$ is positive in that limit. The first row in this figure is for an extremely small value of the noncommutative

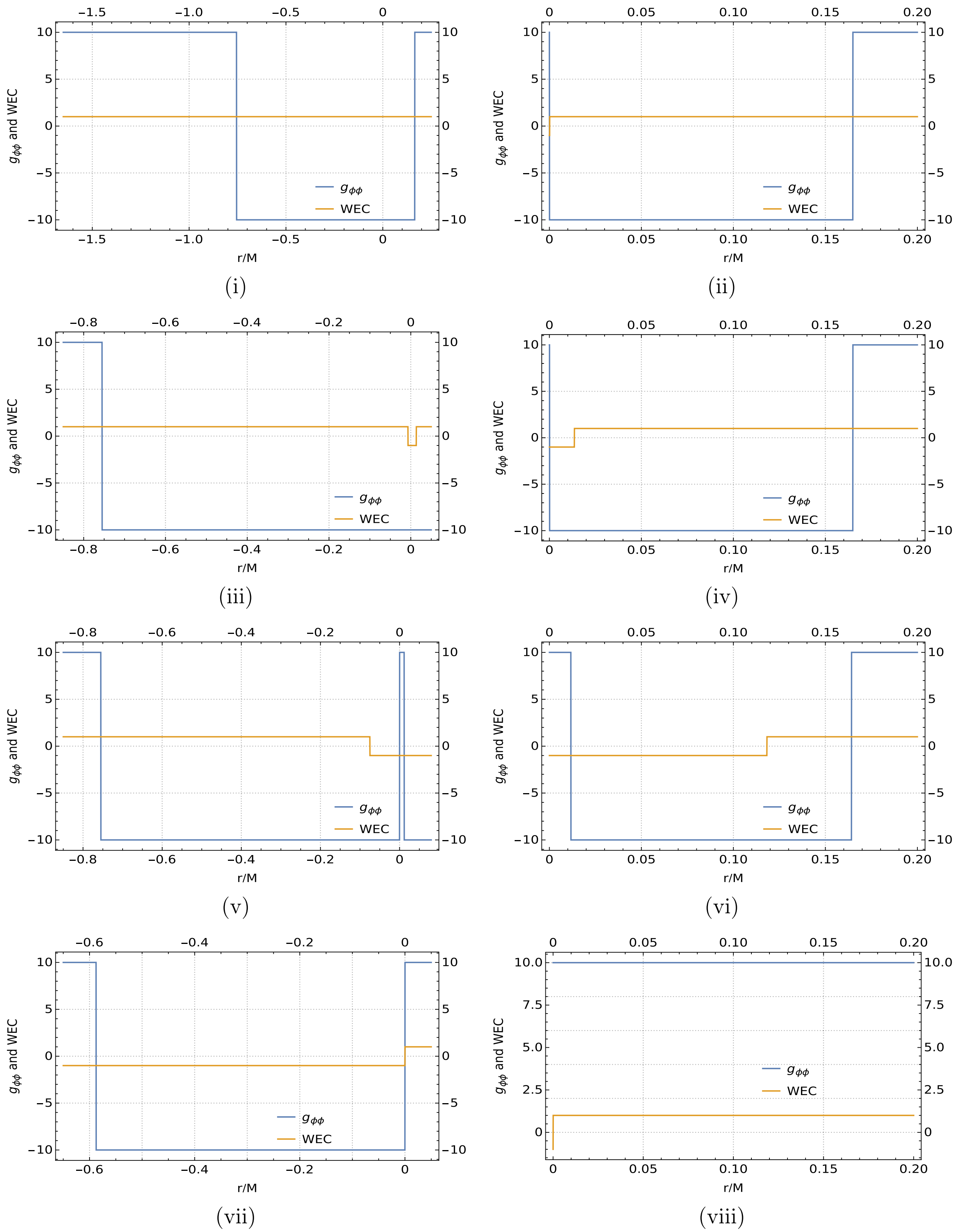

Fig. 3 The behaviour of the signs of the $g_{\phi\phi}$ and WEC function f in the interior of NC Kerr–Newman spacetime across different values of the noncommutativity parameters. In these figures, the left column shows the behaviour for $r < 0$, and the right for $r > 0$. All figures are for $a = 0.5$ and $Q = 0.6$, but vary in the noncommutative parameter Θ across the rows. Figures in the first row are for $\Theta = 10^{-9}$ and thus show asymptotic Kerr–Newman behaviour. The figures in the subsequent rows are for $\Theta = 10^{-5}$, 10^{-3} and 10^{-1} respectively

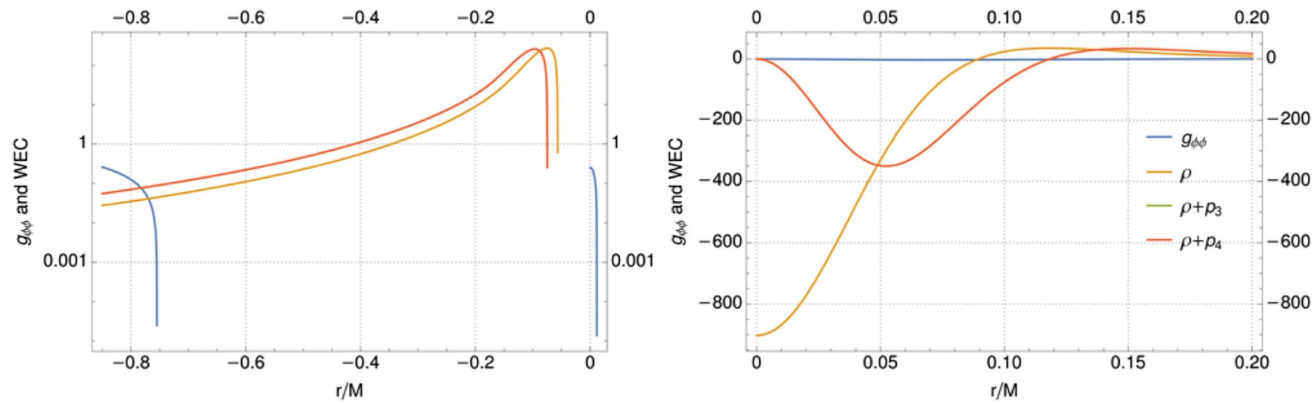

Fig. 4 The behaviour of the metric component $g_{\phi\phi}$ and WEC functions in the interior of NC Kerr–Newman spacetime for the same parameter values and the domain as in (v) and (vi) of Fig. 3. Left plot is for negative r and the right one is for positive r. In the left plot, the y-axis is scaled logarithmically. Therefore, the curves are only plotted where they take positive values

parameter, $\Theta = 10^{-9}$. The violation region of WEC is tiny and is not visible. This is expected, as for minimal values of Θ, the NCKN solution approaches the Kerr–Newman spacetime. For the plots in successive rows, we observe the WEC violating region in the vicinity of $r = 0$, and there is one connected region for each case. From top to bottom, we can see that the width of the CTC region decreases as the noncommutative parameter Θ increases, while the width of the WEC violating region increases. For the completeness we have also given the behaviour of f and $g_{\phi\phi}$ for the parameter values $a = 0.5$, $Q = 0.6$ and $\Theta = 10^{-3}$ in Fig. 4.

The behaviour of the sizes of the region containing CTCs with the WEC violating region has been further investigated. We estimate the *proper* sizes of the respective regions by numerical integration for the proper spatial length:

$$W_\Omega = \int_\Omega \sqrt{g_{rr}}dr \tag{9}$$

Here Ω refers to the domain of integration. For the proper size of the CTC region W_Ω, the domain Ω is all points on the r-axis where $g_{\phi\phi} < 0$. For WEC region, the domain is all points where the WEC is violated ($f < 0$). Thus, the widths are proper distances between the roots of the respective functions between which the functions ($g_{\phi\phi}$ or f) take negative values. The roots are found numerically on the equatorial plane for fixed values of the BH parameters a, Q, Θ. As mentioned earlier, there are two roots each in positive r and negative r domain for $g_{\phi\phi}$, whereas the WEC function f has one root each in the positive and negative r domains. We compute the widths respectively for $g_{\phi\phi}$ and the WEC function f in each domain of interest and sum them.

We then plot in Fig. 5 the total proper widths of the regions containing CTCs (top panel) and the width of the region where WEC is violated (bottom panel), vs the noncommutative parameter Θ for a few values of the BH parameters

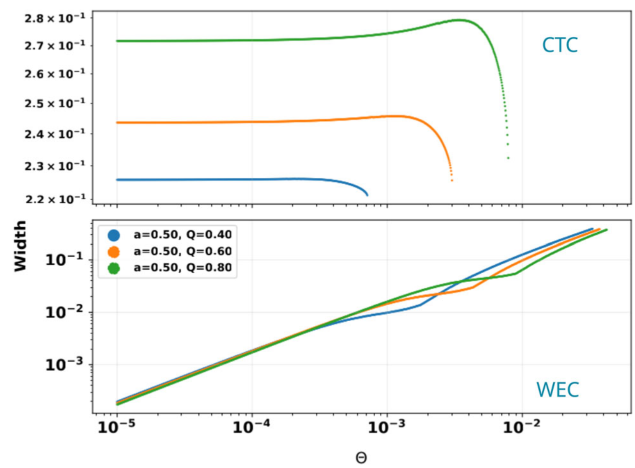

Fig. 5 The sum of the proper widths of the CTC (top panel) and WEC violating (bottom panel) regions in the negative and positive r domains vs. the NC parameter Θ. The global behaviour is that the width of the CTC region (top panel) decreases as the domain where WEC is violated increases (bottom panel)

a and Q. The top panel clearly shows that the width of the region containing CTCs reduces in extent as Θ increases. This result was first noted in [30]. What we are also observing here is that the width of the WEC violation region increases in extent as the dispersion of the effective noncommutative matter (parameterised by Θ) introduced increases (see Eq. (1)). The above observation seems to indicate that the more the extent of WEC violating regions, the lesser the region admitting the CTCs. In this sense, the violation of WEC favours the elimination of CTCs.

We now examine the validity of Hawking's statement for the NCKN spacetime. As stated earlier, according to Hawking's statement, in the spacetime admitting the CTCs, either of the following must hold: (i) curvature singularity exists, (ii) the energy–momentum stress tensor shows pathological

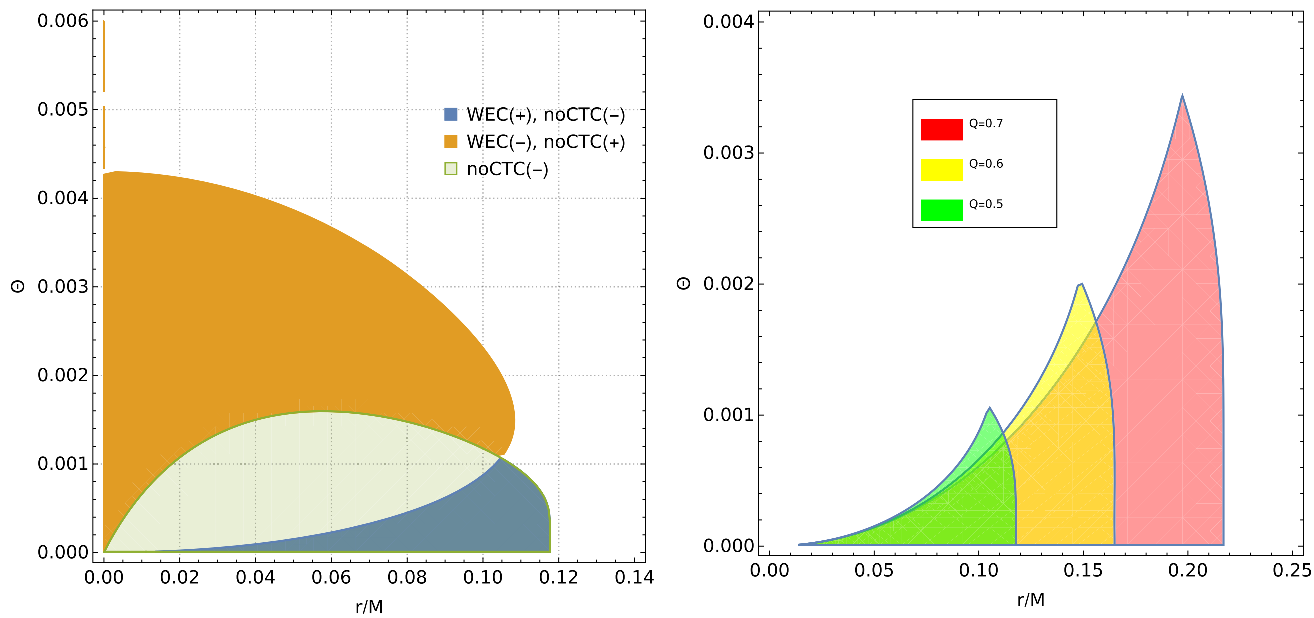

Fig. 6 Left: The three regions in the parameter space: (i) the region where WEC is preserved, but CTCs are present (blue), (ii) the region where WEC is violated but CTCs are absent (orange), and (iii) the region where WEC is violated but CTCs are present. (cream and overlapping with blue). Right: The variation in the shape of the regions in the parameter space where WEC is preserved and CTCs exist, across different values for charge Q and fixed $a = 0.5$. Note that WEC is violated at a given point (r, θ, ϕ) if the value of the NC parameter is large enough. In these regions, CTCs may or may not exist, and is depicted in the left panel of this figure. Also, the blue region in the left panel matches with the green region on the right panel (although shown in different scales)

behaviour at infinity, (iii) WEC must violate. As the spacetime we are considering contains neither curvature singularities nor any pathological behaviour of stress tensor, then for Hawking's statement to hold, the violation of WEC becomes a necessary and sufficient condition for the CTC region to exist. Note that the conjecture is not explicit about where the WEC must be violated in spacetime to admit CTCs. Also, it does not exclude the possibility of having two exclusive regions – (a) the region admitting the CTCs but where the WEC is preserved and (b) the region lacking any CTCs but where the WEC is violated. In this sense, Hawking's statement is a global statement about the existence of CTCs for a given spacetime.

Indeed, in Fig. 6 we found that there are regions in the spacetime where WEC is preserved, while CTCs exist and vice-versa. In the left panel of Fig. 6 we have given the region plots in the $r - \Theta$ space for the fixed value of Q and a. The blue region indicates the portion where WEC is preserved and CTCs are present. This shows that it is possible to find a region in the spacetime where the CTCs exist and WEC is preserved, implying that although Hawking's conjecture holds true globally, there exists at least one sub-region in the spacetime where it gets violated. This is further repeated for other values of Q and a in the right panel of Fig. 6. The orange region (left panel) is where the WEC is violated, and CTCs are absent, which exemplifies the more obvious fact that if WEC is violated in a region, then the conjecture does not imply that CTCs exist. The cream-coloured region (left panel) is the only region where WEC is violated and CTCs are present. Thus, it is the only region where the Hawking conjecture is valid.

To elucidate this point further, we scan the entire parameter space and look for the regions containing CTCs where weak energy conditions are also preserved. In the first two plots of Fig. 7, we highlight the region in the parameter space: $r - \Theta - a$ and $r - \Theta - Q$ where the CTCs are present and also the weak energy condition is preserved. We observe that the situation is similar (to that of WEC) for the strong energy condition (SEC) in Fig. 8. In the top panel, we explore the behaviour of strong energy condition (SEC) in the vicinity of the region containing CTCs. This illustrates the existence of a common region containing CTCs where SEC is preserved in the $r - \Theta$ space. For completeness, we have also presented the $3D$-regions of co-existence of CTCs and strong energy conditions in the bottom panel of Fig. 8.

3 A correlation index

In this section, we define a new way of correlating CTCs and energy consitions and use them to understand the scenatio in NCKN spacetime.

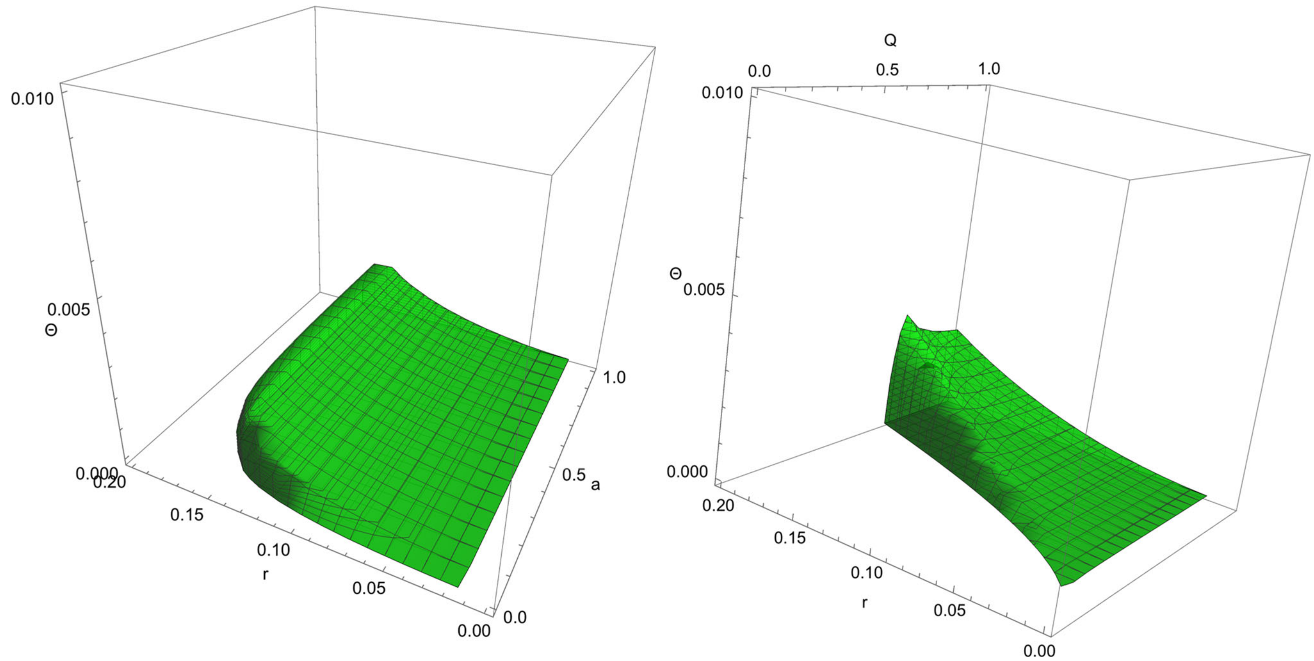

Fig. 7 Left: $3D$-region showing the co-existence of CTCs and weak energy condition in r, Θ, a space with $Q = 0.6$. Right: $3D$-region showing the co-existence of CTCs and weak energy condition in r, Θ, Q space with $a = 0.5$

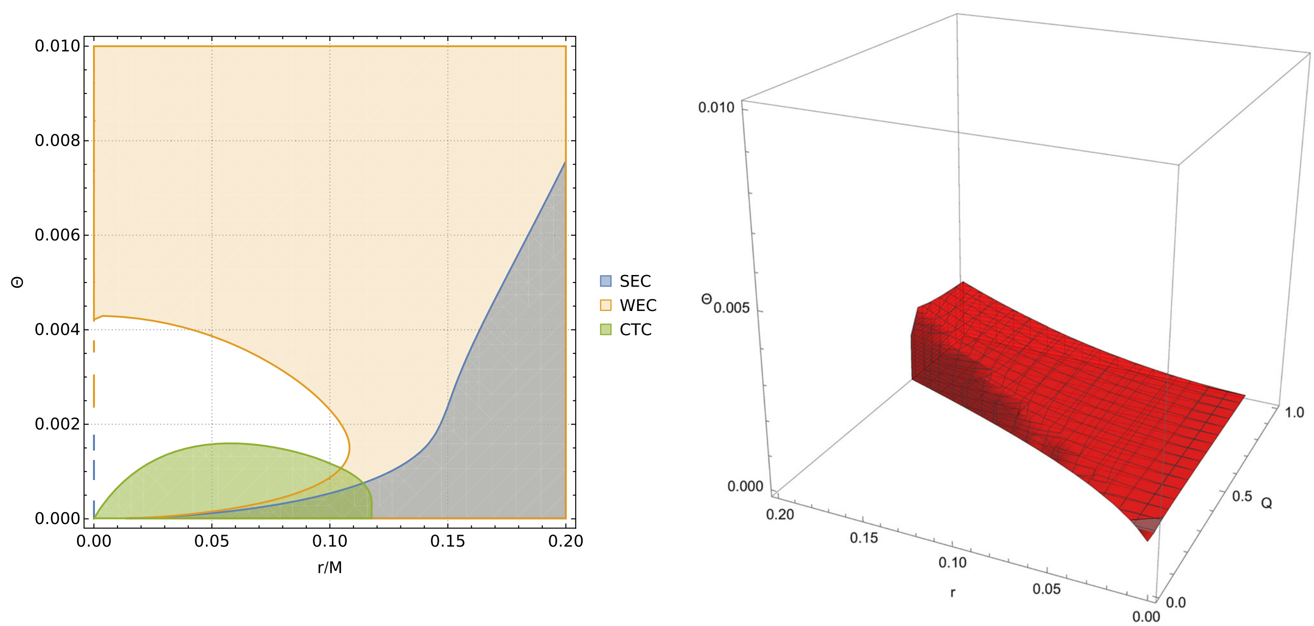

Fig. 8 Left: the region in $r - \Theta$ space containing CTCs where SEC (blue) and WEC (pale orange) are preserved for the parameter values $Q = 0.5$, $a = 0.5$. Right: $3D$-region showing the co-existence of the CTCs and the strong energy conditions in r, Θ, Q space with $a = 0.5$

Our goal is to explore the correlation between CTCs and energy conditions in the NCKN spacetime in this work, and other models in other variants of modified gravity in future. To make the estimate quantitative, we give a general prescription for defining a correlation index in this section.

We observe a common feature among the variants of Kerr–Newman spacetime solutions. The variants are generally dependent on a few parameters $(\mu_1, \mu_2 ... \mu_n)$ that are specific to the model of gravity chosen. Einstein's gravity corresponds to particular values of these parameters. The Kerr–Newman metric has an axial symmetry corresponding to a Killing vector $\xi^{\alpha}_{(\phi)}$ presented in a coordinate chart, say (x^{α}) (here, the subscript ϕ is a label indicating the axial Killing vector field). The CTCs in these coordinates can be singled out by examining the sign of the norm of this Killing vector $\xi^{\alpha}_{(\phi)}\xi_{(\phi)\alpha}$. Wherever $\xi^{\alpha}_{(\phi)}\xi_{(\phi)\alpha}$ becomes negative, we have CTCs. We mention here that our study is restricted only to the consideration of the closed curves that correspond to the integral curves of the Killing vector field ξ_{ϕ}. The modified Kerr–Newman solution is generally dependent on the various parameters $(\mu_1, \mu_2 ... \mu_n)$ and so is the norm of $\xi_{(\phi)})^{\alpha}$. Furthermore, When one computes the expression for the energy conditions, one gets a function of the form $f(x^{\alpha}, \mu_1, \mu_2 ... \mu_n)$. The sign of that function dictates whether a particular energy condition is violated or not. Motivated by this, we *associate* the index for correlation between the existence of CTCs and energy conditions as

$$\mathcal{C} = \frac{\int \frac{f}{|f|}\left(\frac{\xi^{\alpha}_{(\phi)}\xi_{(\phi)\alpha}}{|\xi^{\alpha}_{(\phi)}\xi_{(\phi)\alpha}|} - 1\right)\sqrt{h}\, dX\, d\mu}{\int \left(\frac{\xi^{\alpha}_{(\phi)}\xi_{(\phi)\alpha}}{|\xi^{\alpha}_{(\phi)}\xi_{(\phi)\alpha}|} - 1\right)\sqrt{h}\, dX\, d\mu} \qquad (10)$$

Here dX is a short notation to the relevant spacetime volume element, $d\mu$ stands for volume in the parameter space $d\mu_1 d\mu_2 \cdots d\mu_n$ and $\sqrt{h}$ is the determinant of the induced metric on the relevant hypersurface. If we use the standard Boyer–Lindquist coordinate system (t, r, θ, ϕ), the norm $\xi^{\alpha}_{(\phi)}\xi_{(\phi)\alpha}$ is then equal to the metric component $g_{\phi\phi}$. In the models that we consider below, we restrict our analysis to the equatorial plane and hence we set $\theta = \pi/2$. Due to axial symmetry we can further set $\phi = 0$. Consequently, the index now assumes the form,

$$\mathcal{C} = \frac{\int \frac{f}{|f|}\left(\frac{g_{\phi\phi}}{|g_{\phi\phi}|} - 1\right)\sqrt{h}\, dX\, d\mu}{\int \left(\frac{g_{\phi\phi}}{|g_{\phi\phi}|} - 1\right)\sqrt{h}\, dX\, d\mu} \qquad (11)$$

The motivation for the above definition of the index is stated below. The term $\left(\frac{g_{\phi\phi}}{|g_{\phi\phi}|} - 1\right)$ is zero wherever CTCs are not present and is equal to -2 in regions where CTCs are present. The term $f/|f|$ is $+1$ whenever the relevant energy condition is preserved, as we vary the spacetime coordinates and the parameters μ_i. The term $f/|f|$ will have a value -1 wherever the relevant energy condition is violated. Hence, the numerator of the integrand in Eq. (11) is positive for the parameters for which the CTCs are present, and the relevant energy condition is violated. It is zero wherever CTCs are absent. It is -2 wherever CTCs are present, and the relevant energy conditions are preserved. The denominator of the above expression integrates over the entire parameter range, which includes CTCs, and it gives a vanishing contribution from the points where CTCs are absent. Consequently, the index is ill-defined in situations when there are no CTCs exist anywhere in the spacetime for all parameter values. The resultant index, therefore, takes values between -1 and $+1$. If the index is negative, one can infer that CTCs prefer to exist in the region where the relevant energy condition, denoted by $f(x^{\alpha}, \mu_1, \mu_2 ... \mu_n)$ is violated. A positive value of the index implies that CTCs prefer to live in an energy condition preserved environment, suggesting that eliminating CTCs might require violation of energy conditions.

It is worth mentioning that the hypersurface we have defined is $t = const$ where t is the time in Boyer–Lindquist coordinate system. We have chosen a spacelike hypersurface to define the index since there are CTC regions that have compact support on such a hypersurface that make relevant integrals converge. Note that the time t does not have compact support, and hence the integral involving t would diverge. Due to the existence of a timelike Killing vector, an integral over $t = const$ hypersurface does indeed capture the essence of the correlation. The way we have defined the index allows us to examine the correlation in a subspace of the entire parameter space, making a systematic study feasible and well defined. Here we have evaluated the index for $\theta = \pi/2$ and for a fixed radial coordinate r. The hypersurface $t = const, \theta = \pi/2$ is orthogonal to the Killing vector $\xi_t + \frac{1}{2}a\xi_{\phi}$. This renders the independent coordinate definition of the correlation index.

Now we evaluate the index given by Eq. (11) in a subregion of the parameter space of NCKN spacetime to systematically study the correlation between the existence of CTCs and the weak energy condition. To avoid numerical artefacts while implementing the index formula, care must be taken such that CTCs are present in the integration domain.

3.1 Correlation index for NCKN

Here we evaluate the correlation index as defined in Eq. (11) for the NCKN case for the fixed point with the coordinate $(t, r, \theta = \pi/2, \phi)$ in spacetime. The correlation index is given by

$$\mathcal{C}_r = \frac{\int \frac{f}{|f|}\left(\frac{g_{\phi\phi}}{|g_{\phi\phi}|} - 1\right) da\, dQ\, d\Theta}{\int \left(\frac{g_{\phi\phi}}{|g_{\phi\phi}|} - 1\right) da\, dQ\, d\Theta} \qquad (12)$$

with the function f given by Eq. (8). Note that since we have chosen a particular point in spacetime there is no integral over spacetime coordinates (t, r, θ, ϕ)

In Fig. 9, we present the correlation index at few values of $x = r/M$. The index is calculated by integrating over the parameter space involving a, Q, Θ. The range of x chosen is such that the CTCs are definitely present in some parameter range. The normalized radius ($x = r/M$) is varied from 0.03 to 0.12. The index is evaluated by carrying out an integration

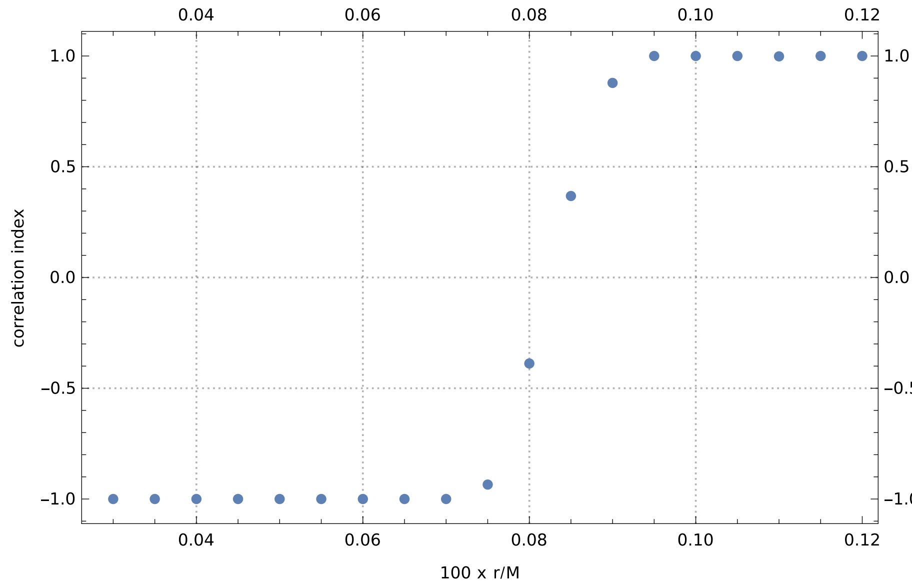

Fig. 9 Indices for correlation between WEC and CTCs in the NCKN geometry for values of x ranging 0.03 to 0.200 (scaled by factor 10^2). The index is calculated over a, Q and Θ

over a, Q and Θ. We see that for smaller values of x, the index is close to -1, and for relatively larger values of x, the index smoothly varies till it becomes $+1$. This implies that for smaller values of x, the CTCs prefer to exist in an environment where WEC tends to get violated. The effect is more prominent in smaller values of x because the WEC violation is required to smoothen out the singularity (which is present if there is no noncommutativity) . The NCKN spacetime asymptotes to the standard Kerr–Newman spacetime for larger x where the energy conditions are not broken. So for larger x, the index smoothly goes to $+1$, which is WEC preserving the environment.

4 Conclusions

Hawking conjectured that nature prevents the appearance of CTCs. The problem due to the presence of CTCs is a fundamental pathology in a manner similar to the pathology of singularity. A concrete proof to the question of whether nature prevents CTCs might have to wait for inputs from quantum gravity in a manner similar to the cosmic censorship hypothesis. We, in this work, address a few aspects that add to our understanding of CTCs. The guiding principle was Hawking's statement regarding the relation between CTCs, singularities and energy conditions. In particular, we study the local validity of Hawking's statement regarding CTCs and energy conditions by defining and computing a correlation index. For the sake of concreteness, we considered the noncommutative Kerr–Newman black hole. This geometry has no curvature singularity, and the energy–momentum tensor has regular behaviour at infinity. This geometry provides an excellent understanding of the dependence between CTCs and energy conditions. To quantify the correlations, we define a correlation index between CTCs and energy conditions that averages over all the parameters of the model in the region containing CTCs.

In order to understand the relationship between the CTCs and WEC, we also carried out a graphical analysis. We described the general structure and location in the spacetime of the regions containing CTCs and the regions where WEC is violated. We found that the CTC region is not a subset of the region where WEC is violated. Further, we computed and contrasted the proper widths of these regions on the equatorial plane for various values of the noncommutative and BH parameters. The general behaviour we found was that the widths of the CTC region decrease while that of the WEC regions decreases with increasing noncommutative parameter values for large enough values. For large enough NC parameter values, the BHs with higher electric charge were found to have higher CTC region widths and lower WEC violating region widths. However, overall, it is clear that noncommutativity opposes the existence of CTCs, as noted previously [30].

Based on the analysis and these results, we could demonstrate that Hawking's statement can have a local and global version. We could show that Hawking's statement can be true globally but need not hold locally. The analysis of the noncommutative Kerr–Newman black hole yields that in the absence of a singularity, the weak energy condition may or may not be violated at the same spacetime point that has a CTC passing through it.

We observe that, based on the correlation index, in the absence of a singularity, the CTCs thrive in a WEC-violating environment close to the smeared-out singularity. In the model that we have examined, we see that for CTCs that are away from the smeared-out singularity (i.e. for large r/M), they prefer to exist in a WEC preserving environment.

The evidence gathered here does not concretely establish the role of WEC in ruling out CTCs, though it indicates the possibility of dependence. More evidence can be gathered by examining other energy conditions too. More solutions of other variants of Kerr Newman and other spacetimes with CTCs need to be studied to draw conclusive evidence. This analysis is also suggestive that violation of energy conditions aids CTC elimination. In effect, this is contrary to the local version of Hawking's statement wherein the WEC is violated in spacetimes with CTCs, but has no singularities or pathological behaviour at infinity.

We strongly believe the study opens pathways to examine Hawking's conjecture. As of now, there is a lack of direction towards a very rigorous mathematical proof of Hawking's conjecture. In such a scenario, the analysis done in this work paves the way toward gathering evidence for the conjecture. Currently, much focus is there on modified gravity owing to unsolved questions in cosmology. Many models in the literature provide excellent 'data points' to gather evidence concerning Hawking's conjecture. We mean that, for each model of modified gravity, one may obtain a variant of spacetime endowed with CTCs (e.g. modified Kerr Newman metrics.).

In this work, we work with a spacetime with no singularities and no pathological behaviour in the energy–momentum tensor at infinity. Therefore, via Hawking's conjecture, the only way the CTCs can be present is through violation of WEC. It would be nice to develop a similar study for the spacetime for which the parameter space is large enough and has a curvature singularity. In such a case, it would be interesting to see if the existence of CTCs requires violating energy conditions.

The study can provide insights and directions towards a more systematic and rigorous study for characterizing CTCs. One can expect these studies to eventually lead to developing theorems for CTCs like the singularity theorems. Another area worth studying is how to extrapolate the aforementioned correlation at the semi-classical level. This might be achieved by replacing $T^{\mu\nu}$ with its expectation value corresponding to appropriately chosen quantum states. We want to address these issues in the near future.

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Author contributions The authors' names are mentioned according to the alphabetical order of their last names.

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Appendix A: Diagonalization of EM tensor

In this appendix we shall give the details of the diagonalization procedure that is used to arrive at Eq. (7) in the main text.

For $\theta = \pi/2$, the energy momentum tensor can be written as

$$
\begin{aligned}
8\pi T^t_t &= -\frac{1}{4r^6}\Big[-4a^2r^3m''(r,\Theta) + \Big(8a^2r^2 + 8r^4\Big)m'(r,\Theta) \\
&\quad +2a^2r^2q''(r,\Theta,Q) + 2r\Big(-4a^2 - 2r^2\Big)q'(r,\Theta,Q) \\
&\quad +\Big(8a^2 + 4r^2\Big)q(r,\Theta,Q)\Big] \\
8\pi T^r_r &= -\frac{r\left[2rm'(r,\Theta) - q'(r,\Theta,Q)\right] + q(r,\Theta,Q)}{r^4} \\
8\pi T^\theta_\theta &= \frac{1}{4r^4}\Big[2r^2\left(-4m'(r,\Theta) - 2rm''(r,\Theta) + q''(r,\Theta,Q)\right) \\
&\quad +4\left(r\left(2rm'(r,\Theta) - q'(r,\Theta,Q)\right) + q(r,\Theta,Q)\right)\Big] \\
8\pi T^\phi_\phi &= -\frac{1}{4r^6}\Big[2r\Big(2r^2\left(a^2+r^2\right) \\
&\qquad \times m''(r,\Theta) + \left(4a^2 + 2r^2\right)q'(r,\Theta,Q)\Big) \\
&\quad -8a^2r^2m'(r,\Theta) - 2r^2\left(a^2+r^2\right)q''(r,\Theta,Q) \\
&\quad +2\left(-4a^2 - 2r^2\right)q(r,\Theta,Q)\Big] \\
8\pi T^t_\phi &= \frac{1}{4r^6}\Big[a\left(a^2+r^2\right)\Big(-4r^3m''(r,\Theta) + 8r^2m'(r,\Theta) \\
&\quad +2r^2q''(r,\Theta,Q) - 8rq'(r,\Theta,Q) + 8q(r,\Theta,Q)\Big)\Big] \\
8\pi T^\phi_t &= \frac{1}{4r^6}\Big[2ar^2\left(4m'(r,\Theta) + 2rm''(r,\Theta) - q''(r,\Theta,Q)\right) \\
&\quad -8a\left(r\left(2rm'(r,\Theta) - q'(r,\Theta,Q)\right) + q(r,\Theta,Q)\right)\Big]
\end{aligned}
$$

Next, we carry out the reduction of the energy momentum tensor to the canonical form. This requires us to find the eigenvalues of the expression

$$det|T_{\mu\nu} - \lambda g_{\mu\nu}| = 0 \tag{13}$$

Using the above one can reduce the energy–momentum tensor to the canonical form. If the eigenvalues are real, then the energy–momentum tensor belongs to type-I according to Hawking and Ellis classification. type-I implies that the energy–momentum tensor and the metric are simultaneously diagonalizable. This implies that one can find a tetrad basis in which the canonical energy–momentum tensor is $T_{ab} = T_{\mu\nu} e^{\mu}_{a} e^{\nu}_{b}$ where (e^{μ}_{a}) are the tetrads which are the simultaneous eigenvectors of type-I energy–momentum tensor and the metric. Thus, the eigenvalues represent the principle pressures and density. For the NCKN case, we show that the energy–momentum tensor can be canonically decomposed to type-I by Hawking classification on the equatorial plane $\theta = \pi/2$. We evaluate the eigenvalues and eigenvectors $(e_{(t)}, e_{(\phi)})$ by focussing on the $t - \phi$ sector of the energy–momentum tensor (the rest of the components are already in the canonical form). The eigenvector $e_{(t)}$ is timelike while $e_{(\phi)}$ is spacelike. The explicit form of these tetrads is furnished by the following expressions

$$e^{t}_{(t)} = \frac{a^2 + r^2}{a} \; ; \; e^{\phi}_{(t)} = 1 \; ; \tag{14}$$

$$e^{t}_{(\phi)} = a \; ; \; e^{\phi}_{(\phi)} = 1 \; ; \; e^{r}_{(t)} = \; ; \; e^{\theta}_{(t)} = \; ; \; e^{r}_{(\phi)} = \; ; \; e^{\theta}_{(\phi)} = 0 \tag{15}$$

Also, as expected, the eigenvectors are orthogonal to each other. After normalizing the eigenvectors and re-expressing the components of the energy–momentum tensor in the $t - \phi$ sector, we finally obtain the following components

$$\begin{aligned} 8\pi \tilde{T}^{t}_{t} &= -\frac{1}{r^4}\Big[q(r,\Theta,Q) + 2r^2 m'(r,\Theta) - r q'(r,\Theta,Q)\Big] \\ 8\pi \tilde{T}^{\phi}_{\phi} &= \frac{1}{r^4}\Big[q(r,\Theta,Q) - r^3 m''(r,\Theta) - r q'(r,\Theta,Q) \\ &\quad + 2r^2 q''(r,\Theta,Q)\Big] \\ 8\pi \tilde{T}^{t}_{\phi} &= 8\pi \tilde{T}^{\phi}_{t} = 0 \end{aligned} \tag{16}$$

while other components remain unchanged.

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