



# On the breakdown of the perturbative interaction picture in Big Crunch/Big Bang or the true reason why perturbative string amplitudes on temporal orbifolds diverge

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**Abstract** We discuss how the perturbative particle paradigm fails in certain background with space-like singularity but asymptotically flat which should admit a S-matrix. The Feynman approach relies on the interaction picture. This approach means that we can interpret interactions as exchanges of particles. Particles are the modes of the quadratic part of the Lagrangian. In certain backgrounds with space-like singularity the interaction Hamiltonian is well defined but the perturbative expansion of the evolution operator through the singularity and the perturbative S matrix do not exist. On the other hand, relying on minisuperspace approximation we argue that the non perturbative evolution operator does exist. The complete breakdown of the perturbative expansion explains why the perturbative computations in the covariant formalism in string theory in temporal orbifold fail, at least at the tree level.

## 1 Introduction

While this paper is mostly on QFT and its behavior on singular spacetimes describing some models of Big Crunch/Big Bang its reason has roots in string theory. String theory, as a promising candidate for a theory of quantum gravity, is supposed to provide a satisfactory description of Big Bang/Big Crunch type singularities, or at least a S matrix in asymptotically flat spaces.

We want therefore to construct and study stringy toy models capable of reproducing a space-like (or null) singularity which appears in space at a specific value of the time coordinate and then disappears.

The easiest way to do so is by generating singularities by quotienting Minkowski with a discrete group with fixed

points, i.e. orbifolding Minkowski. In this way it is possible to produce both space-like singularities and supersymmetric null singularities [1–15] (see also [16–18] for some reviews). Another possible way which is a generalization of the previous orbifolds with null singularity is consider gravitational shock wave backgrounds [19–30].

It happens that in these orbifolds the four tachyon closed string amplitude diverges in some kinematical ranges, more explicitly for the Null Shift Orbifold (which may be made supersymmetric and has a null singularity) we have

$$\mathcal{A}_{4T}^{(closed)} \sim \int_{q \sim \infty} \frac{dq}{|q|} q^{4-\alpha' \mathbf{p}_{\perp t}^2}, \quad (1.1)$$

so the amplitude diverges for  $\alpha' \mathbf{p}_{\perp t}^2 < 4$  where  $\mathbf{p}_{\perp t}$  is the orbifold transverse momentum in  $t$  channel. Until recently this pathological behavior has been interpreted in the literature as “the result of a large gravitational backreaction of the incoming matter into the singularity due to the exchange of a single graviton”. This is not very promising for a theory which should tame quantum gravity.

What has gone unnoticed is that if we perform an analogous computation for the four point open string function we find

$$\mathcal{A}_{4T}^{(open)} \sim \int_{q \sim \infty} \frac{dq}{|q|} q^{1-\alpha' \mathbf{p}_{\perp t}^2} tr(\{T_1, T_2\}\{T_3, T_4\}), \quad (1.2)$$

which is also divergent when for  $\alpha' p_{\perp t}^2 < 1$  [31,32]. This casts doubts on the backreaction as main explanation since we are dealing with open string at tree level. This is further strengthened by the fact that three point amplitudes with massive states may diverge [31] when appropriate polarizations are chosen. For example for the three point function of two tachyons and the first level massive state we find for an

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appropriate massive string polarization

$$\mathcal{A}_{TTM}^{(open)} \sim \int_{u \sim 0} \frac{du}{|u|^{5/2}} \text{tr} (\{T_1, T_2\} T_3). \quad (1.3)$$

In [31] this was interpreted as a non existence of the underlying effective theory. We now revisit this assertion and argue that the effective theory does exist but the usual approach based on the perturbative expansion in the interaction picture completely breaks down.

In this paper we consider what happens when we use perturbation theory in a time dependent background with a space singularity. It is somewhat obvious that we do not expect to find a well behaved perturbation theory because of the singularity. One could expect some kind of pathology like the series being asymptotics. We find a much worse behavior: a complete breakdown of perturbation theory, i.e. perturbation theory does not exist. Let us be more precise. We consider as unperturbed theory the free, non interacting QFT in the given singular time dependent background and then add interactions. We then use the usual interaction picture approach. This approach when used perturbatively naturally leads to Feynman diagrams and a nice particle interpretation of interactions. In the backgrounds we consider all of this suffers from a complete breakdown. There is no perturbative expansion in the usual sense. This prompts the question whether it is perturbation theory which fails or it is the very interacting theory which does not exist. To answer this question we consider the minisuperspace approach, i.e. the consider the QFT reduced to the spacially homogeneous configurations (see [33] for review). In this limit the theory reduces to Quantum Mechanics. We then show that these models do exist. One could wonder whether this reduction is a big limitations and the answer is no since it has been shown [14, 31] that the troubles in perturbation theory stem from these configurations. The main difference with the work from the 80s and 90s is that we are interested in going through the singularity and not giving the boundary conditions at the Big Bang.

This result stresses the importance of treating some sectors as exactly as possible in order to get a perturbation theory for the remaining sectors. Nevertheless it is noteworthy that from our analysis the original Krasner background, which is also a string background admits a good perturbation theory.

Even so we are left with the unanswered question whether it is really consistent to treat QFT on a given singular background without considering the backreaction. It is somewhat likely that the gravitational background and the matter should evolve together, especially in a background which has space singularities. Given the results of this paper it could be sufficient to consider the minisuperspace approximation to get a reasonable approximation. In any case this route is fraught with subtleties like the “problem of time” (see [34] for a review).

The paper is organized as follows.

In Sect. 2 we discuss the background of interest, the generalized Kasner metrics (of which the Boost Orbifold is a very special case) and the simplest interacting field theory, i.e. the scalar field and its minisuperspace approximation.

In Sect. 3 we discuss the simplest example where the perturbative interaction picture breaks completely down: the time dependent harmonic oscillator with  $\Omega^2(t) = \omega^2 + \frac{k}{t^2}$  (for reasons we are going to explain  $k \leq \frac{1}{4}$  so that  $\Omega^2$  may become negative). While this model is natural since it corresponds to, for example, de Sitter modes in conformal time the splitting we perform between the unperturbed Hamiltonian and the perturbative part is somewhat artificial but it is chosen in order to get the simplest example as possible.

In Sect. 4 we consider the interacting theory and we show that generically the perturbation theory of the interacting minisuperspace model does not exist. We then study the minisuperspace model non perturbatively and show that it does exist. The model exhibits two different behaviors: either it is dominated by the combination of kinetic and interaction terms or it is dominated by the time dependent harmonic oscillator (in the appropriate variable) term alone.

Finally in Sect. 5 we discuss what this means for the divergences in string theory. In nuce string theory is well, at least at tree level but the non Hamiltonian perturbation theory has troubles. Moreover we point out that the usual approach to orbifolds used in string theory is not on very sound basis when temporal orbifolds are considered since the orbifold generators are dynamical generators, except for Null Shift Orbifold in light-cone gauge.

## 2 The background

Our starting point is to consider a class of backgrounds which have a space-like singularity and on these backgrounds write down the simplest interacting scalar theory.

Previous results from the analysis of issues in open string amplitudes in these backgrounds [14, 31] hint toward the fact the all troubles derive from special field configurations to which we restrict. In particular this means that we restrict these theories to space independent but time dependent fields in the space-like singularity case.

More precisely this paper we are going to consider the following family of backgrounds.

### 2.1 Kasner-like metrics

The metric we consider is a generalization of the original Kasner metric and reads

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} |t|^{2p(i)} R_{(i)}^2 (dx^i)^2, \quad 0 \leq x^i < 2\pi, \quad (2.1)$$

where we consider  $t \in \mathbb{R}$  and not only  $t > 0$  and therefore we have written  $|t|$  since  $p_{(i)} \in \mathbb{R}$ . We have also considered the  $x^i$  to be compact in order to get a well defined minisuperspace approximation of the scalar field as in Eq. (2.3).

The original Kasner metric corresponds to the case where  $\sum_i p_{(i)} = \sum p_{(i)}^2 = 1$  and space is not compact. It requires that at least one  $p_{(i)}$  is negative when at least two  $p_{(i)}$  are different from zero and corresponds to an empty space-time. Another special case is when only  $p_{(1)} = 1$  and corresponds to Milne space.

All these metrics have a singularity at  $t = 0$  which is the target of our investigation. They have generically also a singularity for  $|t| \rightarrow \infty$  when some  $p$  is negative. When all  $p$  are positive the metric requires repulsive matter.

For generic  $p_{(i)}$ , i.e. not the original Kasner metric this metric is not a consistent string background<sup>1</sup> since  $Ric \neq 0$ .

### 2.2 Interacting scalar models

It is the immediate to write down the action for an interacting real scalar field as

$$S = \int dt \prod_i dx^i \prod_i R_{(i)} |t|^{\sum_i p_{(i)}} \times \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \sum_i \frac{1}{R_{(i)}^2 |t|^{2p_{(i)}}} (\partial_i \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{n} g_n \phi^n \right], \quad n \in 4, 6, \dots \tag{2.2}$$

According to the analysis of string theory on Boost Orbifold [14,31] the problems for this theory derive from the field configurations where the field depends on time only. Restricting to this configuration we get the quantum mechanical model

$$S = \prod_i (2\pi R_{(i)}) \int dt |t|^{2A} \left[ \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{n} g_n \phi^n \right], \tag{2.3}$$

where we have defined  $2A = \sum_i p_{(i)}$  for compactness. We consider only the case where  $A > 0$ .

<sup>1</sup> We can choose as vielbein  $E^t = dt, E^i = R_{(i)} |t|^{p_{(i)}} dx^i$ , then from  $dE^a = \omega^a_b E^b$  we get the only non vanishing spin connection  $\omega^i_t = \omega^t_i = -p_{(i)} \frac{|t|^{p_{(i)}}}{t} dx^i$ . Finally the only non vanishing components of the Riemann form  $R^a_b = d\omega^a_b - \omega^a_c \omega^c_b$  are  $R^t_i = -\frac{p_{(i)}(p_{(i)}-1)}{t^2} E^t E^i$  and  $R^i_j = -\frac{p_{(i)}p_{(j)}}{t^2} E^i E^j$ . Then  $Ric_t = \frac{\sum_i p_{(i)}(p_{(i)}-1)}{t^2}$  and  $Ric_{ij} = \delta_{ij} \frac{1}{t^2} p_{(i)}(1 - \sum_l p_{(l)})$ .

### 3 The simplest example of failure of the perturbative expansion in interaction picture: the time dependent harmonic oscillator

In this section we would like to discuss how the usual perturbative expansion in interaction picture may completely break down when the interaction Hamiltonian has time singularities. This may happen despite the complete model is well defined.

In particular the model we want to consider is

$$L_R = |t|^{2A} \left( \frac{1}{2} \dot{y}^2 - \frac{1}{2} \omega^2 y^2 \right), \tag{3.1}$$

which corresponds to the non interacting scalar on Kasner metrics. Two special cases are  $A = 0$  and  $A = \frac{1}{2}$  and both correspond to the flat space but in Minkowski and Milne (Boost orbifold) coordinates. Upon a change of coordinates as

$$x = |t|^A y, \tag{3.2}$$

we get

$$L_B = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \left( \omega^2 + \frac{k}{t^2} \right) x^2 + \frac{d}{dt} \left( \frac{1}{2} \frac{A}{t} x^2 \right), \quad k = A(1 - A) \in \left( -\infty, \frac{1}{4} \right) \tag{3.3}$$

The total derivative is uninfluential at the classical level while at the quantum it implies a relative time dependent phase for the wave function in the two coordinate systems see Eq. (3.30).

Notice that when  $k$  is negative ( $A > 1$  or  $A < 0$ ) the potential is unbounded from below but despite this the full model is well defined. That this may happen is not a surprise since the hydrogen atom exists and has an unbounded potential. On the other side in the flat space  $A = 0, \frac{1}{2}$  the potential is always bounded from below. In particular the  $A = \frac{1}{2}$  case is the Milne space which is a subset of Minkowski space and even so the model has a singular potential when there is only one space dimension otherwise it is the original Kasner solution.

This model emerges besides the obvious case of the non interacting scalar in Kasner-like metrics mentioned above also in the following cases :

1. The particle or the string in a certain pp-wave background in Brinkmann coordinates that is described by the metric

$$ds_B^2 = -2du dv + \sum_{I=1}^{D-2} A_I (A_I - 1) (x^I)^2 \frac{1}{u^2} du^2 + \sum_{I=1}^{D-2} (dx^I)^2. \tag{3.4}$$

Notice however that a purely gravitational string background, i.e. with trivial dilaton and Kalb–Ramond, must be a Ricci flat background so we need to impose  $\sum_I A_I(A_I - 1) = 0$  if we want a consistent model propagating in this background. The particle action in light-cone gauge  $u = \tau$  reads

$$S_{LC} = \int d\tau \left[ \frac{-1}{e} \dot{v} + \sum_{I=1}^{D-2} \left( \frac{-1}{e} (\dot{x}^I)^2 + \frac{-1}{e} \frac{A_I(A_I - 1)}{\tau^2} (x^I)^2 \right) \right]. \tag{3.5}$$

Since  $e$  is constant on shell, any  $x^I$  has the action (3.3) with  $\omega^2 = 0$ . The case with  $\omega^2 \neq 0$  is recovered when string is considered. In fact the previous  $x^I$  are the string zero modes and the string non zero modes  $x_n^I$  have  $\omega^2 \propto n^2$ .

2. The modes of the scalar field in de Sitter universe in conformal time. If we consider the FLRW metric

$$ds^2 = dt^2 - a^2(t) \sum_{i=1}^{D-1} (dx^i)^2 = a^2(\eta) \left( d\eta^2 - \sum_{i=1}^{D-1} (dx^i)^2 \right), \tag{3.6}$$

with  $d\eta = \frac{1}{a(t)} dt$ . For de Sitter we have  $a_{dS}(t) = e^{Ht}$  so that  $a_{dS}(\eta) = -\frac{1}{H\eta}$  with  $-\infty < \eta < 0^-$ . The real scalar action is then

$$S_{FLRW} = \int dt d^{D-1}x a(t)^D \left[ \frac{1}{2} (\dot{\phi})^2 - \frac{1}{2} a(t)^{-2} (\partial_i \phi)^2 - \frac{1}{2} m^2 \phi^2 \right] = \int d\eta d^{D-1}x \left[ \frac{1}{2} (\dot{\chi})^2 - \frac{1}{2} (\partial_i \chi)^2 - \frac{1}{2} \left( m^2 a^2 - \frac{D-2}{2} \frac{a''(\eta)}{a(\eta)} - \frac{(D-2)(D-4)}{4} \left( \frac{a'(\eta)}{a(\eta)} \right)^2 \right) \chi^2 \right], \tag{3.7}$$

where we defined  $\phi(t, x) = a^{1-\frac{D}{2}}(\eta) \chi(\eta, x^i)$  and  $a'(\eta) = \frac{da(\eta)}{d\eta}$ . Performing the Fourier transform w.r.t. to the space coordinates we get

$$S_{FLRW} = \int d\eta d^{D-1}k \left[ \frac{1}{2} |\tilde{\chi}'(\eta, k)|^2 - \frac{1}{2} \left( k_i^2 + m^2 a^2 - \frac{D-2}{2} \frac{a''(\eta)}{a(\eta)} \right) |\tilde{\chi}(\eta, k)|^2 \right],$$

$$- \frac{(D-2)(D-4)}{4} \left( \frac{a'(\eta)}{a(\eta)} \right)^2 \left| \tilde{\chi}(\eta, k) \right|^2 \tag{3.8}$$

which in de Sitter space becomes

$$S_{dS} = \int d\eta d^{D-1}k \left[ \frac{1}{2} |\tilde{\chi}'(\eta, k)|^2 - \frac{1}{2} \left( k_i^2 + \frac{m^2}{H^2} \frac{1}{\eta^2} - \frac{D(D-2)}{4} \frac{1}{\eta^2} \right) |\tilde{\chi}(\eta, k)|^2 \right], \tag{3.9}$$

which shows that the modes again have action (3.3) but with  $\eta < 0$  so the model we consider is a kind of cyclic de Sitter.

3. The particle in Vaidya metric with linear mass.

### 3.1 Failure of the perturbative expansion of the evolution operator in the interaction picture

Let us now consider the Hamiltonian corresponding to (3.3) as the sum of the usual harmonic oscillator and a quadratic time dependent interaction term. The splitting we perform between the unperturbed Hamiltonian and the perturbative part is somewhat artificial but it is chosen in order to get the simplest example as possible and then discuss the issues in the simplest context.

Explicitly in Schroedinger picture we have

$$H_S(t) = H_{S0}(t) + H_{S1}(t) \tag{3.10}$$

$$H_{S0}(t) = \frac{p_S^2}{2} + \frac{1}{2} \omega^2 x_S^2, \quad H_{S1}(t) = \frac{k}{t^2} x_S^2.$$

Obviously the perturbation Hamiltonian is dominant for small  $t$  and therefore one can expect that perturbation theory be asymptotic as it happens in Stark effect. However we find a complete breakdown of perturbation theory and not an asymptotic series.

The interaction picture is obtained from Schroedinger equation

$$i \frac{\partial}{\partial t} |\psi_S(t, t_0)\rangle = H_S(t) |\psi_S(t, t_0)\rangle, \tag{3.11}$$

by defining

$$|\psi_I(t, t_0)\rangle = U_{0S}(t_0, t) |\psi_S(t, t_0)\rangle, \quad U_{0S}(t_0, t) = T e^{-i \int_{t_0}^t dt' H_{0S}(t')}, \tag{3.12}$$

where  $U_{0S}$  is the evolution operator for the “free” Hamiltonian  $H_{0S}$ . The new state  $|\psi_I(t, t_0)\rangle$  then evolves as

$$i \frac{\partial}{\partial t} |\psi_I(t, t_0)\rangle = H_I(t, t_0) |\psi_I(t, t_0)\rangle,$$

$$H_I(t, t_0) = U_{0S}(t_0, t)H_{1S}(t)U_{0S}(t, t_0). \tag{3.13}$$

The Schroedinger equation in interaction picture has then formal and perturbative solution

$$\begin{aligned} |\psi_I(t, t_0)\rangle &= T e^{-i \int_{t_0}^t dt' H_I(t', t_0)} |\psi_I(t_0, t_0)\rangle \\ &= \left( 1 - i \int_{t_0}^t dt' H_I(t', t_0) + \dots \right) |\psi_I(t_0, t_0)\rangle. \end{aligned} \tag{3.14}$$

If we apply this formalism to our specific case we obtain the interaction Hamiltonian

$$\begin{aligned} H_I(t, t_0) &= -\frac{k}{4\omega t^2} \left( e^{2i\omega(t-t_0)} a_S^{\dagger 2} \right. \\ &\quad \left. + e^{-2i\omega(t-t_0)} a_S^2 - a_S^{\dagger} a_S - a_S a_S^{\dagger} \right), \end{aligned} \tag{3.15}$$

where we have as usual

$$a_S = \frac{p_S - i\omega x_S}{\sqrt{2\omega}}, \quad [a_S^{\dagger}, a_S] = 1,$$

$$U_{0S}(t, t_0) = e^{-i\omega(a_S^{\dagger} a_S + \frac{1}{2})(t-t_0)}. \tag{3.16}$$

We can then build a basis for the Hilbert space  $\{|n\rangle\}_{n \in \mathbb{N}}$  as

$$a_S |0\rangle = 0, \quad |n\rangle = \frac{a_S^{\dagger n}}{\sqrt{n!}} |0\rangle. \tag{3.17}$$

It is then immediate to see that the first order in perturbative expansion for the evolution operator from a negative  $t_0 < 0$  time to a positive time  $t_1 > 0$  is infinite. Explicitly, if we evolve perturbatively from  $|\psi_I(t_0, t_0)\rangle = |n\rangle$  to  $|\psi_I(t_1, t_0)\rangle$  and we try to expand  $|\psi_I(t_1, t_0)\rangle$  on the basis  $\{|m\rangle\}$  we have

$$\begin{aligned} \langle m | \int_{t_0}^{t_1} dt' H_I(t', t_0) |n\rangle &= -\frac{k}{4\omega} \delta_{m,n} (2m+1) \int_{t_0}^{t_1} dt \frac{1}{t^2} \\ &\quad - \frac{k}{4\omega} \delta_{m,n+2} \sqrt{m(m-1)} \int_{t_0}^{t_1} dt \frac{e^{2i\omega(t-t_0)}}{t^2} \\ &\quad - \frac{k}{4\omega} \delta_{m,n-2} \sqrt{(m+2)(m+1)} \int_{t_0}^{t_1} dt \frac{e^{-2i\omega(t-t_0)}}{t^2}. \end{aligned} \tag{3.18}$$

This shows that not only the amplitude is divergent but that we cannot expand  $|\psi_I(t_1, t_0)\rangle$  on the Hilbert basis moreover the divergence cannot be reabsorbed into a c-number shift of the Hamiltonian since all coefficients depend on the states. For later use we notice that to this order of perturbation we have

$$\langle m | \int_{t_0}^{t_1} dt' H_I(t', t_0) |n\rangle = \int_{t_0}^{t_1} \langle m_S(t', t_0) | H_{1S}(t') |n_S(t', t_0)\rangle, \tag{3.19}$$

i.e. we can actually use the Schroedinger states and Hamiltonian without actually computing the corresponding objects in the interaction picture.

### 3.2 The complete theory is well defined: the $H_B$ case

Given the previous failure of the perturbative expansion one can wonder whether the theory exists across the singularity. The answer as we show is affirmative. The same problem has been considered before in [9–12, 22–25, 28, 35] but our point of view is slightly different since this is not the final research target of this paper but we want anyhow to show that we can traverse the singularity and then use this solution for the interacting models.

Even if we are actually interested in adding quartic and higher interactions to  $L_R$  we will perform the analysis for  $L_B$  since it looks more familiar and then map it to  $L_R$  using a time dependent unitary transformation.

The time dependent harmonic oscillator

$$i \partial_t \psi(x, t) = -\frac{1}{2} \partial_x^2 \psi(x, t) + \frac{1}{2} \left( \omega^2 + \frac{k}{t^2} \right) \psi(x, t), \tag{3.20}$$

can be solved exactly using complex classical solutions with a well defined normalization. We review the derivation for completeness in Appendix 1 where we give also more details which are not relevant for the present discussion. The main result is then that the generating function of a possible complete set<sup>2</sup> of wave functions is

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \psi_{n\{t_0\}}(x, t, t_0) &= \sqrt{\frac{1}{2\pi}} \frac{1}{\sqrt{\mathcal{X}(t)}} e^{i \frac{1}{2} \frac{\dot{\mathcal{X}}(t)}{\mathcal{X}(t)} x^2 + \frac{1}{\mathcal{X}(t)} x z - \frac{1}{2} \frac{\mathcal{X}^*(t)}{\mathcal{X}(t)} z^2}, \end{aligned} \tag{3.21}$$

where we have introduced the complex classical solution  $\mathcal{X}(t)$  and its normalization condition

$$\begin{aligned} \ddot{\mathcal{X}}(t) + \Omega^2(t) \mathcal{X}(t) &= 0, \\ \mathcal{X}^* \dot{\mathcal{X}} - \mathcal{X} \dot{\mathcal{X}}^* &= i. \end{aligned} \tag{3.22}$$

We can now solve perturbatively the classical equations of motion around  $t = 0$ .

An issue which arises is the continuation across the singularity but the normalization condition required for the quantum model and “continuity” fix it (see also [36] for the case  $A = \frac{1}{2}$ ).

<sup>2</sup> Different sets are associated with different instantaneous vacua and are obtained by different  $\chi$  which still satisfies the due equations.

Let us start considering the asymptotic behavior for  $t \rightarrow 0^+$  as  $\mathcal{X} \sim t^a$  with  $t > 0$ . It is immediate to find the equation

$$a^2 - a + k = 0 \iff a \in \{A, 1 - A\}, \tag{3.23}$$

so that the leading behavior is

$$\mathcal{X}(t) = c_0(\omega t)^A(1 + O(t^2)) + c_1(\omega t)^{1-A}(1 + O(t^2)), t > 0. \tag{3.24}$$

The normalization condition then implies

$$-(2A - 1)\omega|c_1|^2 \Im\left(\frac{c_0}{c_1}\right) = -\frac{1}{2}. \tag{3.25}$$

Let us consider the case  $A > \frac{1}{2} > 1 - A$  since  $A < \frac{1}{2} < 1 - A$  is obtained by swapping  $A \leftrightarrow 1 - A$ . Then the previous condition implies that the wave functions are normalizable since ( $t > 0$ )

$$\begin{aligned} \psi_0(x, t) &\sim \frac{1}{\sqrt{|\omega t|^{1-A}}} e^{i\frac{1}{2}\left(\frac{1-A}{t} + (2A-1)\omega\frac{c_0}{c_1}|\omega t|^{2(A-1)}\right)x^2} \\ \iff |\psi_0(x, t)|^2 &\sim \frac{1}{|\omega t|^{1-A}} e^{-(2A-1)\omega\Im\left(\frac{c_0}{c_1}\right)|\omega t|^{2(A-1)}x^2}. \end{aligned} \tag{3.26}$$

As discussed in appendix around Eq. (A.27) this is not by chance: the normalization condition on  $\mathcal{X}$  always implies the normalizability of the wave functions.

Let us exam the solution for  $t < 0$ . One would be tempted to write exactly the same Eq. 3.24 with the substitution  $t \rightarrow -t$ . However this would lead to a different normalization condition (3.25). The difference being an overall sign in the left hand side of the normalization equation, i.e.  $+\frac{1}{2}$  in stead of  $-\frac{1}{2}$ . Therefore the proper asymptotic behavior valid for all  $t$  is either

$$\mathcal{X}(t) = c_0|\omega t|^A(1 + O(t^2)) + c_1\omega t|\omega t|^{-A}(1 + O(t^2)),$$

or

$$\begin{aligned} \mathcal{X}(t) \\ = c_0\omega t|\omega t|^{A-1}(1 + O(t^2)) + c_1|\omega t|^{1-A}(1 + O(t^2)). \end{aligned} \tag{3.27}$$

Since this is a classical solution we may expect that the trajectory is continuous then for  $A > 1$  comparing  $t|t|^{-A}$  and  $|t|^{1-A}$  we realize that only the latter is continuous. Hence the true solution is (3.27). Because of this the previous expression for the wave function (3.26) where we took care of distinguish between  $t$  and  $|t|$  is valid for all  $t$  values.

As discussed in Appendix 1 the previous choice can also be obtained regularizing the time dependent pulsation  $\Omega^2(t) = \omega^2 + \frac{k}{t^2}$ .

It is also possible and instructive to use the WKB expansion. We write  $\psi(x, t) = e^{iS(x,t)}$  so that we have to solve the equation

$$\partial_t S(x, t) + \frac{1}{2}(\partial_x S(x, t))^2 + \frac{1}{2}\left(\omega^2 + \frac{k}{t^2}\right) - i\frac{1}{2}\partial_x^2 S(x, t) = 0. \tag{3.28}$$

This is done in Appendix 1.

Notice that (3.26) has two completely different behaviors as  $t \rightarrow 0$ .

$$|\psi_0(x, t)|^2 \sim_{t \rightarrow 0} \begin{cases} 0 & A > 1 \\ \infty & A < 1 \end{cases}. \tag{3.29}$$

This can be understood considering the classical trajectory which behaves as  $x \sim |t|^{min(A, 1-A)}$ . For  $A > 1$  it diverges but the direction depends on the initial  $\dot{x}$  which quantum mechanically cannot be fixed therefore the quantum state is spread over all the possible values of  $x$ . This is shown in Fig. 1a, b. Notice that the classical trajectory (not the complex one used in computing the quantum wave function) is not well defined through  $t = 0$  since we can require the continuity of the trajectory but it is difficult if not impossible to relate the velocity before and after the singularity. On the contrary the quantum theory is well defined since we can find a well defined basis of wave functions.

Differently for  $A < 1$  the classical solution has a fixed point  $x(0) = 0$  and therefore the wave function is a  $\delta(x)$ . This is shown in Fig. 2a, b.

Finally notice that the wildly oscillating phase in (3.26) is not an issue as hypothesized in [9–12], on the contrary as shown in [31] it is a virtue since it helps the convergence of the integrals in the distributional sense (see also [37]).

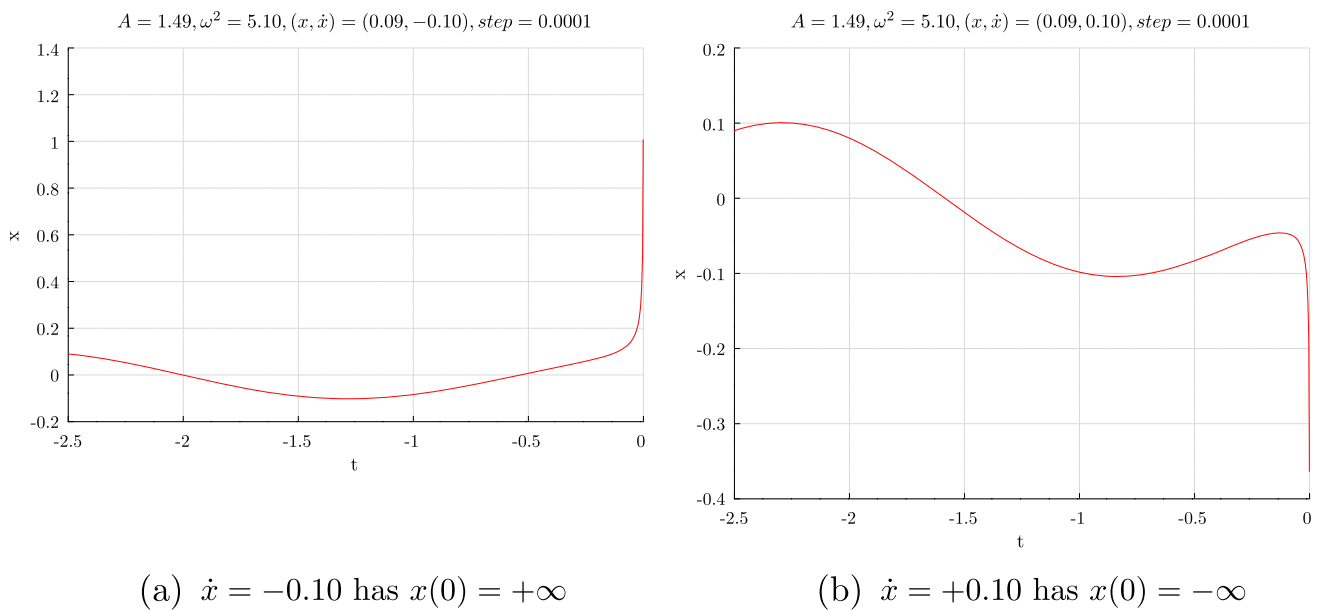
### 3.3 Relation between $L_R$ and $L_B$ non interacting models

While at the classical level the two models are related as described before by a simple change of coordinates and a boundary term, at the quantum level we have

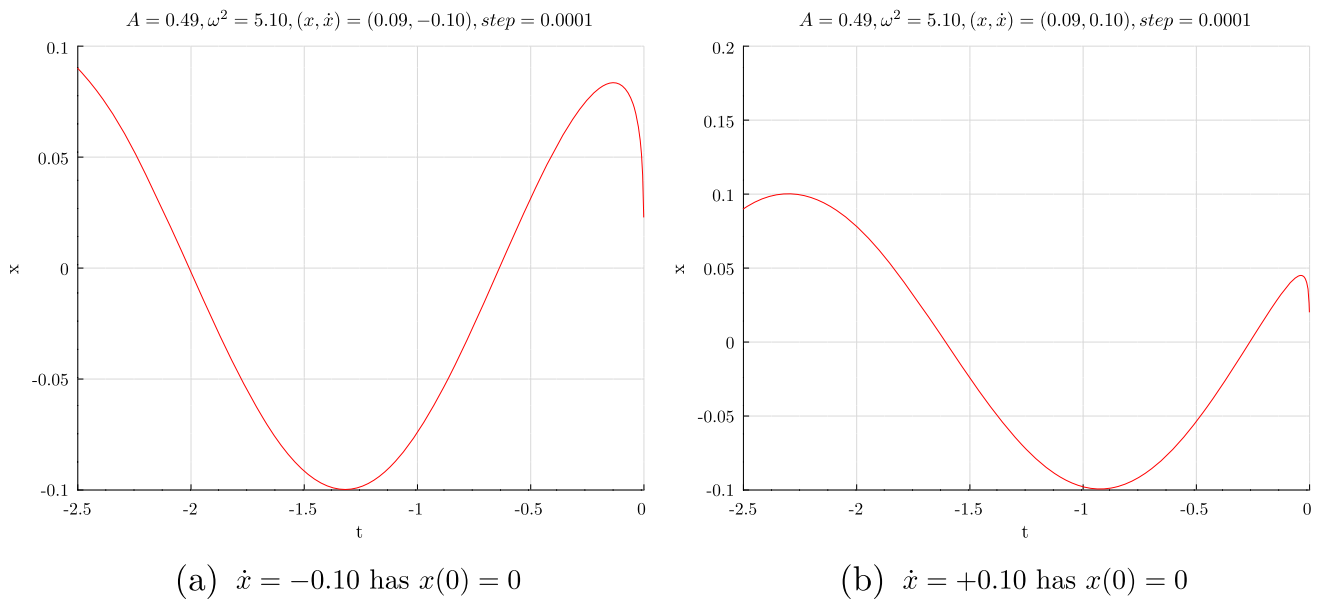
$$\psi_B(x, t) = |t|^{-\frac{1}{2}A} e^{i\frac{1}{2}A\frac{x^2}{t}} \psi_R(y = |t|^{-A}x, t). \tag{3.30}$$

This can be obtained in two different ways. Both start from the Hamiltonians

$$\begin{aligned} H_R &= \frac{p_y^2}{2|t|^{2A}} + \frac{1}{2}\omega^2|t|^{2A}y^2 \\ H_B &= \frac{p^2}{2} + \frac{1}{2}\left(\omega^2 + \frac{k}{t^2}\right)x^2. \end{aligned} \tag{3.31}$$



**Fig. 1** Classical motion for  $L_B$  with  $A > 1$  has two possible asymptotic behaviors



**Fig. 2** Classical motion for  $L_B$  with  $A < 1$  has only one possible asymptotic behavior

The first method is a sequence of transformations on the Schroedinger equation. We first change variables from

$$\begin{cases} x = |t|^A y \\ \tilde{t} = t \end{cases} \Rightarrow \begin{cases} \frac{\partial}{\partial \tilde{t}} = \frac{\partial}{\partial t} + \frac{Ax}{\tilde{t}} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} = |t|^A \frac{\partial}{\partial x} \end{cases} \quad (3.32)$$

Then the  $H_R$  Schroedinger equation becomes

$$i \frac{\partial}{\partial \tilde{t}} \hat{\psi}(x, \tilde{t}) = \left( -\frac{1}{2} \partial_x^2 + \frac{1}{2} \omega^2 x^2 - i \frac{Ax}{\tilde{t}} \partial_x \right) \hat{\psi}(x, \tilde{t}), \quad (3.33)$$

with  $\psi_R(y, t) = \hat{\psi}(x, \tilde{t})$ . However this equation is not a Schrodinger equation since the would be Hamiltonian is not Hermitian because of the term  $-i \frac{Ax}{\tilde{t}} \partial_x$ . To get an Hermitian Hamiltonian we redefine  $\hat{\psi}(x, \tilde{t}) = |\tilde{t}|^{\frac{1}{2}A} \psi_I(x, \tilde{t})$ . Notice that the factor  $|\tilde{t}|^{\frac{1}{2}A}$  is the factor one could expect from the measure due to the change  $x = |t|^A y$ . We get then the intermediate Schroedinger equation

$$i \frac{\partial}{\partial \tilde{t}} \psi_I(x, \tilde{t}) = \left[ \frac{1}{2} \left( -i \partial_x + \frac{Ax}{\tilde{t}} \right)^2 \right]$$

$$+ \frac{1}{2} \left( \omega^2 - \frac{A^2}{2t^2} \right) x^2 \Big] \psi_I(x, \tilde{t}), \tag{3.34}$$

with  $\psi_R(y, t) = |\tilde{t}|^{\frac{1}{2}A} \psi_I(x, \tilde{t})$ . Finally we make a further redefinition as  $\psi_I(x, \tilde{t}) = e^{-i\frac{1}{2}\frac{A^2}{t^2}x^2} \psi_B(x, \tilde{t})$  in order to have a canonical kinetic term. We finally get the desired result

$$i \frac{\partial}{\partial \tilde{t}} \psi_B(x, \tilde{t}) = \left[ -\frac{1}{2} \partial_x^2 + \frac{1}{2} \left( \omega^2 + \frac{A - A^2}{2\tilde{t}^2} \right) x^2 \right] \psi_B(x, \tilde{t}), \tag{3.35}$$

where the relation between  $\psi_R$  and  $\psi_B$  is the one given above in (3.30).

The second method is operatorial. The first step is to use a unitary transformation which implements

$$\begin{cases} x = |t|^A y = U_{R \rightarrow I}^\dagger y U_{R \rightarrow I} \\ p = \frac{p_y}{|t|^A} = U_{R \rightarrow I}^\dagger p_y U_{R \rightarrow I} \end{cases} \Rightarrow U_{R \rightarrow I} = e^{i \ln(|t|^A) \frac{1}{2} \{y, p_y\}} = |t|^{\frac{1}{2}A} |t|^{iA y p_y}. \tag{3.36}$$

We then get the intermediate Hamiltonian<sup>3</sup>

$$\begin{aligned} H_I &= U_{R \rightarrow I}^\dagger H_R U_{R \rightarrow I} + i \dot{U}_{R \rightarrow I}^\dagger U_{R \rightarrow I} \\ &= \frac{1}{2} \left( p_y - \frac{A}{t} y \right)^2 + \frac{1}{2} \left( \omega^2 - \frac{A^2}{t^2} \right) y^2. \end{aligned} \tag{3.37}$$

With a further unitary transformation

$$\begin{cases} y = U_{I \rightarrow B}^\dagger y U_{I \rightarrow B} \\ p_y - \frac{A}{t} y = U_{I \rightarrow B}^\dagger p_y U_{I \rightarrow B} \end{cases} \Rightarrow U_{I \rightarrow B} = e^{-i \frac{1}{2} \frac{A}{t} y^2}, \tag{3.38}$$

used to make the kinetic term canonical we finally get the desired result. Explicitly

$$\begin{aligned} H_B &= U_{I \rightarrow B}^\dagger H_I U_{I \rightarrow B} + i \dot{U}_{I \rightarrow B}^\dagger U_{I \rightarrow B} \\ &= \frac{1}{2} p_y^2 + \frac{1}{2} \left( \omega^2 + \frac{A - A^2}{t^2} \right) y^2, \end{aligned} \tag{3.39}$$

so that

$$|\psi_B(t)\rangle = U_{I \rightarrow B}^\dagger U_{R \rightarrow I}^\dagger |\psi_R(t)\rangle, \tag{3.40}$$

which again reproduces (3.30).

<sup>3</sup> The term  $i \dot{U}^\dagger U$  is obtained from the Schroedinger equation as follows. Set  $|\psi_I(t)\rangle = U^\dagger(t) |\psi_R(t)\rangle$  then from  $i \partial_t |\psi_R(t)\rangle = H_R |\psi_R(t)\rangle$  we get  $i \partial_t |\psi_I(t)\rangle = H_I |\psi_I(t)\rangle$  with  $H_I = U^\dagger H_R U + i \dot{U}^\dagger U$ .

### 3.4 Explicit mapping of the quantum $H_B$ solutions to $H_R$ solutions

Using the explicit mapping in (3.30) we can write the generating function for a complete set of solutions for  $H_R$  as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \psi_{R n(t_0)}(y, t, t_0) \\ &= \sqrt[4]{\frac{1}{2\pi}} \frac{1}{\sqrt{\mathcal{X}_R(t)}} e^{i \frac{1}{2} \frac{\dot{\mathcal{X}}_R(t)}{\mathcal{X}_R(t)} x^2 + \frac{1}{\mathcal{X}_R(t)} x z - \frac{1}{2} \frac{\mathcal{X}_R''(t)}{\mathcal{X}_R(t)} z^2}, \end{aligned} \tag{3.41}$$

where we have introduced the complex classical solution  $\mathcal{X}_R(t) = |t|^{-A} \mathcal{X}(t)$  in analogy to  $y = |t|^{-A} x$ . Its e.o.m and normalization condition follow from the  $\mathcal{X}$  ones and read

$$\begin{aligned} |t|^{-2A} \frac{d}{dt} \left( |t|^{2A} \dot{\mathcal{X}}_R(t) \right) + \omega^2 \mathcal{X}_R(t) &= 0, \\ \mathcal{X}_R^* \dot{\mathcal{X}}_R - \mathcal{X}_R \dot{\mathcal{X}}_R^* &= i |t|^{-2A}. \end{aligned} \tag{3.42}$$

In particular the ‘‘ground state’’ behaves as

$$\begin{aligned} \psi_{R0}(y, t) &\sim |t|^{\frac{1}{2}(A-1)} e^{i \frac{1}{2} \left( (1-A) \text{sgn}(t) |t|^{2A-1} + (2A-1) \frac{c_0}{c_1} \omega^{2A-1} |t|^{2(2A-1)} \right) y^2} \\ &\iff |\psi_{R0}(y, t)|^2 \sim |t|^{2A-1} e^{-(2A-1) \omega \Im \left( \frac{c_0}{c_1} \right) \omega^{2A-1} |t|^{2(2A-1)} y^2}. \end{aligned} \tag{3.43}$$

The wave functions always vanish for  $t \rightarrow 0$  while still being normalizable because the classical particle is diffused on the entire  $y$  axis since  $y \sim |t|^{-2A}$ . This diverges but the direction depends on the initial  $\dot{y}$  which quantum mechanically cannot be fixed.

### 4 Interacting quantum and classical mechanical models

We can now pass to exam what happens when we add interactions to the Kasner metrics. The corresponding quantum mechanical models are

$$L_R = |t|^{2A} \left( \frac{1}{2} \dot{y}^2 - \frac{1}{2} \omega^2 y^2 - \frac{g}{n} y^n \right), \quad g > 0, n \in \{4, 6, \dots\}, \tag{4.1}$$

which become in  $x$  coordinate

$$L_B = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \left( \omega^2 + \frac{k}{t^2} \right) x^2 - \frac{g}{n} \frac{1}{|t|^{A(n-2)}} x^n. \tag{4.2}$$

These models show a strange time dependence in the interaction term which can be explained by noticing that the change from  $y$  to  $x$  in quantum mechanical models cannot be implemented on the metric.

The B models suggest that the interaction is dominant for small  $\omega t$ . This is not evident in R models and it is not always true.



**Table 1** Summary of results for the interacting models ( $n$  even,  $n \geq 4$ )

A	Perturbatively is ok?	Dominating terms in $L$
$2A > \frac{n+2}{n-2}$	NO	$\dot{z}^2 - g_n z^n$ (bounded)
$\frac{n+1}{n-1} < 2A < \frac{n+2}{n-2}$	NO	$\dot{z}^2 + \frac{1}{l^2} z^2$ (unbounded)
$\frac{n+2}{n} < 2A < \frac{n+1}{n-1}$	YES	$\dot{z}^2 + \frac{1}{l^2} z^2$ (unbounded)
$0 < 2A < \frac{n+2}{n}$	YES	$\dot{z}^2 - \frac{1}{l^2} z^2$ (bounded)

Using the results from the previous section on the behavior of the wave function at  $t = 0$  we can now see that the perturbative expansion of the evolution matrix in interaction picture does not exist. Since B models are unitarily equivalent to R models as explicitly shown in (3.40) the results we get for B models are valid for R models. Explicitly for B models we get

$$\begin{aligned} & \int dt' \langle \psi_B(t') | H_{BS1}(t') | \psi_B(t') \rangle \\ & \sim \int dt' \frac{1}{|t|^{A(n-2)}} \int dx x^n |t'|^{-\alpha} e^{-|t'|^{-2\alpha} x^2} \\ & \sim \int dt' \frac{1}{|t'|^{A(n-2)}} \left( \frac{1}{|t'|^{-2\alpha}} \right)^{\frac{n}{2}}, \end{aligned} \tag{4.3}$$

which has an unavoidable divergence for  $A > 1$  and  $-\alpha = A - 1 > 0$ . More precisely the integral is divergent for  $2A > \frac{n+1}{n-1}$ . Anticipating the results (discussed below Eq. (4.10) for the classical case and around Eq. (4.21) for the quantum case) this means that when the behavior is dominated by the interaction, i.e.  $2A > \frac{n+2}{n-2}$  the integral is divergent. This integral may also be divergent when the theory is dominated by the unbounded time dependent harmonic oscillator (in the appropriate variable), i.e.  $\frac{n+1}{n-1} < 2A < \frac{n+2}{n-2}$  (see Eq. (4.16) and Eq. (4.24)). The results are summarized in Table 1. It is noteworthy that the original Krasner background, which is also a string background admits a good perturbation theory since  $2A = 1$ .

4.1 The classical motion

The classical e.o.m for the R models reads

$$|t|^{-2A} \frac{d}{dt} \left( |t|^{2A} \frac{dy}{dt} \right) + \omega^2 y + g y^{n-1} = 0. \tag{4.4}$$

This equation is very close to the Emden-Fowler equation

$$\frac{d}{dt} \left( t^\mu \frac{dy}{dt} \right) + t^\nu y^m = 0. \tag{4.5}$$

This equation is treated in [38] with the result that (with the appropriate range of the parameters  $\mu, \nu$  which can be easily obtained from our treatment) the solution exhibits an oscillating behavior with maxima and minima diverging with a

power law. Instead of the analysis presented there we introduce a different approach which is simpler and clearer based on the action. We apply immediately this approach to the R models whose action is

$$S_R = \int_I dt |t|^{2A} \left( \frac{1}{2} \dot{y}^2 - \frac{1}{2} \omega^2 y^2 - \frac{g}{n} y^n \right), \tag{4.6}$$

where  $I$  is the integration interval. We look for a change of variables as

$$t = \text{sgn}(\tilde{t}) |\tilde{t}|^\beta, \quad y = |\tilde{t}|^\alpha z, \tag{4.7}$$

so that the kinetic term and the interaction term  $z^n$  have coefficients independent of the new time  $\tilde{t}$ . Explicitly we get

$$\begin{aligned} S_R = \int_{\tilde{I}} d\tilde{t} \left\{ \frac{1}{2} \frac{1}{\beta} |\tilde{t}|^{(2A-1)\beta+2\alpha+1} \left( \frac{dz}{d\tilde{t}} - \frac{\alpha}{\tilde{t}} z \right)^2 \right. \\ \left. - \frac{1}{2} \beta \omega^2 |\tilde{t}|^{(2A+1)\beta+2\alpha-1} z^2 - \beta \frac{g}{n} |\tilde{t}|^{(2A+1)\beta+n\alpha-1} z^n \right\}, \end{aligned} \tag{4.8}$$

where  $\tilde{I}$  is the image of the interval  $I$ . We can now require a time independent kinetic and  $z^n$  term imposing

$$(2A - 1)\beta + 2\alpha + 1 = 0, \quad (2A + 1)\beta + n\alpha - 1 = 0, \tag{4.9}$$

which can be solved as

$$\alpha = \frac{4A}{2(n-2)A - (n+2)}, \quad \beta = -\frac{n+2}{2(n-2)A - (n+2)}, \tag{4.10}$$

and get

$$S_R = \int_{\tilde{I}} d\tilde{t} \left\{ \frac{1}{2} \frac{1}{\beta} \left( \frac{dz}{d\tilde{t}} - \frac{\alpha}{\tilde{t}} z \right)^2 - \beta \frac{1}{2} \omega^2 |\tilde{t}|^{(2-n)\alpha} z^2 - \beta \frac{1}{n} g z^n \right\}. \tag{4.11}$$

The previous action can be recast in a more standard form by integrating by part the term proportional to  $\frac{dz}{d\tilde{t}} z = \frac{1}{2} \frac{dz^2}{d\tilde{t}}$  to get

$$\begin{aligned} S_R = & + \frac{1}{2} \frac{\alpha}{\beta} \frac{1}{\tilde{t}} z^2 \Big|_{\tilde{I}} \\ & + \int_{\tilde{I}} d\tilde{t} \left\{ \frac{1}{2} \frac{1}{\beta} \left( \frac{dz}{d\tilde{t}} \right)^2 + \left[ \frac{1}{2} \frac{\alpha(\alpha+1)}{\beta} - \beta \frac{1}{2} \omega^2 \frac{1}{|\tilde{t}|^{(n-2)\alpha}} \right] z^2 - \beta \frac{g}{n} z^n \right\}. \end{aligned} \tag{4.12}$$

If  $\alpha > 0 > \beta$ , i.e.  $2A > \frac{n+2}{n-2}$  the interval around the singularity  $t = 0$   $I = [-\epsilon_1, +\epsilon_2]$  is mapped into an interval around  $|\tilde{t}| = \infty$  as  $\tilde{I} = [-\infty, -\frac{1}{\epsilon_1}] \cup [\frac{1}{\epsilon_2}, +\infty]$  then the  $z^2$  terms are subdominant since  $|\tilde{t}|^{(2A-1)\beta+2\alpha+1} = \frac{1}{\tilde{t}^2}$  and  $|\tilde{t}|^{(2A+1)\beta+2\alpha-1} = \frac{1}{|\tilde{t}|^{(n-2)\alpha}}$ . Moreover the boundary term is finite.

Under the previous choice of  $\alpha, \beta$  we can approximate the action  $S_R$  for the  $I$  around the singularity simply as<sup>4</sup>

$$S_R \sim \int_{\tilde{t}} d\tilde{t} \left\{ \frac{1}{2} \frac{1}{\beta} \left( \frac{dz}{d\tilde{t}} \right)^2 - \beta \frac{g}{n} z^n \right\}. \tag{4.13}$$

Hence the trajectory  $z(\tilde{t})$  is simply oscillating with period

$$\frac{1}{2} P = \frac{1}{\sqrt{2|\beta|E_z}} \left( \frac{nE_z}{|\beta|g} \right)^{\frac{1}{n}} \int_{-1}^{+1} d\zeta \frac{1}{\sqrt{1-\zeta^n}}, \tag{4.14}$$

where  $E_z$  is the system energy.

Despite this nice feature the crossing of the singularity is not very well defined at the classical level since  $t = 0^\pm$  is mapped to  $\tilde{t} = \pm\infty$  and there the particle is spread over the interval  $[-(\frac{nE_z}{|\beta|g})^{\frac{1}{n}}, (\frac{nE_z}{|\beta|g})^{\frac{1}{n}}]$  in  $z$  coordinate and it is not obvious how to match the position at  $\tilde{t} = +\infty$  with the position at  $\tilde{t} = -\infty$ . This is shown in Fig. 3a, b for  $t \rightarrow 0^-$ , i.e. for  $\tilde{t} \rightarrow -\infty$ . And in a smoother case in Fig. 4a, b.

For the case  $\alpha < 0 < \beta$ , i.e.  $0 < 2A < \frac{n+2}{n-2}$  the behavior of the classical motion is dictated by

$$S_R \sim \int_{-|\tilde{\epsilon}_1|}^{-|\tilde{\epsilon}_2|} d\tilde{t} \frac{1}{\beta} \left\{ \frac{1}{2} \left( \frac{dz}{d\tilde{t}} \right)^2 + \frac{1}{2} \alpha(\alpha+1) \frac{1}{|\tilde{t}|^2} z^2 \right\} \\ \sim \int_{-|\tilde{\epsilon}_1|}^{-|\tilde{\epsilon}_2|} d\tilde{t} \frac{1}{\beta} \left\{ \frac{1}{2} \left( \frac{dz}{d\tilde{t}} \right)^2 + \frac{1}{2} \frac{4A(2nA - (n+2))}{(2(n-2)A - (n+2))^2} \frac{1}{|\tilde{t}|^2} z^2 \right\}, \tag{4.15}$$

because the boundary term does not contribute to the e.o.m we find again a time dependent harmonic oscillator as in Eq. (3.3) but with  $A_{eff}$  (where  $k_{eff} = -\alpha(1 + \alpha) = A_{eff}(1 - A_{eff})$ , i.e.  $A_{eff} = -\alpha$ ) which is always real, explicitly

$$\begin{cases} k_{eff} > \frac{1}{4} & A_{eff} \in \mathbb{C} & \text{not possible} \\ 0 < k_{eff} < \frac{1}{4} & 0 \leq A_{eff} \leq 1 & 2A < \frac{n+2}{n} \\ k_{eff} < 0 & A_{eff} > 1 & \frac{n+2}{n} < 2A < \frac{n+2}{n-2} \end{cases}, \tag{4.16}$$

As usual numerics can be tricky and give the wrong impression: compare the Fig. 5a, b with the same solution extended closer to the origin given in Fig. 6a, b. Both for  $t \rightarrow 0^-$ , i.e. for  $\tilde{t} \rightarrow 0^-$

### 4.2 The quantum interacting models exist

We can now exam the question of what happens to the quantum model. We treat only the wave function approach because it is more intuitive.

Despite the fact the classical motion is not very well defined the quantum system seems to be perfectly fine and generically better behaved than the non interacting one. The last sentence means that we can write a normalizable wave

function which generically vanishes at  $t = 0$  but at slower rate than the non interacting, i.e. time dependent quadratic R theory. The adverb generically refers to the fact that there is a “small” range of parameters where system behavior can be mapped to a time dependent harmonic oscillator with unbounded potential.

Another point to stress is that we have found a possible continuation through the singularity it may be that there are other possibilities as in the free case [36].

In order to show that we start with Schroedinger equation for R model

$$i \partial_t \psi(y, t) = \left[ -\frac{1}{2} \frac{1}{|t|^{2A}} \partial_y^2 + |t|^{2A} \left( \frac{1}{2} \omega^2 y^2 + \frac{g}{n} y^n \right) \right] \psi(y, t), \tag{4.17}$$

and following the previous section on the classical motion we perform the same change of variables as in the classic case (4.7)

$$\begin{cases} \tilde{t} = \text{sgn}(t) |t|^{\frac{1}{\beta}} \\ z = |t|^{-\frac{\alpha}{\beta}} y \end{cases} \Rightarrow \begin{cases} \frac{\partial}{\partial \tilde{t}} = \frac{|\tilde{t}|^{-\beta+1}}{\beta} \left( \frac{\partial}{\partial t} - \frac{\alpha z}{\tilde{t}} \frac{\partial}{\partial x} \right) \\ \frac{\partial}{\partial y} = |\tilde{t}|^{-\alpha} \frac{\partial}{\partial z} \end{cases}, \tag{4.18}$$

along with setting  $\psi(y, t) = |\tilde{t}|^{-\frac{1}{2}\alpha} \tilde{\psi}(z, \tilde{t})$ . The choice of the  $\tilde{t}$  power is made considering the invariance of the probability density  $|\psi(y, t)|^2 dy = |\tilde{\psi}(z, \tilde{t})|^2 dz$ . The Schroedinger equation then becomes

$$i \frac{1}{\beta} \frac{\partial}{\partial \tilde{t}} \tilde{\psi}(z, \tilde{t}) = -\frac{1}{2} |\tilde{t}|^{-(2A-1)\beta-2\alpha-1} \frac{\partial^2}{\partial z^2} \tilde{\psi}(z, \tilde{t}) \\ + \frac{g}{n} |\tilde{t}|^{(2A+1)\beta+n\alpha-1} z^n \tilde{\psi}(z, \tilde{t}) \\ + \frac{1}{2} \omega^2 |\tilde{t}|^{(2A+1)\beta+2\alpha-1} z^2 \tilde{\psi}(z, \tilde{t}) \\ + i \frac{\alpha}{2\beta} \frac{1}{\tilde{t}} \left( z \frac{\partial}{\partial z} + \frac{\partial}{\partial z} z \right) \tilde{\psi}(z, \tilde{t}). \tag{4.19}$$

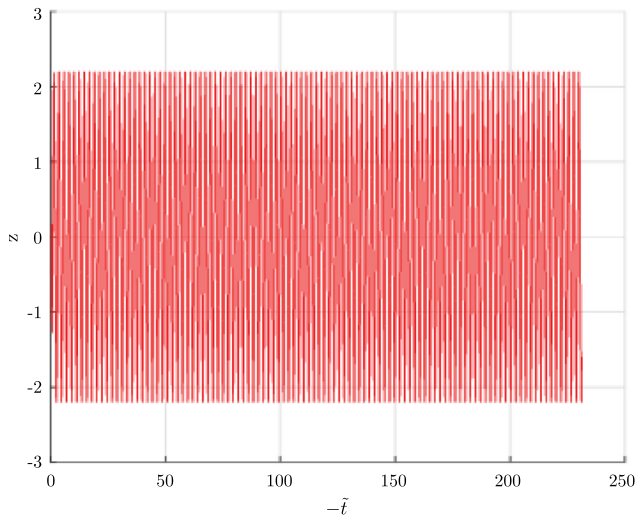
If we require the kinetic and  $z^n$  terms be time independent we get exactly the same solution for  $\alpha, \beta$  as in the classical case (4.10) and the Schroedinger equation becomes

$$i \frac{1}{\beta} \frac{\partial}{\partial \tilde{t}} \tilde{\psi}(z, \tilde{t}) = -\frac{1}{2} \frac{\partial^2}{\partial z^2} \tilde{\psi}(z, \tilde{t}) \\ + \frac{g}{n} z^n \tilde{\psi}(z, \tilde{t}) + \frac{1}{2} \omega^2 \frac{1}{|\tilde{t}|^{(n-2)\alpha}} z^2 \tilde{\psi}(z, \tilde{t}) \\ + i \frac{\alpha}{2\beta} \frac{1}{\tilde{t}} \left( z \frac{\partial}{\partial z} + \frac{\partial}{\partial z} z \right) \tilde{\psi}(z, \tilde{t}), \tag{4.20}$$

which is exactly the Schroedinger equation associated with Eq. (4.11).

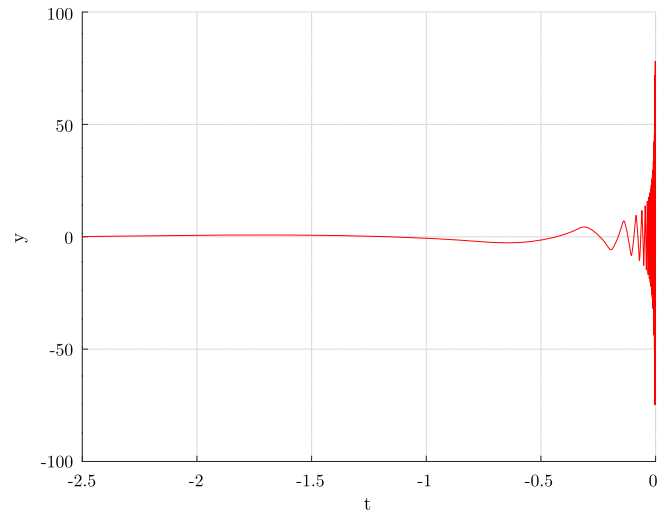
<sup>4</sup> The fact that  $\beta < 0$  is compensated by the orientation of  $\tilde{t}$ .

$$A = 1.49, \omega^2 = 5.10, (z, \dot{z}) = (0.16, -12.16), (g, n) = (0.10, 8), (\tilde{t}_i, \tilde{t}_f) = (0.485762, 231.206479) \text{step} = 0.00010$$



(a) Motion in  $z$  coordinate and  $-\tilde{t}$  time where the singularity is at  $\tilde{t} = +\infty$

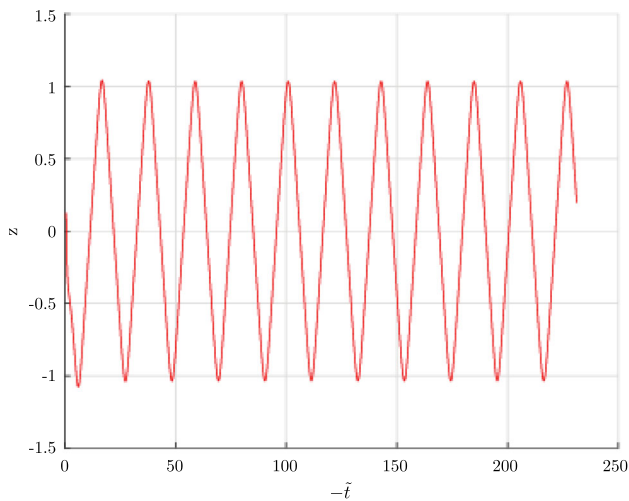
$$A = 1.49, \omega^2 = 5.10, (y, \dot{y}) = (0.09, 1.10), (g, n) = (0.10, 8), (t_i, t_f) = (-2.500000, -0.001000), \text{step} = 0.000100$$



(b) The previous motion in  $y$  coordinate (with remapped initial conditions) and  $t$  time.

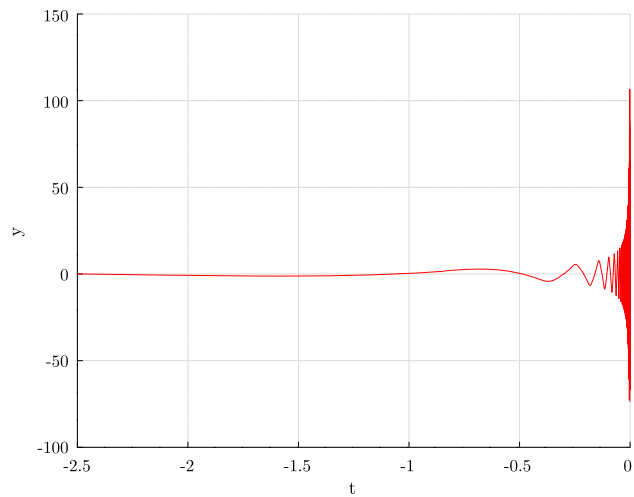
**Fig. 3** Classical motion with  $\alpha > 0$

$$A = 1.49, \omega^2 = 5.10, (z, \dot{z}) = (0.09, 1.10), (g, n) = (0.10, 8), (\tilde{t}_i, \tilde{t}_f) = (0.485762, 231.206479) \text{step} = 0.00010$$



(a) Motion in  $z$  coordinate and  $-\tilde{t}$  time where the singularity is at  $\tilde{t} = +\infty$

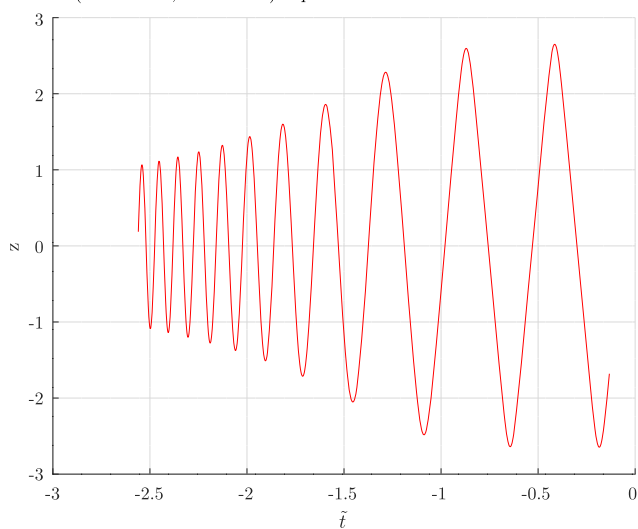
$$A = 1.49, \omega^2 = 5.10, (y, \dot{y}) = (0.05, -1.46), (g, n) = (0.10, 8), (t_i, t_f) = (-2.500000, -0.001000), \text{step} = 0.000100$$



(b) The previous motion in  $y$  coordinate (with remapped initial conditions) and  $t$  time.

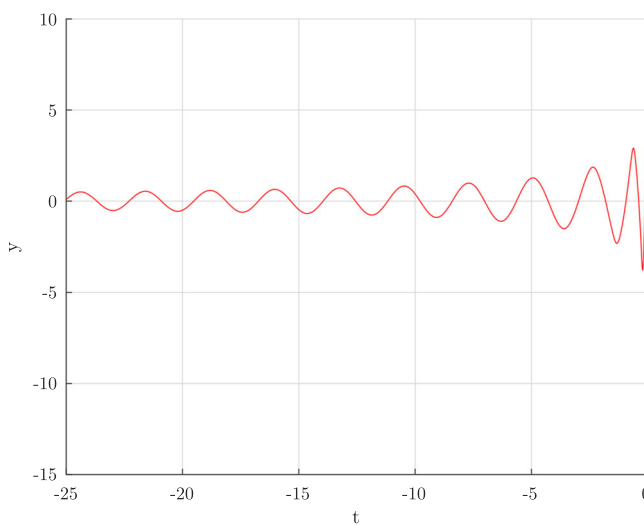
**Fig. 4** Another classical motion with  $\alpha > 0$  with a smoother behavior

$$A = 0.59, \omega^2 = 5.10, (z, \dot{z}) = (0.19, 78.58), (g, n) = (0.10, 8), (\tilde{t}_i, \tilde{t}_f) = (-2.559755, -0.133045) \text{step} = 0.00010$$



(a) Motion in  $z$  coordinate with  $\alpha < 0$  and  $\tilde{t}$  time where the singularity is at  $\tilde{t} = 0$ .

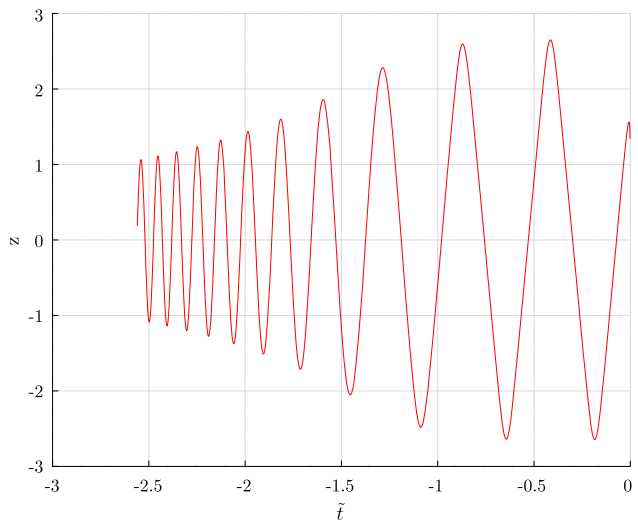
$$A = 0.59, \omega^2 = 5.10, (y, \dot{y}) = (0.09, 1.10), (g, n) = (0.10, 8), (t_i, t_f) = (-25.000000, -0.001000), \text{step} = 0.000100$$



(b) The previous motion (with remapped initial conditions) in  $y$  coordinate and  $t$  time.

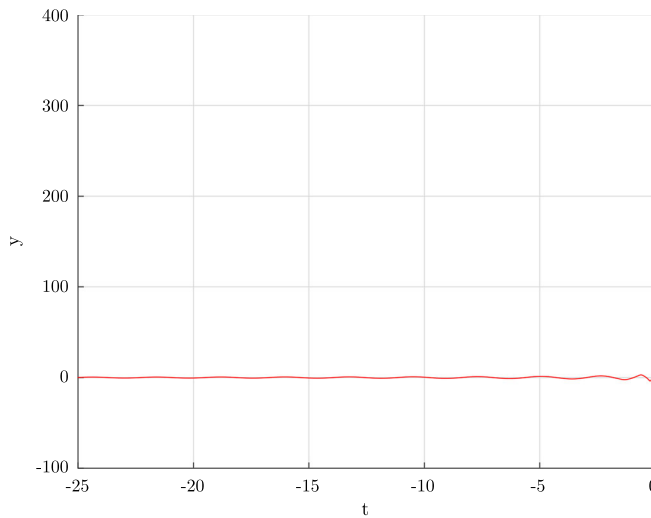
**Fig. 5** Classical motion with  $\alpha < 0$  with a too short integration range to show the expected behavior

$$A = 0.59, \omega^2 = 5.10, (z, \dot{z}) = (0.19, 78.58), (g, n) = (0.10, 8), (\tilde{t}_i, \tilde{t}_f) = (-2.559755, -0.001202) \text{step} = 0.00010$$



(a) Motion in  $z$  coordinate with the same parameters as in figure 5a but with an integration range which extends closer to the origin and shows the proper asymptotic.

$$A = 0.59, \omega^2 = 5.10, (y, \dot{y}) = (0.09, 1.10), (g, n) = (0.10, 8), (t_i, t_f) = (-25.000000, -0.000000), \text{step} = 0.000100$$



(b) The previous motion (with remapped initial conditions) in  $y$  coordinate and  $t$  time.

**Fig. 6** Classical motion with  $\alpha < 0$  with a proper integration range to show the expected behavior

If  $\alpha > 0 > \beta$  the  $z^n$  term is dominating for  $\tilde{t} \rightarrow \pm\infty$  ( $t \rightarrow 0^\pm$ ) as in the classical motion then we get a complete set of wave functions as

$$\psi_k(y, t) \sim_{t \rightarrow 0} |t|^{\frac{\alpha}{2\beta}} \exp\left(-i E_k \beta \frac{\text{sgn}(t)}{|t|^{\frac{1}{\beta}}}\right) \tilde{\psi}_k(z = |t|^{\frac{\alpha}{\beta}} y), \quad (4.21)$$

where  $E_k$  is the  $k$ -th energy eigenvalue of the effective Hamiltonian  $H_{eff} = \frac{1}{2} p_z^2 + \frac{1}{n} g z^n$  and effective time  $t_{eff} = \beta \tilde{t}$ .

The wave functions are normalizable and vanish for  $t \rightarrow 0$  allowing for a nice and “smooth” crossing of the singularity. The vanishing of the wave function can be again interpreted as the fact that the classical particle is spread over all the possible values of  $y$ . Since  $|\frac{\alpha}{2\beta}| = \frac{2A}{n+2}$  the wave functions vanish (generically) slower than the non interacting case and this can be interpreted as the fact that interactions has a better behavior than the non interacting case. Better means that classical interacting particle goes to infinity slower than the free one.

The other case is  $\beta > 0 > \alpha$  as in the classical motion. In this case the  $y$  kinetic term is dominating for  $\tilde{t} \rightarrow 0^\pm$  ( $t \rightarrow 0^\pm$ ). In fact in this limit the Schroedinger equation is

$$i \frac{1}{\beta} \frac{\partial}{\partial \tilde{t}} \tilde{\psi}(z, \tilde{t}) \sim -\frac{1}{2} \frac{\partial^2}{\partial z^2} \tilde{\psi}(z, \tilde{t}) + i \frac{\alpha}{2\beta} \frac{1}{\tilde{t}} \left( z \frac{\partial}{\partial z} + \frac{\partial}{\partial z} z \right) \tilde{\psi}(z, \tilde{t}). \quad (4.22)$$

Redefining  $\tilde{\psi}(z, \tilde{t}) = e^{i \frac{1}{2} \frac{\alpha}{\beta} \frac{z^2}{\tilde{t}}} \Psi(z, \tilde{t})$  we get

$$i \frac{1}{\beta} \frac{\partial}{\partial \tilde{t}} \Psi(z, \tilde{t}) \sim -\frac{1}{2} \frac{\partial^2}{\partial z^2} \Psi(z, \tilde{t}) - \frac{1}{2} \frac{\alpha(\alpha+1)}{(\beta \tilde{t})^2} z^2 \Psi(z, \tilde{t}), \quad (4.23)$$

which is the Schroedinger equation derived from (4.12) and can be seen as a time dependent harmonic oscillator with  $\Omega_{eff}^2 = -\frac{\alpha(1+\alpha)}{t_{eff}^2}$  (so that  $A_{eff} = -\alpha$  as in the classical case) and  $t_{eff} = \beta \tilde{t}$  and therefore it exists as a theory. In particular we get the leading behavior for the “ground state”

$$\begin{aligned} \psi(y, t) &= |\tilde{t}|^{-\frac{1}{2}\alpha} e^{i \frac{1}{2} \frac{\alpha}{\beta} \frac{z^2}{\tilde{t}}} \Psi(z, \tilde{t}) \\ &\sim |t|^{-\frac{2\alpha+1}{\beta}} \exp\left\{ \frac{1}{2} i \left( \frac{2\alpha+1}{\beta} \frac{\text{sgn}(t)}{|t|^{\frac{2\alpha+1}{\beta}}} \right. \right. \\ &\quad \left. \left. - \frac{2\alpha+1}{|\beta|^{2\alpha+1}} \frac{b_0}{b_1} |t|^{-2\frac{2\alpha+1}{\beta}} \right) y^2 \right\}, \end{aligned} \quad (4.24)$$

where  $b_0 = c_0 \omega^{A_{eff}}$  and  $b_1 = c_1 \omega^{1-A_{eff}}$ , i.e. we have reabsorbed the  $\omega$  dependence in (3.27) into the coefficients which must therefore satisfy an equation corresponding to (3.25) without  $\omega$ . Finally notice that  $\frac{2\alpha+1}{\beta} = -\frac{2nA-(n+2)}{n+2}$  so that when perturbation theory breaks down, i.e. when  $2A > \frac{n+2}{n-2}$

( $\alpha < -1$ ) the wave function vanishes when  $t \rightarrow 0$  and the potential is unbounded. Notice that the wave function vanishes when  $t \rightarrow 0$  in a wider range of  $A$  values, i.e.  $2A > \frac{n+2}{n}$  ( $\alpha < -\frac{1}{2}$ ) but not all of them implies a perturbation theory breakdown because the potential is bounded ( $-1 < \alpha < -\frac{1}{2}$ ). See Table 1 for a summary of the behaviors.

### 5 Implications for string theory on temporal orbifolds

All the previous discussion is for the generic Kasner metrics of which the Boost Orbifold is a peculiar case. For the Boost Orbifold where  $A = \frac{1}{2}$  the QFTs considered do not suffer from any breakdown and this is apparently a puzzle because the string on Boost Orbifold has a divergence. The solution of this apparent puzzle is that divergences appear in QFT when higher derivatives interaction terms (induced by massive string states [31]) or non linear sigma model interactions are included.

The reason we did not discuss the quantum mechanical models associated with these QFTs is that either they suffer from Ostrogradskii instability or they are not renormalizable. In any case this is not a limitation since it is easy seen that we suffer of the same issues as the models discussed.

We have then a clear explanation of the origin of the divergences in four point amplitudes as a breakdown of the perturbative expansion. These divergences are also present in three point amplitudes with massive states, i.e. in the lowest order of perturbation theory. The fact that we need to consider the full theory was also partially guessed in [11].

In the full interacting open string at tree level this does not necessarily mean that gravitational backreaction is not going to play any role. In facts in the open string case when solved the issues at tree level it may be well reappear to one loop open string amplitudes. This is however not at all obvious since the previous argument on perturbation theory breakdown applies to closed string as well so the resolution of the issues at the sphere level with three or four punctures could suggest the resolution at the annulus level, i.e. the sphere with two punctures.

Another point worth mentioning is that we have discussed the Boost Orbifold only and not the Null Shift Orbifold. The reason in this case is technical. While for the Boost Orbifold and its generalization the Kasner metric we can reduce the QFT to a quantum mechanical model in the Null Shift Orbifold we can only reduce to a 2d QFT since we need keeping both  $x^\pm$ . Nevertheless we expect the same mechanism to be in action for this case too.

An important point which is worth stressing is that divergences are present in Lagrangian approach, i.e. in the covariant one where the time is integrated over but there is no divergence in the light-cone formalism which is Hamiltonian and

where the time is not integrated [39]. This is the same as the previous quantum mechanical models: the Hamiltonian exists but the perturbation theory does not. Finally notice that this can be shown explicitly for the Null Shift Orbifold which is easily quantized on the light-cone [39]. This observation explains also why the matrix model with light like dilatonic profile [40] and the matrix model for the Null Shift Orbifold [41,42] is well defined.

Since the problem is essentially Lagrangian this is also an issue for Witten string field theory and in general for all the covariant formulations.

So we are left with the issue on how treat this divergences. One possibility is to use the Hamiltonian formalism, for example the light-cone when available. Even if these backgrounds do not possess Poincaré symmetry and the light-cone formalism is well adapted (it is possible to use the light-cone formalism also in other less obvious cases [43]) one could desire to have a covariant formulation in this case too then a possible approach is [14]. Another possibility is to regularize the theory in some way, for example non commutativity can do the job [44].

Finally let us mention that the way of performing the orbifold projection in the temporal orbifold cases used in literature are not on very sound basis since the generators used to write the orbifold projector are dynamical and they change when interactions are switched on. The only clear cut case where this is not the case is the Null Shift Orbifold in light-cone quantization. If we were to use the proper interacting generators there could also be some cancellations which could give raise to finite amplitudes.

## 6 Conclusions

First of all let us discuss what the previous computations imply for QFT and then shortly for string theory since we have discussed string theory in the previous section.

The first and most important point is that interactions can drastically change the fate of the fields under a Big Crunch/Big Bang.

Secondly what happens seems to depend on the details of the interaction, in the models we studied the power of the interaction  $\phi^n$  and the value of  $2A = \sum_i p_i$ . For certain ranges there is no breakdown, in particular it is noteworthy that the original Krasner background, which is also a string background admits a good perturbation.

Thirdly the breakdown of the perturbation theory is a breakdown of Feynman diagram approach, i.e. of the concept of particle. Obviously this happens because of the spacetime region around the singularity and excluding this region, i.e. before and after it the perturbation theory is well defined. Nevertheless this result rises the question of how to treat the  $S$  matrix in these backgrounds, in facts the theory exists and

spaces are asymptotically flat so we could expect to be able to define some kind of  $S$  matrix. Nevertheless it seems that the usual constraints from unitarity must be revisited since near the singularity the concept of particle breaks down.

Finally the previous results seem to point to the importance of minisuperspace approach and pose the question how to extend it to string theory.

For the string theory the main result is that, at least, at the tree level string theory is well for these backgrounds. Whether divergences from backreaction appear at loop level is by now unknown also because we have to find a good way of treating the tree level.

**Data Availability Statement** This manuscript has no associated data or the data will not be deposited. [Authors' comment: All data are in the article.]

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## A Time dependent harmonic oscillator

We will follow essentially Tseytlin et al [45] which refers to [46] but we will be careful in distinguish between Heisenberg and Schrodinger representation and this should make things more clear. For a newer point of view on the problem see also [47].

As usual we define operators in Heisenberg picture as

$$O_H(t, t_0) = U_S^\dagger(t, t_0) O_S(t) U_S(t, t_0), \quad (\text{A.1})$$

so that we get Hamiltonian in Heisenberg picture as ( $m > 0$ )

$$H_H(t, t_0) = \frac{1}{2m} p_H^2(t, t_0) + \frac{1}{2} m \Omega^2(t) x_H^2(t, t_0), \quad (\text{A.2})$$

where in our case

$$\Omega^2(t) = \left( \omega^2 + \frac{A(1-A)}{t^2} \right) = \left( \omega^2 + \frac{k}{t^2} \right). \quad (\text{A.3})$$

We then get the e.o.m

$$\begin{aligned} \dot{x}_H(t, t_0) &= \frac{1}{m} p_H(t, t_0), \\ \dot{p}_H(t, t_0) &= -m \Omega^2(t) x_H(t, t_0), \end{aligned} \quad (\text{A.4})$$

with boundary conditions

$$x_H(t_0, t_0) = x_S, \quad p_H(t_0, t_0) = p_S. \tag{A.5}$$

They imply the second order ODEs

$$\ddot{x}_H + \Omega^2 x_H = \frac{d}{dt} \left( \frac{1}{\Omega^2} p_H \right) + p_H = 0. \tag{A.6}$$

### A.1 Constant Heisenberg creator operator

We now define the operators  $\mathcal{A}_H(t, t_0)$  using the matrix  $\mathcal{M}(t)$  as

$$\begin{aligned} \mathcal{A}_H(t, t_0) &= \mathcal{M}(t) \mathcal{Z}_H(t, t_0) \\ &= \begin{pmatrix} A_H(t, t_0) \\ A_H^\dagger(t, t_0) \end{pmatrix} = i \begin{pmatrix} -\dot{\mathcal{X}}(t) & \frac{1}{m} \mathcal{X}(t) \\ +\dot{\mathcal{X}}^*(t) & -\frac{1}{m} \mathcal{X}^*(t) \end{pmatrix} \begin{pmatrix} x_H(t, t_0) \\ p_H(t, t_0) \end{pmatrix}, \end{aligned} \tag{A.7}$$

where  $\mathcal{X}(t)$  is a complex solution<sup>5</sup> of the classical e.o.m with given normalization<sup>6</sup>

$$\begin{aligned} \ddot{\mathcal{X}}(t) + \Omega^2(t) \mathcal{X}(t) &= 0, \\ \mathcal{X}^* \dot{\mathcal{X}} - \mathcal{X} \dot{\mathcal{X}}^* &= 2i W(\Re \mathcal{X}, \Im \mathcal{X}) = im. \end{aligned} \tag{A.8}$$

Notice that the previous conditions do not fix completely the solution. To fix it we need to choose an instantaneous vacuum, see Appendix 1.

The previous operators satisfy the relations<sup>7</sup>

$$\begin{aligned} [A_H(t, t_0), A_H^\dagger(t, t_0)] &= 1, \\ \frac{dA_H(t, t_0)}{dt} &\equiv \left( \frac{\partial A(t, t_0)}{\partial t} \right)_H + i [H_H(t, t_0), A_H(t, t_0)] = 0, \end{aligned} \tag{A.9}$$

i.e. the canonical commutation relation and the time independence relation.

For later use we note that the inverse of  $\mathcal{M}(t)$  is

$$\mathcal{M}^{-1}(t) = \begin{pmatrix} \frac{1}{m} \mathcal{X}^*(t) & \frac{1}{m} \mathcal{X}(t) \\ +\dot{\mathcal{X}}^*(t) & \dot{\mathcal{X}}(t) \end{pmatrix}. \tag{A.10}$$

<sup>5</sup> As we discuss in Appendix 1 there is a one parameter family of solutions.

<sup>6</sup> Remember that given a second order ODE  $\ddot{y} + a(t)\dot{y} + b(t) = 0$  the Wronskian associated with two solutions  $f(t)$  and  $g(t)$  is  $W(f, g) = f\dot{g} - \dot{f}g$  and it obeys the ODE  $\dot{W} + aW = 0$  therefore  $W = c \exp(-\int dt a(t))$  with  $c$  a constant. In our case  $a(t) = 0$  and the Wronskian is a constant.

<sup>7</sup> Notice that  $\frac{\partial A_H(t, t_0)}{\partial t} \equiv \left( \frac{\partial A(t, t_0)}{\partial t} \right)_H = U_S^\dagger \frac{\partial A_S(t, t_0)}{\partial t} U_S$ . This means that the only reasonable way of computing  $\frac{dA_H(t, t_0)}{dt}$  is to express  $A_H$  in terms of operators whose Schroedinger picture are time independent.

### A.2 Comparing with the usual harmonic oscillator 1

The general solution for the  $\mathcal{X}$  equation for the usual harmonic oscillator is

$$\mathcal{X}(t) = \mathcal{X}_+ e^{i\omega t} + \mathcal{X}_- e^{-i\omega t}, \tag{A.11}$$

then we can compute the constraint

$$\begin{aligned} \mathcal{X} \dot{\mathcal{X}}^* - \mathcal{X}^* \dot{\mathcal{X}} &= -im \\ &= 2i\omega \left( |\mathcal{X}_-|^2 - |\mathcal{X}_+|^2 \right), \end{aligned} \tag{A.12}$$

from which we get the solution

$$\mathcal{X}_+ = \sqrt{\frac{m}{2\omega}} e^{i\omega(t-t_0)}, \quad \mathcal{X}_- = 0. \tag{A.13}$$

Notice that the constraint fixes  $\mathcal{X}_\pm$  up to a phase that we have chosen so that the time invariant Heisenberg operator

$$A_H(t, t_0) = \sqrt{\frac{m}{2\omega}} e^{i\omega(t-t_0)} \left( -i\omega x_H(t, t_0) + \frac{1}{m} p_H(t, t_0) \right), \tag{A.14}$$

matches the corresponding Schroedinger operator for  $t = t_0$ .

### A.3 Hilbert space

We want to construct the Hilbert space of states to be used in Heisenberg formalism, i.e. we want states that do not depend on time.

We notice that acting with  $U_S$  on the  $A_H$  defining equation we get

$$A_H(t, t_0) = \mathcal{M}(t) \mathcal{Z}_H(t, t_0) \implies \mathcal{A}_S(t) = \mathcal{M}(t) \mathcal{Z}_S, \tag{A.15}$$

but because of the boundary condition on  $\mathcal{Z}_H$  we can also write

$$A_H(t_0, t_0) = \mathcal{M}(t_0) \mathcal{Z}_H(t_0, t_0) = \mathcal{M}(t_0) \mathcal{Z}_S = \mathcal{A}_S(t_0), \tag{A.16}$$

then because  $\mathcal{A}_H$  is constant we get the basic result

$$\begin{aligned} A_H(t, t_0) &\equiv U_S^\dagger(t, t_0) \mathcal{A}_S(t) U_S(t, t_0) \\ &= \mathcal{A}_H(t_0, t_0) = \mathcal{A}_S(t_0). \end{aligned} \tag{A.17}$$

Now we can introduce the “vacuum” at time  $t_0$  as

$$A_S(t_0) |0\{t_0\}\rangle = 0, \tag{A.18}$$

and build a basis for the Hilbert space which is characterized by time  $t_0$  as

$$\mathcal{H}_{Bt_0} = \left\{ |n\{t_0\}\rangle = \frac{1}{\sqrt{n!}} A_S^{\dagger n}(t_0) |0\{t_0\}\rangle \right\}. \tag{A.19}$$

### A.4 Time evolution of basis elements and wave functions 1

Given any element of the previous basis we can identify it as a Schroedinger state as

$$|n\{t_0\}; t_0, t_0\rangle_S = |n\{t_0\}\rangle, \tag{A.20}$$

and compute its time evolution as follows.

Let us start with the “vacuum”, and write

$$\begin{aligned} A_S(t_0)|0\{t_0\}; t_0, t_0\rangle_S &= U_S^\dagger(t, t_0)A_S(t)U_S(t, t_0)|0\{t_0\}; t_0, t_0\rangle_S \\ &= U_S^\dagger(t, t_0)A_S(t)|0\{t_0\}; t, t_0\rangle_S = 0 \end{aligned} \tag{A.21}$$

hence we can determine the time evolution of the  $t_0$  vacuum state as

$$A_S(t)|0\{t_0\}; t, t_0\rangle_S = 0, \tag{A.22}$$

from which follows its wave function up to a time dependent normalization

$$\begin{aligned} \left(-\dot{\mathcal{X}}(t)x - \frac{i}{m}\mathcal{X}(t)\partial_x\right)\psi_{0\{t_0\}}(x, t, t_0) \\ = 0 \implies \psi_{0\{t_0\}}(x, t, t_0) = \mathcal{N}(t)e^{i\frac{m}{2}\frac{\dot{\mathcal{X}}(t)}{\mathcal{X}(t)}x^2}. \end{aligned} \tag{A.23}$$

The normalization can be fixed using the Schroedinger equation as

$$\begin{aligned} i\partial_t\psi_{0\{t_0\}}(x, t, t_0) &= \left[i\frac{\dot{\mathcal{N}}}{\mathcal{N}} - \frac{m}{2}\frac{d}{dt}\left(\frac{\dot{\mathcal{X}}}{\mathcal{X}}\right)\right]\psi_{0\{t_0\}}(x, t, t_0) \\ &= H_S(t)\psi_{0\{t_0\}}(x, t, t_0) = \left\{-\frac{1}{2m}\left[im\frac{\dot{\mathcal{X}}}{\mathcal{X}} - m^2\left(\frac{\dot{\mathcal{X}}}{\mathcal{X}}\right)^2x^2\right] \right. \\ &\quad \left. + \frac{1}{2}m\Omega^2x^2\right\}\psi_{0\{t_0\}}(x, t, t_0), \end{aligned} \tag{A.24}$$

and using  $\mathcal{X}$  e.o.m to get

$$\mathcal{N}(t) = \frac{C}{\sqrt{\mathcal{X}(t)}}, \tag{A.25}$$

with  $C$  a constant which can be fixed requiring the normalization of  $\psi_{0\{t_0\}}(x, t, t_0)$  as

$$\begin{aligned} (\psi_{0\{t_0\}}(x, t, t_0), \psi_{0\{t_0\}}(x, t, t_0)) &= \frac{|C|^2}{|\mathcal{X}|}\sqrt{\frac{\pi}{m\mathfrak{S}\left(\frac{\dot{\mathcal{X}}}{\mathcal{X}}\right)}} \\ &= |C|^2\sqrt{\frac{2\pi}{m^2}} = 1, \end{aligned} \tag{A.26}$$

where we have used  $\mathcal{X}$  normalization and e.o.m to write

$$\mathfrak{S}\left(\frac{\dot{\mathcal{X}}}{\mathcal{X}}\right) = \frac{\mathfrak{S}(\dot{\mathcal{X}}\mathcal{X}^*)}{|\mathcal{X}|^2} = \frac{m}{2|\mathcal{X}|^2}, \tag{A.27}$$

where it is interesting to notice that *the chosen  $\mathcal{X}$  normalization allows for the convergence of the integral.* Finally we

can write the normalized wave function as

$$\psi_{0\{t_0\}}(x, t, t_0) = \sqrt{\frac{m^2}{2\pi}}\frac{1}{\sqrt{\mathcal{X}(t)}}e^{i\frac{m}{2}\frac{\dot{\mathcal{X}}(t)}{\mathcal{X}(t)}x^2}. \tag{A.28}$$

### A.5 Comparing with the usual harmonic oscillator 2

Using the results from the previous section and  $\frac{\dot{\mathcal{X}}}{\mathcal{X}} = i\omega$  we get the harmonic oscillator ground state wave function

$$\psi_{0\{t_0\}}(x, t, t_0) = \sqrt{\frac{m\omega}{\pi}}e^{-i\frac{1}{2}\omega(t-t_0)}e^{-\frac{m}{2}\omega x^2}. \tag{A.29}$$

### A.6 Time evolution of basis elements and wave functions 2

To deal with excited states is better to use a generating function and therefore we define

$$\begin{aligned} |z\{t_0\}; t, t_0\rangle_S &= \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\{t_0\}; t, t_0\rangle_S \\ &= e^{zA_S^\dagger(t)}|0\{t_0\}; t, t_0\rangle_S, \end{aligned} \tag{A.30}$$

then we evaluate

$$\begin{aligned} \langle x|z\{t_0\}; t, t_0\rangle_S &= \langle x|U_S(t, t_0)|z\{t_0\}\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\psi_n\{t_0\}(x, t, t_0) \\ &= \langle x|e^{iz\dot{\mathcal{X}}^*(t)x_S}e^{-iz\frac{1}{m}\dot{\mathcal{X}}^*(t)p_S}e^{-i\frac{1}{2}z^*\frac{1}{m}\dot{\mathcal{X}}^*(t)\mathcal{X}^*(t)}|0\{t_0\}; t, t_0\rangle_S \\ &= e^{iz\dot{\mathcal{X}}^*(t)x - i\frac{1}{2}z^*\frac{1}{m}\dot{\mathcal{X}}^*(t)\mathcal{X}^*(t)}\langle x|e^{-z\frac{1}{m}\dot{\mathcal{X}}^*(t)\partial_x}|0\{t_0\}; t, t_0\rangle_S \\ &= \sqrt{\frac{m^2}{2\pi}}\frac{1}{\sqrt{\mathcal{X}(t)}}e^{i\frac{m}{2}\frac{\dot{\mathcal{X}}(t)}{\mathcal{X}(t)}x^2 + \frac{m}{\mathcal{X}(t)}xz - \frac{1}{2}\frac{\dot{\mathcal{X}}^*(t)}{\mathcal{X}(t)}z^2}, \end{aligned} \tag{A.31}$$

upon the use of the  $\mathcal{X}$  normalization condition. It can also be checked that the previous equation satisfy the Schroedinger equation

$$i\partial_t\langle x|z\{t_0\}; t, t_0\rangle_S = \left(-\frac{1}{2m}\partial_x^2 + \frac{1}{2}m\Omega^2(t)x^2\right)\langle x|z\{t_0\}; t, t_0\rangle_S. \tag{A.32}$$

### A.7 Overlaps

Since we want to check that overlaps are well defined we need computing  $\langle n\{t_0\}||l\{t_0\}; t, t_0\rangle_S$  but it is actually simpler to compute

$$\begin{aligned} s(z\{t_0\}; t_0, t_0||w\{t_0\}; t, t_0)_S &= (z\{t_0\}|U_S(t, t_0)|w\{t_0\}) \\ &= \sum_{n,l=0}^{\infty}\frac{z^{*n}}{\sqrt{n!}}\frac{w^l}{\sqrt{l!}}\langle n\{t_0\}||l\{t_0\}; t, t_0\rangle_S, \end{aligned} \tag{A.33}$$

since  $|n\{t_0\}\rangle = |n\{t_0\}; t_0, t_0\rangle_S$ . Performing the explicit  $x$  integral we get



$$\begin{aligned}
 & s(z\{t_0\}; t_0, t_0 | w\{t_0\}; t, t_0) s \\
 &= \sqrt{\frac{-im}{\dot{\mathcal{X}}^*(t_0)\mathcal{X}(t) - \dot{\mathcal{X}}(t)\mathcal{X}^*(t_0)}} e^{\frac{-im}{\dot{\mathcal{X}}^*(t_0)\mathcal{X}(t) - \dot{\mathcal{X}}(t)\mathcal{X}^*(t_0)} \left[ \sqrt{\frac{\mathcal{X}(t)}{\mathcal{X}^*(t_0)}} z^* + \sqrt{\frac{\mathcal{X}^*(t_0)}{\mathcal{X}(t)}} w \right]^2 - \frac{1}{2} \frac{\mathcal{X}(t_0)}{\mathcal{X}^*(t_0)} z^* - \frac{1}{2} \frac{\mathcal{X}^*(t)}{\mathcal{X}(t)} w^2}
 \end{aligned} \tag{A.34}$$

### A.8 Evolution operator in $x$ space

For the same reason as before, i.e. to check the finitness of the regularized string theory we need the kernel or the evolution operator in  $x$  space. We perform the computation using the generating function as follows

$$\begin{aligned}
 \langle x_2, t_2 | x_1, t_1 \rangle &= \langle x_2 | U_S(t_2, t_1) | x_1 \rangle \\
 &= \sum_{n=0}^{\infty} \psi_{n\{t_0\}}(x_2, t_2, t_0) \psi_{n\{t_0\}}^*(x_1, t_1, t_0) \\
 &= \int \frac{d^2z}{\pi} e^{-|z|^2} \langle x_2 | z\{t_0\}; t_2, t_0 \rangle_S \langle z\{t_0\}; t_1, t_0 | x_1 \rangle_S
 \end{aligned} \tag{A.35}$$

where the  $d^2z$  integral is normalized as  $\int \frac{d^2z}{\pi} e^{-|z|^2} = 1$ .  
We get

$$\begin{aligned}
 \langle x_2, t_2 | x_1, t_1 \rangle &= \sqrt{\frac{-im^2}{4\pi \Im(\mathcal{X}(t_2)\mathcal{X}^*(t_1))}} \\
 &\times e^{\frac{im^2}{4\Im(\mathcal{X}(t_2)\mathcal{X}^*(t_1))} \left[ \frac{\mathcal{X}(t_1)}{\mathcal{X}(t_2)} x_2^2 + \frac{\mathcal{X}^*(t_2)}{\mathcal{X}^*(t_1)} x_1^2 - 2x_1x_2 \right]} \\
 &+ i \frac{m}{2} \left[ \frac{\dot{\mathcal{X}}(t_2)}{\mathcal{X}(t_2)} x_2^2 - \frac{\dot{\mathcal{X}}^*(t_1)}{\mathcal{X}^*(t_1)} x_1^2 \right].
 \end{aligned} \tag{A.36}$$

### A.9 Comparing with the usual harmonic oscillator 3

Using the explicit solution for the harmonic oscillator we get

$$\begin{aligned}
 \Im(\mathcal{X}(t_2)\mathcal{X}^*(t_1)) &= \frac{m}{2\omega} \sin \omega(t_2 - t_1), \quad \frac{\mathcal{X}(t_1)}{\mathcal{X}(t_2)} \\
 &= e^{-i\omega(t_2-t_1)},
 \end{aligned} \tag{A.37}$$

then the  $x_2^2$  coefficient becomes

$$\begin{aligned}
 & \frac{im^2}{4\Im(\mathcal{X}(t_2)\mathcal{X}^*(t_1))} \frac{\mathcal{X}(t_1)}{\mathcal{X}(t_2)} + i \frac{m}{2} \frac{\dot{\mathcal{X}}(t_2)}{\mathcal{X}(t_2)} \\
 &= \frac{im\omega}{2 \sin \omega(t_2 - t_1)} \\
 &\times \left[ e^{-i\omega(t_2-t_1)} + i \sin \omega(t_2 - t_1) \right] \\
 &= \frac{im\omega \cos \omega(t_2 - t_1)}{2 \sin \omega(t_2 - t_1)},
 \end{aligned} \tag{A.38}$$

as it should.

## B Complex classical solution for $L_B$

We want to solve the equations (3.22). One possibility is to use the WKB approach, i.e. the adiabatic vacuum approach [48] and write

$$\begin{aligned}
 \mathcal{X}(x) &= \frac{m}{2W(t)} e^{i \int dt' W(t')}, \\
 W^2(t) &= \Omega^2(t) + \delta_1(t) + \frac{\delta_2(t)}{\Omega^2(t)} + O(\Omega^{-4}),
 \end{aligned} \tag{B.1}$$

but this approach singles out  $\Omega$  as a whole while for our purposes we are more interested in singling out  $\omega$ .

### B.1 Perturbative solution for $\mathcal{X}$ in the small $|\omega t|$ limit

We want to solve the classical equation with normalization condition given in (3.22) which we repeat here without setting  $m = 1$

$$\begin{aligned}
 \ddot{\mathcal{X}}(t) + \Omega^2(t)\mathcal{X}(t) &= 0, \\
 \mathcal{X}^* \dot{\mathcal{X}} - \mathcal{X} \dot{\mathcal{X}}^* &= im.
 \end{aligned} \tag{B.2}$$

Actually we are interested in the perturbative solution around  $t = 0$ . This is a second order linear equation and therefore it has two independent solutions. For our purpose it is sufficient to consider the following leading order expansion

$$\mathcal{X}(t) = \begin{cases} c_0(\omega t)^A(1 + O(t^2)) + c_1(\omega t)^{1-A}(1 + O(t^2)) & t > 0 \\ \bar{c}_0(-\omega t)^A(1 + O(t^2)) + \bar{c}_1(-\omega t)^{1-A}(1 + O(t^2)) & t < 0 \end{cases} \tag{B.3}$$

We allow for different coefficients for  $t > 0$  and  $t < 0$  because of the singularity in the differential equation. The normalization condition then implies

$$\begin{aligned}
 -(2A - 1)\omega |c_1|^2 \Im\left(\frac{c_0}{c_1}\right) &= +(2A - 1)\omega |\bar{c}_1|^2 \Im\left(\frac{\bar{c}_0}{\bar{c}_1}\right) \\
 &= -\frac{1}{2}m.
 \end{aligned} \tag{B.4}$$

The issue to solve is the continuation through the singularity  $t = 0$ . Since we deal with a classical solution we can expect that it must be as smooth as possible. For  $A > 1$  (for  $0 < A < 1$  both independent solutions vanish for  $t = 0$  and therefore we take the solution for  $A > 1$  as the the solution for this range) the term  $|t|^{1-A}$  is divergent but it is the best we can do

to get a continuous trajectory. This suggests to set  $c_1 = \bar{c}_1$  and therefore  $c_0 = -\bar{c}_0$  as consequence of the normalization condition. Notice that the discontinuity in the coefficient  $c_0$  does not make  $\mathcal{X}$  discontinuous, only  $\dot{\mathcal{X}}$  is discontinuous. We are therefore led to

$$\mathcal{X}(t) = c_0 \omega t |\omega t|^{A-1} (1 + O(t^2)) + c_1 |\omega t|^{1-A} (1 + O(t^2)). \tag{B.5}$$

The general solution of the normalization condition (B.4) reads

$$\begin{aligned} c_0 &= \sqrt{\frac{m}{2(2A-1)\omega}} \frac{e^{i\alpha}}{\lambda} e^{i\frac{\pi}{4}}, \\ c_1 &= \sqrt{\frac{m}{2(2A-1)\omega}} e^{i\alpha} \lambda e^{-i\frac{\pi}{4}}, \alpha, \lambda \in \mathbb{R}, \end{aligned} \tag{B.6}$$

where  $\alpha$  is a trivial overall phase while  $\lambda$  parameterizes different solutions. Explicitly we can write the normalized complex classical solution as

$$\mathcal{X}(t) = \sqrt{\frac{m}{2(2A-1)\omega}} e^{i\alpha} \left( \lambda e^{-i\frac{\pi}{4}} |t|^{1-A} + \frac{e^{+i\frac{\pi}{4}}}{\lambda} \operatorname{sgn}(t) |t|^A \right) (1 + O(t^2)), \tag{B.7}$$

so that

$$\begin{aligned} \frac{\dot{\mathcal{X}}}{\mathcal{X}} &\sim \frac{1-A}{t} + (2A-1) \frac{c_0}{c_1} \omega |\omega t|^{2(A-1)} \\ &= \frac{1-A}{t} + (2A-1) \frac{1}{\lambda^2} \omega |\omega t|^{2(A-1)}. \end{aligned} \tag{B.8}$$

To understand the role of  $\lambda$  we can compute

$$|\psi_0(x, t)|^2 \sim \frac{1}{|\omega t|^{1-A}} e^{-\frac{\omega^{2A-1}}{\lambda^2} |t|^{2(A-1)} x^2}, \tag{B.9}$$

from which we see that  $\lambda$  parameterizes the instantaneous vacuum, in fact for small  $\omega t_0$  such that  $\Omega(t_0)^2 > 0$  we can compare with the usual harmonic function probability density  $|\psi_{0(h.o)}(x, t)| \sim e^{-m\Omega(t_0)x^2}$ .

### B.2 Continuation through $t = 0$ using a regularized equation

In the previous section we have given a plausible argument on how to continue the solution across the  $t = 0$  singularity based on the continuity. We can make this argument more rigorous by looking to the solution with a regularized  $\Omega^2(t)$ . This argument is more rigorous if one is willing to accept that it is meaningful to regularize  $\Omega^2(t)$  as

$$\Omega^2(t) = \begin{cases} \omega^2 + \frac{k}{t^2} & |t| > \epsilon \\ \omega^2 + \frac{k}{\epsilon^2} & |t| < \epsilon \end{cases}. \tag{B.10}$$

We choose  $\Omega^2(\epsilon) = \omega^2 + \frac{k}{\epsilon^2} < 0$ , i.e. we take  $A > 1$  so that

$$|\Omega(\epsilon)| = \frac{\sqrt{|k|}}{\epsilon} - \frac{1}{2} \frac{\omega^2}{\sqrt{|k|}} \epsilon + O(\epsilon^3). \tag{B.11}$$

Obviously we are not adding anything really new to the previous argument since we are making  $\Omega^2(t)$  finite and continuous and therefore the solution will be finite and continuous across the singularity and therefore unique. It is anyhow interesting to see how the discontinuity in the  $c_1$  coefficient arises.

The general solution for  $|t| < \epsilon$  is

$$\mathcal{X}(t) = c_e \cosh(|\Omega(\epsilon)|t) + c_o \sinh(|\Omega(\epsilon)|t), \tag{B.12}$$

so that the normalization condition (B.4) reads

$$\Im(c_o^* c_e) = -\frac{1}{2} \frac{m}{|\Omega(\epsilon)|}, \tag{B.13}$$

whose general solution is

$$c_e = \sqrt{\frac{m}{2|\Omega(\epsilon)|}} \rho e^{i\beta} e^{-i\frac{\pi}{4}}, \quad c_o = \sqrt{\frac{m}{2|\Omega(\epsilon)|}} \frac{e^{i\beta}}{\rho} e^{+i\frac{\pi}{4}}. \tag{B.14}$$

We can now match the solution at  $t = \epsilon$ . Since the solution for  $t = \epsilon^+$  diverges as  $\mathcal{X}(\epsilon^+) \sim \epsilon^{1-A}$  we have either  $\rho \rightarrow \infty$  or  $\rho \rightarrow 0$ . In the former case we need  $\alpha = \beta$  and get

$$\rho \sim \frac{\lambda \sqrt[4]{|k|} \epsilon^{\frac{1}{2}-A}}{\cosh \sqrt{|k|}}, \tag{B.15}$$

and the solution is essentially even since the odd part is suppressed while in the latter case we need  $\alpha = \beta + \frac{1}{2}\pi$  and get

$$\rho \sim \frac{\lambda \sqrt[4]{|k|} \epsilon^{\frac{1}{2}-A}}{\sinh \sqrt{|k|}}, \tag{B.16}$$

and the solution is essentially odd.

Letting  $A \rightarrow 1^+$ , i.e.  $|k| \rightarrow 0$  such that  $\frac{|k|}{\epsilon^2}$  is kept constant and bigger than  $\omega^2$  we get the usual harmonic oscillator with .... Then only in the even case  $\rho$  has a finite limit while in the odd case  $\rho \sim \epsilon$ .

### C WKB analysis of e.o.m for $L_B$

We want to use the WKB approach to determine the behavior for  $t \rightarrow 0$ . We set  $\psi(x, t) = e^{iS(x,t)}$  so that we want to solve the equation

$$\partial_t S(x, t) + \frac{1}{2} (\partial_x S(x, t))^2 + \frac{1}{2} \left( \omega^2 + \frac{k}{t^2} \right) x^2 - i \frac{1}{2} \partial_x^2 S(x, t) = 0. \tag{C.1}$$

In the limit  $t \rightarrow 0$  we can try to write

$$S(x, t) = \theta(t)t^{\alpha(+)}s_{(+)}(x)(1 + o(1)) + \theta(-t)(-t)^{\alpha(-)}s_{(-)}(x)(1 + o(1)) \tag{C.2}$$

and fix  $a_{(\pm)}$ . In principle we could simply write  $t^a$  for  $t \in \mathbb{R}$  since  $S \in \mathbb{C}$  but this would introduce cuts in the solution and therefore we should discuss which sheet we should use. We prefer to have a well defined leading order.

Notice that we allow for a discontinuity in  $S$  at  $t = 0$  since  $p_x \psi(x, t) = \partial_x S \psi(x, t)$  and the momentum can be discontinuous due to the infinite force.

At the leading order in  $t$  we get

$$a \frac{|t|^a}{t} s_0(x) + \frac{1}{2} |t|^{2a} (s'_0(x))^2 + \frac{1}{2} \frac{k}{t^2} x^2 - i \frac{1}{2} |t|^a s''_0(x) \sim 0, \tag{C.3}$$

where we have dropped the subscript  $(\pm)$  since the equation is the same for both cases. There is a unique solution which requires  $a = -1$ . Then we are left with

$$(s'_0(x))^2 - 2sgn(t)s_0(x) + k = 0. \tag{C.4}$$

This is a special case of Chrystal's equation.<sup>8</sup> The most singular and easiest solution is

$$s_{0\pm}(x) = \frac{1}{2} \alpha_{\pm} x^2, \quad \alpha_{\pm} = sgn(t)\alpha \in \{A, 1 - A\}. \tag{C.5}$$

Since there are two solutions for  $sgn(t)\alpha$  it is still possible that  $s_{(+)}(x)$  differs from  $s_{(-)}(x)$  but it turns out that they are the same since in order to avoid singularities at  $x = 0$  for  $t \neq 0$  in  $S$ . The same constraint implies that we need choosing the smallest  $\alpha$ .

We can therefore simply write the leading order as

$$S(x, t) = \frac{1}{t} s_0(x) + \dots, \quad s_0(x) = \frac{1}{2} \alpha x^2, \quad \alpha \in \{A, 1 - A\}. \tag{C.6}$$

So one could think of setting up an expansion like  $S = \frac{1}{t} s_0(x) + s_1(x) + t s_2(x) + O(t^2)$ . This is possible but does not give the right answer. The equation is non linear and therefore we cannot add solutions hence we must check whether there exist subdominant expansions. Let us therefore write

$$S(x, t) = \frac{1}{t} s_0(x)(1 + O(t)) + t^{\alpha b} s^{[1]}_0(x)(1 + O(t)), \tag{C.7}$$

<sup>8</sup> Chrystal's equation reads

$$\dot{y}^2 + A t \dot{y} + B y + C x^2 = 0.$$

The general solution is

$$x \frac{(z - a)^{a/(a-b)}}{(z - b)^{b/(a-b)}} = k, \quad 4B y = (A^2 - 4C - z^2)x^2,$$

$$\text{and } a, b = \pm \left[ B + \sqrt{(2A + B)^2 - 16C} \right] / 2.$$

and try to fix  $b$ . We do not require the subdominant solution to be regular for  $t = 0$  but we require  $\alpha \Re(b) > -1$  so that the added term is actually subdominant. The equation for  $s^{[1]}_0(x)$  turns out to be

$$s'_0 s^{[1]'}_0 + \alpha b s^{[1]}_0 = 0, \tag{C.8}$$

which has solution

$$s^{[1]}_0(x) = c^{[1]}_0 |x|^{-b}, \tag{C.9}$$

since we do not want singularities in  $x$  we need  $\Re(b) < 0$  which then implies  $\alpha < \frac{1}{|\Re(b)|}$  or equivalently  $A > 1 - \frac{1}{|\Re(b)|}$ .

Finally we can set up the perturbative expansion as

$$S(x, t) = \frac{1}{t} s_0(x) + s_1(x) + t s_2(x) + O(t^2) + \log(|t|) \hat{s}_1 + |t|^{\alpha b} s^{[1]}_0(x) + |t|^{\alpha b} t s^{[1]}_{(+)}(x) + O(t^{\alpha b + 2}) + |t|^{2\alpha b} s^{[2]}_0(x) + |t|^{2\alpha b} t s^{[2]}_1(x) + O(t^{2\alpha b + 2}) + \dots, \tag{C.10}$$

where we added a further logarithmic contribution with constant coefficient  $\hat{s}_1 = (\frac{1}{2}i + \delta)\alpha$  which is necessary for the absence of singularities in  $x = 0$  from  $s_1$  and added double infinite series with power  $|t|^{n\alpha b}$  since as soon as we add  $|t|^{\alpha b}$  we get a term with power  $|t|^{2\alpha b}$  from  $(\partial_x S)^2$ . In the case of non integer power we need paying attention to the definitions of  $s^{[n]}_m$  in order to get equations which do not depend on the sign of  $t$  therefore we write  $|t|^{nb} t^m$ . Finally notice that if we assume this expansion we need not only  $\alpha \Re(b) > -1$  but  $\alpha \Re(b) > -\frac{1}{n}$  so that all added terms are subdominant. This means that  $\alpha \Re(b) > 0$  which together  $\Re(b) < 0$  implies  $\alpha < 0$  or  $A > 1$ . There is however a way out from this constraint which allows  $A < 1$ . The term  $|t|^{2\alpha b}$  from  $(\partial_x S)^2$  can be canceled from a term  $|t|^{2\alpha b} + 1$  from  $\partial_t S$ . If we start this way we see that we need terms  $|t|^{n\alpha b + n - 1}$  only. In this case we need to impose  $n\alpha \Re(b) + n - 1 > -1$  and  $n\alpha \Re(b) + n - 1 > (n - 1)\alpha \Re(b) + (n - 1) - 1$  in order to get a series of subdominant terms. All constraints can be solved by  $\alpha \Re(b) > -1$  so that  $\alpha < \frac{1}{|\Re(b)|}$ .

We now get the equations

$$\begin{aligned} |t|^{\alpha b} / t : 2s'_0 s^{[1]'}_0 + 2\alpha b s^{[1]}_0 &= 0 \\ |t|^{2\alpha b} / t : 2s'_0 s^{[2]'}_0 + 4\alpha b s^{[2]}_0 &= 0 \\ t^{-1} : 2s'_0 s'_1 - i s''_0 + 2\hat{s}_1 &= 0 \\ |t|^{\alpha b} : 2s'_0 s^{[1]'}_1 + 2(\alpha b + 1) s^{[1]}_2 - i s^{[1]''}_0 + 2s'_1 s^{[1]'}_0 &= 0 \\ |t|^{2\alpha b} : 2s'_0 s^{[2]'}_1 + 2(2\alpha b + 1) s^{[2]}_2 - i s^{[2]''}_0 + 2s'_1 s^{[2]'}_0 + (s^{[1]'}_0)^2 &= 0 \\ t^0 : 2s'_0 s'_2 + 2s_2 - i s''_1 + (s'_1)^2 + \omega^2 x^2 &= 0. \end{aligned} \tag{C.11}$$

The solution for  $s^{[1]}_0$  and  $s^{[2]}_0$  read

$$s^{[1]}_0(x) = \alpha c^{[1]}_0 |x|^{-b}, \quad s^{[2]}_0(x) = \alpha c^{[2]}_0 |x|^{-2b}, \tag{C.12}$$

from which one can easily guess the solution for all  $s^{[n]}_0$ . In particular it is possible to set  $s^{[2]}_0(x) = 0$  for having  $A < 1$  too.

The solution for  $s_1, s^{[1]}_1$  and  $s^{[2]}_1$  read

$$\begin{aligned}
 s_1(x) &= \alpha c_1 - \delta \log|x|, \\
 s^{[1]}_1(x) &= \frac{\alpha b c^{[1]}_0 (2\delta - ib - 2i)}{2(2\alpha - 1)} |x|^{-b-2} + \alpha c^{[1]}_1 |x|^{-b-\frac{1}{\alpha}} \\
 s^{[2]}_1(x) &= \frac{\alpha b [c^{[2]}_0 (4\delta - 4ib - 4i) + \alpha b c^{[1]}_0]}{2(2\alpha - 1)} |x|^{-2b-2} \\
 &\quad + \alpha c^{[2]}_1 |x|^{-2b-\frac{1}{\alpha}}.
 \end{aligned}
 \tag{C.13}$$

Finally we get also

$$s_2(x) = -\frac{\omega^2}{2(2\alpha + 1)} x^2 + \frac{\delta(\delta - i)}{2(2\alpha + 1)} \frac{1}{x^2} + \alpha c_2 |x|^{-\frac{1}{\alpha}}.
 \tag{C.14}$$

Assembling all pieces in order to discuss the constraints on the constants we get

$$\begin{aligned}
 S(x, t) &= \frac{1}{t} \left[ \frac{1}{2} \alpha x^2 \right] + \log|t| \left[ \alpha \left( \frac{i}{2} + \delta \right) \right] + [-\delta \log|x|] \\
 &\quad + t \left[ -\frac{\omega^2}{2(2\alpha + 1)} x^2 + \frac{\delta(\delta - i)}{2(2\alpha + 1)} \frac{1}{x^2} - \alpha c_2 |x|^{-\frac{1}{\alpha}} \right] \\
 &\quad + |t|^{\alpha b} \left[ \alpha c^{[1]}_0 |x|^{-b} \right] \\
 &\quad + |t|^{\alpha b} t \left[ \frac{\alpha b c^{[1]}_0 (2\delta - ib - 2i)}{2(2\alpha - 1)} |x|^{-b-2} \right. \\
 &\quad \left. + \alpha c^{[1]}_1 |x|^{-b-\frac{1}{\alpha}} \right] \\
 &\quad + |t|^{2\alpha b} \left[ \alpha c^{[2]}_0 |x|^{-2b} \right] \\
 &\quad + |t|^{2\alpha b} t \left[ \frac{\alpha b [c^{[2]}_0 (4\delta - 4ib - 4i) + \alpha b c^{[1]}_0]}{2(2\alpha - 1)} \right. \\
 &\quad \left. \times |x|^{-2b-2} + \alpha c^{[2]}_1 |x|^{-2b-\frac{1}{\alpha}} \right]
 \end{aligned}
 \tag{C.15}$$

The absence of singularities in  $x = 0$  in the  $O(1)$  term implies  $\delta = 0$ . This is also compatible with the absence of singularities in  $x = 0$  in the  $O(t)$  term.

As discussed below the normalizability of the wave function requires  $c^{[1]}_0 \neq 0$ . Then the  $|t|^{\alpha b} |x|^{-b-2}$  term implies  $b \leq -2$ . From the true solution we know that we have to choose  $b = -2$  and the series expansion of the form  $|t|^{\alpha b+n-1}$ . Why it is so it is not clear at this point.

Let us discuss the normalizability. When  $A > 1$  and therefore  $\alpha = 1 - A < 0$  we get

$$\psi(x, t) = N |t|^{\frac{1}{2}(A-1)} e^{i \left[ -\frac{A-1}{4} \frac{1}{t} x^2 + t \frac{\omega^2}{2(A-1)} x^2 \right]} e^{-i(A-1)c^{[1]}_0 |t|^{A-1} x + \frac{1}{2} c^{[1]}_0 |t|^{2(A-1)} x^2}.
 \tag{C.16}$$

As long as we take  $\Re(c^{[1]}_0) < 0$  (which implies  $c^{[1]}_0 \neq 0$ ) this expression is consistent since the normalization  $N$  is a constant and independent on  $x$  and  $t$  as it follows from

$$\begin{aligned}
 &\int_{-\infty}^{\infty} dx |\psi(x, t)|^2 \\
 &= \int_{-\infty}^{\infty} dx |N|^2 |t|^{(A-1)} e^{2\Re(c^{[1]}_0) |t|^{2(A-1)} x^2 - 2(A-1)\Im(c^{[1]}_0) |t|^{(A-1)} x} \\
 &= |N|^2 \sqrt{\frac{1}{-2\Re(c^{[1]}_0)}} e^{(1-A)^2 \frac{(\Im(c^{[1]}_0))^2}{-2\Re(c^{[1]}_0)}}.
 \end{aligned}
 \tag{C.17}$$

The physical meaning of the vanishing of the wave function for  $t = 0$  is that the particle is diffused uniformly on the entire real axis  $x$ .

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