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Compatibility of Poisson–Lie transformations and symmetries of generalized supergravity equations

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Abstract We investigate two types of transformations that keep NS–NS generalized supergravity equations satisfied: χ-symmetry (20) that shifts dilaton and gauge transformations (30) that change both dilaton and vector field J. Due to these symmetries there is a large set of dilatons and vector fields J that (for a fixed metric and B-field) satisfy generalized supergravity equations but only some of them can be be used as input for Poisson–Lie transformations. Conditions that define the admissible dilatons are given and examples are presented.

Contents

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1 Introduction

Poisson–Lie transformation of solutions of generalized supergravity equations must include dilatons and Killing fields. Formula for transformation of dilaton field Φ accompanying Poisson–Lie transformation [1,2] of sigma model background $\mathcal{F} = \mathcal{G}+\mathcal{B}$ was given in [3] (see also [4–6]). Formula for Poisson–Lie transformation of Killing field J was given in [7,8]. Later it turned out that the the latter formula works well only for non-Abelian T-duality, i.e. for Poisson–Lie transformations of sigma models with isotropic backgrounds, and it was extended in [9] for other type of transformations. Unfortunately, applicability of these formulas is dependent on choice of initial dilatons related by symmetries of solution space of generalized supergravity equations. It turns out that transformed dilatons and vector fields J keep validity of generalized supergravity equations in dependence on the choice on the initial dilaton. Examples of these cases are given below.

Goal of this paper is to discuss compatibility of Poisson–Lie transformation of dilaton and Killing field with transformations that leave invariant (NS–NS part of) generalized supergravity equations and give conditions for applicability of the transformations.

2 Poisson–Lie transformations and generalized supergravity equations

Poisson–Lie duality/plurality is based on the possibility to pass between various decompositions of Drinfel'd double $\mathscr{D}$ that generates background of investigated sigma models. It is a $2d$-dimensional Lie group whose Lie algebra $\mathfrak{d}$ can be decomposed into double cross sum of Lie subalgebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ that are maximally isotropic with respect to non-degenerate symmetric bilinear ad-invariant form $\langle .,. \rangle$. Drinfel'd double with so called Manin triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ and Lie subgroups $\mathscr{G}, \widetilde{\mathscr{G}}$ corresponding to $\mathfrak{g}, \tilde{\mathfrak{g}}$ is denoted by $(\mathscr{G}|\widetilde{\mathscr{G}})$.

Assume that there is d-dimensional Lie group $\mathscr{G}$ whose action on $\mathscr{M}$ is smooth, proper and free. The action of $\mathscr{G}$ is transitive on its orbits, hence we may locally consider $\mathscr{M} \approx (\mathscr{M}/\mathscr{G}) \times \mathscr{G} = \mathscr{N} \times \mathscr{G}$, $\dim \mathscr{M} = \dim \mathscr{N} + \dim \mathscr{G} = n+d$ and introduce adapted coordinates

$$\{y^\mu\} = \{s_\alpha, y^a\}, \tag{1}$$

where y^a are group coordinates and s_α label the orbits of $\mathscr{G}$,

$$\mu = 1, \ldots \dim \mathscr{M}, \quad \alpha = 1, \ldots, n = \dim \mathscr{N}, \\ a = 1, \ldots, d = \dim \mathscr{G}. \tag{2}$$

Coordinates s_α are treated as "spectators" as they do not participate in Poisson–Lie transformations.

Poisson–Lie dualizable sigma models on $\mathscr{N} \times \mathscr{G}$ are given by tensor field

$$\mathcal{F} = \mathcal{F}(y^\mu) = \mathcal{F}(s_\alpha, y^a), \tag{3}$$

that satisfy [1,2]

$$(\mathcal{L}_{v_i}\mathcal{F})_{\mu\nu} = \mathcal{F}_{\mu\kappa} v_j^\kappa \tilde{f}_i^{jk} v_k^\lambda \mathcal{F}_{\lambda\nu}, \quad i = 1, \ldots, \dim \mathscr{G}, \tag{4}$$

where v_i form basis of left-invariant fields on $\mathscr{G}$ and $\tilde{f}_i^{jk}$ are structure coefficients of th Lie group $\widetilde{\mathscr{G}}$. Explicit form of dualizable tensors $\mathcal{F}$ and their Poisson–Lie transformed forms are given in the Appendix.

The NS–NS part of generalized supergravity equations $GSE(\mathcal{F}, \Phi, J)$ read [10,11]

$$0 = R_{\mu\nu} - \frac{1}{4} H_{\mu\rho\sigma} H_\nu^{\ \rho\sigma} + \nabla_\mu X_\nu + \nabla_\nu X_\mu, \tag{5}$$

$$0 = -\frac{1}{2} \nabla^\rho H_{\rho\mu\nu} + X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu, \tag{6}$$

$$0 = R - \frac{1}{12} H_{\rho\sigma\tau} H^{\rho\sigma\tau} + 4\nabla_\mu X^\mu - 4 X_\mu X^\mu \tag{7}$$

where $R_{\mu\nu}$ is Ricci tensor of metric $\mathcal{G}$, $R = R_\mu^{\ \mu}$,

$$H_{\rho\mu\nu} = \partial_\rho \mathcal{B}_{\mu\nu} + \partial_\mu \mathcal{B}_{\nu\rho} + \partial_\nu \mathcal{B}_{\rho\mu} \tag{8}$$

and

$$X_\mu := \partial_\mu \Phi + J^\kappa \mathcal{F}_{\kappa\mu}. \tag{9}$$

For the NS–NS part of generalized supergravity equations (5)–(7) it is not necessary to require that J be Killing vector field of $\mathcal{G}, H, \phi$ even though it is it is required for their full version containing the R-R fields. Goal of this section is to define Poisson–Lie transformations of the fields ϕ, J that keep the Eqs. (5)–(7) satisfied.

Let Drinfel'd double has two decomposition $\mathscr{D} = (\mathscr{G}|\widetilde{\mathscr{G}}) = (\widehat{\mathscr{G}}|\bar{\mathscr{G}})$. Transformation of dilaton under Poisson–Lie T-plurality can be expresed as

$$\Phi^0(s, y) =: \Phi(s, y) - \frac{1}{2} L(s, y) = \widehat{\Phi}(s, \hat{x}) - \frac{1}{2} \widehat{L}(s, \hat{x}) \tag{10}$$

where $\Phi(s, y)$, $\widehat{\Phi}(s, \hat{x})$ are dilatons of the initial and transformed model. Variables y represent coordinates of group $\mathscr{G}$, $\hat{x}$ are coordinates of group $\widehat{\mathscr{G}}$, and terms $L(s, y)$, $\widehat{L}(s, \hat{x})$ read

$$L(s, y) = \ln \left| \frac{(\det \mathcal{G}(s, y))^{1/2}}{\det u(y)} \right|, \\ \widehat{L}(s, \hat{x}) = \ln \left| \frac{(\det \widehat{\mathcal{G}}(s, \hat{x}))^{1/2}}{\det \hat{u}(\hat{x})} \right|, \tag{11}$$

where $\mathcal{G}$ and $\widehat{\mathcal{G}}$ are metrics of sigma models on $\mathscr{N} \times \mathscr{G}$ resp. $\mathscr{N} \times \widehat{\mathscr{G}}$. Matrices $u, \hat{u}$ are components of left-invariant forms of $\mathscr{G}$ and $\widehat{\mathscr{G}}$.

In cases when the invariant dilaton Φ^0 in (10) depends on coordinates y we have to express y in terms of $\hat{x}$ and $\bar{x}$ to get explicit form of transformed dilaton $\widehat{\Phi}(s, \hat{x})$. This can be done by solving relation between two different decompositions of elements of Drinfel'd double $\mathscr{D} = (\mathscr{G}|\widetilde{\mathscr{G}}) = (\widehat{\mathscr{G}}|\bar{\mathscr{G}})$.

$$g(y)\tilde{h}(\tilde{y}) = \widehat{g}(\hat{x})\bar{h}(\bar{x}), \quad g \in \mathscr{G}, \quad \tilde{h} \in \widetilde{\mathscr{G}}, \quad \widehat{g} \in \widehat{\mathscr{G}}, \quad \bar{h} \in \bar{\mathscr{G}} \tag{12}$$

so that

$$y^k = Y^k(\hat{x}, \bar{x}), \quad \tilde{y}^k = \tilde{Y}^k(\hat{x}, \bar{x}). \tag{13}$$

If $\Phi^0(s, y)$ after the insertion (13) into (10) depends linearly on dual-coordinates $\bar{x}_a$

$$\widehat{\Phi}^0(s, \hat{x}, \bar{x}) := \Phi^0(s, Y(\hat{x}, \bar{x})) = \widehat{\Phi}^0(s, \hat{x}) + \bar{d}^a \bar{x}_a \tag{14}$$

then we can transform dilaton Φ and vector field J in the following way (see [7,8]).

$$\widehat{\Phi}(s, \hat{x}) = \widehat{\Phi}^0(s, \hat{x}) + \frac{1}{2} \widehat{L}(s, \hat{x}) \tag{15}$$

$$\widehat{\mathcal{J}}^\alpha = 0, \quad \alpha = 1, \ldots, n = \dim \mathscr{N}, \\ \widehat{\mathcal{J}}^{\dim \mathscr{N} + m}(s, \hat{x}) = \left(\frac{1}{2} \bar{f}^{ab}{}_b - \bar{d}^a \right) \widehat{v}_a{}^m(\hat{x}) \tag{16}$$

The above formulas work well for isometric models, i.e. if $\widetilde{\mathscr{G}}$ is abelian. For some more general type of models transformation of Killing field must be extended to [9]

$$\widehat{\mathcal{J}}^\alpha = 0, \quad \alpha = 1, \ldots, n = \dim \mathscr{N},$$
$$\widehat{\mathcal{J}}^{\dim \mathscr{N}+m}(s, \hat{x}) = \frac{1}{2} \widetilde{f}^{ab}{}_b \left(\frac{\partial \widetilde{y}_a}{\partial \bar{x}_k} \widehat{v}_k{}^m(\hat{x}) - \frac{\partial \widetilde{y}_k}{\partial \hat{x}^a} \widehat{\mathcal{F}}^{km} \right)$$
$$+ \left(\frac{1}{2} \bar{f}^{ab}{}_b - \bar{d}^a \right) \widehat{v}_a{}^m(\hat{x}) \tag{17}$$

where $a, b, k, m = 1, \ldots, \dim \mathscr{G}$, $\widetilde{f}^{ba}{}_c$ and $\bar{f}^{ba}{}_c$ are structure constants of Lie algebras of $\widetilde{\mathscr{G}}$, $\bar{\mathscr{G}}$ and $\widehat{v}_a$ are left-invariant fields of the group $\widehat{\mathscr{G}}$. This modification does not change results of [8,12,13] because those papers deal with isotropic initial models whose corresponding Manin triples $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ are semiabelian, i.e. $\widetilde{f}^{ab}{}_b = 0$.

3 Symmetries of generalized supergravity equations

Let $(\mathcal{F}, \Phi, J)$ satisfy generalized supergravity equations and there is a a symmetry of these equations, i.e. transformation $(\mathcal{F}, \Phi, J) \mapsto (\mathcal{F}', \Phi', J')$ that keep the generalized supergravity equations satisfied

$$GSE(\mathcal{F}, \Phi, J) \Leftrightarrow GSE(\mathcal{F}', \Phi', J'). \tag{18}$$

We shall call this symmetry *Poisson–Lie compatible* if $(\widehat{\mathcal{F}}', \widehat{\Phi}', \widehat{\mathcal{J}}')$ obtained by Poisson–Lie transformation of $(\mathcal{F}', \Phi, J')$ satisfy generalized supergravity equations as well, i.e. if

$$GSE(\mathcal{F}', \Phi', J') \Leftrightarrow GSE(\widehat{\mathcal{F}}', \widehat{\Phi}', \widehat{\mathcal{J}}'). \tag{19}$$

Our aim is finding symmetries compatible with Poisson–Lie transformations (15) and (16) or (17).

3.1 χ-Symmetry

First symmetry we are going to investigate is shift of form X. For torsionless backgrounds, which are all examples below, it is easy to see that if X_μ satisfy the generalized supergravity equations, then

$$X'_\mu := X_\mu + \chi_\mu, \tag{20}$$

where

$$\nabla_\nu \chi_\mu = 0, \quad (X_\mu + 2\,\chi_\mu)\chi^\mu = 0, \tag{21}$$

satisfy the equations as well. Due to the former condition form χ is (locally) exact so that $\chi = d\psi$ and this symmetry is just (t, x-dependent) shift of dilaton

$$\Phi' = \Phi + \psi. \tag{22}$$

Note that the vector field J remains unchanged.

Unfortunately, in many cases χ-symmetries are not Poisson–Lie compatible.

3.1.1 Example 1

Solving the Eq. (21) for flat metric[1]

$$ds^2 = -dt^2 + t^2\, dy_1^2 + t^2\, e^{2y_1} dy_2^2 + t^2\, e^{2y_1} dy_3^2 \tag{23}$$

with coordinates adapted to its Bianchi 5 symmetry

$$[T^1, T^2] = T^2, \quad [T^1, T^3] = T^3, \tag{24}$$

and dilaton $\Phi = 0$ we get

$$\chi_\mu = (C_1\, e^{y_1}, C_1\, t\, e^{y_1}, 0, 0), \quad \Phi' = \psi = C_1\, t\, e^{y_1} + C_0 \tag{25}$$

By nonabelian T-duality given by the matrix

$$C = \begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix} \tag{26}$$

we get (see Appendix)

$$\widehat{\mathcal{F}}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{t^2}{t^4+\hat{x}_2^2+\hat{x}_3^2} & \frac{\hat{x}_2}{t^4+\hat{x}_2^2+\hat{x}_3^2} & \frac{\hat{x}_3}{t^4+\hat{x}_2^2+\hat{x}_3^2} \\ 0 & -\frac{\hat{x}_2}{t^4+\hat{x}_2^2+\hat{x}_3^2} & \frac{t^4+\hat{x}_3^2}{t^2(t^4+\hat{x}_2^2+\hat{x}_3^2)} & -\frac{\hat{x}_2\hat{x}_3}{t^2(t^4+\hat{x}_2^2+\hat{x}_3^2)} \\ 0 & -\frac{\hat{x}_3}{t^4+\hat{x}_2^2+\hat{x}_3^2} & -\frac{\hat{x}_2\hat{x}_3}{t^2(t^4+\hat{x}_2^2+\hat{x}_3^2)} & \frac{t^4+\hat{x}_2^2}{t^2(t^4+\hat{x}_2^2+\hat{x}_3^2)} \end{pmatrix} \tag{27}$$

but if $C_1 \neq 0$ *we cannot apply formulas* (15) *and* (16) *or* (17) *for the non-Abelian T-duality* (and some other Poisson–Lie T-pluralities) as the condition (14) for its application does not hold because

$$\widehat{\Phi}^{0\,\prime}(t, \hat{x}, \bar{x}) = C_1\, t\, e^{\bar{x}_1} + C_0 + \bar{x}_1 - \frac{3}{2} \log t.$$

[1] We work with four-dimensional models invariant w.r.t. three-dimensional groups so that $dim \mathscr{N} = 1$, and spectator is denoted as t.

3.1.2 Example 2

Background

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-y_1} y_1 & e^{-y_1} \\ 0 & e^{-y_1} y_1 & e^{-2y_1} & 0 \\ 0 & e^{-y_1} & 0 & 0 \end{pmatrix} \quad (28)$$

that corresponds to the flat metric adapted to Bianchi 4 symmetry satisfy generalized supergravity equations together with vanishing X-form. Solving the equations (21) we get

$$\Phi' = \psi = C_0 + C_1 t + C_2 e^{-y_1} + C_3 \left(y_1 + e^{-y_1} y_2\right) + C_4 \left(\frac{1}{2} e^{-y_1} y_2^2 + (y_1 - 1)\, y_2 - \frac{e^{y_1}}{2} + y_3\right) \quad (29)$$

where the latter condition of (21) implies

$${C_1}^2 + {C_3}^2 - 2\, C_2 C_4 = 0.$$

The background (28), dilaton (29) and vanishing J satisfy generalized supergravity equations but once again we cannot apply formulas (15) and (16) or (17) for non-Abelian T-duality as the condition (14) is satisfied only if $\Phi' = C_0$.

3.2 Gauge transformations

Another symmetry of NS–NS generalized supergravity equations is gauge transformation[2]

$$\mathcal{F}_\Lambda := \mathcal{F}, \quad \Phi_\Lambda := \Phi + \Lambda \quad J_\Lambda := J - d\Lambda . \mathcal{F}^{-1} \quad (30)$$

where Λ is arbitrary (differentiable) function of (s, y). It leaves X invariant, $X = X_\Lambda$, so that

$$GSE(\mathcal{F}_\Lambda, \Phi_\Lambda, J_\Lambda) \Leftrightarrow GSE(\mathcal{F}, \Phi, J). \quad (31)$$

Contrary to this, formulas (15) and (16) or (17) for Poisson–Lie transformation of dilaton and Killing field *do not provide solution of generalized supergravity equations for arbitrary* Λ.

$$GSE(\mathcal{F}_\Lambda, \Phi_\Lambda, J_\Lambda) <\neq> GSE(\widehat{\mathcal{F}}_\Lambda, \widehat{\Phi}_\Lambda, \widehat{J}_\Lambda).$$

3.2.1 Example 3, trivial - identical transformation of flat background

Let us investigate the simplest Poisson–Lie transformation – identity of the flat model (23) with $\mathcal{B}$-field vanishing. This background together with

$$\Phi = 0, \quad J^\mu = (0, 0, 0, 0), \quad X_\mu = (0, 0, 0, 0), \quad (32)$$

obviously satisfy generalized supergravity equations. Besides that, identical Poisson–Lie transformation of (32) gives the same fields ϕ, J, X.

However, generalized supergravity equations are satisfied also for

$$\Phi_\Lambda = \Lambda, \quad J^\mu_\Lambda = -\partial_\nu \Lambda\, \mathcal{G}^{\nu\mu}, \quad \mathcal{F}_\Lambda = \mathcal{F}, \quad X_\Lambda = (0, 0, 0, 0) \quad (33)$$

where Λ is arbitrary function of (t, y_1, y_2, y_3). Choosing for example $\Lambda = y_1$ we get

$$\Phi = y_1, \; J^\mu = \left(0, \frac{-1}{t^2}, 0, 0\right). \quad (34)$$

Applying formulas (15) and (16) for identical Poisson–Lie transformation to (34) we find

$$\widehat{\Phi} = \hat{x}_1, \quad \widehat{J}^\mu = (0, 0, 0, 0), \quad \widehat{\mathcal{F}}_\Lambda = \mathcal{F}, \quad \widehat{X}_\Lambda = (0, 1, 0, 0)$$

and generalized supergravity equations are not satisfied.

This simple example shows that *Poisson–Lie transformations are not in general compatible with gauge transformations* and that the condition (14) is not sufficient for applicability of the formulas (15) and (16) or (17).

Note that the field J in (34) in the latter case is not Killing field of the flat metric. It may give a clue for restriction of the gauge transformations.

4 Choice of Poisson–Lie compatible gauge

We have seen that in spite of the fact that due to symmetries there can be quite large set of dilatons and vector fields J that satisfy generalized supergravity equations, only very limited subset of them are Poisson–Lie compatible. By inspection of the formulas (15) and (16) one can see that the problem is in fulfilling the condition (14) for invariant dilaton Φ^0 shifted both by χ-symmetry and gauge transformation

$$\Phi^{0\,\prime}(s, y) = \Phi^0(s, y) + \psi(s, y) + \Lambda(s, y). \quad (35)$$

Fortunately, we can use the arbitrariness of the gauge function Λ to satisfy the condition (14). On the other hand, we know from the Example 3.2.1 that gauge transformations in general are not compatible with Poisson–Lie transformation so that they must be further restricted.

Condition that the field J is Killing vector of the background $\mathcal{F} = \mathcal{G} + \mathcal{B}$ (up to an exact 2-form) is not necessary for satisfying NS–NS generalized supergravity equations but it is required for their full version containing the R–R fields [11]. Beside that, dilaton must also be invariant in direction

[2] Note that this symmetry is different from the the transformation of B-field $B_\lambda = B + d\lambda$ investigated e.g. in [14,15]

of J. Therefore, if $(\mathcal{F}, \Phi, J)$ satisfy generalized supergravity equations (5)–(7) we will require for Λ

$$\mathcal{L}_{J_\Lambda}\mathcal{G} = 0, \quad \mathcal{L}_{J_\Lambda}\mathcal{B} = d\omega, \quad \mathcal{L}_{J_\Lambda}\Phi = 0 \tag{36}$$

where and $J_\Lambda := J - d\Lambda.\mathcal{F}^{-1}$. It turns out that *these additional conditions together with (14) are sufficient for compatibility of Poisson–Lie transformations with symmetries introduced in the Sect. 3.*

4.1 Example 1: continued

Let $\mathcal{B}$-field is vanishing, background is given by the flat metric (23), vanishing J-field and dilaton obtained by χ-symmetry and gauge transformation

$$\Phi'(t, y) = C_1\, t\, e^{y_1} + C_0 + \Lambda(t, y_1, y_2, y_3). \tag{37}$$

They satisfy generalized supergravity equations. Condition (14) for non-Abelian T-duality is fulfilled for

$$\Lambda = -C_1\, t\, e^{y_1} + \Lambda_0(t) + \lambda_1 y_1 + \lambda_2\, y_2 + \lambda_3\, y_3. \tag{38}$$

Requiring that J_Λ is Killing vector of flat metric (23), and dilaton (37) we get

$$\Lambda_0(t) = \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

This gauge is compatible with T-duality of background (23) dilaton (37) and vanishing J-field. This means that only the trivial dilaton $\Phi' = C_0$ can be dualized by formulas (15), (16). By the non-Abelian T-dual given by (26) we get the background (27) and formulas (15), (16) yield [16]

$$\widehat{\Phi} = -\frac{1}{2}\log\left(-t^2\left(t^4 + \hat{x}_2^2 + \hat{x}_3^2\right)\right), \quad \widehat{J}^\mu = (0, 2, 0, 0). \tag{39}$$

Generalized supergravity equations are satisfied for these fields.

Let us note that repeating the Poisson–Lie transformation given by (26) on the tensor field (27) we return to flat metric (23) but dual dilaton and vector J given by(15) and (17) are of the form (34) that differ from the initial ones (32) by gauge transformation $\Lambda = y_1$.

4.2 Example 2: continued

Let $\mathcal{B}$-field is vanishing, background is given by the flat metric (28), vanishing J-field and by dilaton

$$\Phi' = C_0 + C_1\, t + C_2 e^{-y_1} + C_3\left(y_1 + e^{-y_1} y_2\right) \tag{40}$$
$$+ C_4\left(\frac{1}{2}e^{-y_1} y_2^2 + (y_1 - 1)\, y_2 - \frac{e^{y_1}}{2} + y_3\right) + \Lambda(t, y_1, y_2, y_3). \tag{41}$$

They satisfy generalized supergravity equations. Condition (14) for T-duality given by (26) is fulfilled if

$$\Lambda = c_0 + c_1 y_2 + c_2 y_3 + \frac{1}{2}\left(C_4 e^{y_1} - e^{-y_1}\left(2C_2 + 2C_3 y_2 + C_4 y_2^2\right)\right) \tag{42}$$
$$+ y_1\,(c_4 - C_4 y_2) + \Lambda_0(t). \tag{43}$$

Requiring that J_Λ is Killing vector of flat metric (28), and dilaton (40) one gets

$$c_1 = -c_2 = C_4, \quad c_4 = -C_3, \quad \Lambda_0(t) = c_5 - C_1 t.$$

This gauge eliminates χ-symmetry shift in (40) up to constant and only the trivial dilaton $\Phi' = C_0$ can be dualized by formulas (15), (16). We get

$$\widehat{\Phi} = \widehat{C}_0 - \frac{1}{2}\log\left(\hat{x}_3{}^2 - 1\right), \quad \widehat{J} = (0, -2, 0, 0) \tag{44}$$

that together with

$$\widehat{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\hat{x}_3} \\ 0 & 0 & 1 & \frac{\hat{x}_3-\hat{x}_2}{\hat{x}_3-1} \\ 0 & \frac{1}{\hat{x}_3+1} & \frac{\hat{x}_3-\hat{x}_2}{\hat{x}_3+1} & \frac{(\hat{x}_2-\hat{x}_3)^2}{\hat{x}_3{}^2-1} \end{pmatrix}, \tag{45}$$

obtained by (55)–(58), satisfy generalized supergravity equations.

On the other hand, let us note that there are Poisson–Lie transformations that impose weaker restriction on the gauge transformations and therefore admit a wider subset of dilatons that can be Poisson–Lie transformed. It is for example Poisson–Lie T-plurality $(4|1) \to (6_{-1}|2)$ of (28), (40) given by

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{46}$$

Condition (14) then gives $C_1 = C_3 = C_4 = 0$ and

$$\Lambda = \Lambda_1(t, y_1, y_3) + \Lambda_2(y_1, y_3) + \Lambda_3(y_1) + c_1 y_2. \tag{47}$$

Requiring further that J_Λ is Killing vector of flat metric (28), and dilaton (40), i.e. satisfies (36), one gets finally

$$\Lambda = c_2 + c_3 e^{-y_1}, \quad \Phi' = C_0 + C_2\, e^{-y_1} + c_2 + c_3 e^{-y_1} \tag{48}$$

All these dilatons can be pluralized. Poisson–Lie T-plurality induced by (46) then gives

$$\widehat{\mathcal{F}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\hat{x}_1^2 & -e^{-x_1}\hat{x}_1 & e^{\hat{x}_1} \\ 0 & e^{-\hat{x}_1}\hat{x}_1 & e^{-2\hat{x}_1} & 0 \\ 0 & e^{\hat{x}_1} & 0 & 0 \end{pmatrix} \tag{49}$$

$$\widehat{\Phi} = C_0 + (C_2 + c_3)\, e^{\hat{x}_1}, \quad \widehat{J} = (0, 0, 0, 0) \tag{50}$$

and generalized supergravity equations are satisfied.

5 Conclusion

We have investigated two types of transformations that keep NS–NS part of generalized supergravity equations satisfied. They are χ-symmetry (20) for torsionless sigma models, that shifts dilaton only, and gauge transformations (30) that change both dilaton and vector field J but leave the form

$$X_\mu := \partial_\mu \Phi + J^\kappa \mathcal{F}_{\kappa\mu}.$$

invariant. Due to these symmetries there is a large set of dilatons and vector fields J that satisfy generalized supergravity equations for fixed tensor field $\mathcal{F}$.

We have shown that Poisson–Lie transformations are not in general compatible with the above mentioned symmetries - see Examples 1,2,3. In other words, formulas (15), (16) or (17) for Poisson–Lie transformations of dilatons and vector fields J can be applied only to a rather narrow subset of dilatons in order that the transformed fields satisfy generalized supergravity equations.

The applicability of the formulas (15), (16) or (17) is restricted

1. By the condition (14) requiring that the invariant dilaton Φ^0 given by (10) is linear in the dual coordinates $\bar{x}$.
2. By the conditions (36) that fixes the admissible gauges, namely, that gauge transformed vector field J_Λ is Killing vector of metric, torsion and dilaton. It is interesting that this condition of Poisson–Lie compatibility is identical with condition for full generalized supergravity equations containing the R-R fields.

Within these restrictions Poisson–Lie transformations keep generalized supergravity equations satisfied – see Examples 1,2 continued. Typically it chooses just one dilaton and Killing vector field but it is not a rule as shown in the Sect. 4.2. We have checked several other cases of Poisson–Lie duality/plurality with equal results.

Besides that we have found that twice applied T-duality produces identical dilatons but vector fields J only up to a gauge transformation. It means that, diferently from transformations of tensor fields (55)–(58), the above mentioned formulas do not provide true representation of $O(d, d)$.

Appendix: Poisson–Lie transformations of the tensor field

For many Drinfel'd doubles several decompositions may exist. Suppose that we have sigma model on $\mathcal{N} \times \mathcal{G}$ and tensor field $\mathcal{F}$ satisfies Eq. (4). Let Drinfel'd double $\mathcal{D} = (\mathcal{G}|\widetilde{\mathcal{G}})$ splits into another pair of Lie subgroups $\widehat{\mathcal{G}}$ and $\bar{\mathcal{G}}$ so that $(\mathcal{G}|\widetilde{\mathcal{G}}) = (\widehat{\mathcal{G}}|\bar{\mathcal{G}})$. Then we can apply the full framework of Poisson–Lie T-plurality [1,3] and find tensor field $\widehat{\mathcal{F}}$ for sigma model on $\mathcal{N} \times \widehat{\mathcal{G}}$ in the following way.

Poisson–Lie dualizable sigma models on $\mathcal{N} \times \mathcal{G}$ satisfying (4) are given by tensor field $\mathcal{F}$ of the form

$$\mathcal{F}(s, y) = \mathcal{E}(y) \cdot (\mathbf{1}_{n+d} + E(s) \cdot \Pi(y))^{-1} \cdot E(s) \cdot \mathcal{E}^T(y) \tag{51}$$

where $E(s)$ is spectator-dependent $(n+d) \times (n+d)$ matrix. Denoting generators of Manin triple $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ as $T, \widetilde{T}$, matrix $\Pi(y)$ is given by submatrices $a(y)$ and $b(y)$ of the adjoint representation

$$ad_{g^{-1}}(\widetilde{T}) = b(y) \cdot T + a^{-1}(y) \cdot \widetilde{T}$$

as

$$\Pi(y) = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & b(y) \cdot a^{-1}(y) \end{pmatrix}.$$

Matrix $\mathcal{E}(y)$ reads

$$\mathcal{E}(y) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & e(y) \end{pmatrix} \tag{52}$$

where $e(y)$ is $d \times d$ matrix of components of right-invariant Maurer–Cartan form $(dg)g^{-1}$ on $\mathcal{G}$.

Manin triples $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$ and $(\mathfrak{d}, \hat{\mathfrak{g}}, \bar{\mathfrak{g}})$ are two decompositions of Lie algebra $\mathfrak{d}$ into double cross sum of subalgebras that are maximally isotropic with respect to $\langle ., . \rangle$. Pairs of mutually dual bases $T_a \in \mathfrak{g}$, $\widetilde{T}^a \in \tilde{\mathfrak{g}}$ and $\widehat{T}_a \in \hat{\mathfrak{g}}$, $\bar{T}^a \in \bar{\mathfrak{g}}$, $a = 1, \ldots, d$, then must be related by transformation

$$\begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix} = C \cdot \begin{pmatrix} T \\ \widetilde{T} \end{pmatrix} \tag{53}$$

where C is an invertible $2d \times 2d$ matrix. (Non-Abelian) T-duality is obtained by

$$C = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}.$$

Poisson–Lie T-plurality is given by $d \times d$ matrices P, Q, R, S such that

$$\begin{pmatrix} T \\ \widetilde{T} \end{pmatrix} = C^{-1} \cdot \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \cdot \begin{pmatrix} \widehat{T} \\ \bar{T} \end{pmatrix}. \tag{54}$$

For the following formulas it is convenient to extend matrices P, Q, R, S to $(n+d) \times (n+d)$ matrices

$$\mathcal{P} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & P \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & Q \end{pmatrix},$$
$$\mathcal{R} = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & R \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & S \end{pmatrix}$$

to accommodate the spectator fields.

Sigma model on $\mathscr{N} \times \widehat{\mathscr{G}}$ obtained from (51) via Poisson–Lie T-plurality is given by tensor field

$$\widehat{\mathcal{F}}(s, \hat{x}) = \widehat{\mathcal{E}}(\hat{x}) \cdot \widehat{E}(s, \hat{x}) \cdot \widehat{\mathcal{E}}^T(\hat{x}), \qquad \widehat{\mathcal{E}}(\hat{x}) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & \widehat{e}(\hat{x}) \end{pmatrix}, \tag{55}$$

where $\widehat{e}(\hat{x})$ is $d \times d$ matrix of components of right-invariant Maurer–Cartan form $(d\hat{g})\hat{g}^{-1}$ on $\widehat{\mathscr{G}}$ and

$$\widehat{E}(s, \hat{x}) = \left(\mathbf{1}_{n+d} + \widehat{E}(s) \cdot \widehat{\Pi}(\hat{x})\right)^{-1} \cdot$$
$$\widehat{E}(s) = \left(\widehat{E}^{-1}(s) + \widehat{\Pi}(\hat{x})\right)^{-1}. \tag{56}$$

The matrix $\widehat{E}(s)$ is obtained from $E(s)$ in (51) by formula

$$\widehat{E}(s) = (\mathcal{P} + E(s) \cdot \mathcal{R})^{-1} \cdot (\mathcal{Q} + E(s) \cdot \mathcal{S}), \tag{57}$$

and

$$\widehat{\Pi}(\hat{x}) = \begin{pmatrix} \mathbf{0}_n & 0 \\ 0 & \widehat{b}(\hat{x}) \cdot \widehat{a}^{-1}(\hat{x}) \end{pmatrix},$$
$$ad_{\hat{g}^{-1}}(\bar{T}) = \widehat{b}(\hat{x}) \cdot \widehat{T} + \widehat{a}^{-1}(\hat{x}) \cdot \bar{T}. \tag{58}$$

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: As purely theoretical paper it does not need any supplemental data.]

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