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General relativity versus dark matter for rotating galaxies

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Abstract A very general class of axially symmetric metrics in general relativity (GR) that includes rotations is used to discuss the dynamics of rotationally supported galaxies. The exact vacuum solutions of the Einstein equations for this extended Weyl class of metrics allow us to rigorously deduce the following: (i) GR rotational velocity always exceeds the Newtonian velocity (thanks to Lenz's law in GR). (ii) A non-vanishing intrinsic angular momentum (J) for a galaxy demands the asymptotic constancy of the Weyl (vectorial) length parameter (a)—a behaviour identical to that found for the Kerr metric. (iii) Asymptotic constancy of the same parameter a also demands a plateau in the rotational velocity. Unlike the Kerr metric, the extended Weyl metric can and has been continued within the galaxy, and it has been shown under what conditions Gauß and Ampére laws emerge along with Ludwig's extended gravito-electromagnetism (GEM) theory with its attendant non-linear rate equations for the velocity field. Better estimates (than that from the Newtonian theory) for the escape velocity of the Sun have been presented.

1 Introduction

The velocities of the ionized gases circling many galaxies, as a function of the distance from their centre (the *rotation curves*), do not appear to follow a Kepler law and drop as $1/\sqrt{r}$, but on the contrary tend to reach a plateau velocity (v_φ). This experimental fact, first discovered by Vera Rubin [1,2] in the 1980s and confirmed by many later observations, poses one of the most interesting theoretical questions of today's physics. The most common explanation is to suppose that the observed mass and volume of a galaxy is only a small part of the total, with the rest being a vast (spherically symmetric) distribution of hypothetical dark matter (DM), which interacts only gravitationally. This is the basis of the widely accepted ΛCDM model [3] (cosmological constant plus cold dark matter) that is itself anchored upon general relativity with a cosmological constant suitably chosen to yield a cosmologically flat universe. By contrast, its applications to the detailed phenomenology of rotating galaxies in the DM framework are based on the Newtonian theory.

The DM model when applied to (rotating) galaxies has its problems. First of all, in spite of extensive searches, no trace of this mysterious DM has been found. Secondly, there is an empirical but successful relation, the (baryonic) Opik–Tully–Fischer law [4–6], between the velocity of the gas at the edge of the optical region ($\bar{v}$) and the visible—hence baryonic—mass of the galaxy: ($M_{baryonic} \propto \bar{v}^4$), even in cases with substantial DM. We recall that in DM, an asymptotic v_φ is generated by the *dark mass*, not the baryonic mass, and Salucci has succeeded in finding more general relationships [7]. Thirdly, there has been no satisfactory explanation offered—in DM—for the magnitude of the intrinsic angular momentum (J_z) of a galaxy. By contrast, in GR, we can compute J_z in terms of the rotation velocity and the baryonic mass-current density that only extends over the visible size of any galaxy [8].

What we propose and show in this paper, building on previous work by other authors [9–17], is that GR, when appropriately applied, is perfectly capable of explaining the observed phenomena above, provided one takes into account the finite size (and a non-spherical mass distribution) of most galaxies and the basic fact that they rotate and radiate gravitational radiation.

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To be concrete, let us consider our own galaxy [18]. The Milky Way has a diameter of 25 kiloparsecs and a thickness of 2 kiloparsecs, with a visible baryonic mass of about $(1\div 2.5)\times 10^{11} M_{\odot}$. The considerably non-spherical geometry fixes the (stable) axis of rotation, and our galaxy acquires a rotational velocity of about 200 km/s at the edge (of the diameter). Rotations bring about a well-known but oft forgotten fundamental difference between the Newtonian theory and GR.

In the Newtonian theory, *there is no dependence of the gravitational field upon the rotation of a body* [19]. In GR, on the other hand, the rotation of a system makes the metric non-diagonal (i.e., the time-space component $g_{0,i} \propto A_i$ becomes non-zero and a 3-vector-field A_i is generated). A *preferred* direction (in space) is thus chosen and the *sense* of rotation (clockwise or anti-clockwise) established and fixed. This leads to the introduction of parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) violating but $\mathcal{PT}$ conserving terms. Thus, a geo-magnetic field $\mathbf{B} = \nabla \wedge \mathbf{A}$ emerges (already at the linearized level in GR) that gives rise to the GEM (geo-electromagnetic) theory of Thirring and Lense [20–22]. (The ensuing Lense–Thirring effect was beautifully confirmed experimentally in Ref. [23].) An angular momentum $\mathbf{J}$ is generated (through the non-diagonal term). These issues are discussed in detail in later sections.

The paper is organized as follows. In Sect. 2, we anchor our formalism upon the most general class of stationary, axially symmetric metrics found by Weyl [24,25]. In this section, we discuss the Einstein equations valid in the vacuum (i.e., outside the galaxy). In Sect. 3, we consider the choice of the matter energy–momentum density appropriate for a galaxy that is supported entirely by rotations with zero pressure. The nature of the solutions of the Einstein equations for the matter within the galaxy are explored. In Sect. 4, we highlight a key role that Lenz's law plays in always boosting the rotation velocity. In Sect. 5, we continue our discussion of Ludwig's extended GEM theory arising from the exact Weyl type constraints. We discuss the affinity between the Weyl class of metrics and the specialized Kerr metric in Sect. 6, in particular the appearance of an angular momentum whose value is computed for both. It is important to note that the Schwarzschild metric has zero angular momentum simply because it is spherical and thus lacks a vector field fixing a direction in space. In Sect. 8, we briefly discuss some alternatives to DM that have been proposed in the literature. The paper is concluded in Sect. 9 with a summary of results obtained, work in progress, and future prospects.

2 The Weyl metric

We shall write the axially symmetric Weyl metric for a cylindrically symmetric space-time [26], with coordinates (ct, φ, ρ, z), including explicitly the rotation term (see for example Ref. [19]):

$$\begin{aligned} ds^2 &= -e^{2U}(cdt - a d\varphi)^2 + e^{-2U}\rho^2 d\varphi^2 \\ &\quad + e^{2\nu-2U}(d\rho^2 + dz^2), \\ g_{\mu\nu} &= \begin{pmatrix} -e^{2U} & e^{2U}a & 0 & 0 \\ e^{2U}a & -e^{2U}a^2 + e^{-2U}\rho^2 & 0 & 0 \\ 0 & 0 & e^{2\nu-2U} & 0 \\ 0 & 0 & 0 & e^{2\nu-2U} \end{pmatrix}; \\ g &= \det g_{\mu\nu} = -e^{4\nu-4U}\rho^2; \end{aligned} \tag{2.1}$$

the inverse metric has the form:

$$g^{\mu\nu} = \begin{pmatrix} \frac{e^{2U}a^2}{\rho^2} - e^{-2U} & \frac{e^{2U}a}{\rho^2} & 0 & 0 \\ \frac{e^{2U}a}{\rho^2} & \frac{e^{2U}}{\rho^2} & 0 & 0 \\ 0 & 0 & e^{2U-2\nu} & 0 \\ 0 & 0 & 0 & e^{2U-2\nu} \end{pmatrix};$$

and the invariant (spatial) volume element reads

$$\begin{aligned} dV &= (d\rho)(d\varphi)(dz)\sqrt{-g} = e^{-2(U-\nu)}(\rho d\rho dz d\varphi); \\ dV &\geq dV_{\text{flat}}. \end{aligned} \tag{2.2}$$

Below, we list some salient aspects of the above axially symmetric metric.

1. U, a, and ν are functions only of $\rho = \sqrt{x^2 + y^2}$ and z, independent of φ. Hence, there are two Killing vectors of the system, one time-like and the other space-like (outside of the horizon).
2. The function U is related to the Newtonian potential Φ through $e^{2U} = 1 + 2\Phi/c^2$.
3. The function a would be related to the angular momentum of the system.
4. The gravito-magnetic potential field $A_\phi = ca/\rho$ is a vector potential $= (0, ca/\rho, 0)$.
5. The three potential fields (U, a, and ν) characterizing the metric are not all independent. The Einstein equations in the vacuum, that is, outside the boundaries of a confined system such as a galaxy, impose the following *exact* non-linear differential constraints on these functions [19]:

$$\begin{aligned} &R_{\mu\nu} = 0; \text{ in the vacuum of the system implies:} \\ &\left[\frac{\partial^2 U}{\partial \rho^2} + \frac{\partial U}{\rho \partial \rho} + \frac{\partial^2 U}{\partial z^2}\right] = -\frac{e^{4U}}{2\rho^2} \\ &\times\left[\left(\frac{\partial a}{\partial \rho}\right)^2 + \left(\frac{\partial a}{\partial z}\right)^2\right]; (i) \\ &\frac{\partial}{\partial z}\left(\frac{e^{4U}}{\rho}\frac{\partial a}{\partial z}\right) + \frac{\partial}{\partial \rho}\left(\frac{e^{4U}}{\rho}\frac{\partial a}{\partial \rho}\right) = 0; (ii) \\ &\text{and } \frac{\partial \nu}{\rho \partial \rho} = \left[\left(\frac{\partial U}{\partial \rho}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2\right] \end{aligned}$$

$$-\frac{e^{4U}}{4\rho^2}\Big[\Big(\frac{\partial a}{\partial\rho}\Big)^2-\Big(\frac{\partial a}{\partial z}\Big)^2\Big];\,(iii)$$
$$\Big(\frac{\partial\nu}{\rho\partial z}\Big)=2\Big(\frac{\partial U}{\partial\rho}\Big)\Big(\frac{\partial U}{\partial z}\Big)-\Big(\frac{e^{4U}}{2\rho^2}\Big)\Big(\frac{\partial a}{\partial\rho}\Big)\Big(\frac{\partial a}{\partial z}\Big);\,(iv) \tag{2.3}$$

N.B.: Since U and a begin at order G, ν begins at second order (i.e., is of order G^2). Once U and a satisfy the top two equations relating them, Eqs. (2.3(i),(ii)), a solution for ν exists since the last two Eqs. (2.3(iii),(iv)) become the integrability conditions for it; $\nu\to 0$ as $\rho\to 0$ for any z.

6. The inequality in Eq. (2.2) that tells us that the invariant spatial volume element is larger than its value in the flat limit is useful for proving bounds on integrals of (positive definite) integrands in gravitational asymptotic perturbation theory such as that developed by Landau–Lifshitz [26] and by Weinberg [8].
7. A test particle in this axially symmetric metric would have two constants of motion, which we shall indicate as $p_0=E/c$ for time translations, and $p_\phi=J/c$ for rotational motion in the xy plane. We shall write $E=\gamma mc^2$, or $E=mc^2+\mathcal{E}_{NR}$ to study the non-relativistic limit.

We now write the geodesic equation for a test particle of mass m for the above metric. The simplest formalism that extends to a Riemannian space blessed with a metric is through the action principle. Calling the action S, m the mass, and τ the proper time, we have

$$\begin{aligned}&\mathrm{d}S=-(mc^2)\mathrm{d}\tau;\;(\mathrm{d}S)^2=(mc)^2(c\mathrm{d}\tau)^2;\\&\text{let } p_\mu=\frac{\partial S}{\partial x^\mu};\;\text{Hamilton}-\text{Jacobi equation implies:}\\&\quad g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu}=-(mc)^2;\\&\text{we have }\;p_\mu p_\nu g^{\mu\nu}=-(mc)^2.\end{aligned}\tag{2.4}$$

As stated earlier, an axially symmetric system has two conserved quantities, the energy E and the component of angular momentum J_z, say, for rotational motion in the xy-plane. Hence, the dependence on time interval (t) and that on φ can be prescribed as

$$\begin{aligned}&S(ct;\rho;\varphi;z)=-Et+J\varphi+\hat{S}(\rho;z);\\&-\frac{\partial S}{\partial ct}=E/c;\;\frac{\partial S}{\partial\varphi}=J;\;\frac{\partial S}{\partial\rho}=p_\rho;\;\frac{\partial S}{\partial z}=p_z.\end{aligned}\tag{2.5}$$

Thus, for the Weyl metric, we have

$$\begin{aligned}&(mc)^2=\Big(\frac{E}{c}\Big)^2\Big[e^{-2U}-\Big(\frac{a}{\rho}\Big)^2e^{2U}\Big]-2\frac{a}{\rho}\frac{J}{\rho}\frac{E}{c}e^{2U}\\&-\Big(\frac{J}{\rho}\Big)^2e^{2U}-e^{2(U-\nu)}[p_\rho^2+p_z^2];\quad(i)\\&\Big(\frac{E}{c}\Big)^2e^{-2U}-\Big[\frac{J}{\rho}+\frac{a}{\rho}\frac{E}{c}\Big]^2e^{2U}=(mc)^2\\&\quad+e^{2(U-\nu)}[p_\rho^2+p_z^2];\quad(ii)\\&\text{Or}:\;\Big[\frac{E}{c}\Big(e^{-U}+\frac{a}{\rho}e^U\Big)+\frac{J}{\rho}e^U\Big]\\&\times\Big[\frac{E}{c}\Big(e^{-U}-\frac{a}{\rho}e^U\Big)-\frac{J}{\rho}e^U\Big]\\&=(mc)^2+e^{2(U-\nu)}(p_\rho^2+p_z^2).\;\;(iii)\end{aligned}\tag{2.6}$$

Let $E=mc^2\gamma$, and as both E and J are constants of motion, we can define a reduced (a dimensional) angular momentum, i.e., angular momentum per unit energy per unit ρ (the perpendicular distance, or the impact parameter), $j\equiv(Jc/E\rho)$, and through it a *rotational* velocity $v_\varphi\equiv(jc)$. Similarly, the rotational parameter a from the metric can be employed to define a *vector potential*, $A_\varphi\equiv(ca/\rho)$, that has the dimensions of a velocity. With these definitions, Eq. (2.6;(ii)) reads:

$$\begin{aligned}&J=\rho\Big(\frac{E}{c}\Big)j;\;v_\varphi=(cj);\,a=\rho\Big(\frac{A_\varphi}{c}\Big);\,\pi_\varphi\equiv(v_\varphi+A_\phi);\\&\gamma^2\Big[e^{-2U}-\Big(\frac{\pi_\varphi}{c}\Big)^2e^{2U}\Big]=1+e^{2(U-\nu)}\Big[\frac{(p_\rho^2+p_z^2)}{(mc)^2}\Big].\end{aligned}\tag{2.7}$$

For a galaxy supported totally by rotations along φ, which is the focus of this paper, we set $p_z=0$ and $p_\rho=0$. Then the above equation is reduced to

$$\begin{aligned}&\gamma=\frac{1}{\sqrt{[e^{-2U}-(\pi_\varphi/c)^2e^{2U}]}};\\&\text{keeping leading terms only}:\;\gamma\approx\frac{1}{\sqrt{[1-2U-(\pi_\varphi/c)^2]}};\\&\text{test particle energy}:\;E=\gamma(mc^2)\approx\;mc^2+\mathcal{E}_{NR};\\&\mathcal{E}_{NR}=m\Phi+\frac{m}{2}\pi_\varphi^2;\;\pi_\varphi=(v_\varphi+A_\varphi).(ii)\end{aligned}\tag{2.8}$$

Equation (2.8(ii)) clearly shows what the Newtonian theory leaves out that GR supplies, i.e., the vector potential A_φ, which in turn generates the GEM magnetic field. The lack of the dynamics generated by mass current density in the Newtonian theory is a serious lacuna that has important consequences. We discuss one such important improvement that GR provides.

As $U<0$, the particle will remain bound so long as $|v_\varphi+A_\varphi|<\sqrt{-2\Phi}$ and not $v_\varphi<\sqrt{-2\Phi}$ (their values at the coordinates ρ,z in question) as the Newtonian theory asserts.

This leads to the well-known quandary when one computes—using Newtonian gravity—the escape velocity of our Sun were it to escape from our galaxy. The mean rotational velocity of our Sun is about 220 km/s and it is approximately 8.2 kiloparsecs away from the centre of our galaxy. There is apparently very little (baryonic) mass beyond this distance. Thus, Newtonian theory for the Sun's escape velocity predicts $\sqrt{2}\times(220)\approx$ 310 km/s [27] in the vicinity of our Sun, and experimental astrophysicists estimate the Sun's escape velocity to be between (500 ÷ 550) km/s [27].

In GEM, by contrast, the escape velocity reads: $v_{\text{escape}} \approx -A_\varphi + \sqrt{-2\Phi}$. As we shall discuss later in more detail, Lenz's law (reminding us that all masses attract so that the GEM magnetic field obeys the *left-hand rule*) forces us to have $A_\varphi < 0$, thus boosting the escape velocity (see Sect. 4). From a simple phenomenology of the Milky Way, we estimate the magnetic term to add about 200 km/sec, thereby bringing the escape velocity much closer to its estimated experimental value. A quantitative analysis of this matter shall be presented in a later work.

Having delineated a few important aspects that distinguish GR from the Newtonian theory regarding the dynamics of a rotation-supported galaxy, let us return to a discussion of the exact Weyl constraints.

At first glance, Eqs. (2.3(i–iv)) appear quite opaque and daunting, but they acquire a physically more appealing aspect through the following *dictionary* in terms of the GEM electric **E** and magnetic **B** fields of order G, along with a higher-order field $\hat{\mathbf{B}}$ that is of order G^2. They are defined as follows:

$$\begin{aligned}
&\mathbf{E} = (E_\rho, 0, E_z) = \Big(-\frac{\partial\Phi}{\partial\rho}, 0, -\frac{\partial\Phi}{\partial z}\Big) = -\nabla\,\Phi;\ (i)\\
&\mathbf{B} = (B_\rho, 0, B_z) = \Big(-\frac{\partial A_\varphi}{\partial z}, 0, \frac{\partial A_\varphi}{\partial\rho}\Big) = \nabla\wedge\mathbf{A};\ (ii)\\
&\hat{\mathbf{B}} = (\hat{B}_\rho, 0, \hat{B}_z) = \Big(-\frac{1}{\rho}\frac{\partial\nu}{\partial z}, 0, \frac{1}{\rho}\frac{\partial\nu}{\partial\rho}\Big);\ (iii);\\
&\text{thus, we have}:\ \hat{\mathbf{B}}^2 = \frac{1}{\rho^2}(\nu_\rho^2 + \nu_z^2);\ (iv);\\
&\&\ -\rho(\nabla\wedge\hat{\mathbf{B}})_\varphi = \nu_{\rho\rho} + \nu_{zz} - \frac{1}{\rho}\nu_\rho;\ (v).
\end{aligned} \tag{2.9}$$

Before considering the equations they obey, let us pause to say a few words about the genesis of the nomenclature in Eq. (2.9). This electromagnetic analogy was first noticed and Eqs. (2.9(i–ii)) used by Thirring. His initial purpose was to compute the gravitational field inside a hollow rotating sphere (in linearized GR). Later, with Lense, he extended the analysis of the effect of proper rotation of a central body on the motion of other celestial bodies, which led to the discovery of the Lense–Thirring effect [23]. In a set of three beautiful papers, Ludwig [15–17] has extended GEM by including additional field energy (that is second order in G) and obtained a closed set of non-linear equations for the rotational velocity (v_φ) in terms of the Newtonian velocity (via its acceleration) and the matter distribution within the galaxy. We shall return to discuss them in a later section and show that they are indeed reproduced in the appropriate limit.

In terms of the field variables defined in Eq. (2.9), the Weyl equations—in the vacuum—given in Eq. (2.3) read:

$$\begin{aligned}
&\nabla\cdot\mathbf{E} = -\Big(\frac{2}{c^2}\Big)e^{-2U}\mathbf{E}^2 + \Big(\frac{c^2}{2}\Big)e^{6U}\mathbf{B}^2;\ (i);\\
&\nabla\wedge\mathbf{B} = -\Big(\frac{4}{c^2}\Big)\Big(\mathbf{E}\wedge\mathbf{B}\Big);\ (ii);\\
&\hat{\mathbf{B}}_\rho = \Big(\frac{\rho}{c^2}\Big)\Big[E_z^2 - E_\rho^2\Big] + \frac{e^{4U}}{4}\Big[B_\rho^2 - B_z^2\Big];\ (iii);\\
&\hat{\mathbf{B}}_z = 2\Big(\frac{\rho}{c^2}\Big)\Big(E_\rho E_z\Big) + \Big(\frac{e^{4U}}{2}\Big)\Big(B_\rho B_z\Big);\ (iv).
\end{aligned} \tag{2.10}$$

Within the galaxy, the *Gauß law* in Eq. (2.10(i)) will obtain the mass density term on the right-hand side ($-4\pi\rho_m$). Similarly the *Ampere law* in Eq. (2.10(ii)) will obtain the mass current density ($-4\pi\rho_m v_\varphi$) when we continue the solution within the galaxy. On the other hand, Eqs. (2.10(iii–iv)) remain valid both inside and outside of the galaxy, due to our choice of the matter energy–momentum density, as discussed later in Sect. 3 in detail.

The various exponentials in these expressions add on higher-order polynomials in the Newtonian potential due to the non-linearity of GR. In all four equations above, the quadratic terms in **E** and **B** appear; these are easily interpretable as different components of the field energy–momentum density.

An attentive reader might wonder how (and why) one can possibly succeed in describing the dynamics of a spin-2 gravitational field in terms of just the GEM-electric and magnetic (spin-1 vector) fields. The answer to this question lies in the non-linearity of GR. Already at the second order (in G), there are constraints between the **E**-field (whose longitudinal part is defined through the gradient of the Newtonian potential Φ and whose transverse part arises through the time derivative of the transverse part of the vector potential, $\partial\mathbf{A}_T/\partial t$) and there are constraints between them, see Eqs. (2.3(i–ii)). Further on, at order G^2, a subsidiary field ν appears in the metric as well as in the equations of motion, which is completely constrained by the behaviour of the GEM fields and the boundary condition that $\nu(\rho = 0; z) \equiv 0$. Thus, in the far-field region, once the origin is appropriately chosen, the gravitational field is limited to its two degrees of freedom and its multipole expansion beginning with the quadrupole. Not so in the near field within or in the vicinity of the galaxy, where both longitudinal and transverse fields are present, with constraints between them playing a crucial role in limiting the dynamics, as the following discussion illustrates.

The assumption that there is no motion along the (radial) ρ-direction or along the z-direction brings in constraints for the dynamical system. Weinberg's Eq. (9.12) [8] gives the following expression for a particle's (spatial) acceleration $\mathcal{A}^i$ ($i = 2, 3, 4$ with coordinates labelled as $x^\mu : (x^1 = ct, x^2 = \varphi; x^3 = \rho; x^4 = z)$

$$\begin{aligned}
\mathcal{A}^i = &-\Gamma^i_{1,1} - 2\Gamma^i_{1,j}\Big(\frac{\mathrm{d}x^j}{\mathrm{d}t}\Big) - \Gamma^i_{j,k}\Big(\frac{\mathrm{d}x^j}{\mathrm{d}t}\Big)\Big(\frac{\mathrm{d}x^k}{\mathrm{d}t}\Big)\\
&+\Big(\frac{\mathrm{d}x^i}{\mathrm{d}t}\Big)\Big(\Gamma^1_{1,1} + 2\Gamma^1_{1,j}\Big(\frac{\mathrm{d}x^j}{\mathrm{d}t}\Big) + \Gamma^1_{j,k}\Big(\frac{\mathrm{d}x^j}{\mathrm{d}t}\Big)\Big(\frac{\mathrm{d}x^k}{\mathrm{d}t}\Big)\Big).
\end{aligned} \tag{2.11}$$

Assuming only circular motion (about the z-axis), we have non-vanishing velocity only along the φ-axis: $\mathrm{d}\varphi/\mathrm{d}t = v/\rho$ and $\mathrm{d}x^i/\mathrm{d}t = 0$ for $i = 3, 4$. Under this premise, the accelerations along the 3- and 4-axes must vanish as well:

$$\begin{aligned}(i)\ \mathcal{A}^\rho &= -c^2 e^{4U-2\nu} U_{,\rho} + c e^{4U-2\nu}\Big(\frac{v}{\rho}\Big)\Big(a_{,\rho} + 2aU_{,\rho}\Big)\\ &\quad - e^{-2\nu}\Big(\frac{v}{\rho}\Big)^2 - \rho + e^{4U} a a_{,\rho} + \rho^2 U_{,\rho} + e^{4U} a^2 U_{,\rho} = 0;\\ (ii)\ \mathcal{A}^z &= -c^2 e^{4U-2\nu} U_{,z} + c e^{4U-2\nu}\big(\frac{v}{\rho}\big)[a_{,z} + 2aU_{,z}]\\ &\quad - e^{-2\nu}\Big(\frac{v}{\rho}\Big)^2\Big(\rho^2 U_{,z} + e^{4U} a^2 U_{,z} + e^{4U} a a_{,z}\Big) = 0. \end{aligned} \tag{2.12}$$

Equation (2.12), along with Eqs. (2.3(i,ii)), allows us to obtain an exact non-linear, first-order differential equation for the velocity field $\beta(\rho, z = 0) = v(\rho, z = 0)/c$ on the equatorial plane in terms of the (normalized dimensionless) Newtonian (velocity squared) defined as usual $g(\rho) = (\rho/c^2)(\partial\Phi(\rho, 0)/\partial\rho)$, where $\Phi(\rho, 0)$ is the Newtonian potential in the equatorial plane. We relegate this rather complicated expression to Appendix A. Here we shall illustrate the strategy employed to derive the result valid to the lowest non-vanishing order. To the desired order of accuracy, Eq. (2.12) yields the following expressions for a, ρ, and $a_{,z}$:

$$\begin{aligned}\frac{a_{,\rho}}{\rho} &= -\Big(\frac{\beta}{\rho}\Big) + \Big(\frac{1}{\beta} + \beta\Big)\Big(\frac{\Phi_{,\rho}}{c^2}\Big);\\ \frac{a_{,z}}{\rho} &= +\Big(\frac{1}{\beta} + \beta\Big)\Big(\frac{\Phi_{,z}}{c^2}\Big).\end{aligned} \tag{2.13}$$

We can thus eliminate $a_{,\rho}; a_{,z}$ in Eq. [2.3(ii)], to obtain an expression for the second derivatives of U. To the desired order of accuracy:

$$\Big[e^{4U}\Big(\frac{1}{\beta} + \beta\Big)U_{,z}\Big]_{,z} + \Big[e^{4U}\Big\{-\frac{\beta}{\rho} + \Big(\frac{1}{\beta} + \beta\Big)U_{,\rho}\Big\}\Big]_{,\rho} = 0; \tag{2.14}$$

keeping only terms linear in the U-field:

$$\begin{aligned}\Big(\frac{1}{\beta} + \beta\Big)[U_{,\rho,\rho} + U_{,z,z}] &= \Big(\frac{1-\beta^2}{\beta^2}\Big)(\beta_{,z} U_{,z})\\ &\quad + \Big(\frac{1-\beta^2}{\beta^2}\Big)(\beta_{,\rho} U_{,\rho}) - \frac{\beta}{\rho^2} + \frac{\beta_{,\rho}}{\rho};\end{aligned} \tag{2.15}$$

thus:

$$\begin{aligned}\Big[U_{,\rho,\rho} + U_{,z,z} + \Big(\frac{U_{,\rho}}{\rho}\Big)\Big] &= \Big[\frac{1-\beta^2}{\beta(1+\beta^2)}\Big](\beta_{,z} U_{,z})\\ -\Big(\frac{\beta^2}{\rho^2(1+\beta^2)}\Big) &+ \Big(\frac{\rho\beta_{,\rho}}{\rho^2}\Big)\Big[\frac{\beta^2 + (1-\beta^2)g(\rho, z)}{\beta(1+\beta^2)}\Big]\\ + \frac{g(\rho, z)}{\rho^2}.\end{aligned} \tag{2.16}$$

According to Eq. ((2.3)(i)), the left-hand side is of order G^2, outside the galaxy. Thus, to linear order in G, we have at $z = 0$ upon using the up-down symmetry, for the rate of increase of $\beta(\rho)$ (outside the galaxy)

$$\Big(\rho\frac{\partial\beta}{\partial\rho}\Big) = \beta\Big[\frac{\beta^2 - g(\rho)(1-\beta^2)}{\beta^2 + g(\rho(1+\beta^2))}\Big]. \tag{2.17}$$

Equation (2.17) is of course only valid outside the galaxy. It agrees exactly with Ludwig's Eq. (4.13) [15] when his solution is continued to outside the galaxy where the matter density term $f = 0$.

It is easy to obtain the rate equation inside the galaxy (to linear order) upon including the matter density term on the right-hand side of Eq. (2.3(i)). To lowest order, the (two-dimensional) Laplacian of U receives the matter field contribution $(4\pi G\rho_m)$. Explicitly, inside the galaxy, we have

$$\begin{aligned}&\nabla^2 U(\rho, z) = \Big(\frac{4\pi G\rho_m(\rho, z)}{c^2}\Big) + \text{terms of order } G^2;\\ &\text{define for } z = 0;\ f(\rho) = \Big(\frac{4\pi G\rho_m(\rho, z = 0)\rho^2}{c^2}\Big);\\ &\text{eq.2.14} \rightarrow\ (f - g) + \frac{\beta^2}{1+\beta^2} = \frac{1}{\beta(1+\beta^2)}\Big(\rho\frac{\partial\beta}{\partial\rho}\Big)\\ &\Big[\beta^2 + g(1-\beta^2)\Big];\\ &\Big(\rho\frac{\partial\beta}{\partial\rho}\Big) = \beta\Big[\frac{\beta^2 + (1-\beta^2)(f-g)}{\beta^2 + g(1+\beta^2)}\Big].\end{aligned} \tag{2.18}$$

This essentially reproduces Ludwig's result inside the galaxy and reduces to Eq. (2.17) outside the galaxy for which $f = 0$.

3 Matter energy–momentum density

Within the boundaries of the galaxy, the dynamics of course changes:

$$E_{\mu\nu}(\rho, z) = R_{\mu\nu} - \Big(\frac{1}{2}\Big)R\, g_{\mu\nu} = \Big(\frac{8\pi G}{c^4}\Big)T_{\mu\nu}; \tag{3.1}$$

and thus we need a model for the energy–momentum density of the rotating galaxy and a choice for the metric inside. Hoping that no confusion ensues, we shall continue to use the same form of the metric as given in Eq. (2.1). The simplest and most commonly used model for matter is that of *free dust* with in general an equation of state relating the mass density to the pressure. We shall further assume that our galaxy has zero-pressure, which implies that it is *totally* supported by rotations around its stable axis, with no further extraneous motion. Choosing the axis of rotation along the z-axis (with an angular velocity $\dot{\varphi}$), our extremely simplifying assumptions allow us to restrict the matter energy–momentum density to the following form (with coordinates (o, φ, ρ, z)):

$$T^{\mu\nu} = \rho_m u^\mu u^\nu;$$

$$u^\mu(\rho,z) = (\gamma c)\Big(1, \frac{\beta}{\rho}, 0, 0\Big);$$
$$u_o = -(\gamma c)e^{2U}\Big[1-\beta\frac{a}{\rho}\Big];\ u_\varphi = (\gamma c)$$
$$\Big[e^{2U}a\Big(1-\beta\frac{a}{\rho}\Big) + (\beta\rho)e^{-2U}\Big];\ u_\rho = 0;\ u_z = 0;$$
$$\text{the trace}:\ T^\mu_\mu = -(\rho_m c^2) \Rightarrow$$
$$\Big[\frac{1}{\gamma^2}\Big] = \Big[\Big(1-\beta\frac{a}{\rho}\Big)^2 e^{2U} - \beta^2 e^{-2U}\Big]. \tag{3.2}$$

While lack of motion along the ρ (radial) and z (vertical) directions simplifies the structure of the matter energy–momentum density tensor from a (4×4) matrix to a (2×2) matrix form, this simplification also brings some unexpected peculiarities:

1. Even though the reduced matrix $T_{\mu\nu}$ is real-Hermitian, it is non-diagonal, and because it is factorizable, its determinant is zero. We recall that in the general case, this matrix has four eigenvalues: a positive definite (time-like) mass density, with three (space-like) pressures (p_1, p_2, p_3 along its principal axes). By setting all pressures p_i to zero, we have made the matrix *singular*, with the lone non-vanishing eigenvalue the scalar (generally invariant) mass density $\rho_m c^2$.
2. For any finite β, the Lorentz factor γ in Eq. (3.2) does not reduce to its expected value $(1-\beta^2)^{-1/2}$ unless the rotation parameter $a \to 0$. However, if we let $a = 0$, the metric becomes *diagonal*, since then $g_{o\varphi} = 0$, thereby rendering the (matter+field) angular momentum zero. Clearly, this is unphysical and thus unacceptable. We must have $a \neq 0$ (it can be positive or negative, of course).
3. In the expression for γ, the linear term in β induced by a non-vanishing length parameter $a \neq 0$ would exceed the expected β^2 correction unless, *for any value of* $\rho \leq \rho_{edge}$ *within the galaxy*, $2|a(\rho)/\rho| < \beta(\rho)$. In short, β cannot be too small if the rotational velocity alone has to support a galaxy with zero internal pressure.
4. The metric and its first derivatives must be matched at the boundary for their inside versus outside values.

Thus, β just outside cannot be too small either. A clear indication from GR that Newtonian values for β are becoming too small at the edge must be supplemented by (the mass current density) contributions to stabilize the system.

To emphasize the affinity and the difference between Einstein gravity and electromagnetism, and partly to follow the works by Ludwig [15–17], it is convenient to write the Einstein equations for this metric in terms of the three vectors $\mathbf{E}, \mathbf{B}, \hat{\mathbf{B}}$ defined earlier. Overall we have a dictionary with which we can write the Einstein equations

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{3.3}$$

We have:

$$R = g^{\mu\nu}R_{\mu\nu} = \frac{8\pi G}{c^4}g^{\mu\nu}T_{\mu\nu} = e^{2U-2\nu}$$
$$\Big(2\nabla^2 U + \frac{e^{4U}}{\rho^2}(a^2_{,\rho}+a^2_{,z}) - 2(\nu_{,\rho,\rho}+a_{,zz}+U^2_\rho+U^2_{,z})\Big)$$
$$= e^{2U-2\nu}\Big(-2\frac{e^{-2U}}{c^2}\nabla\cdot\mathbf{E} - 4\frac{e^{-4U}}{c^4}\mathbf{E}^2 + 16\frac{e^{4U}}{c^2}\mathbf{B}^2$$
$$+2\rho(\nabla\wedge\hat{\mathbf{B}})_\varphi + \hat{\mathbf{B}}_\rho\Big)$$

and therefore a “Gauß law”

$$\nabla\cdot\mathbf{E} = -4\pi G\rho_m e^{2\nu}\Big(1+e^{-2U}(\beta\gamma)^2\Big) - 2\frac{e^{-2U}}{c^2}\mathbf{E}^2$$
$$+8e^{6U}\mathbf{B}^2 + \rho c^2 e^{2U}(\nabla\wedge\hat{\mathbf{B}})_\varphi - \frac{1}{2}c^2 e^{2U}\hat{\mathbf{B}}_z. \tag{3.4}$$

To single out the non-diagonal part of $E_{\mu\nu}$ in terms of the matter current density $\mathbf{J}_m = \rho_m\mathbf{v}_\varphi$, we consider the combination

$$aE_{ct\,ct} + E_{ct\,\varphi} = \frac{8\pi G}{c^2}(aT_{ct\,ct} + T_{ct\,\varphi})$$
$$= \frac{8\pi G}{c^2}\Big[-(J_m)_\varphi\gamma^2\rho\Big(1-\frac{a}{\rho}\beta\Big)\Big]$$
$$= -\frac{1}{2}e^{4U-2\nu}\Big[a_{,\rho,\rho}+a_{,z,z} - \frac{1}{\rho}a_{,\rho} + 4(a_{,\rho}U_{,\rho}+a_{,z}U_{,z})\Big]$$
$$= \frac{2\rho}{c}e^{4U-2\nu}\Big((\nabla\wedge\mathbf{B})_\varphi - 4(\mathbf{E}\wedge\mathbf{B})_\varphi\Big). \tag{3.5}$$

and therefore an “Ampère law” emerges:

$$\nabla\wedge\mathbf{B} = \frac{4\pi G}{c}e^{-4U+2\nu}\Big[-\mathbf{J}_m\gamma^2\Big(1-\frac{a}{\rho})\beta\Big)\Big] + \frac{c}{2\rho}e^{-4U+2\nu}\mathbf{E}\wedge\mathbf{B} \tag{3.6}$$

For the convenience of the reader, in Appendix B, we have reproduced some details of the traditional iterative scheme in GR (developed over a century ago). Anyone interested can readily compare the higher-order contributions as they arise from the perturbative scheme with the exact Einstein–Weyl equations.

Neglecting the higher-order term in G and (special) relativistic corrections, we can summarize Gauß and Ampère law as:

$$\nabla\cdot\mathbf{E} = -4\pi G\rho_m, \qquad \nabla\wedge\mathbf{B} = -\frac{4\pi G}{c}\mathbf{J}_m. \tag{3.7}$$

It is important to note (and very useful to remember to implement) the negative sign of the matter fields on the right-hand side of Eqs. (3.7), especially in the Ampére, law which leads to a *left-hand rule* for the GEM magnetic field. Precisely because gravitation has only attraction (unlike E&M that has both), the Lenz law for gravity implies that there is a net boost to the acceleration due to other masses. We illustrate in Sect. 4 that the model obeying Lenz's law produces a rotation velocity curve consistent with mass-to-luminosity data, whereas another model, while successful in producing the rotation curve, was inconsistent with the light intensity data.

4 Lenz's law always boosts rotational velocities for stable galaxies

An attentive reader might rightly wonder why there is always a counter-rotating GEM magnetic field produced by the velocity field of material masses. Such is not always the case in Maxwellian electrodynamics because both attractive and repulsive forces are generated, as both positive and negative charges exist in Maxwell's electromagnetic theory. In GEM, however, the force is always attractive [28,29]. For the problem at hand, it is most easily seen in the equation for the GEM magnetic field

$$\nabla \times \mathbf{B} = -\Big(\frac{4\pi G}{c^2}\Big)\rho \mathbf{v} + \frac{\partial \mathbf{E}}{c^2 \partial t} e \qquad (4.1)$$

The minus sign in the first term on the right-hand side of Eq. (7.1) tells us that the magnetic field induced on the left side (due to the velocity field) always follows the *left-hand rule*. In standard electrodynamics with different signs of charge, Lenz's law implies that a negatively charged electron in a beam of co-moving electrons loses momentum due to other negatively charged electrons in the beam. On the other hand, the same Lenz law implies that an electron gains momentum if there are, say, positively charged parallel moving protons. In GEM, there is only attraction between masses, and thus the situation is similar to that between an electron and a proton. Ergo, Lenz's law implies that there is always an increase in the rotational velocity of galaxies due to GEM. In the following sections, we shall explicitly confirm that the resultant rotational velocity is indeed boosted through a GEM magnetic term $B_z < 0$.

We discuss it below and show that the model obeying Lenz's law produces a rotation velocity curve consistent with mass-to-luminosity data, whereas another model, while successful in producing the rotation curve, was inconsistent with the light intensity data.

The example of galaxy NGC 1560 has been discussed at length in [15] using two different parameterizations, which we shall call model I and model II:

$$\begin{aligned}
&\text{model I}: R_s = 7 \times 10^{-6}\, kpc;\ a = 0.373\, kpc;\\
&\quad b = 0.300\, kpc;\\
&\text{normalization point}: \beta(8.29\, kpc)\\
&\quad = 2.67 \times 10^{-4};\\
&\text{model II}: R_s = 1.46 \times 10^{-6}\, kpc;\ a = 7.19\, kpc;\\
&b = 0.567\, kpc;\\
&\text{normalization point}: \beta(8.29\, kpc) = 2.67 \times 10^{-4}. \qquad (4.2)
\end{aligned}$$

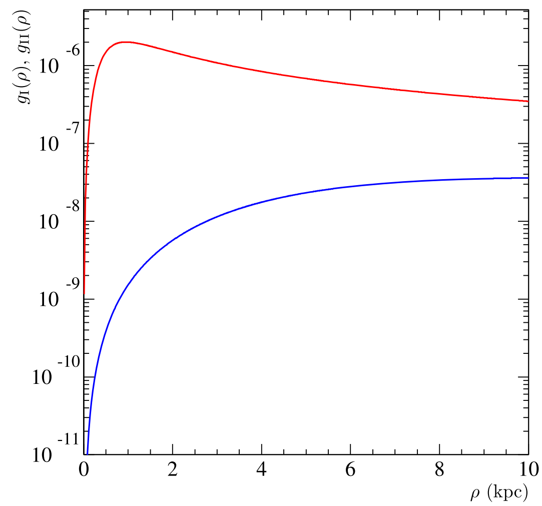

Fig. 1 Newtonian g-functions for the two models as defined in Eq.(4.4) in the text are shown in this figure with g_I in red and g_{II} in blue

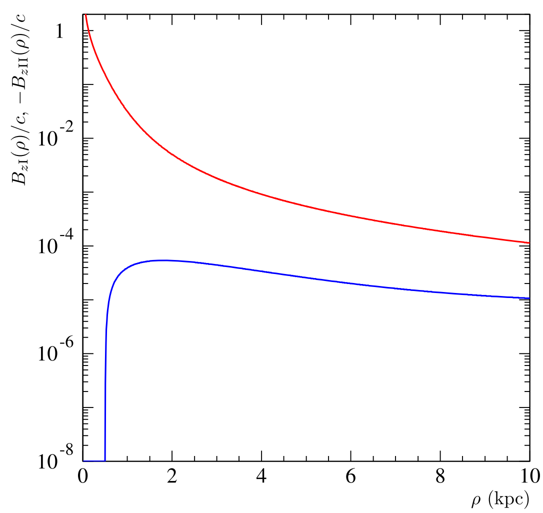

Fig. 2 B_z for model I and $-B_z$ for model II are shown in this figure. Model I has the wrong sign while Model II has the correct sign according to Lenz's law

They both produce roughly the same $\beta(\rho)$. To illustrate our point as simply as possible, we made a simple interpolation of the numerical result that Ludwig found from his rate equation. The interpolation reads

$$\beta(\rho) \approx (2.64 \times 10^{-4})\Big[\frac{\rho^2}{(\rho^2 + 2.92)}\Big];\ \text{(all distances in kpc)}. \qquad (4.3)$$

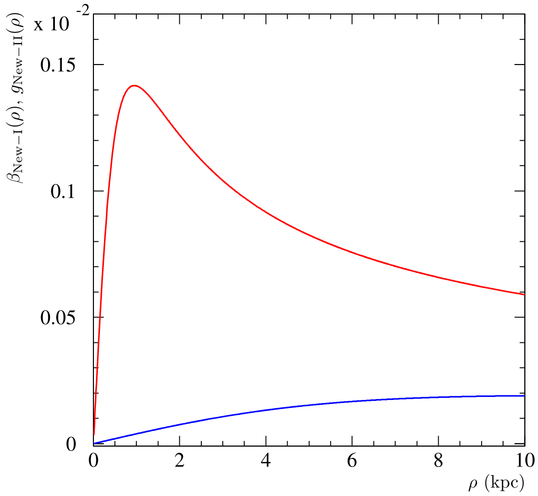

Fig. 3 We show the normalized Newtonian velocities β-functions, g_I in red and g_{II} in blue

The Newtonian g-functions for the two models are as follows:

$$g_I(\rho) = (3.5 \times 10^{-6})\left[\frac{\rho^2}{(\rho^2+0.45)^{3/2}}\right];$$
$$g_{II}(\rho) = (7.3 \times 10^{-7})\left[\frac{\rho^2}{(\rho^2+60.17)^{3/2}}\right]. \tag{4.4}$$

These are shown in Fig. 1. The GEM magnetic field is defined as

$$\frac{B_z}{c} = \frac{g(\rho)-\beta^2}{\beta\rho}. \tag{4.5}$$

For model I, $B_z > 0$, and for model II, $B_z < 0$. In Fig. 2, we show the magnetic fields, Bz_I for model I and $-B_{z,II}$ for model II. Lenz's law is not obeyed in model I, but it is in model II. In Fig. 3, we show the corresponding Newtonian velocities.

Ludwig's model II obeys Lenz's law and at the same time is also consistent with the mass-to-luminosity data, whereas model I does not agree with the mass-to-luminosity data. This shows the efficacy of Lenz's law in limiting the class of solutions.

5 More on rotation velocity and the Tully–Fisher law

As discussed in Sect. 4, the induced GEM magnetic field **B** is always counter-rotating (follows the left-hand rule) with respect to the velocity field of material masses that produce it. Also, as shown earlier, the Einstein–Weyl equations acquire the form of Gau-like and Ampère-like laws, even at the linearized level.

Upon assuming that $\mathbf{A}_g = A_\varphi\hat{\varphi}$; $\mathbf{v} = v\,\hat{\varphi}$ and that we are in stationary conditions, the equations (in cylindrical coordinates) read [15]:

$$\phi_g = \left(\frac{\Phi}{c^2}\right);$$
$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(r\frac{\partial\phi_g}{\partial\rho}\right) + \frac{\partial^2\phi_g}{\partial z^2} = \nabla^2\phi_g = 4\pi G\rho_m;$$
$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial\rho}\right) + \frac{\partial^2 A_\varphi}{\partial z^2} = \frac{4\pi G}{c^2}\rho_m\, v. \tag{5.1}$$

The assumption is that $v(\rho, z)$ continuously describes the motion of the rotating matter inside the galaxy and the motion of the ionized gas that circles round it. While the geodesic equations for the (spatial) acceleration of a particle $\mathcal{A}^i$ have been shown to be non-linear and complicated, we want to limit our discussion here and consider only equatorial circular motion around the z-axis with $d\varphi/dt = v/\rho$ and $\mathcal{A}^\rho = \mathcal{A}^z = 0$. Under these provisions, to lowest order we have the Lorentz force equations:

$$\frac{\partial\Phi}{\partial\rho} - \frac{v^2}{\rho} = \frac{v}{\rho}\frac{\partial(ca)}{\partial\rho}, \qquad \frac{\partial\Phi}{\partial z} = \frac{v}{\rho}\frac{\partial(ca)}{\partial z} \quad \leftrightarrow$$
$$E_z - vB_\rho = 0; \quad E_\rho + vB_z = -\frac{v^2}{\rho};$$
$$\text{define a magnetic velocity term: } \beta_{\text{mag}} \equiv \frac{\rho(-B_z)}{c} \geq 0;$$
$$\text{thus, with } g \text{ the Newtonian velocity squared:}$$
$$\beta^2 = g + (\beta\beta_{\text{mag}}) \geq g;\ (i)$$
$$\beta = (\frac{1}{2})\left[\beta_{\text{mag}} + \sqrt{(4g+\beta^2_{\text{mag}})}\,\right];\ (ii) \tag{5.2}$$

Thus, as we proposed to show in Sect. 1, GR with its inherent Lenz's law does indeed produce the remarkable result that the rotational velocity always exceeds its Newtonian value: $\left(\beta^2 \geq g \text{ Eq. } (5.2(\text{i}))\right)$.

To put it in perspective, this relationship is amply confirmed through 2700 data points from 153 SPARC galaxies. For details, we refer the reader to Ref. [6], especially its Fig. 3.

We have also shown that up to the order of required accuracy, Ludwig's rate equations for the rotation velocity emerge from the Weyl metric, thereby giving strong support to Ludwig's computational program. We shall return to it in Sect. 7.

A simple qualitative argument for constant asymptotic velocity can be deduced from these equations, with a Newtonian term augmented by the magnetic term. At small distances from the centre, the Newtonian term dominates, but as one proceeds further towards the edge of the galaxy, the picture changes dramatically due to the onset of the magnetic term.

If we consider our own galaxy, the Newtonian velocity has, roughly speaking, two bumps, and then it goes down in the Keplerian fashion as $1/\sqrt{\rho}$. If we simply add a magnetic term that begins from zero and grows near the edge to produce a constant (negative) vector potential A_φ in obedience to Lenz's law, we have the desired result of a constant rotational velocity, the now well-established result, first found experimentally by Vera Rubin.

We also notice that the same asymptotically constant vector potential allows us to obtain a reasonable estimate for both the rotation velocity and the angular momentum of our galaxy.

For our galaxy, the maximum of the Newtonian term coincides approximately with the onset of asymptotic velocity, $\beta^2(\infty) = (\frac{R_s}{2R_{\text{edge}}})$, where the Schwarzschild radius $R_s = (2GM/c^2)$, with M denoting the baryonic mass (plus that of the gravitational field). For a pillbox-like galaxy, $V = (\pi R^2_{\text{edge}})h,\ M = (\rho_m V)$, so that $\beta^2(\infty) \sim (\frac{M}{M^{1/2}}) \sim M^{1/2}$, reproducing the Tully–Fisher law: $M \propto \beta^4$.

6 Weyl class of metrics and the particular Kerr metric

We wish to investigate the similarities and differences between the large-distance behaviour of the Weyl class of metrics and the particular one of the Kerr solution of the Einstein equations [19]. This solution apparently describes a rotating black hole in terms of a mass M and a (constant) length parameter a that is known to be linearly related to its angular momentum.

Taking $\hat{z}$ as the axis of rotation, $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$, at large distance, the Kerr metric asymptotic behaviour is given by p. 240 of Ref. [8]):

$$h_{ij} \to -\frac{R_s}{r^3} x_i x_j, \quad h_{0i} \to \frac{R_s}{r^2}\Big(x_i + \frac{1}{r}(\mathbf{a} \wedge \mathbf{x})_i\Big),$$
$$R_s \equiv \frac{2GM}{c^2}.\ \mathbf{a} = (0, 0, a),\ i, j = 1, 2, 3 \tag{6.1}$$

As amply discussed in Appendix B, this is quite generally all that one needs in order to calculate the total mass and angular momentum. For the Kerr metric (6.1), Eq. (9.12) yields $E_{tot} = Mc^2,\ \mathbf{J} = (Mc)\mathbf{a}$, as expected. If $a = 0$, the Kerr metric coincides with the Schwarzschild metric and $\mathbf{J}$ is zero. We can see that for the system to have a finite angular momentum—and a rotating galaxy certainly has that—it is crucial that the space-time part of $h_{\mu\nu}$ does not vanish asymptotically beyond $1/r^2$.

Let us now consider the general class of Weyl's axially symmetric metrics as in Sect. 2 focusing on their space-time part in the equatorial plane (i.e., at $z = 0$, so that $\rho = r$), and we have:

$$g_{0,\varphi}(r) = \frac{a(r)}{c} e^{2U(r)}, \text{can be written in pseudo}$$
$$-\text{Euclidean coordinates as the special case of}$$
$$g_{0,i} = \epsilon_{ijk} a_j x_k (\frac{e^{2U}}{r^2});$$
$$\text{with Weyl's being the special case } \mathbf{a} = (0, 0, a);$$
$$\text{Expanding in perturbation theory:}$$

$$g_{0,i} = g^{(1)}_{0,i} + g^{(2)}_{0,i} = \epsilon_{ijk} \frac{a_j x_k}{r^2}(1 + 2U(r) + \ldots),$$
$$\text{with } g^{(1)}_{0,i} = \epsilon_{ijk}\frac{a_j x_k}{r^2}\ ;\ and\ g^{(2)}_{0,i} = \epsilon_{ijk}\frac{a_j x_k}{r^2}(2U(r); \tag{6.2}$$

We are interested in the second part ($g^{(2)}_{0,i}$) that relates to the angular momentum ($\mathbf{J}$) of the system. Asymptotically, we have (see [8]) for the second term,

$$g^{(2)}_{0,i} = \Big(\frac{2G}{r^3}\Big)(\mathbf{r} \times \mathbf{J}\Big)_{\mathbf{i}};$$
$$\text{we find } J_z = (Mc)a; \tag{6.3}$$

exactly the same as that for the Kerr metric, provided we associate the (constant) Kerr length parameter a with the (asymptotic) Weyl length parameter a.

The implication is that a finite value of the total (material plus that of the gravitational field) angular momentum of the galaxy requires that the rotational velocity is asmyptotic to a constant value and vice versa.

A mental picture of what is happening may be formed through the following rough guide about the Weyl parameter a. For small r, a increases from zero linearly until the edge, beyond which—while continuous at the edge—it eventually becomes a constant. At very large r, as expected, the GEM magnetic field ($-B_z \to 1/r$), as all radiation fields do.

7 Ludwig's non-linear differential equation for the velocity field

Whereas in Sect. 5 Eq. (5.2) we have tried to keep our equations *linear* by keeping both the Newtonian and the magnetic contributions at the same level, the strategy followed by Ludwig [15] (see also [30,31]) has been to eliminate the magnetic term entirely, at the expense of course of ending up with a non-linear equation for the velocity field. Below we follow his formalism to pinpoint a few aspects.

As stated in the last paragraph, we can use Eq. (5.1) to eliminate A_φ from the expression of the Ampère law, which becomes

$$\frac{\partial}{\partial\rho}\Big(\frac{1}{v}\frac{\partial\phi}{\partial\rho} - \frac{v}{\rho}\Big) + \frac{\partial}{\partial z}\Big(\frac{1}{v}\frac{\partial\phi}{\partial z}\Big) = \frac{4\pi G}{c^2}\rho_m v. \tag{7.1}$$

This equation multiplied by v and subtracted from the expression of Gauß' law given earlier, eliminates the double derivatives and yields:

$$4\pi G\rho_m\Big(1 - \frac{v^2}{c^2}\Big) = \Big(\frac{1}{\rho} + \frac{1}{v}\frac{\partial v}{\partial\rho}\Big)\frac{\partial\phi_g}{\partial\rho} + \frac{1}{v}\frac{\partial v}{\partial z}\frac{\partial\phi_g}{\partial z} + v\frac{\partial}{\partial\rho}\frac{v}{\rho} \tag{7.2}$$

a non-linear first-order differential equation for $v(\rho, z)$ for given $\rho(\rho, z)_m,\ \Phi_g(\rho, z)$. In the equatorial plane $z = 0$, by the up-down symmetry we can drop the $\frac{\partial \phi_g}{\partial z}$; then:

$$\left(\beta^2 + \rho \frac{\partial \varphi}{\partial \rho}\right) \rho \frac{\partial \beta}{\partial \rho} = \frac{\beta}{\rho}$$
$$\left[\left(\beta^2 - \rho \frac{\partial \varphi}{\partial \rho}\right) + \frac{4\pi G \rho_m}{c^2} \rho^2 (1 - \beta^2)\right];$$
$$\beta = \frac{v(\rho, 0)}{c}, \varphi = \frac{\phi_g}{c^2}.$$

Outside the galaxy, where $\rho(\rho, 0)_m = 0$, the equation becomes

$$\frac{\rho}{\beta} \frac{\partial \beta}{\partial \rho} = \frac{\beta^2 - \rho \frac{\partial \varphi}{\partial \rho}}{\beta^2 + \rho \frac{\partial \varphi}{\partial \rho}} = \frac{\beta^2 - g(\rho)}{(\beta^2 + g(\rho))}. \tag{7.3}$$

This equation shows the key role played by the GEM magnetic field, which is now:

$$\frac{B_z}{c} = \frac{\rho \frac{\partial \varphi}{\partial \rho} - \beta^2}{\beta \rho} = \frac{g(\rho) - \beta^2}{\beta \rho}. \tag{7.4}$$

Equation (7.3) is an elegant rate equation for the velocity outside the galaxy. However, in any phenomenology, care must be taken to ensure that the GEM magnetic field employed (see Eq. (7.4) $B_z < 0$ is indeed negative. A counter example has already been provided in Sect. 4.

8 Alternatives to ΛCDM

In his excellent review of the phenomenology of rotationally supported galaxies, Salucci warns us *to end our fascination with the ΛCDM weakly interacting massive particles scenario* and concludes, on a somber note: *It seems impossible to explain the observational evidences gathered so far in a simple dark matter framework.* Through a *reverse engineering* approach, he arrives at the notion of luminous-dark matter interactions along with interactions between different types of DM particles all transcending the simple notion of non-interacting DM particles.

Instead of DM, the MOND approach [32] that modifies Newtonian potential at large distances has been invoked to discuss the observational baryonic Tully–Fischer and Trimble relations, and an extension of GR with further degrees of freedom has been proposed in Ref. [33] for an explanation of baryonic Tully–Fischer relations. Higher-order GR in Ref. [34] and a natural approach to extend Newtonian gravity in Ref. [35] have been proposed as alternatives to dark matter. Also, a tensorial formulation of GR has recently been invoked for the possible replacement of DM [36].

By contrast to the alternate schemes mentioned above, Ludwig's extended GEM theory is able to avoid the Keplerian falloff of the rotation velocity at large distances from the centre of a rotating galaxy through the geomagnetic field generated by the matter current density.

9 Conclusions and future prospects

Here we first summarize results obtained, then describe research in progress and close with prospects for the future.

1. Our work began with the most general framework in GR to discuss rotationally supported galaxies. Fortunately, there is the Weyl class of axisymmetric metrics for whom the solutions to the Einstein–Weyl equations in the vacuum are known in terms of a few differential equations. Even more fortunately, for what we call the extended Weyl class that includes rotations explicitly, exact differential equations are also known.
2. Unlike the Kerr metric, the Weyl metric can easily be (and has been) continued within the galaxy and physically meaningful results obtained,
3. Armed with exact solutions, it became possible to show how Gauß and Ampére laws emerged and under what conditions Ludwig's extended GEM theory and his non-linear rate equations for the rotation velocity field could be deduced.
4. Using the century-old iterative procedure in GR and further elaborated by Weinberg, we could discuss the value of the mass M (baryonic mass + that of the gravitational field) and that of the intrinsic angular momentum J of a rotationally supported galaxy. The extended Weyl metric analysis allowed us to rigorously conclude that Weyl's (vectorial) length parameter a must have a finite limit to obtain a finite J. As the same parameter also controls the asymptotic limit of the rotation velocity, we can conclude that GR is indeed capable of obtaining a flat plateau in the rotation velocity.
5. We have attempted an alternative strategy to that of Ludwig as far as the phenomenology of the rotation curves is concerned. Ludwig eliminated the magnetic contribution to obtain his non-linear rate equation for the velocity field in terms of the input from the Newtonian potential and the mass distribution within the galaxy. Instead, we kept the Newtonian input and the magnetic input together—thus our velocity equations remained linear. This allowed us to provide a clear physical picture: at small distances, the velocity is basically described by the Newtonian term, and as it begins to fall off, it is supported near the edge by essentially a constant vector potential. It also brought to focus the crucial role of Lenz's law and the left-hand rule for the GEM magnetic field.
6. As by-products of our analysis, we have deduced a few other practical results: (i) imposition of Lenz's law implies the rigorous inequality: $\beta^2 \geq g$, the Newto-

nian value, a result supported by 2700 data points from 153 rotating galaxies; (ii) a better estimate ($\geq$ 500 km/s) for our Sun's escape velocity from our galaxy; (iii) an easy-to-remember pneumonic for the asymptotic velocity $\beta^2 \approx (R_s/(2R_{\rm edge}))$; (iv) how Tully–Fisher law emerges from a rotating *pill-box* galaxy; (v) simple dimensional analysis implies $J \propto M^{7/4}$ if Tully–Fischer holds.

Our present focus is fourfold:

A: A satisfactory GR description of the deflection of light from large galaxies and from galaxy clusters.
B: To obtain a better understanding of the Tully–Fischer law ($M \propto \beta^4$) and the Virginia Trimble law ($J \propto M^{1.9}$), the latter covering data that run over 50 orders of magnitude [37].
C: A comprehensive phenomenology of the rotation curves with realistic densities and more refined Newtonian inputs.
D: Testing our conjecture that spiral arms in rotating galaxies such as ours are generated dynamically through non-linear effects inherent in GR.

On the broader horizon, it is reasonable to hope for further yet more brilliant advances in astrophysical observations (for example, via renewed investigations involving Hanbury–Brown–Twiss techniques) so as to reduce the error bars in rotation curves. Only then would it be possible to truly distinguish between different theoretical models.

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Appendix A: Exact non-linear expression for the velocity field

The exact expressions for $a_{,\rho}$ and $a_{,z}$ read

$$\frac{a_{,\rho}}{\rho} = -\Big(\frac{we^{-4U}}{\rho}\Big) + \Big(\frac{1}{w} + we^{-4U}\Big)U_{,\rho};$$
$$\frac{a_{,z}}{\rho} = +\Big(\frac{1}{w} + we^{-4U}\Big)U_{,z};$$
$$\text{where}: \ w = \frac{\beta}{1-\beta(\frac{a}{\rho})}. \tag{9.1}$$

Following exactly the steps described in Eqs. (2.13–2.17) *et sec.* in Sect. 3, we find two expressions for $[U_{,\rho,\rho} + U_{,z,z}]$, which we equate and find

$$-\frac{U_{,\rho}}{\rho} - \Big(\frac{e^{4U}}{2}\Big)$$
$$\Big\{\Big(\frac{1}{w}+we^{-4U}\Big)^2 U_{,z}^2 + \Big[-\frac{we^{-4U}}{\rho} + \Big(\frac{1}{w} + we^{-4U}\Big)U_{,\rho}\Big]^2\Big\}$$
$$= \Big(\frac{we^{-4U}}{1+w^2e^{-4U}}\Big)\Big(\frac{w}{\rho}\Big)_{,\rho} - \Big(\ln\Big(\frac{e^{4U}}{w} + w\Big)\Big)_{,\rho} U_{,\rho}$$
$$-\Big(\ln\Big(\frac{e^{4U}}{w} + w\Big)\Big)_{,z} U_{,z}. \tag{9.2}$$

Once again, on the equatorial plane $z = 0$, using the up-down symmetry, we can drop all terms such as $w_{,z}$ and $U_{,z}$ and thus remain with

$$-\frac{U_{,\rho}}{\rho} - \Big(\frac{e^{4U}}{2}\Big)\Big[-\frac{we^{-4U}}{\rho} + \Big(\frac{1}{w} + we^{-4U}\Big)U_{,\rho}\Big]^2$$
$$= \Big(\frac{we^{-4U}}{1+w^2e^{-4U}}\Big)\Big(\frac{w}{\rho}\Big)_{,\rho} - \Big(\ln\Big[\frac{e^{4U}}{w} + w\Big]\Big)_{,\rho} U_{,\rho};$$
$$\Big(\frac{w^2e^{-4U}}{\rho}\Big)U_{,\rho} + \frac{e^{-2U}\beta^2}{w^2\gamma^2} = \Big(\frac{we^{-4U}}{1+w^2e^{-4U}}\Big)\Big(\frac{w_{,\rho}}{\rho}\Big)$$
$$-\Big(\ln\Big[\frac{e^{4U}}{w} + w\Big]\Big)_{,\rho} U_{,\rho} + \Big(\frac{e^{4U}}{2}\Big)\Big(\frac{1}{w} + we^{-4U}\Big)^2 U_{,\rho}^2. \tag{9.3}$$

Appendix B: Iterative computational procedure in GR

Over the past century, a detailed program [often dubbed post-Newtonian, post-post Newtonian, etc.] was developed to systematically compute the metric, the Ricci tensor, and the like in a perturbation expansion in powers of the Newton constant G. The procedure is somewhat involved but technically straightforward albeit cumbersome. And it does require the introduction of a non-tensorial object first introduced by Einstein and which he called the pseudo-energy momentum tensor for the gravitational field. It was formalized by Landau and Lifshitz [26] and is amply discussed in the excellent text-books such as those by Weinberg [8] and by Stephani [19]. In order not to duplicate some long expressions, we shall refer the reader to these references abbreviated as (L&L), W, or S.

A few words are in order as to the reason for this appendix. While well known to physicists of the last generation, our own experience has been that the detailed procedures are largely forgotten by a vast majority of practicing physicists. Thus, to bring out the differences with the traditional post-Newtonian theory and to stress the importance of what is involved in the very definition of the *far field*, we here review the iterative formalism in some detail. Another point to stress here is that the exact Weyl solutions for the vacuum that are discussed here appear to be analytically continuable within the system (say, a galaxy) and exchanges of energy–momentum emerge at order G^2. Thus, a diligent reader can compare for herself the exact results with pieces constructed from higher-order iterative solutions.

Consider the Einstein equation with its prescribed source, a matter energy–momentum tensor that is limited in its spatial and temporal extent.

$$
\begin{aligned}
&G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\\
&R = \Big(\frac{8\pi G}{c^4}\Big) T_{\mu\nu};\ R_{\mu\nu} = \Big(\frac{8\pi G}{c^4}\Big)\Big[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\Big];\ (i)\\
&T^{\nu}_{\mu;\nu} = \frac{1}{\sqrt{-g}} \frac{\partial (T^{\nu}_{\mu}\sqrt{-g})}{\partial x^{\nu}} - \frac{1}{2}\Big(\frac{\partial g_{\nu\lambda}}{\partial x^{\mu}}\Big) T^{\nu\lambda} = 0;\ (ii)\\
&R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu};\ (iii)\\
&R^{(1)}_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} = \frac{1}{2} g^{\lambda\sigma}\\
&[\frac{\partial^2 g_{\mu\lambda}}{\partial x^{\nu}\partial x^{\sigma}} + \frac{\partial^2 g_{\nu\lambda}}{\partial x^{\mu}\partial x^{\sigma}} - \frac{\partial^2 g_{\lambda\sigma}}{\partial x^{\mu}\partial x^{\nu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\lambda}\partial x^{\sigma}}];\ (iv)\\
&R^{(2)}_{\mu\nu} = \Big[\Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}\lambda\sigma - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}\Big](v). \qquad (9.4)
\end{aligned}
$$

Note that the vanishing of the covariant divergence of the (material) energy–momentum tensor T^{ν}_{μ} as given in Eq. (9.4(ii)) does not lead to a local energy–momentum—or angular momentum—conservation law. This reflects the physical fact that in a gravitational field, the 4-momentum of the matter field alone is not conserved, but rather the 4-momentum of matter plus that of the gravitational field; the latter is not included in T^{ν}_{μ}. Thus, one defines a pseudo energy–momentum tensor $t^{\mu\nu}$ for the gravitational field [38], so that the following condition holds:

$$
\partial_{\nu}(T^{\mu\nu} + t^{\mu\nu}) = 0; \qquad (9.5)
$$

We know that $t^{\mu\nu}$ is not a tensor; ordinary derivative in Eq. (9.5) confirms this fact. However, we can devise a recipe so that, asymptotically, the fields are Lorentz-covariant. Below are the steps of the perturbative recipe:

Step I:

We know that there exists a space-time point at which all the Γs can be made to vanish (the first derivatives of the metric but not the metric itself can be made to vanish). But this implies the following:

(a) Through Eq. (9.4(ii)), the last term disappears, and the determinant of the metric can be taken out of the partial derivative in the first term, rendering the covariant derivative to an ordinary derivative, i.e., $\partial_{\nu} T^{\mu\nu} = 0$ at this point.

(b) Simultaneously, we learn from Eq. (9.4(v)) that $R^{(2)}_{\mu\nu}$ vanishes at this point. Thus the entire Einstein Eq. (9.4(i)) is reduced (at this space-time point) to

$$
G_{\mu\nu} \to G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{(1)} = (\frac{8\pi G}{c^4}) T_{\mu\nu}. \qquad (9.6)
$$

Consider the special case (certainly valid for weak gravity) that the metric can be expanded around its flat Minkowski limit $\eta_{\mu\nu}$ and for computational simplicity choose *harmonic coordinates* (indices being raised and lowered by $\eta_{\mu\nu}$).

$$
\begin{aligned}
&\text{harmonic coordinates}: g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0;\ \text{picking the gauge}:\\
&\text{coordinate conditions};\ (i)\\
&g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu};\ \bar{h}_{\mu\nu} = h_{\mu\nu} - \Big(\frac{1}{2}\Big) h \eta_{\mu\nu};\ h = h^{\mu}_{\mu} = -\bar{h};\\
&g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \Big(\frac{1}{2}\Big)\eta_{\mu\nu}\bar{h};\\
&\text{So}:\ R^{(1)}_{\mu\nu} = \frac{1}{2}(\bar{h}^{\lambda}{}_{\mu,\nu,\lambda} + \bar{h}^{\lambda}{}_{\nu,\mu,\lambda} - (\partial_{\lambda}\partial^{\lambda})\bar{h}_{\mu\nu}\\
&+\frac{1}{2}\eta_{\mu\nu}(\partial_{\lambda}\partial^{\lambda})\bar{h});\\
&\text{and}:\ G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(1)} = -\frac{1}{2}(\partial_{\lambda}\partial^{\lambda})\bar{h}_{\mu\nu} + \mathcal{X}_{\mu\nu};\\
&\text{where}\ \mathcal{X}_{\mu\nu} = \frac{1}{2}(\bar{h}^{\lambda}{}_{\mu,\nu,\lambda} + \bar{h}^{\lambda}{}_{\nu,\mu,\lambda} - \eta_{\mu\nu}\bar{h}^{\lambda\sigma}{}_{,\lambda,\sigma}). \qquad (9.7)
\end{aligned}
$$

We can eliminate $\mathcal{X}_{\mu\nu}$ through the following 4-coordinate condition choice allowed by Eq. (9.7(i)). (Details can be checked via Eqs. (Stephani 13.8–13.14)):

$$
\begin{aligned}
&\text{harmonic coordinates defined by}:\\
&\text{curvilinear D'Alembertian}(x^{\mu}) = \frac{1}{\sqrt{-g}}(\sqrt{-g}\ g^{\nu\lambda} x^{\mu}, \lambda)_{,\nu}\\
&= 0;\ (i)\ \text{implies}\ (\sqrt{-g}\ g^{\mu\nu})_{,\nu} = 0;\ (ii)\\
&\text{thus, if}\ \sqrt{-g}\ g^{\mu\nu} = \eta^{\mu\nu} - \tilde{\bar{h}}^{\mu\nu},\ \text{we have}\ \tilde{\bar{h}}^{\mu\nu}{}_{,\nu} = 0.(iii) \qquad (9.8)
\end{aligned}
$$

To accomplish it, we need to make a coordinate change as follows:

$$
\text{Let}\ \bar{x}^{\mu} = x^{\mu} + b^{\mu};
$$

implying a change in the metric $\bar{g}^{\mu\nu} = g^{\lambda\sigma}(\delta^{\mu}_{\lambda} + b^{\mu}_{,\lambda})(\delta^{\nu}_{\sigma} + b^{\nu}_{,\sigma}) \approx g^{\mu\nu} + g^{\lambda\nu}b^{\mu}_{,\lambda} + g^{\lambda\mu}b^{\nu}_{,\lambda}$; (9.9)

and a change in the metric determinant $\bar{g} = |\bar{g}^{\mu\nu}|^{-1} \approx g(1 + 2b^{\lambda}_{,\lambda})^{-1}$;

$\tilde{\bar{h}}^{\mu\nu} = \bar{h}^{\mu\nu} - b^{\mu\ ,\nu} - b^{\nu\ ,\mu} + \eta^{\mu\nu}b^{\lambda}_{,\lambda}$;

imposing $\bar{h}^{\mu\nu}_{\ ,\nu} = (\partial_\sigma\partial^\sigma)b^{\mu}$;

so that $G^{(1)}_{\mu\nu} \rightarrow \Big(-\frac{1}{2}\Big)(\partial_\sigma\partial^\sigma)$

$(\bar{h}_{\mu\nu} - b_{\mu\ ,\nu} - b_{\nu\ ,\mu} + \eta_{\mu\nu}b^{\lambda}_{,\lambda}) = \Big(-\frac{1}{2}\Big)(\partial_\sigma\partial^\sigma)\tilde{\bar{h}}_{\mu\nu}$;

and $\tilde{\bar{h}}^{\mu\nu}_{\ ,\nu} = 0$; (9.10)

by virtue of Eq. (9.8(iii)). This completes the proof and we have (dropping the tilde and the bar on h), in harmonic coordinates:

$$G^{(1)}_{\mu\nu} = \Big(-\frac{1}{2}\Big)(\partial_\sigma\partial^\sigma)h_{\mu\nu} = \Big(\frac{8\pi G}{c^4}\Big)T_{\mu\nu};\ (i)$$

$$(\partial_\sigma\partial^\sigma)h_{\mu\nu} = \Big(-\frac{16\pi G}{c^4}\Big)T_{\mu\nu}.(ii)$$

Step II:

By definition

$G^{(1)}_{\mu\nu} = \Big(\frac{8\pi G}{c^4}\Big)[T_{\mu\nu} + t_{\mu\nu}]$;

(i) where the gravitational energy − momentum tensor is given by

$t_{\mu\nu} = -\Big(\frac{c^4}{8\pi G}\Big)[G_{\mu\nu} - G^{(1)}_{\mu\nu}]$;

(ii) the total matter + gravitational energy − momentum tensor

$\tau^{\lambda\sigma} \equiv \eta^{\lambda\mu}\eta^{\sigma\nu}(T_{\mu\nu} + t_{\mu\nu})$;

that is locally conserved $\tau^{\lambda\sigma}_{\ ,\sigma} = 0$. (9.11)

Let us note here the convention that indices for quantities such as $h_{\mu\nu}$; $G^{(1)}_{\mu\nu}$ and $\frac{\partial}{\partial x^\lambda}$ are raised and lowered by the η's, whereas on true tensors such as $R_{\mu\nu}$ are raised and lowered with g's as usual.

Weinberg Chapter 7 (Sec. 6) describes in detail the definition of total momentum, total energy, and the angular momentum, as well as the computational strategy for a perturbative expansion of $t_{\mu\nu}$. We list some of them below for reference:

total 4 − momentum $P^{\mu} = \int_V \tau^{o\mu}(d^3x)$; *and* $\tau^{i\nu}$

is the flux;

total Angular − momentum density and flux

$M^{\mu\nu\lambda} = \tau^{\mu\lambda}x^{\nu} - \tau^{\mu\nu}x^{\lambda}$;

$\partial_\mu M^{\mu\nu\lambda} = 0$; since $\tau^{\nu\lambda} = \tau^{\lambda\nu}$ *and* $\tau^{\mu\nu}_{\ ,\nu} = 0$;

total Angular − Momentum

$J^{\nu\lambda} = -J^{\lambda\nu} = \int_V (d^3x)M^{o\nu\lambda}$ (a constant if no surface terms);

(9.12)

In Weinberg (Eqs. (7.6.14–15)), a power series for $t_{\mu\nu}$ in h is developed up to terms of order h^2:

$$t_{\mu\nu} = \Big(\frac{c^4}{8\pi G}\Big)\Big[-\Big(\frac{1}{2}\Big)h_{\mu\nu}R^{(1)\lambda}_{\ \ \lambda} + \frac{1}{2}\eta_{\mu\nu}h^{\lambda\sigma}R^{(1)}_{\lambda\sigma} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}R^{(2)}_{\lambda\sigma}\Big] + \bigcirc(h^3); \quad (9.13)$$

where $R^{(2)}$ is given by the terms of order h^2 in Eq. (9.4(v)) and written out in detail in Weinberg (Eq.(7.6.15)).

Far away from the finite material system that produces the gravitational field, $T^{\mu\nu}$ vanishes, and since $t_{\mu\nu}$ is of order h^2, the source terms on the rhs of Eq. (9.11) are confined to a finite region. Thus, we expect them to behave as electrostatic potentials or as in Newtonian gravitational theory. Typically, we expect for large distances from the source that

$$h_{\mu\nu} \rightarrow \bigcirc\Big(\frac{1}{r}\Big);\ \frac{\partial h_{\mu\nu}}{\partial x^\lambda} \rightarrow \bigcirc\Big(\frac{1}{r^2}\Big);\ \frac{\partial h_{\mu\nu}}{\partial x^\lambda \partial x^\sigma} \rightarrow \bigcirc\Big(\frac{1}{r^3}\Big);$$

$$\text{Hence } t_{\mu\nu} \rightarrow \bigcirc\Big(\frac{1}{r^4}\Big); \quad (9.14)$$

so that the integrals for the total momentum and energy as given in Eqs. (9.12) should converge. In fact, there are very simple expressions for the total energy and the angular momentum of a finite system (to linear order in the metric perturbations):

$$E_{\text{Total}} = P^{o}c = -\Big(\frac{c^4}{16\pi G}\Big)\int\Big[\frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j}\Big](n_i r^2 d\Omega);$$

$$J^{jk} = -\Big(\frac{c^3}{16\pi G}\Big)\int K_{ijk}(n_i r^2 d\Omega);$$

n_i is the outward normal;

$$K_{ijk} = (-x_j h_{ok,i} + x_k h_{oj,i}) + (x_j h_{ki,o} - x_k h_{ji,o}) + (h_{ok}\delta_{ij} - h_{oj}\delta_{ik});\ J_1 = J^{23}, \ldots \quad (9.15)$$

For the above computations, only the asymptotic behaviour of the metric is required (at large distances from the source). It should also be noted that, while the total energy can be proven to be positive (provided there is a mass in the system), total angular momentum is strictly zero unless (asymptotically) either (i) the purely spatial metric is time-dependent or (ii) there is a non-trivial (i.e., non-removable by a coordinate transformation) h_{oi}. As important examples, one finds by explicit calculation that for both the Schwarzschild metric and the Kerr metric, the total energy $E_{\text{total}} = Mc^2$. On the other hand, the total angular momentum for the Schwarzschild case is zero, whereas for the Kerr metric, $J = (Mc)a$, where a is a length parameter associated with rotations.

Of course, as Weinberg explicitly cautions, Eq. (9.14) need not always be true. He gives the standard example of a system that has been continuously radiating energy (as gravitational waves) and so the total energy is indeed infinite: It shows up theoretically in that various derivatives become all of the same order, violating Eq. (9.14).

While the above is evidently acceptable on physics grounds, there are other more subtle effects that can invalidate or certainly modify the *reasonable*-sounding estimates provided by Eq. (9.14) augmented by our deep-seated Newtonian bias. One of them concerns rotations [39]. Simply because rotation about a fixed axis differentiates between clockwise and anti-clockwise motion, suppose the rotation is about the z-axis confined to the $(x\,y)$ plane, and if the system is axially symmetric, J_z would be conserved. It would appear that PT would be conserved but not P or T separately, because by assumption, our system is rotating with respect to an external inertial observer. For rotations that are measurable in the *far field*, traditional power counting methods need to be critically examined.

For the problem at hand i.e., the dynamics of rotation-supported galaxies, it is obviously not only convenient but appears mandatory that the kernel of the *perturbative solution* include not only the Newtonian potential U but also Weyl's rotation field a explicitly. Technically, this means that the "asymptotic" metric is not Galilean but augmented by the Weyl field in such a manner that a finite total angular momentum of the system is simply reproduced.

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38. Such a quandry arises in all non-Abelian theories, such as SU(2) of weak interactions or SU(3) of color interactions and of course in GR if considered as a gauge theory such as GL(4,R) for example. The central point is that the vector potentials, the gluons say in QCD, $A_\mu^{(c)}(x)$ carry color charge (as the index c makes evident)

just as the quarks (the matter fields of QCD) carry color charge. Similarly, in GR, the role of charge being played by "mass" and the role of the gauge fields is played by the Γ's that carry the world indices that are "rotated" in a coordinate transformation. It is only in Abelian theories such as the Maxwell field where the vector potential does not carry a "charge" only the charged matter fields do. Thus Gauss law in QCD for the color-electric field reads $(\mathcal{D}ivE)^a = \rho^a$; $\mathbf{Div} \cdot \mathbf{E}^a + igf^a_{bc}\mathbf{A}^b \cdot \mathbf{E}^c = \rho^a$; (7.24) where ρ^a is the quark color charge density; $\mathbf{A}^a$ are the gluon-fields carrying color charge ($a = 1, 2, ...8$). The analog of the pseudo-tensor being discussed in the text is mirrored here if one considers $(-igf^a_{bc}\mathbf{A}^b \cdot \mathbf{E}^c)$ as the gluon color field. It is important to note that while ρ^a is a true gauge-covariant quantity, the color charge field of the gluon defined in the last sentence is not

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