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The role of Pryce's spin and coordinate operators in the theory of massive Dirac fermions

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Abstract It is shown that the components of Pryce's spin operator of Dirac's theory are $SU(2)$ generators of a representation carried by the space of Pauli's spinors determining the polarization of the plane wave solutions of Dirac's equation. These operators are conserved via Noether's theorem such that new conserved polarization operators can be defined for various polarizations. The corresponding one-particle operators of quantum theory are derived showing how these are related to the isometry generators of the massive Dirac fermions of any polarization, including momentum-dependent ones. In this manner, the problem of separating conserved spin and orbital angular momentum operators is solved naturally. Moreover, the operator proposed by Pryce as a mass-center coordinate is studied, showing that after quantization, this becomes in fact the dipole one-particle operator. As an example, the quantities determining the principal one-particle operators are derived for the first time in a momentum-helicity basis.

1 Introduction

The historical problem of finding a good spin operator of Dirac's theory comes from the fact that the Pauli spin part of the total angular momentum is not conserved separately via Noether's theorem. This problem was studied by many authors, giving rise to a rich literature, but without arriving at a commonly accepted solution (see for instance, Refs. [1,2] and the literature indicated therein). Nevertheless, there exists a privileged conserved self-adjoint spin operator satisfying all our exigences. This was found for the first time by Pryce [2] in association with a mass-center operator and redefined later with the help of a Foldy–Wouthuysen transformation in momentum representation (**p**-rep.) [3] (presented in Appendix A).

Remaining outside this debate we tried to study how the polarization of Dirac's field can be changed by applying suitable operators in configuration rep. (**x**-rep.). The polarization is determined by Pauli spinors that enter in the structure of the plane wave solutions of Dirac's equation, offering supplemental $SU(2)$ degrees of freedom but which are less studied so far. This is because of the difficulties in finding suitable operators able to transform the Pauli spinors only without affecting other quantities. Fortunately, we have found a spectral rep. of a class of integral operators, allowing us to define such transformations (transfs.) acting in **x**-rep. but changing the Pauli spinors of the plane wave solutions in **p**-rep. The generators of these transfs. close an $SU(2)$ algebra and, in addition, are conserved via Noether's theorem just as the components (comps.) of the spin operator one looks for. As the action of these operators can be calculated in **p**-rep., we arrived at a surprising result: the $SU(2)$ generators acting on Pauli's spinors are just the comps. of the spin operator proposed by Pryce and redefined then by Foldy and Wouthuysen in this rep. In fact, we found an alternative definition of the same spin operator but in a new framework, allowing us to study how the principal operators of relativistic quantum mechanics (RQM) depend on polarization before and after quantization when these become the one-particle operators of the quantum field theory (QFT).

Our principal objective here is to present the theory of Dirac's free field in this framework, focusing on the role of polarization in determining the form, action and physical meaning of principal operators of RQM and QFT. When the Pauli spinors depend on momentum, as in the case of the widely used momentum-helicity basis, we say that the polarization is *peculiar*. Otherwise, we have a *common* polarization, independent of momentum as, for example, in the momentum-spin basis. We present here general results concerning the principal operators for any polarization, first in RQM and then, after quantization, in QFT.

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The first novelty here is the spectral rep. allowing us to define the spin operator in **x**-rep. as an integral operator having a kernel whose Fourier transform is just the Pryce spin operator in **p**-rep. Moreover, we give a new general definition of the polarization operator that holds even in the case of peculiar polarization getting a natural physical meaning after quantization. Similarly, we derive the action in **x**-rep. of the associated coordinate operator defined by Pryce in **p**-rep. pointing out that this depends linearly on time and defining the corresponding velocity operator. Furthermore, performing the quantization, we obtain for the first time the one-particle spin and polarization operators as well as the isometry generators for any peculiar polarization. We show that these operators depend on new Pauli-type momentum-dependent matrices and "covariant" momentum derivatives. Moreover, we point out that the total angular momentum operator of QFT is split into a new orbital angular momentum and spin operators, each one being conserved separately. However, the surprise is the operator defined by Pryce as a mass-center position vector, which becomes after quantization a time-dependent *dipole* operator whose velocity is known as the classical current. The final original results we present here are the aforementioned matrices and derivatives determining the form of the one-particle operators in a momentum-helicity basis.

We start in the next section briefly revisiting the covariant Dirac free field, its symmetries, and the relativistic scalar product in **x**-rep. In Sect. 3, devoted to **p**-rep., we first present the general properties of the mode spinors of this rep., focusing then on the equivalence of the covariant rep. with an orthogonal sum of a pair of Wigner's unitary and irreducible ones, induced by the rep. of spin half of the $SU(2)$ group. Here, we obtain the structure of the mode spinors of any peculiar polarization and the transfs. of the wave functions in **p**-rep. showing how the generators of covariant and induced reps. are related among themselves. In the first part of Sect. 4 we introduce our new spectral rep., helping us to define the spin and polarization operators in the next part and to analyze the associated coordinate operator in the last one.

These results are obtained in **x** and **p** reps. of RQM where there are problems in interpreting the anti-particle terms. By a lucky chance, the framework adopted here allows us to apply easily the Bogolyubov method of quantization, transforming the expectation values of the operators of RQM into the one-particle operators of QFT. In Sect. 5, devoted to this procedure, we give the principal one-particle operators, showing how the total angular momentum is split into spin and orbital angular momentum conserved one-particle operators. The last part of this section is devoted to the associated coordinate operator that after quantization becomes the dipole one-particle operator evolving linearly in time thanks to the conserved classical current.

In Sect. 6 we give as an example the operators of QFT in a momentum-helicity basis for which we derive for the first time the aforementioned Pauli-type matrices and momentum derivatives. Moreover, we discuss the difference between our polarization operator and the helicity one, showing that these operators give eigenvalues with opposite signs in the antiparticle sector. Finally, we show how the isometry generators look when we turn back to RQM but constructed as the one-particle restriction of QFT. We observe that in the space of Pauli spinors defining one-particle states in **p**-rep., the spin and polarization operators of the relativistic approach are just the original Pauli ones of the non-relativistic theory.

The last section in which we present our concluding remarks is followed by two appendices. In the first appendix, we give some technical details concerning the boost matrices, projection operators, and the Foldy–Wouthuysen transf. laying out the Pryce spin operator. In the second appendix, we discuss briefly the role of the induced reps. in RQM.

2 Massive Dirac field

Let us start with the Minkowski space-time, (M, η), having the metric $\eta = \text{diag}(1, -1, -1, -1)$ and Cartesian coordinates x^μ ($\alpha, \beta, ...\mu, \nu... = 0, 1, 2, 3$). The isometries of M are transfs. of the Poincaré group $\mathscr{P}_+^\uparrow = T(4) \circledS L_+^\uparrow$ [4], $(\Lambda, a) : x \to x' = \Lambda x + a$, formed by transfs. $\Lambda \in L_+^\uparrow$ of the orthochronous proper Lorentz group, preserving the metric η, and four-dimensional translations $a \in T(4)$. The universal covering group of the Poincaré one, $\tilde{\mathscr{P}}_+^\uparrow = T(4) \circledS SL(2, \mathbb{C})$, includes transfs. $\lambda \in SL(2, \mathbb{C})$ related to those of the Lorentz group through the canonical homomorphism, $\lambda \to \Lambda(\lambda)$ [4].

The *covariant* Dirac field, $\psi : M \to \mathscr{V}_D$, is locally defined over M with values in the vector spaces $\mathscr{V}_D$ carrying the finite-dimensional rep. $\rho_D = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of the $SL(2, \mathbb{C})$ group where one defines the Dirac γ matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, giving rise to $SL(2, \mathbb{C})$ generators. Here, we consider exclusively the chiral representation (with diagonal γ^5) in which the transfs.

$$\lambda(\omega) = \exp\left(-\frac{i}{2}\omega^{\alpha\beta} s_{\alpha\beta}\right) \in \rho_D\,, \quad s^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]\,, \tag{1}$$

with real-valued parameters, $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$, are reducible to the subspaces of the irreducible reps. $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of ρ_D [4,5]. We denote by $r = \text{diag}(\hat{r}, \hat{r}) \in \rho_D[SU(2)]$ the transfs. we call here simply rotations, for which we use the Cayley–Klein parameters $\theta^i = \frac{1}{2}\varepsilon_{ijk}\omega^{jk}$ and the generators $s_i = \frac{1}{2}\varepsilon_{ijk}s^{jk} = \text{diag}(\hat{s}_i, \hat{s}_i)$, where $\hat{s}_i = \frac{1}{2}\sigma_i$ are the comps. of the Pauli spin operator depending on Pauli's matrices σ_i ($i, j, k... = 1, 2, 3$). Similarly, the transfs. $l = \mathrm{diag}(\hat{l}, \hat{l}^{-1}) \in \rho_D[SL(2,\mathbb{C})/SU(2)]$, called here the Lorentz boosts, are generated by the matrices $s_{0i} = \mathrm{diag}(i\hat{s}_i, -i\hat{s}_i)$ whose parameters are denoted by $\tau^i = \omega^{0i}$. Summarizing, we may write

$$r(\theta) = \mathrm{diag}(\hat{r}(\theta), \hat{r}(\theta)), \quad \hat{r}(\theta) = e^{-i\theta^i \hat{s}_i} = e^{-\frac{i}{2}\theta^i \sigma_i}, \quad (2)$$

$$l(\tau) = \mathrm{diag}(\hat{l}(\tau), \hat{l}^{-1}(\tau)), \quad \hat{l}(\tau) = e^{\tau^i \hat{s}_i} = e^{\frac{1}{2}\tau^i \sigma_i}. \quad (3)$$

In the covariant parameterization, the associated Lorentz transfs. may be expanded as

$$\Lambda^{\mu\cdot}_{\cdot\nu}[\lambda(\omega)] = \delta^\mu_\nu + \omega^{\mu\cdot}_{\cdot\nu} + \omega^{\mu\cdot}_{\cdot\alpha}\omega^{\alpha\cdot}_{\cdot\nu} + \cdots, \quad (4)$$

as it results from the canonical homomorphism [4] .

The massive Dirac field ψ of mass m and its Dirac adjoint, $\overline{\psi} = \psi^+\gamma^0$, are canonical variables of the action

$$\mathscr{S}[\psi] = \int d^4x \mathscr{L}_D(\psi, \overline{\psi}), \quad (5)$$

defined by the Lagrangian density,

$$\mathscr{L}_D(\psi) = \frac{i}{2}[\overline{\psi}\gamma^\alpha\partial_\alpha\psi - (\overline{\partial_\alpha\psi})\gamma^\alpha\psi] - m\overline{\psi}\psi. \quad (6)$$

This action gives rise to the Dirac equation,

$$E_D\psi = (i\gamma^\mu\partial_\mu - m)\psi = 0, \quad (7)$$

and the form of the relativistic scalar product

$$\langle\psi, \psi'\rangle_D = \int d^3x \overline{\psi}(x)\gamma^0\psi'(x) = \int d^3x \psi^+(x)\psi'(x), \quad (8)$$

related to the conserved electric charge.

The Dirac field transforms under isometries according to the *covariant* rep. $T : (\lambda, a) \to T_{\lambda,a}$ of the group $\tilde{\mathscr{P}}^\uparrow_+$, as [4]

$$(T_{\lambda,a}\psi)(x) = \lambda\psi\left(\Lambda(\lambda)^{-1}(x-a)\right). \quad (9)$$

The well-known basis generators of this rep.,

$$P_\mu = -i\left.\frac{\partial T_{1,a}}{\partial a^\mu}\right|_{a=0}, \quad J_{\mu\nu} = i\left.\frac{\partial T_{\lambda(\omega),0}}{\partial\omega^{\mu\nu}}\right|_{\omega=0}, \quad (10)$$

may be rewritten in vector notation, separating the momentum comps. and energy operator, $P^i = -i\partial_i$ and $H = P_0 = i\partial_t$, and denoting the $SL(2,\mathbb{C})$ generators as

$$J_i = \frac{1}{2}\varepsilon_{ijk}J_{jk} = -i\varepsilon_{ijk}\underline{x}^j\partial_k + s_i, \quad (11)$$

$$K_i = J_{0i} = i(\underline{x}^i\partial_t + t\partial_i) + s_{0i}, \quad (12)$$

where the comps. $\underline{x}^i$ of the coordinate vector operator $\underline{\mathbf{x}}$ act as $(\underline{x}^i\psi)(x) = x^i\psi(x)$. The set $\{H, P^i, J_i, K_i\}$ represents the usual basis of the Lie algebra $\mathrm{Lie}(T)$ of the rep. T [4].

The action (5) is invariant under isometries, such that the scalar product (8) is also invariant,

$$\langle T_{A,a}\psi, T_{A,a}\psi'\rangle_D = \langle\psi, \psi'\rangle_D, \quad (13)$$

because the generators $X \in \mathrm{Lie}(T)$ are self-adjoint, obeying $\langle X\psi, \psi'\rangle_D = \langle\psi, X\psi'\rangle_D$, as the $SL(2,\mathbb{C})$ generators of the rep. ρ_D are Dirac self-adjoint, $\overline{s}_{\mu\nu} = s_{\mu\nu}$. All these generators are conserved via Noether's theorem in the sense that their expectation values $\langle\psi, X\psi\rangle_D$ are independent of time. Therefore, we may conclude that in this framework, the covariant rep. T behaves as a *unitary* one with respect to the relativistic scalar product (8).

Of special interest is the total angular momentum operator $\mathbf{J} = \underline{\mathbf{x}} \wedge \mathbf{P} + \mathbf{s}$, defined by Eq. (11), which is formed by the orbital term $\underline{\mathbf{x}} \wedge \mathbf{P}$ associated to the reducible Pauli spin operator $\mathbf{s}$. Unfortunately, these operators are not conserved separately in the sense of the above definition, such that they do not have a correct physical meaning in special relativity. For this reason, one seeks a *conserved* spin operator $\mathbf{S}$ associated to a suitable coordinate operator, $\mathbf{X}$, allowing the new splitting

$$\mathbf{J} = \underline{\mathbf{x}} \wedge \mathbf{P} + \mathbf{s} = \mathbf{X} \wedge \mathbf{P} + \mathbf{S}, \quad (14)$$

whose orbital term, $\mathbf{X} \wedge \mathbf{P}$, also has to be conserved. This is the principal problem we discuss in this paper focusing on Pryce's spin and coordinate operators.

The invariants of the Dirac field are the eigenvalues of Casimir operators of the rep. T that read [5]

$$C_1 = P_\mu P^\mu \sim m^2, \quad (15)$$

$$C_2 = -\eta_{\mu\nu}W^\mu W^\nu \sim m^2 s(s+1), \quad s = \tfrac{1}{2}, \quad (16)$$

where the Pauli–Lubanski operator [4],

$$W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}, \quad (17)$$

has the components

$$W_0 = J_i P^i = s_i P^i, \quad W_i = H J_i + \varepsilon_{ijk}P_j K_k, \quad (18)$$

as we use $\varepsilon^{0123} = -\varepsilon_{0123} = -1$. These operators are considered by many authors as the comps. of a covariant four-dimensional spin operator of RQM as long as W_0 is just the helicity operator [6]. This is the only differential operator able to define a polarization, but leads to minor difficulties at the level of QFT, as we shall later see. For this reason, we believe that a more convenient polarization operator must come from the algebra $\mathrm{Lie}(T)$.

3 Momentum representation

The Dirac equation allows plane wave solutions that form a basis of **p**-rep., in which the commuting operators H and P^i are diagonal. This system of operators is incomplete, as a polarization operator is missing. This will be added after discussing the problem of finding conserved spin and polarization operators. For the time being we assume that the arbitrary Pauli spinors entering in the structure of the rest frame plane waves define implicitly the polarization.

3.1 Frequencies separation

The general solutions of the free Dirac equation may be expanded in terms of usual mode spinors (or fundamental spinors), $U_{\mathbf{p},\sigma}$ and $V_{\mathbf{p},\sigma} = CU^*_{\mathbf{p},\sigma}$, of positive and negative frequencies, respectively, related through the charge conjugation defined by the matrix $C = C^{-1} = i\gamma^2$ and acting as [7]

$$\gamma^{\mu *} = -C\gamma^\mu C \;\to\; s^*_{\mu\nu} = -Cs_{\mu\nu}C \;\to\; \lambda^* = C\lambda C\,. \tag{19}$$

The plane wave mode spinors satisfy the Dirac equation and the eigenvalues problems,

$$HU_{\mathbf{p},\sigma} = E(p)U_{\mathbf{p},\sigma} \quad HV_{\mathbf{p},\sigma} = -E(p)V_{\mathbf{p},s\sigma}\,, \tag{20}$$
$$P^iU_{\mathbf{p},\sigma} = p^i\,U_{\mathbf{p},\sigma}\,, \quad P^iV_{\mathbf{p},\sigma} = -p^i\,V_{\mathbf{p},\sigma}\,, \tag{21}$$

where $E(p) = \sqrt{m^2+p^2}$ ($p = |\mathbf{p}|$) is the relativistic energy. Then the general solution of the Dirac equation, we call here simply the Dirac free field, can be expanded as [7,8]

$$\begin{aligned}\psi(x) &= \psi^+(x) + \psi^-(x)\\ &= \int d^3p \sum_\sigma \left[U_{\mathbf{p},\sigma}(x)\alpha_\sigma(\mathbf{p}) + V_{\mathbf{p},\sigma}(x)\beta^*_\sigma(\mathbf{p})\right]\,,\end{aligned} \tag{22}$$

in terms of mode functions in $\mathbf{p}$-rep., α_σ and β_σ, of particles and antiparticles, respectively, of arbitrary *polarization* $\sigma = \pm\frac{1}{2}$ that may be defined in various manners, as we shall show after studying the spin operators. In this manner, the space of Dirac's mode spinors, $\mathscr{F}_D = \mathscr{F}^+_D \oplus \mathscr{F}^-_D$, is split into two orthogonal subspaces of mode spinors of positive and negative frequencies, respectively.

The mode spinors prepared at the initial time $t_0 = 0$ have the general form

$$U_{\mathbf{p},\sigma}(x) = u_\sigma(\mathbf{p})\frac{1}{(2\pi)^{\frac{3}{2}}}\,e^{-iE(p)t+i\mathbf{p}\cdot\mathbf{x}}\,, \tag{23}$$
$$V_{\mathbf{p},\sigma}(x) = v_\sigma(\mathbf{p})\frac{1}{(2\pi)^{\frac{3}{2}}}\,e^{iE(p)t-i\mathbf{p}\cdot\mathbf{x}}\,, \tag{24}$$

where the spinors $u_\sigma(\mathbf{p})$ and $v_\sigma(\mathbf{p}) = Cu_{s\sigma}(\mathbf{p})^*$ must be normalized in order to obtain the orthonormalization relations

$$\langle U_{\mathbf{p},\sigma}, U_{\mathbf{p}',\sigma'}\rangle_D = \langle V_{\mathbf{p},\sigma}, V_{\mathbf{p}',\sigma'}\rangle_D = \delta_{\sigma\sigma'}\delta^3(\mathbf{p}-\mathbf{p}')\,, \tag{25}$$
$$\langle U_{\mathbf{p},\sigma}, V_{\mathbf{p}',\sigma'}\rangle_D = \langle V_{\mathbf{p},\sigma}, U_{\mathbf{p}',\sigma'}\rangle_D = 0\,, \tag{26}$$

and the completeness condition,

$$\begin{aligned}\int d^3p \sum_\sigma &\Big[U_{\mathbf{p},\sigma}(t,\mathbf{x})U^+_{\mathbf{p},\sigma}(t,\mathbf{x}')\\ &+V_{\mathbf{p},\sigma}(t,\mathbf{x})V^+_{\mathbf{p},\sigma}(t,\mathbf{x}')\Big] = \delta^3(\mathbf{x}-\mathbf{x}')\,.\end{aligned} \tag{27}$$

Moreover, Eqs. (25) and (26) help us to write the inversion formulas

$$\alpha_\sigma(\mathbf{p}) = \langle U_{\mathbf{p},\sigma}, \psi\rangle_D\,, \quad \beta_\sigma(\mathbf{p}) = \langle \psi, V_{\mathbf{p},\sigma}\rangle_D\,, \tag{28}$$

we need in applications.

In RQM, the physical meaning of the field ψ is encapsulated in the wave functions in $\mathbf{p}$-rep., α_σ and β_σ, that form the Pauli spinors,

$$\alpha = \begin{pmatrix}\alpha_{\frac{1}{2}}\\ \alpha_{-\frac{1}{2}}\end{pmatrix} \in \mathscr{F}_\alpha\,, \quad \beta = \begin{pmatrix}\beta_{\frac{1}{2}}\\ \beta_{-\frac{1}{2}}\end{pmatrix} \in \mathscr{F}_\beta\,, \tag{29}$$

of the Hilbert spaces $\mathscr{F}_\alpha \sim \mathscr{F}_\beta$ equipped with scalar products,

$$\begin{aligned}\langle\alpha,\alpha'\rangle &= \int d^3p\,\alpha^+(\mathbf{p})\alpha'(\mathbf{p})\\ &= \int d^3p \sum_\sigma \alpha^*_\sigma(\mathbf{p})\alpha'_\sigma(\mathbf{p})\,,\end{aligned} \tag{30}$$

and similarly for the spinors β, such that we can write

$$\langle\psi,\psi'\rangle_D = \langle\alpha,\alpha'\rangle + \langle\beta,\beta'\rangle\,, \tag{31}$$

after using Eqs. (25) and (26).

We remind the reader that the differential operators in $\mathbf{x}$-rep., $F(i\partial_\mu) : \mathscr{F}^\pm_D \to \mathscr{F}^\pm_D$, give rise to multiplicative operators in $\mathbf{p}$-rep., $\tilde{F}(p^\mu)$, acting differently on the mode spinors U and V,

$$\begin{aligned}\left[F(i\partial_\mu)\psi\right](x) = \int d^3p \sum_\sigma &\Big[\tilde{F}(p^\mu)U_{\mathbf{p},\sigma}(x)\alpha_\sigma(\mathbf{p})\\ &+\tilde{F}(-p^\mu)V_{\mathbf{p},\sigma}(x)\beta^*_\sigma(\mathbf{p})\Big]\,.\end{aligned} \tag{32}$$

We say that these operators are diagonal on $\mathscr{F}_D$, as they do not mix the mode spinors of different frequencies. For example, the Dirac Hamiltonian operator $H_D = -i\gamma^0\gamma^i\partial_i + m\gamma^0$ acts as

$$(H_DU_{\mathbf{p},\sigma})(x) = \tilde{H}_D(\mathbf{p})U_{\mathbf{p},\sigma}(x) = E(p)U_{\mathbf{p},\sigma}(x)\,, \tag{33}$$
$$(H_DV_{\mathbf{p},\sigma})(x) = \tilde{H}_D(-\mathbf{p})V_{\mathbf{p},\sigma}(x) = -E(p)V_{\mathbf{p},\sigma}(x)\,, \tag{34}$$

where

$$\tilde{H}_D(\mathbf{p}) = m\gamma^0 + \gamma^0\gamma^ip^i = E(p)\left[\tilde{\Pi}_+(\mathbf{p}) - \tilde{\Pi}_-(\mathbf{p})\right]\,, \tag{35}$$

is the Hamiltonian operator in $\mathbf{p}$-rep. expressed in terms of the projection operators (A.6) and (A.7).

3.2 Relating covariant and induced representations

For developing our approach, we need to work simultaneously in $\mathbf{x}$ and $\mathbf{p}$ reps. relating the covariant rep. of $\mathbf{x}$-rep. to the Wigner-induced reps. transforming the spinors (29). Moreover, the Wigner method allows us to construct the mode spinors separating the Pauli spinors whose degrees of freedom have to be studied here.

The wave functions of **p**-rep. (29) are defined on orbits in momentum space, $\Omega_{\mathring{p}} = \{\mathbf{p}|\mathbf{p} = \Lambda\mathring{p}, \Lambda \in L_+^\uparrow\}$, that may be built by applying Lorentz transfs. on a *representative* momentum $\mathring{p}$ [9,10]. In the case of massive particles we discuss here, the representative momentum is just the rest frame one, $\mathring{p} = (m, 0, 0, 0)$. The rotations that leave $\mathring{p}$ invariant, $R\mathring{p} = \mathring{p}$, form the *stable* group $SO(3) \subset L_+^\uparrow$ of $\mathring{p}$ whose universal covering group $SU(2)$ is called the *little* group associated to the representative momentum $\mathring{p}$.

For each momentum **p** there exist a set of transfs. $L_{\mathbf{p}}$ generating it as $\mathbf{p} = L_{\mathbf{p}}\,\mathring{p}$. These transfs. are defined up to arbitrary rotations $R(\mathbf{p})$ which may depend on **p**, as these do not change the representative momentum, $L_{\mathbf{p}}R(\mathbf{p})\,\mathring{p} = L_{\mathbf{p}}\,\mathring{p}$. This means that the orbit $\Omega_{\mathring{p}}$ is in fact a homogeneous space $L_+^\uparrow/SO(3)$. Consequently, the corresponding transfs. $\lambda_{\mathbf{p}} \in \rho_D$ which satisfy $\Lambda(\lambda_{\mathbf{p}}) = L_{\mathbf{p}}$ and $\lambda_{\mathbf{p}=0} = 1 \in \rho_D$ have the general form $\lambda_{\mathbf{p}} = l_{\mathbf{p}} r(\mathbf{p})$, being constituted by genuine Lorentz boosts $l_{\mathbf{p}} \in \rho_D[SL(2,\mathbb{C})/SU(2)]$ defined by Eq. (A.1) and arbitrary rotations $r(\mathbf{p}) \in \rho_D[SU(2)]$ satisfying $r(\mathbf{p}=0) = 1 \in \rho_D$.

Hereby, we see that the functions (29) are defined on the orbit $\Omega_{\mathring{p}}$ where the invariant measure is [4]

$$\mu(\mathbf{p}) = \mu(\Lambda\mathbf{p}) = \frac{d^3p}{E(p)}, \quad \forall\Lambda \in L_+^\uparrow. \tag{36}$$

Therefore, $\sqrt{E}\alpha$ and $\sqrt{E}\beta$ are square integrable functions of the Hilbert space $\mathscr{L}^2(\Omega_{\mathring{p}}, \mu, \mathscr{V}_P)$ equipped with an invariant scalar product that can be put in the form (30). In practice, it is convenient to work with the spaces $\mathscr{F}_\alpha \sim \mathscr{F}_\beta \sim \mathscr{L}^2(\Omega_{\mathring{p}}, d^3p, \mathscr{V}_P)$ which are isometric with $\mathscr{L}^2(\Omega_{\mathring{p}}, \mu, \mathscr{V}_P)$.

For investigating how these functions transform under isometries, one assumes that there exists a rep. $\tilde{T} : (\lambda, a) \to \tilde{T}_{\lambda,a}$, carried by the spaces $\mathscr{F}_\alpha$ and $\mathscr{F}_\beta$, associated to the rep. T as [4,5,11]

$$(T_{\lambda,a}\psi)(x) = \int d^3p \sum_\sigma \Big[U_{\mathbf{p},\sigma}(x)(\tilde{T}_{\lambda,a}\,\alpha)_\sigma(\mathbf{p}) + V_{\mathbf{p},\sigma}(x)(\tilde{T}_{\lambda,a}\,\beta)^*_\sigma(\mathbf{p})\Big]. \tag{37}$$

Taking into account that the covariant rep. T is defined by Eq. (9), we use the identity $(\Lambda x)\cdot p = x\cdot(\Lambda^{-1}p)$ and the invariant measure, $d^3p E(p)^{-1} = d^3p' E(p')^{-1}$, for changing the integration variable,

$$\mathbf{p} \to \mathbf{p}' = \Lambda(\lambda)^{-1}\mathbf{p}, \tag{38}$$

finding the action of the operators $\tilde{T}$ [4,5],

$$\sum_{\sigma'} u_{\sigma'}(\mathbf{p})(\tilde{T}_{\lambda,a}\,\alpha)_{\sigma'}(\mathbf{p}) = \frac{E(p')}{E(p)} \sum_\sigma \lambda u_\sigma(\mathbf{p}')\,\alpha_\sigma(\mathbf{p}')\,e^{ia\cdot p}, \tag{39}$$

$$\sum_{\sigma'} v_{\sigma'}(\mathbf{p})(\tilde{T}_{\lambda,a}\,\beta^*)_{\sigma'}(\mathbf{p}) = \frac{E(p')}{E(p)} \sum_\sigma \lambda v_\sigma(\mathbf{p}')\,\beta^*_\sigma(\mathbf{p}')\,e^{-ia\cdot p}, \tag{40}$$

where $a\cdot p = a_\mu p^\mu = E(p)a^0 - \mathbf{p}\cdot\mathbf{a}$.

Furthermore, according to Wigner's general method, we introduce the spinors [5],

$$u_\sigma(\mathbf{p}) = N(\mathrm{p})\lambda_{\mathbf{p}}\,\mathring{u}_\sigma = N(p)l_{\mathbf{p}}r(\mathbf{p})\,\mathring{u}_\sigma, \tag{41}$$

$$v_\sigma(\mathbf{p}) = Cu^*_\sigma(\mathbf{p}) = N(p)\lambda_{\mathbf{p}}\mathring{v}_\sigma = N(p)l_{\mathbf{p}}r(\mathbf{p})\mathring{v}_\sigma, \tag{42}$$

where $N(p) \in \mathbb{R}$ satisfying $N(0) = 1$ is a normalization factor. The rest frame spinors $\mathring{u}_\sigma = u_\sigma(0)$ and $\mathring{v}_\sigma = v_\sigma(0) = C\mathring{u}^*_\sigma$ are solutions of the Dirac equation in the rest frame where this equation reduces to the eigenvalues problems of the matrix γ^0,

$$\gamma^0\mathring{u}_\sigma = \mathring{u}_\sigma, \quad \gamma^0\mathring{v}_\sigma = -\mathring{v}_\sigma. \tag{43}$$

Then the Wigner spinors (41) and (42) are solutions of the Dirac equation in any frame of **p**-rep. Indeed, observing that the matrix $\gamma p = E(p)\gamma^0 - \gamma^i p^i$ satisfies $\gamma p\,\lambda_{\mathbf{p}} = m\lambda_{\mathbf{p}}\gamma^0$, we obtain the Dirac equations in **p**-rep.,

$$(\gamma p - m)u_\sigma(\mathbf{p}) = 0, \quad (\gamma p + m)v_\sigma(\mathbf{p}) = 0, \tag{44}$$

after exploiting Eq. (43).

The matrices $\frac{1\pm\gamma^0}{2}$ form an orthogonal system of projection matrices such that the spinor subspaces $\frac{1+\gamma^0}{2}\mathscr{V}_D$ and $\frac{1-\gamma^0}{2}\mathscr{V}_D$ are orthogonal. Moreover, we assume that all these spinors are normalized, $\mathring{u}^+_\sigma\mathring{u}_{\sigma'} = \mathring{v}^+_\sigma\mathring{v}_{\sigma'} = \delta_{\sigma\sigma'}$, forming complete systems on their subspaces,

$$\sum_\sigma \mathring{u}_\sigma\mathring{u}^+_\sigma = \frac{1+\gamma^0}{2}, \quad \sum_\sigma \mathring{v}_\sigma\mathring{v}^+_\sigma = \frac{1-\gamma^0}{2}. \tag{45}$$

We have now the opportunity of introducing the Pauli spinors we need for studying the polarization, assuming that in the chiral rep. of the Dirac matrices (with diagonal γ^5) we may express the momentum-dependent spinors as

$$\mathring{u}_\sigma(\mathbf{p}) = r(\mathbf{p})\mathring{u}_\sigma = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi_\sigma(\mathbf{p})\\ \xi_\sigma(\mathbf{p})\end{pmatrix}, \tag{46}$$

$$\mathring{v}_\sigma(\mathbf{p}) = r(\mathbf{p})\mathring{v}_\sigma = \frac{1}{\sqrt{2}}\begin{pmatrix}\eta_\sigma(\mathbf{p})\\ -\eta_\sigma(\mathbf{p})\end{pmatrix}, \tag{47}$$

in terms of Pauli spinors $\xi_\sigma(\mathbf{p})$ and $\eta_\sigma(\mathbf{p}) = i\sigma_2\xi_\sigma(\mathbf{p})^*$ that form related bases that are orthogonal,

$$\xi_\sigma^+(\mathbf{p})\xi_{\sigma'}(\mathbf{p}) = \eta_\sigma^+(\mathbf{p})\eta_{\sigma'}(\mathbf{p}) = \delta_{\sigma\sigma'}, \tag{48}$$

and complete systems,

$$\sum_\sigma \xi_\sigma(\mathbf{p})\xi_\sigma^+(\mathbf{p}) = \sum_\sigma \eta_\sigma(\mathbf{p})\eta_\sigma^+(\mathbf{p}) = 1_{2\times 2}. \tag{49}$$

The functions $\xi_\sigma : \mathbb{R}^3_p \to \mathscr{V}_P$ remain arbitrary, representing the polarization degrees of freedom which will be determined after defining the polarization operators. As mentioned before, when these spinors depend explicitly on $\mathbf{p}$, we say that we have a *peculiar* polarization, while a polarization independent of $\mathbf{p}$ will be referred to as *common* polarization.

Finally, by setting the normalization factor as

$$N(p) = \sqrt{\frac{m}{E(p)}}, \tag{50}$$

we obtain the expressions of the spinors (23) and (24),

$$U_{\mathbf{p},\sigma}(t,\mathbf{x}) = N(p)l_{\mathbf{p}}\,\mathring{u}_\sigma(\mathbf{p})\frac{1}{(2\pi)^{\frac{3}{2}}}\,e^{-iE(p)t+i\mathbf{p}\cdot\mathbf{x}}, \tag{51}$$

$$V_{\mathbf{p},\sigma}(t,\mathbf{x}) = N(p)l_{\mathbf{p}}\mathring{v}_\sigma(\mathbf{p})\frac{1}{(2\pi)^{\frac{3}{2}}}\,e^{iE(p)t-i\mathbf{p}\cdot\mathbf{x}}, \tag{52}$$

that satisfy Eqs. (25), (26) and (27) .

With these preparations we arrive at the principal result of Wigner's approach, showing that the Pauli spinors α and β transform alike under Wigner's rep. $\tilde{T}$ acting as [4,5,9]

$$(\tilde{T}_{\lambda,a}\,\alpha)_\sigma(\mathbf{p}) = \sqrt{\frac{E(p')}{E(p)}}e^{ia\cdot p}\sum_{\sigma'} D_{\sigma\sigma'}(\lambda,\mathbf{p})\alpha_{\sigma'}(\mathbf{p}'), \tag{53}$$

where the matrix elements

$$D_{\sigma\sigma'}(\lambda,\mathbf{p}) = \mathring{u}^+(\mathbf{p})_\sigma w(\lambda,\mathbf{p})\mathring{u}_{\sigma'}(\mathbf{p}'), \tag{54}$$

depend on the spinors (46) and Wigner transfs. $w(\lambda,\mathbf{p}) = l_{\mathbf{p}}^{-1}\lambda\, l_{\mathbf{p}'}$. We observe that the corresponding Lorentz transf. $\Lambda[w(\lambda,\mathbf{p})] = L_{\mathbf{p}}^{-1}\Lambda(\lambda)L_{\mathbf{p}'}$ leaves invariant the representative momentum, as we have $\Lambda[w(\lambda,\mathbf{p})]\mathring{p} = L_{\mathbf{p}}^{-1}\Lambda(\lambda)\mathbf{p}' = L_{\mathbf{p}}^{-1}\mathbf{p} = \mathring{p}$, such that $\Lambda[w(\lambda,\mathbf{p})] \in SO(3) \to w(\lambda,\mathbf{p}) \in SU(2)$. Therefore, we may write the definitive form of the matrix elements (54) as

$$D_{\sigma\sigma'}(\lambda,\mathbf{p}) = \xi_\sigma^+(\mathbf{p})\hat{l}_{\mathbf{p}}^{-1}\hat{\lambda}\,\hat{l}_{\mathbf{p}'}\xi_{\sigma'}(\mathbf{p}'). \tag{55}$$

Obviously, the matrices $D(\lambda,\mathbf{p})$ form the irreducible rep. of spin $s = \frac{1}{2}$ of the $SU(2)$ group, which means that the Wigner irreducible reps. $\tilde{T}$ are *induced* by the subgroup $T(4)\circledS SU(2)$ [4,5,9], as we show in Appendix B. For the antiparticle spinors β^*, we obtain the matrix elements

$$\mathring{v}_{s\sigma}^+(\mathbf{p})w(\lambda,\mathbf{p})\mathring{v}_{s\sigma'}(\mathbf{p}') = \left(\mathring{u}_\sigma^+(\mathbf{p})w(\lambda,\mathbf{p})\mathring{u}_{\sigma'}(\mathbf{p}')\right)^* = [D_{\sigma\sigma'}(\lambda,\mathbf{p})]^*, \tag{56}$$

showing that the spinors α and β transform alike under isometries. Note that the form of the matrix elements (55) explicitly involves the Pauli spinors, helping us to study their dependence on polarization.

The Wigner rep. $\tilde{T}$ is irreducible, as the matrices D are irreducible. Moreover, these are unitary with respect to the scalar product (30) [9,10],

$$\langle\tilde{T}_{\lambda,a}\alpha, \tilde{T}_{\lambda,a}\alpha'\rangle = \langle\alpha,\alpha'\rangle, \tag{57}$$

and similarly for β. As the covariant reps. are unitary with respect to the scalar product (8) which can be decomposed as in Eq. (31), we conclude that the expansion (22) establishes the unitary equivalence, $T = \tilde{T}\oplus\tilde{T}$, of the covariant rep. with the orthogonal sum of Wigner's unitary irreducible reps. [10]. This means that the generators $\tilde{X}\in \mathrm{Lie}(\tilde{T})$ defined as

$$\tilde{P}_\mu = -\,i\left.\frac{\partial\tilde{T}_{1,a}}{\partial a^\mu}\right|_{a=0}, \quad \tilde{J}_{\mu\nu} = i\left.\frac{\partial\tilde{T}_{\lambda(\omega),0}}{\partial\omega^{\mu\nu}}\right|_{\omega=0}, \tag{58}$$

are related to the corresponding generators $X\in\mathrm{Lie}(T)$, such that

$$(X\psi)(x) = \int d^3p\sum_\sigma\Big[U_{\mathbf{p},\sigma}(x)(\tilde{X}\,\alpha)_\sigma(\mathbf{p}) - V_{\mathbf{p},\sigma}(x)(\tilde{X}\,\beta)^*_\sigma(\mathbf{p})\Big], \tag{59}$$

as we deduce deriving Eq. (37) with respect to a group parameter $\zeta\in(\omega,a)$ in $\zeta = 0$.

4 Looking for spin and coordinate operators

The next step is to define the polarization, looking for operators acting on the space of Pauli spinors $\mathscr{V}_P$. As the representative momentum corresponds to a set of rest frames related among themselves through $SO(3)$ rotations of a stable group, we observe that the space of Pauli's spinors has similar degrees of freedom governed by the $SU(2)$ little group. These degrees of freedom deserve to be investigated, as a symmetry neglected so far. For this purpose, it is convenient to re-denote the Dirac field (22) by $\psi_\xi(x)$ and the mode spinors (51) and (52) by $U_{\mathbf{p},\xi_\sigma}$ and $V_{\mathbf{p},\eta_\sigma}$, respectively, explicitly pointing out their dependence on Pauli's spinors.

On the other hand, we take into account that there are no differential or multiplicative operators acting directly on the Pauli spinors without affecting other quantities. Therefore, these must be more general operators as the integral ones defined by kernels whose Fourier transforms are usual operators of the **p**-rep. used in various applications. In what follows, we consider such operators for constructing the spectral reps. we need in our investigation.

4.1 Spectral representation of integral operators

We focus on the integral operators, $Z : \mathscr{F}_D \to \mathscr{F}_D$, whose action,

$$(Z\psi)(x) = \int d^4x' \mathscr{K}_Z(x-x')\psi(x')\,, \tag{60}$$

is determined by the kernels $\mathscr{K}_Z$ which are 4×4 matrices depending on $x-x'$. These operators are linear, forming an algebra in which the multiplication is defined by the convolution,

$$\mathscr{K}_{Z_1Z_2}(x-x') = \int d^4x'' \mathscr{K}_{Z_1}(x-x'')\mathscr{K}_{Z_2}(x''-x')\,, \tag{61}$$

denoted as $\mathscr{K}_{Z_1Z_2} = \mathcal{K}_{Z_1} * \mathscr{K}_{Z_2}$. The identity operator has the kernel $\mathscr{K}_1(x-x') = \delta^4(x-x')$. An operator Z is invertible if there exists an operator Z^{-1} such that $\mathscr{K}_Z * \mathscr{K}_{Z^{-1}} = \mathscr{K}_{Z^{-1}} * \mathscr{K}_Z = \mathscr{K}_1$. For any integral operator Z, we may write the bracket

$$\langle\psi, Z\psi'\rangle_D = \int d^4x\, d^4x' \psi^+(x)\mathcal{K}_Z(x-x')\psi(x') \tag{62}$$

observing that Z is self-adjoint with respect to this scalar product only if $\mathscr{K}_Z(x) = \mathscr{K}_Z^+(-x)$. Note that the multiplicative or differential operators are particular cases of integral ones. For example, the derivatives ∂_μ can be seen as integral operators having the kernels $\mathscr{K}_{\partial_\mu}(x) = \partial_\mu\delta^4(x)$.

Of special interest are the equal-time operators Y whose kernels have the form

$$\mathscr{K}_Y(x-x') = \delta(t-t')\mathscr{K}_Y(\mathbf{x}-\mathbf{x}')\,, \tag{63}$$

acting as

$$(Y\psi)(t,\mathbf{x}) = \int d^3x' \mathscr{K}_Y(\mathbf{x}-\mathbf{x}')\psi(t,\mathbf{x}')\,, \tag{64}$$

without involving the time. In this case, the kernels allow the three-dimensional Fourier rep.,

$$\mathscr{K}_Y(\mathbf{x}) = \int d^3p \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^3}\tilde{Y}(\mathbf{p})\,, \tag{65}$$

where $\tilde{Y}(\mathbf{p})$ is the operator in **p**-rep. of RQM corresponding to Y. Then the action (64) on a field (22) can be written as

$$(Y\psi_\xi)(x) = \int d^3p \sum_\sigma \Big[\tilde{Y}(\mathbf{p})U_{\mathbf{p},\xi_\sigma}(x)\alpha_\sigma(\mathbf{p}) + \tilde{Y}(-\mathbf{p})V_{\mathbf{p},\eta_\sigma}(x)\beta^*_\sigma(\mathbf{p})\Big]\,. \tag{66}$$

We thus see that all these integral operators are diagonal on $\mathscr{F}_D$, acting separately on $\mathscr{F}_D^\pm$ without mixing mode spinors of different frequencies.

In what follows we consider a special type of such equal-time operator, denoted by Y_ξ, acting alike on the spinors α and β, defined by kernels that allow the spectral rep.

$$\mathscr{K}_{Y_\xi}(\mathbf{x}-\mathbf{x}') = \int d^3p \sum_{\sigma,\sigma'} \Big[U_{\mathbf{p},\xi_\sigma}(t,\mathbf{x})\tilde{y}_{\sigma\sigma'}U^+_{\mathbf{p},\xi_{\sigma'}}(t,\mathbf{x}') + V_{\mathbf{p},\eta_\sigma}(t,\mathbf{x})\tilde{y}^*_{\sigma\sigma'}V^+_{\mathbf{p},\eta_{\sigma'}}(t,\mathbf{x}')\Big]\,, \tag{67}$$

in terms of mode spinors (51) and (52) depending on arbitrary Pauli spinors $\xi \subset \mathscr{V}_P$ that may depend on **p**. The action of these operators can be calculated easily in **p**-rep. using the orthogonality properties (25) and (26). Then it turns out the action in **x**-rep.,

$$\begin{aligned}(Y_\xi\psi_\xi)(x) &= \int d^3x' \mathscr{K}_{Y_\xi}(\mathbf{x}-\mathbf{x}')\psi_\xi(t,\mathbf{x}')\\ &= \int d^3p \sum_{\sigma,\sigma'}\big[U_{\mathbf{p},\xi_\sigma}(t,\mathbf{x})\tilde{y}_{\sigma\sigma'}\alpha_{\sigma'}(\mathbf{p}) + V_{\mathbf{p},\eta_\sigma}(t,\mathbf{x})\tilde{y}^*_{\sigma\sigma'}\beta^*_{\sigma'}(\mathbf{p})\big]\,,\end{aligned} \tag{68}$$

related to the action of associated matrix operator $\tilde{y}(\mathbf{p})$, transforming alike the spinors α and β,

$$(\tilde{y}\alpha)_\sigma(\mathbf{p}) = \langle U_{\mathbf{p},\xi_\sigma}, Y_\xi\psi_\xi\rangle_D = \tilde{y}_{\sigma\sigma'}\alpha_{\sigma'}(\mathbf{p})\,, \tag{69}$$

$$(\tilde{y}\beta)_\sigma(\mathbf{p}) = \langle Y_\xi\psi_\xi, V_{\mathbf{p},\eta_\sigma}\rangle_D = \tilde{y}_{\sigma\sigma'}\beta_{\sigma'}(\mathbf{p})\,. \tag{70}$$

The special form of these operators allows us to derive their expectation values,

$$\langle\psi_\xi, Y_\xi\psi_\xi\rangle_D = \langle\alpha, \tilde{y}\,\alpha\rangle + \langle\beta, \tilde{y}^+\beta\rangle\,, \tag{71}$$

exploiting Eqs. (25) and (26). Here we see that an operator Y_ξ is self-adjoint if the matrix $\tilde{y}$ is Hermitian, $\tilde{y}_{\sigma\sigma'} = \tilde{y}^*_{\sigma'\sigma}$.

We must stress that the dependence of the operator Y_ξ on ξ is not an impediment, as we know what happens when we change the spinor basis. Thus, by using Eqs. (75) and (76), we find that $Y_\xi \to Y_{\hat{r}\xi} \Rightarrow \tilde{y} \to D(\hat{r})\tilde{y}$ if we keep unchanged the spinors α and β encapsulating the physical meaning.

4.2 Spin and polarization

For exploiting the spin degrees of freedom, we start with an arbitrary orthonormal basis $\xi \subset \mathscr{V}_P$, satisfying Eqs. (48) and (49), whose spinors may depend on **p** but without denoting this explicitly. The rotations $\hat{r} \in SU(2)$ of the little group transform this basis as

$$\hat{r}\,\xi_\sigma = \sum_{\sigma'}\xi_{\sigma'}D_{\sigma'\sigma}(\hat{r}) \Rightarrow r\,\mathring{u}_\sigma = \sum_{\sigma'}\mathring{u}_{\sigma'}D_{\sigma'\sigma}(\hat{r})\,, \tag{72}$$

$$\hat{r}\,\eta_\sigma = \sum_{\sigma'}\eta_{\sigma'}D^*_{\sigma'\sigma}(\hat{r}) \Rightarrow r\,\mathring{v}_\sigma = \sum_{\sigma'}\mathring{v}_{\sigma'}D^*_{\sigma'\sigma}(\hat{r})\,, \tag{73}$$

where $r = \mathrm{diag}(\hat{r}, \hat{r}) \in \rho_D$ is an arbitrary rotation corresponding to $\hat{r}$ for which we use the traditional notation

$$D_{\sigma'\sigma}(\hat{r}) = \xi^+_{\sigma'}\hat{r}\xi_\sigma \,. \tag{74}$$

We thus obtain the transfs. of mode spinors,

$$U_{\mathbf{p},\hat{r}\xi_\sigma}(x) = \sum_{\sigma'} U_{\mathbf{p},\xi_{\sigma'}}(x)D_{\sigma'\sigma}(\hat{r}) \,, \tag{75}$$

$$V_{\mathbf{p},\hat{r}\eta_\sigma}(x) = \sum_{\sigma'} V_{\mathbf{p},\eta_{\sigma'}}(x)D^*_{\sigma'\sigma}(\hat{r}) \,, \tag{76}$$

which give the transformed free field $\psi_{\hat{r}\xi}$ according to the expansion (22).

Under such circumstances, we may look for a rep., $\mathfrak{R} : \hat{r} \to \mathfrak{R}(\hat{r})$, of the little group $SU(2)$ with values in a set of operators $\mathfrak{S} = \{\mathfrak{R}(\hat{r})|\hat{r} \in SU(2)\}$ able to rotate the Pauli spinors but without depending on the spinor basis ξ or affecting other quantities. These operators can be constructed as integral operators with kernels of the form (67), where

$$\tilde{y} = D(\hat{r}) \;\Rightarrow\; \tilde{y}_{\sigma\sigma'} = D_{\sigma\sigma'}(\hat{r}) = \xi^+_\sigma \hat{r}\xi_{\sigma'} \,. \tag{77}$$

Then, by substituting these matrices in Eq. (68) and applying the identities (75) and (76), we find the desired action

$$\begin{aligned}&[\mathfrak{R}(\hat{r})\psi_\xi](t, \mathbf{x})\\ &\quad = \int d^3x' \mathscr{K}_{\mathfrak{R}(\hat{r})}(\mathbf{x} - \mathbf{x}')\psi_\xi(t, \mathbf{x}') = \psi_{\hat{r}\xi}(t, \mathbf{x}) \,,\end{aligned} \tag{78}$$

upon the basis of Pauli spinors. In addition, the general rule (71) allows us to derive the expectation values of these operators

$$\langle\psi_\xi, \mathfrak{R}(\hat{r})\psi_\xi\rangle_D = \langle\alpha, D(\hat{r})\alpha\rangle + \langle\beta, D^+(\hat{r})\beta\rangle \,, \tag{79}$$

which depend explicitly on ξ through the matrix $D(\hat{r})$.

Furthermore, we consider the mode spinors (51) and (52), the action of $SU(2)$ rotations (72) and (73), and the identities (A.4) and (A.5), deducing that the integral operators $\mathfrak{R}(\hat{r}) \in \mathfrak{S}$ have kernels of the form (65) whose Fourier transforms read

$$\begin{aligned}\tilde{\mathfrak{R}}(\hat{r}, \mathbf{p}) &= \frac{m}{E(p)}\left[l_\mathbf{p}\, r\, \frac{1+\gamma^0}{2} l_\mathbf{p} + l_\mathbf{p}^{-1} r\, \frac{1-\gamma^0}{2} l_\mathbf{p}^{-1}\right]\\ &= l_\mathbf{p} r\, l_\mathbf{p}^{-1}\tilde{\Pi}_+(\mathbf{p}) + l_\mathbf{p}^{-1} r\, l_\mathbf{p}\tilde{\Pi}_-(\mathbf{p}) \,,\end{aligned} \tag{80}$$

where $r = \mathrm{diag}(\hat{r}, \hat{r}) \in \rho_D$, while $\tilde{\Pi}_\pm(\mathbf{p})$ are the projection operators (A.6) and (A.7). The operator $\mathfrak{R}(\hat{r})$ is independent of the spinor bases under consideration, as the spinors $\mathring{u}(\mathbf{p})$ and $\mathring{v}(\mathbf{p})$ satisfy similar relations as (45), because $r(\mathbf{p})$ commutes with γ^0.

We thus defined the set $\mathfrak{S}$ of operators whose properties can be studied in **p**-rep., as their Fourier transforms obey the same algebra,

$$\mathfrak{R} = \mathfrak{R}_1\mathfrak{R}_2 \;\Rightarrow\; \mathscr{K}_\mathfrak{R} = \mathscr{K}_{\mathfrak{R}_1} * \mathscr{K}_{\mathfrak{R}_2} \;\Rightarrow\; \tilde{\mathfrak{R}}(\mathbf{p}) = \tilde{\mathfrak{R}}_{1(\mathbf{p})}\tilde{\mathfrak{R}}_{2(\mathbf{p})} \,. \tag{81}$$

By then using the identities (A.4) and (A.5), after a little calculation, we verify that

$$\tilde{\mathfrak{R}}(\hat{r}, \mathbf{p})\tilde{\mathfrak{R}}(\hat{r}', \mathbf{p}) = \tilde{\mathfrak{R}}(\hat{r}\hat{r}', \mathbf{p}) \,, \tag{82}$$

observing that for $\hat{r} = 1_{2\times2}$ we have

$$\tilde{\mathfrak{R}}(1_{2\times2}, \mathbf{p}) = \tilde{\Pi}_{+(\mathbf{p})} + \tilde{\Pi}_{-(\mathbf{p})} = 1 \in \rho_D \,. \tag{83}$$

We conclude that the set $\mathfrak{S}$ forms just the $SU(2)$ rep. we are looking for. We thus arrive at our principal objective, namely, the definition of spin operator.

Definition 1 The spin operator is the vector operator **S** whose components form a canonical basis of the algebra $\mathrm{Lie}(\mathfrak{R}) \sim su(2)$.

Starting with the transf. $\mathfrak{R}(\hat{r}(\theta))$, depending on the rotation (2), we derive the spin comps.,

$$S_i = i\,\frac{\partial\mathfrak{R}(\hat{r}(\theta))}{\partial\theta^i}\bigg|_{\theta^i=0} \,, \tag{84}$$

finding that they are integral operators acting as

$$\begin{aligned}[S_i\psi_\xi](t, \mathbf{x}) &= \int d^3x' \mathscr{K}_{S_i}(\mathbf{x} - \mathbf{x}')\psi_\xi(t, \mathbf{x}')\\ &= \psi_{\hat{s}_i\xi}(t, \mathbf{x}) \,,\end{aligned} \tag{85}$$

through kernels having as Fourier transforms the spin comps. in **p**-rep.,

$$\begin{aligned}\tilde{S}_i(\mathbf{p}) &= \frac{m}{E(p)}\left[l_\mathbf{p}\, s_i\, \frac{1+\gamma^0}{2} l_\mathbf{p} + l_\mathbf{p}^{-1} s_i\, \frac{1-\gamma^0}{2} l_\mathbf{p}^{-1}\right]\\ &= s_i(\mathbf{p})\tilde{\Pi}_+(\mathbf{p}) + s_i(-\mathbf{p})\tilde{\Pi}_-(\mathbf{p}) \,,\end{aligned} \tag{86}$$

where $s_i(\mathbf{p}) = l_\mathbf{p} s_i l_\mathbf{p}^{-1}$ are the comps. of the transformed reducible Pauli spin operator. In the rest frame we have $\mathbf{p} = 0 \Rightarrow \tilde{\mathbf{S}}(0) = \mathbf{s}(0) = \mathbf{s}$.

Surprisingly, after a little calculation, we find that the operators (86) are just the comps. of Pryce's spin operator,

$$\tilde{S}_i(\mathbf{p}) = \frac{m}{E(p)}s_i + \frac{p^i(\mathbf{s}\cdot\mathbf{p})}{E(p)(E(p)+m)} + \frac{i}{2E(p)}\varepsilon_{ijk}p^j\gamma^k \,, \tag{87}$$

found long ago (see the third of Eqs. (6.7) of Ref. [12]) in association with a would-be relativistic mass-center coordinate operator. The same spin operator was defined alternatively as in Eq. (A.10) such that it becomes the Pauli one in the frame where the Hamiltonian is $\gamma^0E(p)$, instead of the rest frame, as in the case of our definition. However, the principal novelty of our definition is in pointing out that the spin comps. are the generators of a Noetherian symmetry. Thus, we conclude that Definition 1 is different from those of Pryce or Foldy and Wouthuysen.

By definition, the operators (86) generate the $su(2)$ algebra,

$$\left[\tilde{S}_i(\mathbf{p}), \tilde{S}_j(\mathbf{p})\right] = i\varepsilon_{ijk}\tilde{S}_k(\mathbf{p}) \;\Rightarrow\; \left[S_i, S_j\right] = i\varepsilon_{ijk}S_k \,, \tag{88}$$

are self-adjoint and conserved, commuting with the Dirac Hamiltonian in **p**-rep. (35). The action of these operators on the mode spinors (51) and (52) can be derived as in Eq. (66), obtaining

$$(S_i U_{\mathbf{p},\xi_\sigma})(x) = \tilde{S}_i(\mathbf{p}) U_{\mathbf{p},\xi_\sigma}(x) = U_{\mathbf{p},\hat{s}_i\xi_\sigma}(x)\,, \tag{89}$$

$$(S_i V_{\mathbf{p},\eta_\sigma})(x) = \tilde{S}_i(-\mathbf{p}) V_{\mathbf{p},\eta_\sigma}(x) = V_{\mathbf{p},\hat{s}_i\eta_\sigma}(x)\,, \tag{90}$$

after using the form (86) and the identities (A.4) and (A.5).

In other respects, the polarization is given by the pair of related spinors $\xi_\sigma(\mathbf{p})$ and $\eta_\sigma(\mathbf{p})$ assumed to satisfy the eigenvalue problems

$$\hat{s}_i n^i(\mathbf{p})\xi_\sigma(\mathbf{p}) = \sigma\,\xi_\sigma(\mathbf{p}) \to \hat{s}_i n^i(\mathbf{p})\eta_\sigma(\mathbf{p}) = -\sigma\,\eta_\sigma(\mathbf{p}), \tag{91}$$

where the unit vector $\mathbf{n}(\mathbf{p})$ gives the peculiar direction with respect to which the peculiar polarization is measured. Under such circumstances, we may define a convenient polarization operator appropriate to the present framework.

Definition 2 The polarization operator is the integral operator W whose kernel has the Fourier transform

$$\tilde{W}(\mathbf{p}) = w(\mathbf{p})\Pi_+(\mathbf{p}) + w(-\mathbf{p})\tilde{\Pi}_-(\mathbf{p})\,, \tag{92}$$

where $w(\mathbf{p}) = l_\mathbf{p} s_i n^i(\mathbf{p}) l_\mathbf{p}^{-1}$.

This operator is conserved, commuting with the Hamiltonian operator (35). Moreover, it commutes with P^i and H acting on the mode spinors constructed with the spinors of Eq. (91) as

$$\begin{aligned}(W U_{\mathbf{p},\xi_\sigma(\mathbf{p})})(x) &= \tilde{W}(\mathbf{p}) U_{\mathbf{p},\xi_\sigma(\mathbf{p})}(x) = U_{\mathbf{p},\hat{s}_i n^i(\mathbf{p})\xi_\sigma(\mathbf{p})}(x)\\ &= \sigma U_{\mathbf{p},\xi_\sigma(\mathbf{p})}(x)\,,\end{aligned} \tag{93}$$

$$\begin{aligned}(W V_{\mathbf{p},\eta_\sigma(\mathbf{p})})(x) &= \tilde{W}(-\mathbf{p}) V_{\mathbf{p},\eta_\sigma(\mathbf{p})}(x) = V_{\mathbf{p},\hat{s}_i n^i(\mathbf{p})\eta_\sigma(\mathbf{p})}(x)\\ &= -\sigma V_{\mathbf{p},\eta_\sigma(\mathbf{p})}(x)\,.\end{aligned} \tag{94}$$

These eigenvalue problems convince us that W is just the operator we need for completing the system of commuting operators as $\{H, P^1, P^2, P^3, W\}$ for defining properly the **p**-reps. of RQM.

As the comps. of a spin operator are integral operators, we understand that W remains an operator of this type even in the case of common polarization when $\mathbf{n}$ and ξ are independent of $\mathbf{p}$, and consequently we may write $W = \mathbf{S}\cdot\mathbf{n}$. The well-known example is the momentum-spin basis [7], where $\mathbf{n} = \mathbf{e}_3$, while $W = S_3$ defines the basis spinors

$$\xi_{\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_{-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix} \tag{95}$$

used in various applications [7,8,11]. Therefore, we may say that there are no situations in which the polarization might be properly defined by a differential polarization operator as, for example, the helicity one, W_0. We shall later discuss the differences between these two operators when we study the polarization and spin operators in the momentum-helicity basis.

4.3 Associated coordinate operator

Looking for a mass-center position vector, Pryce assumed that this has the form $\mathbf{X} = \underline{\mathbf{x}} + \delta\mathbf{X}$ focusing on the correction $\delta\mathbf{X}$ which, according to the identity (14), must satisfy $\delta\mathbf{X} \wedge \mathbf{P} + \mathbf{S} = \mathbf{s}$. Analyzing various hypotheses, Pryce concluded that $\delta\mathbf{X}$ is an integral operator whose comps. have kernels given by the Fourier transforms [12]

$$\delta\tilde{X}^i(\mathbf{p}) = \frac{i\gamma^i}{2E(p)} + \frac{\varepsilon_{ijk}p^j s_k}{E(p)(E(p)+m)} - \frac{ip^i\gamma^j p^j}{2E(p)^2(E(p)+m)}\,, \tag{96}$$

which satisfies the desired identity $\delta\tilde{\mathbf{X}}(\mathbf{p}) \wedge \mathbf{p} + \tilde{\mathbf{S}}(\mathbf{p}) = \mathbf{s}$. These are self-adjoint operators commuting with $\mathbf{P}$ such that

$$[X^i, P^j] = [\underline{x}^i, P^j] = i\delta_{ij} 1 \in \rho_D\,. \tag{97}$$

Other properties including commutation relations have to be studied after quantization in order to avoid tedious calculations in **p**-rep.

Then, by using suitable codes on a computer, it is not difficult to verify that these operators can be put in the form

$$\delta\tilde{X}^i(\mathbf{p}) = \delta x^i(\mathbf{p})\tilde{\Pi}_+(\mathbf{p}) + \delta x^i(-\mathbf{p})\tilde{\Pi}_-(\mathbf{p})\,, \tag{98}$$

where

$$\delta x^i(\mathbf{p}) = -i\frac{1}{N(p)}\partial_{p^i}\left(N(p)l_\mathbf{p}\right)l_\mathbf{p}^{-1}\,. \tag{99}$$

Furthermore, observing that these operators derive only the factor $N(p)l_\mathbf{p}$ of the mode spinors (51) and (52), we may write their action as

$$\begin{aligned}\left(\delta X^i U_{\mathbf{p},\xi_\sigma}\right)(t,\mathbf{x}) &= \delta\tilde{X}^i(\mathbf{p}) U_{\mathbf{p},\xi_\sigma}(t,\mathbf{x}) = -i\partial_{p^i} U_{\mathbf{p},\xi_\sigma}(t,\mathbf{x})\\ &\quad - x^i U_{\mathbf{p},\xi_\sigma}(t,\mathbf{x}) + \frac{tp^i}{E(p)} U_{\mathbf{p},\xi_\sigma}(t,\mathbf{x})\\ &\quad + \sum_{\sigma'} U_{\mathbf{p},\xi_{\sigma'}}(t,\mathbf{x})\Omega_{i\,\sigma'\sigma}(\mathbf{p})\,,\end{aligned} \tag{100}$$

$$\begin{aligned}\left(\delta X^i V_{\mathbf{p},\eta_\sigma}\right)(t,\mathbf{x}) &= \delta\tilde{X}^i(-\mathbf{p}) V_{\mathbf{p},\eta_\sigma}(t,\mathbf{x}) = i\partial_{p^i} V_{\mathbf{p},\eta_\sigma}(t,\mathbf{x})\\ &\quad - x^i V_{\mathbf{p},\eta_\sigma}(t,\mathbf{x}) + \frac{tp^i}{E(p)} V_{\mathbf{p},\eta_\sigma}(t,\mathbf{x})\\ &\quad - \sum_{\sigma'} V_{\mathbf{p},\eta_{\sigma'}}(t,\mathbf{x})\Omega^*_{i\,\sigma'\sigma}(\mathbf{p})\,.\end{aligned} \tag{101}$$

Here, we use the artifice

$$\partial_{p^i}\xi_\sigma(\mathbf{p}) = \sum_{\sigma'}\xi_{\sigma'}(\mathbf{p})\Omega_{i\,\sigma',\sigma}(\mathbf{p})\,, \tag{102}$$

and similarly for the spinors $\eta_\sigma(\mathbf{p})$, denoting

$$\Omega_{i\,\sigma\sigma'}(\mathbf{p}) = \xi_\sigma^+(\mathbf{p})\left[\partial_{p^i}\xi_{\sigma'}(\mathbf{p})\right] = \left\{\eta_\sigma^+(\mathbf{p})\left[\partial_{p^i}\eta_{\sigma'}(\mathbf{p})\right]\right\}^*\,, \tag{103}$$

and observing that $\Omega_{i\,\sigma\sigma'}(\mathbf{p}) = -\Omega^*_{i\,\sigma'\sigma}(\mathbf{p})$, which means that the matrices $i\Omega_i$ are Hermitian.

Hereby we understand that the Pryce coordinate operator depends linearly on time as $\mathbf{X}(t) = \underline{\mathbf{x}} + \delta\mathbf{X}(t) = \mathbf{X} + \mathbf{V}t$. From Eqs. (100) and (101), and applying the Green theorem in the integral over momenta, we find the actions of these operators,

$$(X^i\psi_\xi)(x) = \int d^3p \sum_\sigma \Big[U_{\mathbf{p},\xi_\sigma}(x) i\tilde{\partial}_i \alpha_\sigma(\mathbf{p}) - V_{\mathbf{p},\xi_\sigma}(x) i\tilde{\partial}_i \beta^*_{\sigma'}(\mathbf{p})\Big], \tag{104}$$

$$(V^i\psi_\xi)(x) = \int d^3p \frac{p^i}{E(p)} \sum_\sigma \big[U_{\mathbf{p},\xi_\sigma}(x)\alpha_\sigma(\mathbf{p}) + V_{\mathbf{p},\eta_\sigma}(x)\beta^*_\sigma(\mathbf{p})\big], \tag{105}$$

written in terms of the "covariant" derivatives

$$\tilde{\partial}_i \alpha_\sigma(\mathbf{p}) = \partial_{p^i}\alpha_\sigma(\mathbf{p}) + \sum_{\sigma'} \Omega_{i\,\sigma\sigma'}(\mathbf{p})\alpha_{\sigma'}(\mathbf{p}), \tag{106}$$

where the matrices $\Omega_i(\mathbf{p})$ play the role of connection. These derivatives commute among themselves, $[\tilde{\partial}_i, \tilde{\partial}_j] = 0$, ensuring that $[X^i, X^j] = 0$.

The operator $\mathbf{X}$ is the Pryce coordinate operator at $t = 0$, while $\mathbf{V}$ is a conserved velocity. Bearing in mind that in RQM the antiparticle terms cannot be properly interpreted, we may ask whether the Pryce assumption that these kinetic quantities describe the inertial motion of the mass center is correct. We shall find the answer after performing the quantization.

5 One-particle operators of quantum theory

The principal benefit of our approach based on spectral reps. is the relation between the operator actions in **x** and **p**-reps., allowing us at any time to derive the expectation values of operators defined in **p**-rep. We thus have the opportunity to apply the Bogolyubov method for quantizing the Pryce operators and deriving the isometry generators of the massive Dirac fermions of arbitrary polarization.

5.1 Quantization and diagonal operators

Adopting here the Bogolyubov method of quantization [13], we first replace the functions in **p**-rep. with field operators, $(\alpha, \alpha^*) \to (\mathfrak{a}, \mathfrak{a}^\dagger)$ and $(\beta, \beta^*) \to (\mathfrak{b}, \mathfrak{b}^\dagger)$, satisfying canonical anti-commutation relations among which the non-vanishing ones are

$$\Big\{\mathfrak{a}_\sigma(\mathbf{p}), \mathfrak{a}^\dagger_{\sigma'}(\mathbf{p}')\Big\} = \Big\{\mathfrak{b}_\sigma(\mathbf{p}), \mathfrak{b}^\dagger_{\sigma'}(\mathbf{p}')\Big\} = \delta_{\sigma\sigma'}\delta^3(\mathbf{p} - \mathbf{p}'). \tag{107}$$

The Dirac free field ψ (written hereafter without the index ξ) thus becomes a field operator, while the expectation value of any time-dependent operator $A(t)$ becomes the one-particle operator,

$$A(t) \to \mathsf{A} = :\langle\psi, A(t)\psi\rangle_D :|_{t=0}, \tag{108}$$

calculated respecting the normal ordering of the operator products [7] at the initial time $t = 0$ when we assume that the quantization is performed.

We thus obtain a basis of operator algebra formed by field and one-particle operators which have the obvious properties

$$\big[\mathsf{A}, \psi(x)\big] = -(A\psi)(x), \quad \big[\mathsf{A}, \mathsf{B}\big] =: \langle\psi, [A, B]\psi\rangle_D :, \tag{109}$$

preserving the structures of Lie algebras. Note that the quantization does not take over other algebraic properties from RQM, as the product of two one-particle operators is generally no longer an operator of this type.

The quantization reveals the physical meaning of the quantum observables of RQM, transforming them into the one-particle operators of QFT. The simplest example is the identity operator $1 \in \rho_D$ of RQM, i.e., the generator of the gauge group $U(1)_{em}$, becoming through quantization the conserved charge operator,

$$\mathsf{Q} = :\langle\psi, \psi\rangle_D := \mathsf{Q}_+ + \mathsf{Q}_- = \int d^3p \sum_\sigma \Big[\mathfrak{a}^\dagger_\sigma(\mathbf{p})\mathfrak{a}_\sigma(\mathbf{p}) - \mathfrak{b}^\dagger_\sigma(\mathbf{p})\mathfrak{b}_\sigma(\mathbf{p})\Big], \tag{110}$$

where the particle and antiparticle charge operators are given just by the projection operators (A.6) and (A.7) as

$$\mathsf{Q}_\pm =: \langle\psi, \Pi_\pm\psi\rangle_D : . \tag{111}$$

Then the operator of number of particles,

$$\mathsf{N} = \mathsf{Q}_+ - \mathsf{Q}_- =: \langle\psi, (\Pi_+ - \Pi_-)\psi\rangle_D :, \tag{112}$$

is related to the operator whose kernel has the Fourier transform $E(p)^{-1}\tilde{H}_D(\mathbf{p})$.

In the basis of mode spinors in which the commuting operators $\{H, P^1, P^2, P^3, W\}$ are diagonal, we derive the corresponding one-particle operators,

$$\mathsf{H} = :\langle\psi, H\psi\rangle_D : = \int d^3p\, E(p) \sum_\sigma \Big[\mathfrak{a}^\dagger_\sigma(\mathbf{p})\mathfrak{a}_\sigma(\mathbf{p}) + \mathfrak{b}^\dagger_\sigma(\mathbf{p})\mathfrak{b}_\sigma(\mathbf{p})\Big], \tag{113}$$

$$\mathsf{P}^i = :\langle\psi, P^i\psi\rangle_D : = \int d^3p\, p^i \sum_\sigma \Big[\mathfrak{a}^\dagger_\sigma(\mathbf{p})\mathfrak{a}_\sigma(\mathbf{p}) + \mathfrak{b}^\dagger_\sigma(\mathbf{p})\mathfrak{b}_\sigma(\mathbf{p})\Big], \tag{114}$$

$$\mathsf{W} = :\langle\psi, W\psi\rangle_D := = \frac{1}{2}\int d^3p \sum_\sigma \sigma \Big[\mathfrak{a}^\dagger_\sigma(\mathbf{p})\mathfrak{a}_\sigma(\mathbf{p}) + \mathfrak{b}^\dagger_\sigma(\mathbf{p})\mathfrak{b}_\sigma(\mathbf{p})\Big], \tag{115}$$

which commute among themselves and with Q. We thus obtain the complete system $\{\mathsf{H}, \mathsf{P}^1, \mathsf{P}^2, \mathsf{P}^3, \mathsf{W}, \mathsf{Q}\}$ determining the bases of the Fock state space.

The energy operator H generates the unitary operators of time translations, giving the operators at any time,

$$\mathsf{U}(t)=e^{i\mathsf{H}t}:\mathsf{A}\ \to\ \mathsf{A}(t)=\mathsf{U}(t)\mathsf{A}\mathsf{U}^{\dagger}(t)\,, \tag{116}$$

apart from the *conserved* ones which commute with H.

5.2 Spin operator and $SL(2,\mathbb{C})$ generators

For applying the same method to Pryce's spin operator, we look for expectation values of its comps. S_i at the level of RQM. These can be found by deriving with respect to Cayley–Klein parameters θ^i the expectation values (79), where we substitute $D(\hat{r})\to D[\hat{r}(\theta)]$ with $\hat{r}(\theta)$ given by Eq. (2). As the quantization changes the wrong sign of the anti-particle term, we obtain the comps. of the spin operator,

$$\mathsf{S}_i=:\langle\psi,S_i\psi\rangle_D:=\frac{1}{2}\int d^3p\sum_{\sigma,\sigma'}\Sigma_{i\,\sigma\sigma'}(\mathbf{p})\Big[\mathfrak{a}^{\dagger}_{\sigma}(\mathbf{p})\mathfrak{a}_{\sigma'}(\mathbf{p})+\mathfrak{b}^{\dagger}_{\sigma}(\mathbf{p})\mathfrak{b}_{\sigma'}(\mathbf{p})\Big]\,, \tag{117}$$

where we denote by

$$\Sigma_{i\,\sigma\sigma'}(\mathbf{p})=\xi^{+}_{\sigma}(\mathbf{p})\sigma_i\,\xi_{\sigma'}(\mathbf{p}) \tag{118}$$

the matrix elements of Pauli's operators in the ξ basis in which the quantum field ψ was defined. The operators S_i are self-adjoint and form the canonical basis of an operator valued rep. of the $su(2)\sim so(3)$ algebra.

Let us now see how the spin operator defined above is involved in the structure of the $SL(2,\mathbb{C})$ generators. We start with the expectation values,

$$\langle\psi,X\psi\rangle_D=\langle\alpha,\tilde{X}\alpha\rangle-\langle\beta,\tilde{X}\beta\rangle\,, \tag{119}$$

derived from Eq. (59) for any pair of related generators $X\in\mathrm{Lie}(T)$ and $\tilde{X}\in\mathrm{Lie}(\tilde{T})$ corresponding to the same group parameter. Then, applying the quantization, we change the wrong relative sign $(-)$ in Eq. (119) after restoring the normal ordering of operator products, obtaining correct forms of one-particle operators. The $SL(2,\mathbb{C})$ generators may be derived by using our parameterizations (2) and (3).

For deriving the rotation generators, we take $\hat{\lambda}=\hat{r}(\theta)$, observing that the transformed momentum (38) can now be expanded as $p'^i=p^i+\varepsilon_{ijk}p^j\theta^k+\cdots$, according to the general rule (4). Introducing these quantities in Eq. (55) and deriving the transf. (53) with respect to the Cayley–Klein parameters θ^i in $\theta^i=0$, it turns out

$$\mathsf{J}_i=:\langle\psi,J_i\psi\rangle_D:=\mathsf{L}_i+\mathsf{S}_i\,, \tag{120}$$

where the comps. S_i of the spin operator are defined by Eq. (117). The associated orbital angular momentum operator has the comps.

$$\mathsf{L}_i=-i\int d^3p\,\varepsilon_{ijk}p^j\sum_{\sigma}\Big[\mathfrak{a}^{\dagger}_{\sigma}(\mathbf{p})\tilde{\partial}_k\mathfrak{a}_{\sigma}(\mathbf{p})+\mathfrak{b}^{\dagger}_{\sigma}(\mathbf{p})\tilde{\partial}_k\mathfrak{b}_{\sigma}(\mathbf{p})\Big]\,, \tag{121}$$

where the derivatives $\tilde{\partial}_i$ are defined by Eq. (106).

The commutation relations of these operators can be easily derived using Eqs. (48) and (49), identities of the form

$$\tilde{\partial}_i\mathfrak{a}_{\sigma}(\mathbf{p})=\sum_{\sigma'}\xi^{+}_{\sigma}(\mathbf{p})\partial_{p^i}\left[\xi_{\sigma'}(\mathbf{p})\mathfrak{a}_{\sigma'}(\mathbf{p})\right]\,, \tag{122}$$

and taking into account that $[\tilde{\partial}_i,\Sigma_i]=0$. We thus find that the operators L_i and S_i form the bases of two *independent* $su(2)\sim so(3)$ algebras commuting each other, $\left[\mathsf{L}_i,\mathsf{S}_j\right]=0$. Moreover, these operators are *conserved* separately, each one commuting with the Hamiltonian operator,

$$\left[\mathsf{H},\mathsf{L}_i\right]=0\,,\qquad\left[\mathsf{H},\mathsf{S}_i\right]=0\,. \tag{123}$$

Therefore, we may conclude that the Pryce spin operator of RQM gives just the conserved one-particles spin operator we need in QFT.

The generators of the Lorentz boosts can be found by choosing $\hat{\lambda}=\hat{l}(\tau)$ as in Eq. (3), observing that now $p'^i=p^i+\tau^iE(p)+\cdots$ and deriving Eq. (53) with respect to τ^i in $\tau=0$. After a few manipulations, we derive the operators at the initial time $t=0$,

$$\begin{aligned}\mathsf{K}_i&=:\langle\psi,K_i\psi\rangle_D:\\&=\int d^3p\sum_{\sigma,\sigma'}k_{i\,\sigma\sigma'}(\mathbf{p})\Big[\mathfrak{a}^{\dagger}_{\sigma}(\mathbf{p})\mathfrak{a}_{\sigma'}(\mathbf{p})+\mathfrak{b}^{\dagger}_{\sigma}(\mathbf{p})\mathfrak{b}_{\sigma'}(\mathbf{p})\Big]\\&\quad+i\int d^3p\,E(p)\sum_{\sigma}\Big[\mathfrak{a}^{\dagger}_{\sigma}(\mathbf{p})\tilde{\partial}_i\mathfrak{a}_{\sigma}(\mathbf{p})+\mathfrak{b}^{\dagger}_{\sigma}(\mathbf{p})\tilde{\partial}_i\mathfrak{b}_{\sigma}(\mathbf{p})\Big]\,,\end{aligned} \tag{124}$$

which depend on the matrices

$$k_i(\mathbf{p})=\frac{1}{2(E(p)+m)}\varepsilon_{ijk}p^j\,\Sigma_k(\mathbf{p})\,. \tag{125}$$

The operators (124) are self-adjoint but they are not conserved, satisfying the commutation relations of the $sl(2,\mathbb{C})$ algebra,

$$\left[\mathsf{H},\mathsf{K}_i\right]=-i\mathsf{P}^i\,,\quad\left[\mathsf{P}^i,\mathsf{K}_j\right]=-i\delta^i_j\mathsf{H}\,, \tag{126}$$

and evolving as

$$\mathsf{K}_i(t)=\mathsf{U}(t)\mathsf{K}_i\mathsf{U}^{\dagger}(t)=\mathsf{K}_i+\mathsf{P}^i\,t\,. \tag{127}$$

On the other hand, we must specify that the operators $\mathsf{K}_i(t)$ cannot be split as the total angular momentum. They satisfy the canonical relations

$$\left[\mathsf{J}_i,\mathsf{K}_j(t)\right]=i\varepsilon_{ijk}\mathsf{K}_k(t)\,, \tag{128}$$

$$\left[\mathsf{K}_i(t),\mathsf{K}_j(t)\right]=-i\varepsilon_{ijk}\mathsf{J}_k\,, \tag{129}$$

but without giving relevant commutators with L_i or S_i. This means that the splitting (120) cannot be extended to the entire $sl(2,\mathbb{C})$ algebra.

We thus derived the self-adjoint basis generators of a family of unitary reps. of the group $\tilde{\mathscr{P}}_+^\uparrow$, with values in a set of one-particle operators which are determined by the bases of Pauli spinors ξ we chose for describing polarization.

5.3 Coordinate operators

For performing the quantization of Pryce's coordinate operator, we start with the expectation values derived, according to Eqs. (104) and (105), as

$$\langle\psi, X^i\psi\rangle_D = \langle\alpha,\tilde{\partial}_i\alpha\rangle + \langle\beta,\tilde{\partial}_i\beta\rangle \tag{130}$$

$$\langle\psi, V^i\psi\rangle_D = \int d^3p\,\frac{p^i}{E(p)}\left[\alpha^+(\mathbf{p})\alpha(\mathbf{p}) + \beta^+(\mathbf{p})\beta(\mathbf{p})\right]. \tag{131}$$

As usual, the quantization changes the sign of antiparticle terms such that we obtain the one-particle operators

$$\mathsf{X}^i = :\langle\psi, X^i\psi\rangle_D := \mathsf{X}^i_+ + \mathsf{X}^i_- = i\int d^3p\sum_\sigma\left[\mathfrak{a}^\dagger_\sigma(\mathbf{p})\tilde{\partial}_i\mathfrak{a}_\sigma(\mathbf{p}) - \mathfrak{b}^\dagger_\sigma(\mathbf{p})\tilde{\partial}_i\mathfrak{b}_\sigma(\mathbf{p})\right], \tag{132}$$

$$\mathsf{V}^i = :\langle\psi, V^i\psi\rangle_D := \mathsf{V}^i_+ + \mathsf{V}^i_- = \int d^3p\,\frac{p^i}{E(p)}\sum_\sigma\left[\mathfrak{a}^\dagger_\sigma(\mathbf{p})\mathfrak{a}_\sigma(\mathbf{p}) - \mathfrak{b}^\dagger_\sigma(\mathbf{p})\mathfrak{b}_\sigma(\mathbf{p})\right]. \tag{133}$$

We thus have the surprise of seeing that the would-be mass-center operator proposed by Pryce becomes under quantization the charge center one or, in other words, the *dipole* operator at the time $t=0$. Accordingly, we understand that V^i are the comps. of the classical current vector operator.

This interpretation is confirmed by the commutation relations we briefly inspect in what follows. We start with

$$[\mathsf{H},\mathsf{X}^i] = -i\mathsf{V}^i\,,\quad [\mathsf{H},\mathsf{V}^i]=0\,, \tag{134}$$

showing that the comps. V^i are conserved, while the dipole ones evolve as

$$\mathsf{X}^i(t) = \mathsf{U}(t)\mathsf{X}^i\mathsf{U}^\dagger(t) = \mathsf{X}^i + \mathsf{V}^i t\,. \tag{135}$$

Moreover, we can verify that $\mathsf{X}^i(t)$ are comps. of a $SO(3)$ vector operator,

$$\left[\mathsf{L}_i,\mathsf{X}^j(t)\right] = i\varepsilon_{ijk}\mathsf{X}^k(t)\,,\quad \left[\mathsf{S}_i,\mathsf{X}^j(t)\right]=0\,, \tag{136}$$

which satisfy the canonical coordinate-momentum commutation relations,

$$\left[\mathsf{X}^i(t),\mathsf{X}^j(t)\right]=0\,,\quad \left[\mathsf{X}^i(t),\mathsf{P}^j\right] = i\delta_{ij}\mathsf{Q}\,, \tag{137}$$

in accordance with Eq. (97) and our interpretation, as through quantization, $1\in\rho_D$ becomes the charge operator Q.

However, we did not say which may be the real mass-center operator. Even though there are many definitions of mass center, we can easily construct only the version of position vectors weighted by rest masses (as in definition (a) of Ref. [12]), defining *ad hoc* its comps. and those of the mass-center velocity as

$$\mathsf{X}^i_{MC} = \mathsf{X}^i_+ - \mathsf{X}^i_-\,,\quad \mathsf{V}^i_{MC} = \mathsf{V}^i_+ - \mathsf{V}^i_-\,, \tag{138}$$

such that $[\mathsf{H},\mathsf{X}^i_{MC}] = -i\mathsf{V}^i_{MC}$ gives the inertial motion

$$\mathsf{X}^i_{MC}(t) = \mathsf{U}(t)\mathsf{X}^i_{MC}\mathsf{U}^\dagger(t) = \mathsf{X}^i_{MC} + \mathsf{V}^i_{MC}t\,. \tag{139}$$

These operators satisfy similar commutation relations as the dipole one, apart from the last of Eq. (137), which now reads

$$\left[\mathsf{X}^i_{MC}(t),\mathsf{P}^j\right] = i\delta_{ij}\mathsf{N}\,, \tag{140}$$

indicating that they describe the kinematics of the center of rest masses which are the same for particles and antiparticles of any momenta. In view of the above results, we are skeptical that other coordinate operators simultaneously satisfying similar commutation relations could be derived. Nevertheless, we do not exclude the possibility of finding new mass-center operators relaxing the canonical conditions as, for example, in the case of spin-induced non-commutativity [14].

The problem which remains open is how the corresponding mass-center operator of RQM may be defined in $\mathbf{x}$ and $\mathbf{p}$ reps. Our preliminary calculations indicate that there exists a spectral rep. solving this problem, but the calculations are quite complicated, exceeding the scope of this paper. We hope to discuss this problem and other versions of mass center in a further investigation.

6 Example: momentum-helicity basis

The only peculiar polarization used so far is the helicity giving rise to the momentum-helicity basis in which the spinors $\xi_\sigma(\mathbf{p})$ and $\eta_\sigma(\mathbf{p}) = i\sigma_2\xi^*_\sigma(\mathbf{p})$ satisfy the related eigenvalue problems

$$\hat{s}_i n^i_p\xi_\sigma(\mathbf{p}) = \sigma\,\xi_\sigma(\mathbf{p}) \;\to\; \hat{s}_i n^i_p\eta_\sigma(\mathbf{p}) = -\sigma\,\eta_\sigma(\mathbf{p}), \tag{141}$$

where $\mathbf{n}_p = \frac{\mathbf{p}}{p}$ is the unit vector of $\mathbf{p}$. One obtains these spinors, transforming the spin basis (95) as

$$\xi_\sigma(\mathbf{p}) = \hat{r}_h(\mathbf{p})\xi_\sigma \;\to\; \eta_\sigma(\mathbf{p}) = \hat{r}_h(\mathbf{p})\eta_\sigma\,, \tag{142}$$

with the help of the $SU(2)$ rotation

$$\hat{r}_h(\mathbf{p}) = \sqrt{\frac{p+p^3}{2p}}\left[1_{2\times2} - i\,\frac{p^1\sigma_2 - p^2\sigma_1}{p+p^3}\right]. \tag{143}$$

The associated $SO(3)$ rotation, $R(\hat{r}_h(\mathbf{p}))$, transforms the polarization direction $\mathbf{e}_3$ of the spin basis into the helicity one, $\mathbf{n}_p$. In our framework, we find that the corresponding transf. of the mode spinors is performed by the integral operator $\mathfrak{R}_h$ whose kernel has the Fourier transform

$$\tilde{\mathfrak{R}}_h(\mathbf{p}) = l_{\mathbf{p}} r_h(\mathbf{p}) l_{\mathbf{p}}^{-1} \tilde{\Pi}_+(\mathbf{p}) + l_{\mathbf{p}}^{-1} r_h(-\mathbf{p}) l_{\mathbf{p}} \tilde{\Pi}_-(\mathbf{p}), \tag{144}$$

where $r_h(\mathbf{p}) = \mathrm{diag}(\hat{r}_h(\mathbf{p}), \hat{r}_h(\mathbf{p})) \in \rho_D$. This transf. is complicated but can be controlled by using algebraic codes on a computer.

6.1 Principal operators

In this basis, the momentum and energy operators which do not depend on polarization lead to the one-particle operators (113) and (114), respectively. In contrast, the polarization operator (92) with $\mathbf{n}(\mathbf{p}) = \mathbf{n}_p$ can be put in the specific form

$$\tilde{W}(\mathbf{p}) = s_i n_p^i \left[\tilde{\Pi}_+(\mathbf{p}) - \tilde{\Pi}_-(\mathbf{p})\right] = s_i n_p^i \frac{\tilde{H}_D(\mathbf{p})}{E(p)}, \tag{145}$$

as $\mathbf{n}(-\mathbf{p}) = -\mathbf{n}_p$, and $s_i n_p^i$ commutes with $l_{\mathbf{p}}$. Then the mode spinors constructed using helicity spinors satisfy the eigenvalue problems (93) and (94) which guarantee the convenient quantization (115).

On the other hand, here we may use the comps. of the Pauli–Lubanski operator interpreted as a covariant four-vector spin operator. Its 0th comp. is the helicity operator

$$W_0 = J_i P^i = s^i P^i \Rightarrow \tilde{W}_0(\mathbf{p}) = \tilde{S}_i(\mathbf{p}) p^i = s_i p^i, \tag{146}$$

whose action on the mode spinors,

$$\begin{aligned}(W_0 U_{\mathbf{p},\xi_\sigma(\mathbf{p})})(x) &= \tilde{W}_0(\mathbf{p}) U_{\mathbf{p},\xi_\sigma(\mathbf{p})}(x) \\ &= \sigma p U_{\mathbf{p},\xi_\sigma(\mathbf{p})}(x) \qquad (147)\\ (W_0 V_{\mathbf{p},\eta_\sigma(\mathbf{p})})(x) &= \tilde{W}_0(-\mathbf{p}) V_{\mathbf{p},\eta_\sigma(\mathbf{p})}(x) \\ &= \sigma p V_{\mathbf{p},\eta_\sigma(\mathbf{p})}(x), \qquad (148)\end{aligned}$$

calculated as in the previous case, is different from that of W, as the eigenvalue of Eq. (148) is σp instead of $-\sigma p$. This difference comes from the term $E(p)^{-1}\tilde{H}_D(\mathbf{p})$ of Eq. (145), which ensures suitable eigenvalues of the operator W. However, the inverse antiparticle eigenvalues of the helicity operator are not a major impediment, such that in QFT, we may use either the operator W defined by Eq. (115) or the helicity one,

$$\mathsf{W}_0 = \frac{1}{2}\int d^3p\, p \sum_\sigma \sigma \left[\mathfrak{a}_\sigma^\dagger(\mathbf{p})\mathfrak{a}_\sigma(\mathbf{p}) - \mathfrak{b}_\sigma^\dagger(\mathbf{p})\mathfrak{b}_\sigma(\mathbf{p})\right], \tag{149}$$

bearing in mind its specific action.

For writing down the spin comps. (117), we derive the matrices (118) in this basis,

$$\begin{aligned}\Sigma_1(\mathbf{p}) &= \frac{p^1}{p}\sigma_3 - p^1\frac{p^1\sigma_1 + p^2\sigma_2}{p(p+p^3)} + \sigma_1, \\ \Sigma_2(\mathbf{p}) &= \frac{p^2}{p}\sigma_3 - p^2\frac{p^1\sigma_1 + p^2\sigma_2}{p(p+p^3)} + \sigma_2, \\ \Sigma_3(\mathbf{p}) &= \frac{p^3}{p}\sigma_3 - \frac{p^1\sigma_1 + p^2\sigma_2}{p}, \qquad (150)\end{aligned}$$

which satisfy $p^i\Sigma_i(\mathbf{p}) = p\sigma_3$. The form of the "covariant" derivatives $\tilde{\partial}_i = \partial_{p^i} 1_{2\times2} + \Omega_i(\mathbf{p})$ is determined by the matrices (103) that read

$$\begin{aligned}\Omega_1(\mathbf{p}) &= \frac{-i}{2p^2(p+p^3)}\Big[p^1p^2\sigma_1 + pp^2\sigma_3 \\ &\quad + (pp^3 + p^{2^2} + p^{3^2})\sigma_2\Big], \\ \Omega_2(\mathbf{p}) &= \frac{i}{2p^2(p+p^3)}\Big[p^1p^2\sigma_2 + pp^1\sigma_3 \\ &\quad + (pp^3 + p^{1^2} + p^{3^2})\sigma_1\Big], \\ \Omega_3(\mathbf{p}) &= \frac{i}{2p^2}\left(p^1\sigma_2 - p^2\sigma_1\right) \qquad (151)\end{aligned}$$

and satisfy $p^i\Omega_i(\mathbf{p}) = 0$. We thus obtain apparently complicated matrices Σ_i and Ω_i but whose algebra is the same as in the momentum-spin basis where $\Omega_i = 0$ and $\Sigma_i = \sigma_i$. The identities (122) help us to show that Σ_j and $\tilde{\partial}_i$ satisfy the same commutation relations as σ_i and ∂_{p^i}. Using these matrices, we may derive for the first time all the isometry generators and kinetic operators acting on the Fock state space of QFT in the momentum-helicity basis.

6.2 One-particle relativistic quantum mechanics

In applications, we may turn back to RQM but considered now as the one-particle restriction of QFT. For example, in the normalized one-particle state

$$|\alpha\rangle = \int d^3p \sum_\sigma \alpha_\sigma(\mathbf{p})\mathfrak{a}_\sigma^\dagger(\mathbf{p})|0\rangle, \quad \langle\alpha|\alpha\rangle = 1, \tag{152}$$

defined by the wave functions $\alpha_\sigma(\mathbf{p})$ that form the normalized Pauli spinor $\alpha \in \mathscr{F}_\alpha$ as in Eq. (29), we may calculate the expectation value of any generator X as

$$\langle\alpha|\mathsf{X}|\alpha\rangle = \langle\alpha, \tilde{X}\alpha\rangle, \tag{153}$$

where $\tilde{X} \in \mathrm{Lie}(\tilde{T})$ is the generator of RQM corresponding to X. The list of these generators can be written down, according to Eqs. (113–125), but omitting the explicit dependence on

p,

$$\tilde{P}^i = p^i\,, \qquad \tilde{H} = E\,, \qquad \tilde{W} = \frac{1}{2}\sigma_3\,, \quad \tilde{W}_0 = \frac{p}{2}\sigma_3\,,$$
$$\tilde{J}_i = \tilde{L}_i + \tilde{S}_i\,, \quad \tilde{S}_i = \frac{1}{2}\,\Sigma_i\,, \quad \tilde{L}_i = -i\varepsilon_{ijk}p^j\tilde{\partial}_k\,,$$
$$\tilde{K}_i = iE\tilde{\partial}_i + \frac{1}{2(E+m)}\,\varepsilon_{ijk}p^j\,\Sigma_k\,. \tag{154}$$

Similarly we find $\tilde{Q} = 1$ and the kinetic operators

$$\tilde{X}^i = \tilde{X}^i_{MC} = i\tilde{\partial}_i\,, \quad \tilde{V}^i = \tilde{V}^i_{MC} = \frac{p^i}{E}\,. \tag{155}$$

of the particle inertial motion.

More algebra may be performed resorting to algebraic codes on a computer. We thus obtain the space comps. of the Pauli–Lubanski operator,

$$\tilde{W}_i = E\tilde{J}_i + \varepsilon_{ijk}p^j\tilde{K}_k = m\tilde{S}_i + \frac{p^i}{E+m}\tilde{W}_0\,, \tag{156}$$

and verify that the second Casimir operator (16) in this rep. gives the expected invariant $\tilde{C}_2 = -\eta^{\mu\nu}\tilde{W}_\mu\tilde{W}_\nu = \frac{3}{4}m^2$. Note that in this rep., the first invariant (15) is implicit, as E is just the relativistic energy.

This example shows that the structure and properties of the spin, angular momenta and polarization operators defined here are close to the original non-relativistic Pauli's spin theory in **p**-rep. However, this happens only in the particle sector, as for antiparticles there is a discrepancy because of the operators $\tilde{Q}$, $\tilde{W}_0$, $\tilde{X}^i$, and $\tilde{V}^i$ which change their signs.

7 Concluding remarks

We have shown that the comps. of the spin operator proposed by Pryce in **p**-rep. are Fourier transforms of the kernels of integral operators generating a rep. of the little group $SU(2)$ carried by the space of Pauli's spinors determining the polarization. Therefore, these operators are conserved via Noether's theorem, allowing us to define the conserved polarization operator (92).

After quantization, the spin operator becomes the desired conserved one-particle operator of comps. (117), splitting naturally the total angular momentum into two conserved parts, i.e., this spin operator and the associated conserved angular momentum of comps. (121). Therefore, we must accept that this is the correct spin operator we need for defining and controlling the polarization in special relativistic QFT. The action of the corresponding polarization operator (115) defined here for the first time confirms this interpretation.

In contrast, the associated Pryce coordinate operator defined initially in **p**-rep. as a mass-center one becomes after quantization the dipole operator (132) evolving linearly in time thanks to the conserved classical current (133). Nevertheless, a mass-center position and velocity operators (138) can be written by hand but corresponding to another definition of mass center, different from that considered by Pryce [12]. Thus, we again verify that the correct physical meaning of the relativistic observables can be found only in QFT.

As an application, we derived the matrices (150) and (151) we need for writing down the isometry generators and kinetic operators in the momentum-helicity basis. Turning back to the RQM seen as a one-particle restriction of QFT, we obtained the operators (154) that are close to those of the original non-relativistic Pauli theory. We believe that this is a significant physical argument in favor of our approach.

Unfortunately, we do not have other examples of peculiar polarization, as the helicity is the only one used so far. We hope that our spin and polarization operators defined here will provide the opportunity to define new types of peculiar polarization that could be observed in further experiments.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical paper which does not use or produce data.]

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Appendix A: Boosts and Foldy–Wouthuysen transformations

The standard Lorentz boosts of ρ_D are transfs. of the form (3) with parameters $\tau^i = -\frac{p^i}{p}\tanh^{-1}\frac{p}{E(p)}$ that read [5]

$$l_{\mathbf{p}} = \frac{E(p) + m + \gamma^0\gamma^i p^i}{\sqrt{2m(E(p)+m)}} \in \rho_D\,, \tag{A.1}$$

giving rise to the boosts $L_{\mathbf{p}} = \Lambda(l_{\mathbf{p}}) \in L^\uparrow_+$ with the matrix elements [4]

$$(L_{\mathbf{p}})^{0}_{\cdot 0} = \frac{E(p)}{m}\,, \quad (L_{\mathbf{p}})^{0}_{\cdot i} = (L_{\mathbf{p}})^{i}_{\cdot 0} = \frac{p^i}{m}\,,$$
$$(L_{\mathbf{p}})^{i}_{\cdot j} = \delta_{ij} + \frac{p^i p^j}{m(E(p)+m)}\,. \tag{A.2}$$

The matrices (A.1) satisfy $l_{\mathbf{p}} = l_{\mathbf{p}}^{+}$ and $l_{\mathbf{p}}^{-1} = l_{-\mathbf{p}} = \gamma^0 l_{\mathbf{p}} \gamma^0$, and

$$l_{\mathbf{p}}^2 = \frac{E(p) + \gamma^0\gamma^i p^i}{m}\,, \qquad l_{-\mathbf{p}}^2 = \frac{E(p) - \gamma^0\gamma^i p^i}{m}\,, \tag{A.3}$$

giving rise to the following identities

$$\frac{1+\gamma^0}{2} l_{\mathbf{p}}^2 \frac{1+\gamma^0}{2} = \frac{E(p)}{m}\frac{1+\gamma^0}{2}\,, \tag{A.4}$$

$$\frac{1-\gamma^0}{2} l_{-\mathbf{p}}^2 \frac{1-\gamma^0}{2} = \frac{E(p)}{m}\frac{1-\gamma^0}{2}\,. \tag{A.5}$$

which help us to recover the integral operators Π_+ and Π_- defined by Pryce [12] whose kernels have the Fourier transforms

$$\tilde{\Pi}_+(\mathbf{p}) = \frac{m}{E(p)} l_{\mathbf{p}} \frac{1+\gamma^0}{2} l_{\mathbf{p}} = \frac{1}{2}\left(1 + \frac{\tilde{H}_D(\mathbf{p})}{E(p)}\right)\,, \tag{A.6}$$

$$\tilde{\Pi}_-(\mathbf{p}) = \frac{m}{E(p)} l_{\mathbf{p}}^{-1} \frac{1-\gamma^0}{2} l_{\mathbf{p}}^{-1} = \frac{1}{2}\left(1 - \frac{\tilde{H}_D(\mathbf{p})}{E(p)}\right)\,, \tag{A.7}$$

where $\tilde{H}_D(\mathbf{p})$ is given by Eq. (35). It is not difficult to verify that Π_+ and Π_- form a complete system of orthogonal projection operators satisfying $\Pi_+^2 = \Pi_+\,,\ \Pi_-^2 = \Pi_-\,,\ \Pi_+\Pi_- = \Pi_-\Pi_+ = 0$ and $\Pi_+ + \Pi_- = 1 \in \rho_D$. According to Eqs. (33) and (34), we find that these operators separate the mode spinors of positive and negative frequencies as $\Pi_+\mathscr{F}_D = \mathscr{F}_D^+$ and $\Pi_-\mathscr{F}_D = \mathscr{F}_D^-$ [12].

Looking for a unitary transf. able to bring the Hamiltonian $\tilde{H}_D(\mathbf{p})$ in diagonal form, Foldy and Wouthuysen found the unitary transf. [3]

$$U_{FW}(\mathbf{p}) = U_{FW}^{+}(-\mathbf{p}) = \frac{E(p) + m + \gamma^i p^i}{\sqrt{2E(p)(E(p)+m)}} \tag{A.8}$$

acting as

$$U_{FW}(\mathbf{p})\tilde{H}_D(\mathbf{p})U_{FW}(-\mathbf{p}) = \gamma^0 E(p)\,. \tag{A.9}$$

Assuming that in the frame where the Hamiltonian is diagonal the spin operator is the Pauli one, $\mathbf{s}$, and applying the inverse transf.,

$$\tilde{\mathbf{S}}(\mathbf{p}) = U_{FW}(-\mathbf{p})\mathbf{s}\,U_{FW}(\mathbf{p})\,, \tag{A.10}$$

it turns out just the Pryce spin operator whose comps. are given by Eq. (86). For this reason, $\tilde{\mathbf{S}}(\mathbf{p})$ is often called the Foldy–Wouthuysen spin operator. In fact, the spin operator is the same but defined in two different ways: either indirectly in association with Pryce's coordinate operator or through the transf. (A.10). Note that both these definitions are different from that we propose here for the same operator.

Appendix B: Induced representations

The induced reps. are a tool for constructing unitary reps. of a local-compact group in terms of unitary ones of a compact subgroup [9,10]. Given a local-compact group G, a subgroup H, and the function $\phi : G \to \mathscr{V}$, with values in a vector space $\mathscr{V}$, one says that the natural rep. $\pi(g)\phi(x) = \phi(g^{-1}x)$ is induced by the rep. τ of the group H if [5]

$$\phi(xh^{-1}) = \tau(h)\phi(x)\,, \quad \forall x \in G\,, \quad h \in H\,. \tag{B.1}$$

Bearing in mind that a Haar measure can be defined at any time on the cosset space G/H, it is convenient to consider the new function $\hat{\phi} = \phi \circ \chi : G/H \to \mathscr{V}$ defined with the help of an arbitrary function $\chi : G/H \to G$. If τ is an unitary rep., then the induced rep. is unitary, transforming the functions $\hat{\phi} \in \mathscr{L}^2(G/H, \mu, \mathscr{V})$ but preserving the scalar product of this Hilbert space [10]. An induced rep. is irreducible if the rep. τ is irreducible. When fermions must be studied, then instead of G and H, we consider their universal covering groups, $\tilde{G}$ and the little group $\tilde{H}$ [5].

In RQM, the wave functions of $\mathbf{p}$-rep. transform under translations by simple multiplications with phase factors such that we may restrict ourselves to the reps. of the groups $G = L_+^{\uparrow}$ or $\tilde{G} = SL(2,\mathbb{C})$. The wave functions are defined on orbits associated to representative momenta as in Sect. 3.2. Each orbit $\Omega_{\mathring{p}}$ is isomorphic with the cosset space $L_+^{\uparrow}/H$ where H is the stable group of the representative momentum $\mathring{p}$. Therefore, we can define the mapping $\chi : \Omega_{\mathring{p}} \to G$ such that $p = \chi(p)\mathring{p}$. Then the natural action of any $\Lambda \in L_+^{\uparrow}$ can be written as

$$\Lambda : \hat{\phi}(p) \to \phi\left(\Lambda^{-1}\chi(p)\right) = \phi\left(\chi(\Lambda^{-1}p)W(\Lambda, p)^{-1}\right)\,, \tag{B.2}$$

where $W(\Lambda, p) = \chi^{-1}(p)\Lambda\chi(\Lambda^{-1}p) \in H$ is a Wigner transf. of stable group, as $W(\Lambda, p)\mathring{p} = \mathring{p}$. Finally, we find the transformation rule

$$\Lambda : \hat{\phi}(p) \to \tau(W(\Lambda, p))\hat{\phi}(\Lambda^{-1}p) \tag{B.3}$$

resulted from condition (B.1).

For massive fermions discussed in Sect. 3.2, we chose $\chi(p) = L_{\mathbf{p}}$ defined by Eq. (A.2). In this case, the little group $\tilde{H} = SU(2)$ is compact, allowing finite-dimensional unitary and irreducible reps. We use here only the rep. of spin $\frac{1}{2}$ such that $\sqrt{E}\alpha$ and $\sqrt{E}\beta$ are functions of the Hilbert space $\mathscr{L}^2(\Omega_{\mathring{p}}, \mu, \mathscr{V}_P)$ with the invariant measure (36). These functions transform as in Eq. (B.3), such that the functions α and β transform according to the rule (53) deduced indirectly from Wigner's approach. Thus, for massive particles, the covariance is solved in terms of unitary reps. with a natural physical meaning.

However, the group $SO(3)$ is the only compact subgroup of the group $L_+^{\uparrow}$, such that for other orbits, there are difficulties. In the case of massless particles, the representative momentum $\mathring{p} = (1, 0, 0, 1)$ has the stable group $H = E(2)$ formed by $SO(2)$ rotations and two nilpotent translations [15] whose effect must be eliminated in order to keep only the unitary reps. of the subgroup $SO(2)$. This can be done resorting to some supplemental restrictions, for example, keeping only the left-handed components of the neutrino or setting the Coulomb gauge of the Maxwell field. For tachyons having the stable group $H = SO(2, 1)$, there are no solutions.

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