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Observational constraints on the fractal cosmology

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Abstract In this paper, we explore a fractal model of the universe proposed by Calcagni (J High Energy Phys 03:120, 2010) for a power-counting renormalizable field theory living in a fractal spacetime. Considering a timelike fractal profile, we derived field equations in fractal cosmology, in order to explore the structure formation and the expansion history in fractal universe. Numerical investigations based on matter power spectra diagrams report higher structure growth in fractal cosmology, being in contrast to local galaxy surveys. Additionally, according to the evolution of Hubble parameter diagrams, it can be understood that Hubble constant would decrease in fractal cosmology, which is also incompatible with low redshift estimations of H_0. So, concerning primary numerical studies, it seems that fractal cosmology is not capable to alleviate the tensions between local and global observational probes. Then, in pursuance of more accurate results, we constrain the fractal cosmology by observational data, including Planck cosmic microwave background (CMB), weak lensing, supernovae, baryon acoustic oscillations (BAO), and redshift-space distortions (RSD) data. The derived constraints on fractal dimension β indicate that there is no considerable deviation from standard model of cosmology.

1 Introduction

Recent observational data from type Ia supernovae (SNeIa) [1,2], the cosmic microwave background (CMB) anisotropies [3–5], large scale structures [6–8], and baryon acoustic oscillations (BAO) [9–11], confirm ΛCDM as the concordance model of cosmology. On the other hand, ΛCDM model which is based on the theory of general relativity (GR), encounters some observational discrepancies, principally σ_8 and H_0 tensions. To be more specific, there is a disagreement between low-redshift determinations and Planck CMB measurements of matter perturbation amplitude, σ_8 [5,12]. Moreover, direct measurements of Hubble constant are in significant tension with Planck observations [5,13–16]. So, inconsistencies between local and global data, motivate cosmologists to prob beyond ΛCDM model. Accordingly, it is suggested to consider some corrections on GR, describing as modified theory of gravity [17–20].

On the other hand, Calcagni proposed an effective quantum field theory which is power counting renormalizable and Lorentz invariant, living in a fractal universe [21,22]. The fractal nature firstly introduced by Mandelbrot in 1983 [23] suggests conditional cosmological principle in fractal universe, where the universe appears the same from every galaxy. Thereafter, in 1986, Linde [24] propounded a model of an eternally existing chaotic inflationary universe, explaining a fractal cosmology. Accordingly, there are several investigations on the theory of fractal cosmology in literatures. Rassem and Ahmed [25] in 1996 considered a nonhomogeneous cosmological model with a fractal distribution of matter which evolves to a homogeneous universe as time passes. The conditional cosmological principle in fractal cosmology is discussed in [26,27]. Calcagni [28] studies Multi-fractal geometry. Thermodynamics of the apparent horizon in a fractal universe is explored in [29]. In order to find more theoretical studies on fractal cosmology refer to e.g. [30–34]. Furthermore, it is interesting to explore fractal models with cosmological data as discussed in [35–40] (also see [41] for a review). Correspondingly, in the present work we are going to investigate the fractal universe in background and perturbation levels, as well as studying observational constraints on parameters of fractal cosmology.

The paper is organized as follows. Section 2 is dedicated to field equations in a fractal universe. In Sect. 3, we study the fractal model numerically, and further we constrain the model

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with current observational data in Sect. 4. We summarize our results in Sect. 5.

2 Field equations in a fractal universe

The total action in a fractal spacetime is given by [21]

$$S = \frac{1}{16\pi G}\int \mathrm{d}\varrho(x)\,\sqrt{-g}\big(R-2\Lambda-\omega\partial_\mu v\partial^\mu v\big)+S_m, \tag{1}$$

where ω is the fractal parameter, v is the fractional function, and $\mathrm{d}\varrho(x)$ is a Lebesgue–Stieltjes measure. It is possible to derive field equations from action (1) similar to scalar–tensor theories. Thus, in a fractal universe we obtain [21]

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}(R-2\Lambda)+g_{\mu\nu}\frac{\Box v}{v}-\frac{\nabla_\mu\nabla_\nu v}{v}$$
$$+\,\omega\Big(\frac{1}{2}g_{\mu\nu}\partial_\sigma v\partial^\sigma v-\partial_\mu v\partial_\nu v\Big)=8\pi G T_{\mu\nu}. \tag{2}$$

Furthermore, continuity equation in a fractal spacetime takes the form [21]

$$\nabla_\mu\big(vT^\mu_\nu\big)-\partial_\nu v\mathscr{L}_m=0. \tag{3}$$

It should be noted that for $v=1$, standard equations in GR will be recovered. Here, we focus on a timelike fractal, then v is only time dependent given by

$$v=H_0^\beta\Big(\frac{a}{a_0}\Big)^\beta, \tag{4}$$

in which $\beta=4(1-\alpha)$ is the fractal dimension, and the parameter α ranges as $0<\alpha\leq 1$.

We consider a fractal universe with the following flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric in the synchronous gauge

$$\mathrm{d}s^2=a^2(\tau)\Big(-\mathrm{d}\tau^2+\big(\delta_{ij}+h_{ij}\big)\mathrm{d}x^i\mathrm{d}x^j\Big), \tag{5}$$

in which

$$h_{ij}(\mathbf{x},\tau)=\int \mathrm{d}^3k\, e^{i\mathbf{k}.\mathbf{x}}\Big(\hat{k}_i\hat{k}_j h(\mathbf{k},\tau)+\Big(\hat{k}_i\hat{k}_j-\frac{1}{3}\delta_{ij}\Big)6\eta(\mathbf{k},\tau)\Big),$$

with scalar perturbations h and η, and $\mathbf{k}=k\hat{k}$ [42]. Then, considering the energy content of the universe as a perfect fluid with $T_{\mu\nu}=\big(\rho+p\big)u_\mu u_\nu+g_{\mu\nu}p$, field equations in background level take the form

$$H^2\Big(1+\beta-\frac{1}{6}\omega H_0^{2\beta}\beta^2\Big(\frac{a}{a_0}\Big)^{2\beta}\Big)=\frac{8\pi G}{3}\sum_i\bar{\rho}_i, \tag{6}$$

$$(\beta+2)\frac{H'}{a}+H^2\Big(3+2\beta+\beta^2+\frac{1}{2}\omega H_0^{2\beta}\beta^2\Big(\frac{a}{a_0}\Big)^{2\beta}\Big)$$
$$=-8\pi G\sum_i\bar{p}_i, \tag{7}$$

where a prime indicates a deviation with respect to the conformal time. It can be easily seen that, $\beta=0$ restores field equations in standard cosmology. According to Eq. (6), total density parameter can be find as

$$\Omega_{\rm tot}=1+\beta-\frac{1}{6}\omega H_0^{2\beta}\beta^2\Big(\frac{a}{a_0}\Big)^{2\beta}, \tag{8}$$

where we have considered a universe filled with radiation (R), baryons (B), dark matter (DM) and cosmological constant (Λ). Also field equations to linear order of perturbations can be written as

$$\frac{a'}{a}\Big(1+\frac{1}{2}\beta\Big)h'-2k^2\eta=8\pi Ga^2\sum_i\delta\rho_i, \tag{9}$$

$$k^2\eta'=4\pi Ga^2\sum_i\big(\bar{\rho}_i+\bar{p}_i\big)\theta_i, \tag{10}$$

$$\frac{1}{2}h''+3\eta''+\frac{a'}{a}\big(h'+6\eta'\big)\Big(1+\frac{1}{2}\beta\Big)-k^2\eta=0, \tag{11}$$

$$\frac{a'}{a}\Big(2+\beta\Big)h'+h''-2k^2\eta=-24\pi Ga^2\sum_i\delta p_i. \tag{12}$$

In addition, regarding Eq. (3), conservation equations of fractal cosmology for ith component of the universe in background and perturbation levels become

$$\bar{\rho}_i'+(3+\beta)\frac{a'}{a}\big(\bar{\rho}_i+\bar{p}_i\big)=0, \tag{13}$$

$$\delta_i'=-\frac{a'}{a}(3+\beta)\Big[\delta_i\big(c_{si}^2-w_i\big)$$
$$+\big(c_{si}^2-c_{ai}^2\big)\frac{a'}{a}(3+\beta)\big(1+w_i\big)\frac{\theta_i}{k^2}\Big]$$
$$-\big(1+w_i\big)\theta_i-\frac{1}{2}\big(1+w_i\big)h', \tag{14}$$

$$\theta_i'=\theta_i\Big[-\frac{a'}{a}(4+\beta)+\frac{a'}{a}(3+\beta)\big(1+w_i+c_{si}^2-c_{ai}^2\big)\Big]$$
$$+\frac{k^2c_{si}^2}{1+w_i}\delta_i. \tag{15}$$

Then, choosing $\beta = 0$ would recover equations in concordance ΛCDM model.

Now that we have described main equations in fractal cosmology, it is possible to derive observational constraints on fractal model, using a modified version of the CLASS[1] code [43], and also applying an MCMC[2] approach via the Monte Python code [44,45].

3 Numerical results

In this section, we modify the CLASS cosmological Boltzmann code according to the field equations of fractal model outlined in Sect. 2. Correspondingly, we take into account the Planck 2018 data [5] (i.e. $\Omega_{B,0}h^2 = 0.02242$, $\Omega_{DM,0}h^2 = 0.11933$, $H_0 = 67.66\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $A_s = 2.105 \times 10^{-9}$, and $\tau_{\rm reio} = 0.0561$) in our numerical investigation.

In Fig. 1 we depict the CMB temperature power spectra in fractal cosmology compared to ΛCDM model. Upper panels display the CMB power spectra for different values of β, where we have considered $\omega = 0$. In lower panels the CMB temperature anisotropy diagrams are illustrated for different values of ω, while regarding $\beta = 0.04$. Furthermore, for a better comparison we have shown the relative ratio with respect to $\omega = 0$ in the bottom-right panel, which indicates that the effect of ω is not significant in CMB power spectra.

Figure 2 demonstrates the matter power spectra of fractal model in comparison with standard cosmological model. In left panel we see matter power spectra for different values of β, where ω is considered to be zero. Right panel shows the power spectra for different values of ω, in which we have fixed β to be 0.04. Considering this figure, fractal cosmology predicts a higher growth of structure which is in conflict with local probes of large scale structures. In addition, this plot explains that the fractal parameter ω has no remarkable influence on structure formation.

Moreover, the evolution of Hubble parameter in fractal cosmology is depicted in Fig. 3. Accordingly, we can conclude that Hubble tension is aggravated in fractal model, and also the expansion history would not be affected by the fractal parameter ω.

4 Constraints from observational data

In this part, we employ the MCMC sampling package Monte Python to investigate observational constraints on fractal model. In this respect, we use the following set of cosmological parameters: $\{100\,\Omega_{B,0}h^2, \Omega_{DM,0}h^2, 100\,\theta_s, \ln(10^{10}A_s), n_s, \tau_{\rm reio}, \beta\}$, where $\Omega_{B,0}h^2$ and $\Omega_{DM,0}h^2$ stand for the baryon and cold dark matter densities respectively, θ_s represents the ratio of the sound horizon to the angular diameter distance at decoupling, A_s indicates the amplitude of the primordial scalar perturbation spectrum, n_s stands for the scalar spectral index, $\tau_{\rm reio}$ is the optical depth to reionization, and β is the fractal dimension. We have also put constraints on four derived parameters which are reionization redshift ($z_{\rm reio}$), the matter density parameter ($\Omega_{M,0}$), the Hubble constant (H_0), and the root-mean-square mass fluctuations on scales of $8\,h^{-1}$ Mpc (σ_8). Additionally, preliminary numerical explorations consider the prior range $[0, 0.04]$ for the fractal dimension. It is worth mentioning that, since the fractal parameter ω is irrelevant to cosmological observable (based on numerical results in Sect. 3), we fix ω to be zero in MCMC analysis.

The dataset we use to constrain fractal model includes the Planck likelihood with Planck 2018 data (containing high-l TT,TE,EE, low-l EE, low-l TT, and lensing) [5], the Planck-SZ likelihood for the Sunyaev–Zeldovich effect measured by Planck [46,47], the CFHTLenS likelihood with the weak lensing data [48,49], the Pantheon likelihood with the supernovae data [50], the BAO likelihood with the baryon acoustic oscillations data [51,52], and the BAORSD likelihood for BAO and redshift-space distortions (RSD) measurements [53,54].

The best fit and 1σ and 2σ confidence intervals for cosmological parameters of fractal model and also ΛCDM as the reference model, from "Planck + Planck-SZ + CFHTLenS + Pantheon + BAO + BAORSD" dataset are summarized in Table 1.

Correspondingly, contour plots for some selected parameters of fractal model compared to ΛCDM are displayed in Fig. 4.

Considering the obtained constraints on fractal dimension, observational data detect no significant deviation from ΛCDM model.

Moreover, the Akaike information criterion (AIC) defined as $\mathrm{AIC} = -2\ln\mathscr{L}_{\rm max} + 2K$ where $\mathscr{L}_{\rm max}$ is the maximum likelihood function and K indicates the number of free parameters [55,56], results in $\mathrm{AIC}_{(\Lambda\mathrm{CDM})} = 3847.12$ and $\mathrm{AIC}_{(\mathrm{fractal})} = 3849.70$, then consequently $\Delta AIC = 2.58$. Thus, ΛCDM provides a better fit to observations, while fractal model is still allowed.

5 Conclusions

We have considered a fractal model of the universe introduced by Calcagni [21,22], which is a power counting renormalizable and also Lorentz invariant model. Taking into account a timelike fractal, we have investigated the influence of fractal model on observables, mainly power spectra and Hubble constant, by using a modified version of the publicly available CLASS code. Considering matter power spectra diagrams in Fig. 2, there is an enhancement in structure

[1] Cosmic Linear Anisotropy Solving System.

[2] Markov Chain Monte Carlo.

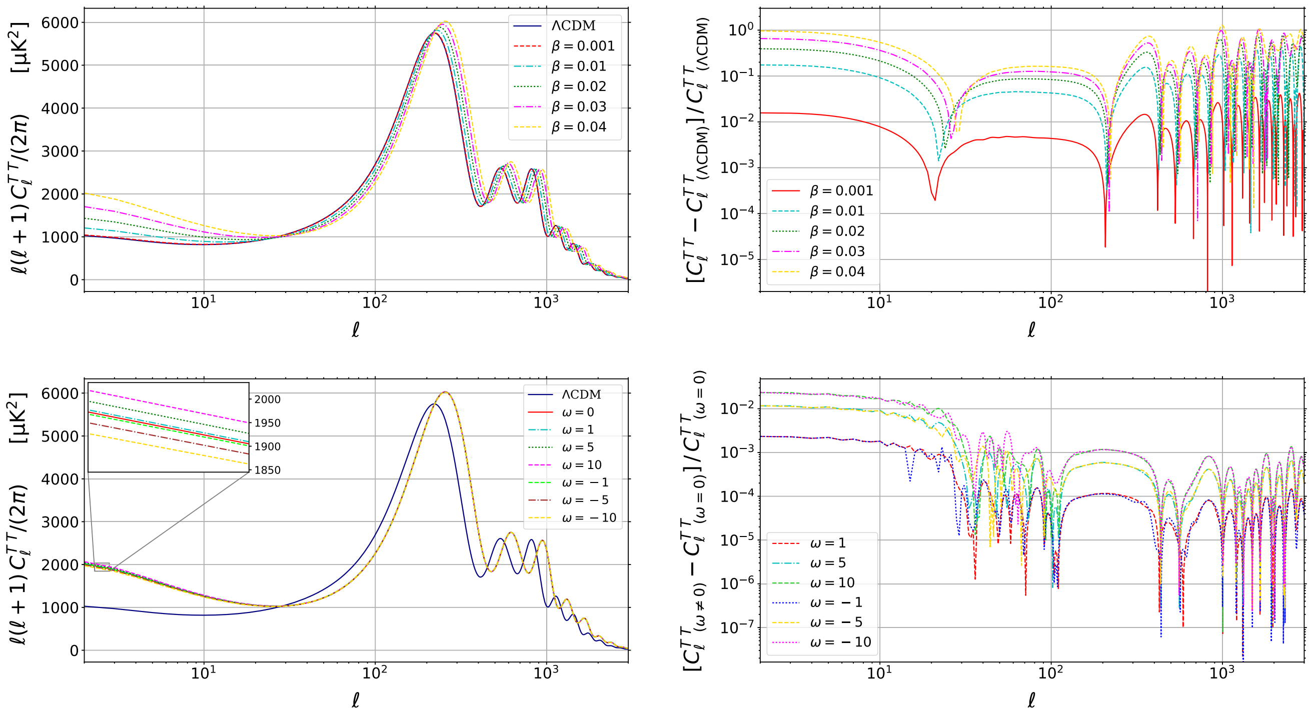

Fig. 1 Upper panels show the CMB power spectra diagrams (left) and their relative ratio with respect to standard cosmological model (right) for different values of β, considering $\omega = 0$. Lower panels show the CMB power spectra diagrams for different values of ω, considering $\beta = 0.04$, compared to ΛCDM model (left) and their relative ratio with respect to the case $\omega = 0$ (right)

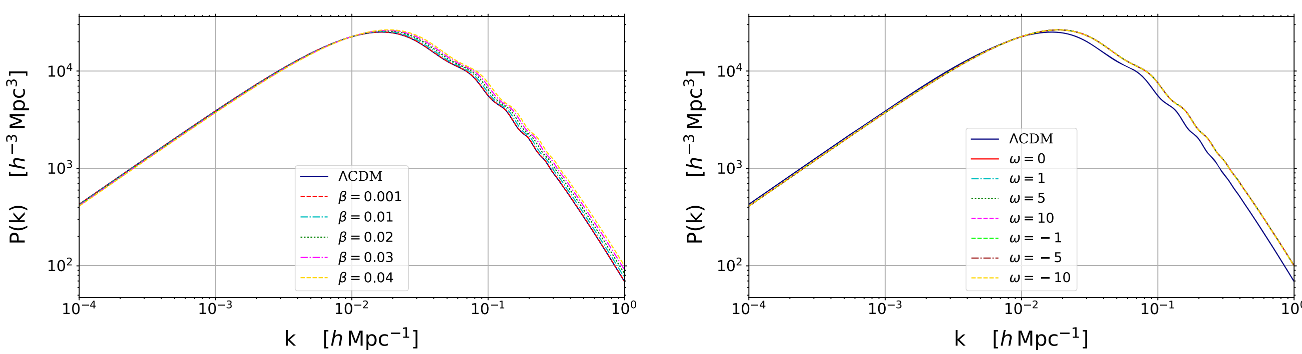

Fig. 2 Matter power spectra diagrams for different values of β compared to ΛCDM model, considering $\omega = 0$ (left), and analogous diagrams for different values of ω, where $\beta = 0.04$ (right)

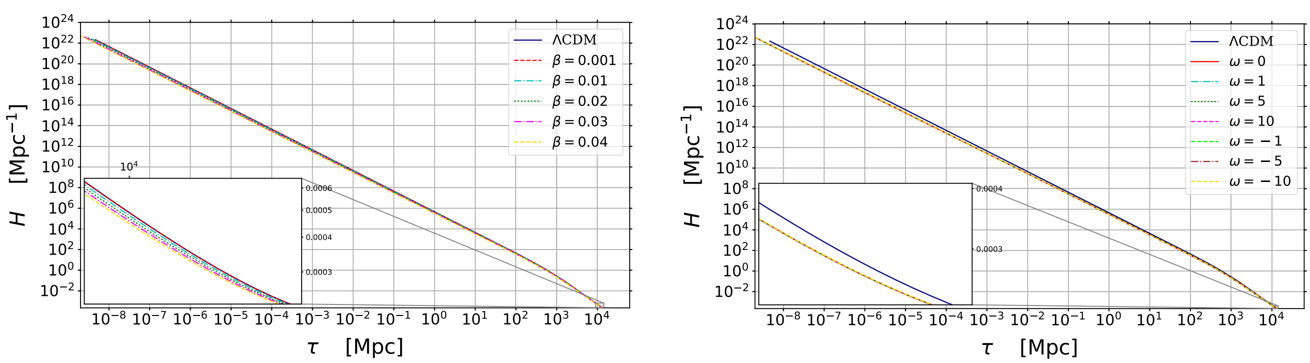

Fig. 3 Hubble parameter in term of conformal time for different values of β compared to ΛCDM model, where $\omega = 0$ (left), and analogous diagrams for different values of ω, regarding $\beta = 0.04$ (right)

Table 1 Best fit values and 68% and 95% confidence limits for cosmological parameters from "Planck + Planck-SZ + CFHTLenS + Pantheon + BAO + BAORSD" data set for ΛCDM and fractal model

Parameter	ΛCDM		Fractal model	
	Best fit	68% and 95% limits	Best fit	68% and 95% limits
$100\,\Omega_{B,0}h^2$	2.261	$2.263^{+0.012+0.026}_{-0.013-0.025}$	2.274	$2.265^{+0.014+0.028}_{-0.015-0.027}$
$\Omega_{DM,0}h^2$	0.1163	$0.1164^{+0.00078+0.0015}_{-0.00079-0.0015}$	0.1164	$0.1168^{+0.00086+0.0017}_{-0.00093-0.0018}$
$100\,\theta_s$	1.042	$1.042^{+0.00029+0.00055}_{-0.00026-0.00053}$	1.042	$1.042^{+0.00030+0.00055}_{-0.00028-0.00054}$
$\ln(10^{10}A_s)$	3.034	$3.024^{+0.010+0.023}_{-0.014-0.021}$	3.023	$3.025^{+0.0091+0.022}_{-0.014-0.021}$
n_s	0.9712	$0.9719^{+0.0036+0.0072}_{-0.0039-0.0074}$	0.9724	$0.9720^{+0.0036+0.0074}_{-0.0038-0.0072}$
τ_{reio}	0.05358	$0.04963^{+0.0041+0.010}_{-0.0074-0.0096}$	0.04706	$0.04919^{+0.0039+0.010}_{-0.0077-0.0092}$
β	–	–	0.0003854	$0.0005421^{+0.00012+0.00092}_{-0.00054-0.00054}$
z_{reio}	7.502	$7.084^{+0.50+1.0}_{-0.69-1.0}$	6.811	$7.046^{+0.49+1.0}_{-0.70-1.0}$
$\Omega_{M,0}$	0.2871	$0.2876^{+0.0043+0.0086}_{-0.0044-0.0086}$	0.2828	$0.2841^{+0.0056+0.010}_{-0.0050-0.012}$
H_0 [km s^{-1} Mpc^{-1}]	69.56	$69.54^{+0.37+0.73}_{-0.36-0.71}$	70.16	$70.08^{+0.46+1.3}_{-0.68-1.1}$
σ_8	0.8079	$0.8044^{+0.0045+0.0096}_{-0.0051-0.0091}$	0.8063	$0.8092^{+0.0049+0.012}_{-0.0067-0.011}$

growth for non-zero values of fractal dimension β, being incompatible with low redshift structure formation. Moreover, according to Fig. 3, the current value of Hubble parameter decreases in fractal cosmology, which is inconsistent with local determinations of H_0. Consequently, primary numerical results indicate that fractal cosmology has no positive impact on relieving cosmological tensions.

Furthermore, we put constraints on the parameters of the fractal universe by utilizing current observations, chiefly

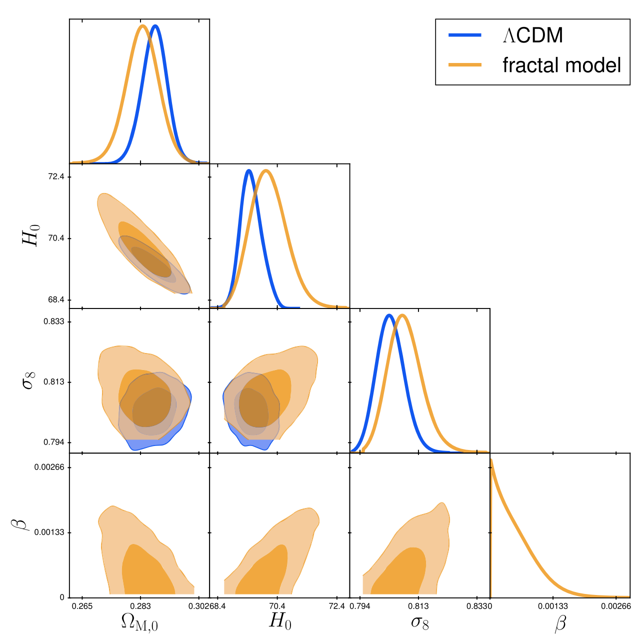

Fig. 4 The one-dimensional posterior distribution and two-dimensional posterior contours with 68% and 95% confidence limits for the selected cosmological parameters of fractal model (orange) compared to ΛCDM (blue)

Planck CMB, weak lensing, supernovae, BAO, and RSD data. Numerical results based on MCMC analysis, detect no significant departure from standard cosmological model. On the other hand, the model selection criterion AIC affirms that ΛCDM model is more favored by observations, however, the fractal model can not be ruled out.

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: Observational data we have used in this paper are publicly available in Refs. [5,46–54]. The modified version of the CLASS code is available under reasonable request.]

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