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Scaled affine quantization of φ_4^4 in the low temperature limit

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Abstract We prove through Monte Carlo analysis that the covariant Euclidean scalar field theory, φ_n^r, where r denotes the power of the interaction term and $n = s + 1$ where s is the spatial dimension and 1 adds imaginary time, such that $r = n = 4$ can be acceptably quantized using scaled affine quantization and the resulting theory is nontrivial and renormalizable even at low temperatures in the highly quantum regime.

1 Introduction

The classical limit only imposes a constraint on the quantum theory of a given system so there is no reason why the classical limit should determine the quantum theory uniquely. Accordingly, it is worthwhile to look for alternative quantization recipes, such as affine quantization. We recently showed [1–4] that a covariant Euclidean scalar field quantization, henceforth denoted φ_n^r, where r is the power of the interaction term and $n = s + 1$, where s is the spatial dimension and 1 adds imaginary time, such that $r = 2n/(n - 2)$, e.g., $r = n = 4$, can be acceptably quantized using scaled *affine quantization* (AQ) [5,6] and the resulting theory is nontrivial, unlike what happens using the usual canonical quantization (CQ) [7–10].[1] In such studies the temperature was kept constant throughout the whole analysis. It is therefore important

to study the behavior of the system as we allow temperature to become lower and lower thereby approaching the extreme quantum regime.

The present study will show, through a path integral Monte Carlo (MC) analysis, that as the temperature is lowered the renormalized mass is almost unaffected but the renormalized coupling constant diminishes. Nonetheless at any given temperature, even in the low temperature, strongly quantum, regime, the scaled AQ model appears to be renormalizable showing a non-free behavior in the continuum limit. This success of scaled AQ suggests that for the φ_4^4 field theory the more common CQ should be replaced by the less known AQ.

2 Affine quantization field theory

For a single scalar field, with spacial degrees of freedom $x = (x_1, x_2, \ldots, x_s)$, $\varphi(x)$ with canonical momentum $\pi(x)$, the classical affine variables are $\kappa(x) \equiv \pi(x) \varphi(x)$ and $\varphi(x) \neq 0$. The reason we insist that $\varphi(x) \neq 0$ is because if $\varphi(x) = 0$ then $\kappa(x) = 0$ and $\pi(x)$ can not help.

We next introduce the classical Hamiltonian expressed in affine variables. This leads us to

$$\mathcal{H}(\kappa, \varphi) = \int \{\tfrac{1}{2}[\kappa(x)^2 \varphi(x)^{-2}$$
$$+ (\nabla\varphi(x))^2 + m^2 \varphi(x)^2] + g \, \varphi(x)^r\} \, d^s x, \quad (1)$$

where r is a positive, even, integer and $g \geq 0$ is the bare coupling constant such that for $g \to 0$ we fall into the free field theory. With these variables we do not let $\varphi(x) = \infty$ otherwise $\varphi(x)^{-2} = 0$ which is not fair to $\kappa(x)$ and, as we already observed, we must forbid also $\varphi(x) = 0$ which would admit $\varphi(x)^{-2} = \infty$ giving again an undetermined kinetic term. Therefore the AQ bounds $0 < |\varphi(x)| < \infty$ *forbid any nonrenormalizability* which is otherwise possible for CQ [7–10].

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[1] In a CQ covariant model the interaction term $g \int \phi(x)^r d^n x$ has a power r/n per integration. This should be compared with the kinetic term $\int [\nabla\phi(x)]^2 d^n x$ which has a power $2/(n-2)$ per integration. Now, since we work in a finite volume region, if $r/n > 2/(n-2)$ then the domain where the CQ action is finite $\mathcal{D}_{g>0} \subset \mathcal{D}_{g=0}$ and the domains change because of reducing g back to zero will only retain the smallest version of the domain by continuity, and that will not be the theory you started out with so that the CQ model is trivial. Models for which $r > 2n/(n - 2)$ have been also recently correctly quantized, as for example φ_3^{12} [11,12].

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The quantum affine operators are the scalar field $\hat{\varphi}(x) = \varphi(x)$ and the *dilation* operator $\hat{\kappa}(x) = [\hat{\varphi}(x)\hat{\pi}(x) + \hat{\pi}(x)\hat{\varphi}(x)]/2$ where the momentum operator is $\hat{\pi}(x) = -i\hbar\delta/\delta\varphi(x)$. Accordingly for the self adjoint kinetic term $\hat{\kappa}(x)\hat{\varphi}(x)^{-2}\hat{\kappa}(x) = \hat{\pi}(x)^2 + (3/4)\hbar\delta(0)^{2s}\varphi(x)^{-2}$ and one finds for the quantum Hamiltonian operator

$$\hat{H}(\hat{\kappa}, \hat{\varphi}) = \int \left\{ \frac{1}{2}[\hat{\pi}(x)^2 + (\nabla\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^r + \frac{3}{8}\hbar^2\frac{\delta(0)^{2s}}{\varphi(x)^2} \right\} d^s x. \tag{2}$$

The affine action is found adding time, $x_0 = ct$, where c is the speed of light constant and t is imaginary time, so that $\mathcal{S} = \int_0^\beta H\,dx_0$, with H the semi-classical Hamiltonian corresponding to the one of Eq. (2), will then read

$$\mathcal{S}[\varphi] = \int_0^\beta dx_0 \int_{L^s} d^s x \left\{ \frac{1}{2}\left[\sum_{\mu=0}^s \left(\frac{\partial\varphi(x)}{\partial x_\mu}\right)^2 + m^2\varphi(x)^2 \right] + g\,\varphi(x)^r + \frac{3}{8}\hbar\frac{\delta(0)^{2s}}{\varphi(x)^2} \right\}, \tag{3}$$

where with an abuse of notation we here use x for $(x_0, x_1, x_2, \ldots, x_s)$ and $\beta = 1/k_B T$, with k_B the Boltzmann's constant, is the inverse temperature. At low temperatures the quantum effects become more relevant and this is the regime we are interested in this work.

The vacuum expectation value of an observable $\mathcal{O}[\varphi]$ will then be given by the following expression

$$\langle\mathcal{O}\rangle = \frac{\int \mathcal{O}[\varphi]\exp(-\mathcal{S}[\varphi])\,\mathcal{D}\varphi(x)}{\int \exp(-\mathcal{S}[\varphi])\,\mathcal{D}\varphi(x)}, \tag{4}$$

where the functional integrals will be calculated on a lattice using the path integral Monte Carlo method as explained further on.

3 Lattice formulation of the field theory

The theory considers a real scalar field φ taking the value $\varphi(x)$ on each site of a periodic n-dimensional lattice, with $n = s + 1$ space-time dimensions, of lattice spacing a, the ultraviolet cutoff, and spacial periodicity $L = Na$ and temporal periodicity $\beta = N_0 a$. The field path is a closed loop on an n-dimensional surface of an $(n + 1)$-dimensional β-cylinder. We used a lattice formulation of the AQ field theory of Eq. (3) (also studied in Eq. (8) of [1]) using the scaling $\varphi \to a^{-s/2}\varphi$ and $g \to a^{s(r-2)/2}g$ which is necessary[2] to

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[2] Note that from a physical point of view one never has to worry about the mathematical divergence since the lattice spacing will necessarily have a lower bound. For example at an atomic level one will have $a \gtrsim 1$Å. In other words the continuum limit will never be a mathematical one.

eliminate the Dirac delta factor $\delta(0) = a^{-1}$ divergent in the continuum limit $a \to 0$. The affine action for the field (in the *primitive approximation* [13]) has then the following valid discretization

$$S[\varphi]/a = \frac{1}{2}\left\{ \sum_{x,\mu} a^{-2}[\varphi(x) - \varphi(x + e_\mu)]^2 + m^2\sum_x \varphi(x)^2 \right\} + \sum_x g\,\varphi(x)^r + \frac{3}{8}\sum_x \frac{\hbar^2}{\varphi(x)^2}, \tag{5}$$

where e_μ is a vector of length a in the $+\mu$ direction with $\mu = 0, 1, 2, \ldots, s$. We will have $\mathcal{S} \approx S$.

In this work we are interested in reaching the continuum limit by taking Na fixed and letting $N \to \infty$ at fixed volume L^s. The absolute temperature $T = 1/k_B\beta$ is allowed to vary so that the number of discretization points for the imaginary time interval $[0, \beta]$ will be $N_0 = \beta/a$. We are here interested in the $N_0 \gg N$ (or $\beta \gg L$) regime.

3.1 Monte Carlo results

We performed a path integral MC [13–16] calculation for the AQ field theory described by the action of Eq. (5). We calculated the renormalized coupling constant g_R and mass m_R defined in Eqs. (11) and (13) of [1] respectively, measuring them in the path integral MC through vacuum expectation values like in Eq. (4). In particular

$$m_R^2 = \frac{p_0^2\langle|\tilde{\varphi}(p_0)|^2\rangle}{\langle\tilde{\varphi}(0)^2\rangle - \langle|\tilde{\varphi}(p_0)|^2\rangle}, \tag{6}$$

and at zero momentum

$$g_R = \frac{3\langle\tilde{\varphi}(0)^2\rangle^2 - \langle\tilde{\varphi}(0)^4\rangle}{\langle\tilde{\varphi}(0)^2\rangle^2}, \tag{7}$$

where $\tilde{\varphi}(p) = \int d^n x\, e^{ip\cdot x}\varphi(x)$ is the Fourier transform of the field and we choose the 4-momentum p_0 with one spacial component equal to $2\pi/Na$ and all other components equal to zero.

In our previous studies [1,4] we set $L = \beta = 1$. Here we will consider $L = 1$ and $\beta \gg L$ instead. As usual we will impose periodic boundary conditions both in space and in imaginary time. We will use natural units $c = \hbar = k_B = 1$ throughout the whole analysis.

Following Freedman et al. [7], we fix (within 10%) the renormalized mass $m_R \approx 3$, tuning appropriately the bare mass m by trial and error, and we measure the renormalized coupling constant g_R at various values of the bare coupling g. We found that the renormalized mass is almost independent on β. So we chose the same values of m for all the temperatures studied. But the renormalized coupling g_R diminishes as β and/or m increase. It is then convenient to define a
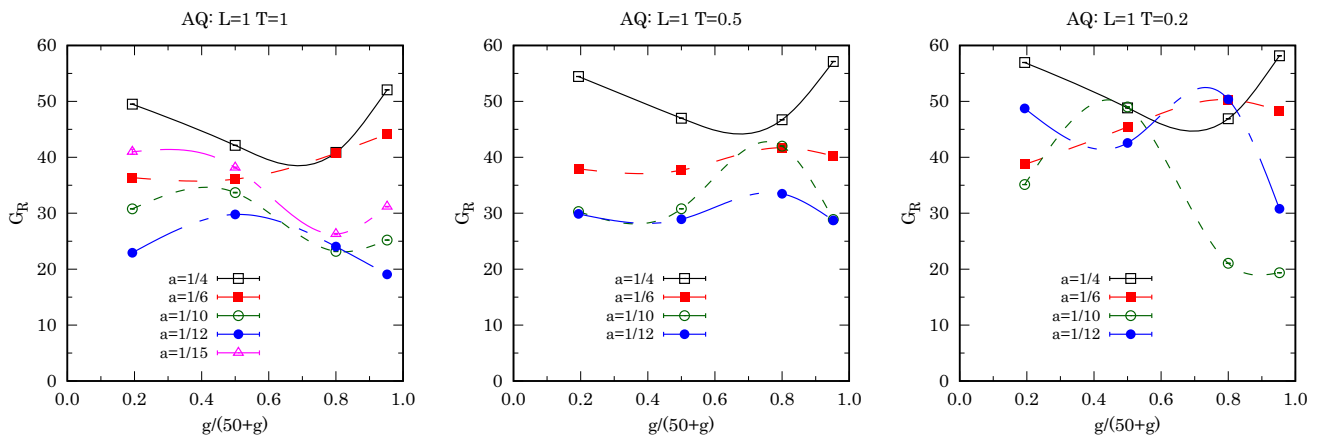
Fig. 1 The left panel is for AQ with $L = 1$ and $T = 1$, The central panel is for AQ with $L = 1$ and $T = 0.5$, and the right panel is for AQ with $L = 1$ and $T = 0.2$. We show the renormalized coupling constant G_R, defined in the text, as a function of $g/(50 + g)$ for decreasing values of the lattice spacing a. The renormalized mass was kept fixed to $m_R \approx 3$ (within 10%) in all cases. The statistical errors in the Monte Carlo were in all cases smaller than the symbols used. The main source of uncertainty is nonetheless the indirect one stemming from the unavoidable difficulty of keeping the renormalized mass constant throughout all cases. The lines connecting the points are just a guide for the eye

second renormalized coupling constant which is less dependent on β, L, and m. Following Freedman et al. [7] we set $G_R = g_R m_R^n L^s \beta$.

We chose two low temperatures (the case $T = 1$ had already been studied in Ref. [4]), namely an intermediate one $T = 0.5$ and an extreme one $T = 0.2$. In each case we study the continuum limit by choosing decreasing values of a, namely $a = 1/4, 1/6, 1/10, 1/12$ and $1/15$ corresponding respectively to $N_0 = 1/Ta = 8, 12, 20, 24, 30$ for $T = 0.5$ and to $N_0 = 20, 30, 50, 60, 75$ for $T = 0.2$. In each run we used 3×10^7 MC steps, where one step consists in $N^s N_0$ Metropolis [14] configuration moves of each field component, reaching equilibrium after 10% of the largest a run to 50% of the smallest a run. In our simulations we used block averages and estimated the statistical errors using the jakknife method (described in Section 3.6 of [17]) to take into account of the correlation time. It took roughly 25 days of computer time for the $T = 0.2$, $a = 1/12$ run to complete. In Fig. 1 we show the numerical results.

From the figure we can see how at all temperatures and all bare coupling constants G_R tends to stay far from zero as we approach the continuum limit $a \rightarrow 0$. Moreover, with respect to the case $T = 1$, already studied in Ref. [4], where the value for G_R tends to revert its trend to decrease for a decrease of the lattice spacing only for an ultraviolet cutoff as small as $a = 1/15$, now we find that at $T = 0.5$ this inversion happens already for $a = 1/10$ at least at intermediate bare coupling and at $T = 0.2$ already for $a = 1/6$. This had to be expected on general grounds because it is impossible to distinguish time from the other spacial components just by looking at the action expression (5) and the $T = 1$, $a = 1/15$ case has a total of $15^4 = 50625$ lattice points which is very close to

the total lattice points of the case $T = 0.2$, $a = 1/10$ which are $10^3 50 = 50000$. We are just choosing an hyperrectangle instead of an hypercube periodic lattice. Nonetheless there is a strong indication that our scaled AQ model is indeed non-free in the continuum thus resulting renormalizable, unlike the corresponding CQ model.[3] And the more so at lower temperatures. We can therefore infer that the same should continue to hold also in the $T \rightarrow 0$, ground state, limit.

4 Conclusions

In conclusion we studied the renormalizability property of one real scalar covariant Euclidean field quantized through scaled affine quantization (AQ) with the path integral Monte Carlo method on a lattice permeating the whole spacetime. We therefore used periodic spacial boundary conditions at finite unit volume to simulate an infinite volume system and in measuring the renormalized mass and coupling constant of the model we also enforced periodic temporal boundary conditions which are necessary in order to determine the required vacuum expectation values. The periodicity on the imaginary time, i.e. the inverse temperature $\beta = 1/T$, was chosen at increasing values equal to 1, 2, 5. Keeping fixed the renormalized mass, our numerical results for the renor-

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[3] For a comparison with the corresponding scaled CQ results see Ref. [18] and for the unscaled CQ ones see Ref. [1].

malized coupling constant showed how this has a non monotonically decreasing behavior with respect to a decreasing lattice spacing. This remains true even at low temperature thus proving the renormalizability of the model even when the temperature is lowered in the extreme quantum regime. We therefore suspect that the non triviality still holds for the ground state.

On general grounds we should accept affine quantization as a way to remove infinities, which are mathematical but not physical, from the field theory. In fact just by looking at the kinetic term in Eq. (1) we can say that if φ is allowed to become infinity (or zero) then κ cannot help. If κ becomes infinite then φ cannot help. $\kappa = 0$ is allowed so that $\pi = 0$. When π and φ were alone, as in the canonical quantization picture, they could allow mathematical infinities. In a physical (or Monte Carlo) measure of an observable there is no space for mathematical infinities.

For the Higgs sector of the Standard Model, the low energy properties are very specific and, so far, observation confirms that they are well described by canonical φ^4. It is certainly true that canonical quantization (CQ) of φ^4 does not reach down to distances of the order of the Planck length – in that realm, anyway, gravity cannot be dealt with classically – so affine quantization (AQ) may be used to solve this problem.

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