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Duality between operator ordering factor and massless scalar field

Dongshan He[1], **Qing-yu Cai**[2,3,4,a]

[1] College of Physics and Electronic Engineering, Xianyang Normal University, Xianyang 712000, China
[2] Center for Theoretical Physics, Hainan University, Haikou 570228, China
[3] School of Information and Communication Engineering, Hainan University, Haikou 570228, China
[4] Peng Huanwu Center for Fundamental Theory, Hefei 230026, Anhui, China

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Abstract In order to investigate the role of quantum effects in the evolution of the universe, one can either use the Wheeler–DeWitt equation (WDWE) that contains an operator ordering factor, or add an item called massless scalar field to WDWE. In this paper, we study the relationship between operator ordering factor and massless scalar field, by applying de Broglie–Bohm quantum trajectory approach to WDWE. In theory, the evolution of the universe is determined by action, i.e., the phase part of the wavefunction of the universe. For the case of operator ordering factor and the case of massless scalar field, the functions that determine the phase part of the wavefunction of the universe satisfy the same differential equation, both in the minisuperspace model and in the Kantowski–Sachs model. This shows the equivalence of using operator ordering factor or massless scalar field to study evolution of the universe. Since there is no accelerating solution of WDWE with operator ordering factor for a grownup universe in the minisuperspace model, the equivalence of the operator ordering factor and the massless scalar field rules out the possibility of a massless scalar field as the candidate for dark energy, if the current universe is indeed homogeneous and isotropic.

1 Introduction

In quantum cosmology, the universe is described by the wavefunction rather than classical spacetime [1–3]. The wavefunction of the universe that contains all the information of the universe should satisfy the quantum gravity equation, or called Wheeler–DeWitt equation (WDWE) [4]. In the quantization procedure, there are some ambiguities when the momenta are replaced by differential operators, which introduces an operator ordering factor p into WDWE [1]. In order to accurately predict the experimental observation results, the ambiguity in the WDWE must be eliminated, i.e., the operator ordering factor should be determined with some constraints. A "natural" choice of the ordering factor is $p = 1$, which makes the derivative piece become a Laplacian in the supermetric [5,6], and Vilenkin got the exact solution of the WDWE in this case [7]. In a simple vacuum universe model, by supposing the universe wavefunction is finite, authors showed that the factor should be chosen as $p = -2$ [8]. Vieira et al. obtained a boundary condition for the ordering parameter $0 \leq p \leq 2$ [9]. As a matter of fact, the effect of the operator ordering is only significant when the universe is very small [7,10,11], so the factor ordering problem cannot be resolved by means of the semiclassical limit [12].

Scalar fields are often used in quantum cosmology to describe matter in cosmology [1,3]. One can also study the evolution of the universe driven by scalar fields. Generally, one can't get the exact solution of WDWE coupled with a scalar field. In slow-roll model, the scalar field approximates a constant, so one can get an inflation solution. Specifically, massless scalar fields as a solvable model have been widely studied [13–16]. Mithani and Vilenkin shows that a massless scalar field ϕ can play the role of a "clock" [15]. Pinto-Neto et al. investigated the effect of massless scalar fields in the evolution of the universe [17]. However, it is not clear whether we should use massless scalar fields or use operator ordering factor to study the evolution of the universe in quantum cosmology.

In this paper, we will investigate the relationship between operator ordering factor and massless scalar field. Since WDWE is time independent, one cannot use it to study the evolution law of the universe directly. In order to gain evolution laws of the universe, de Broglie–Bohm quantum trajectory approach can be applied to WDWE. In de Broglie–Bohm

[a] e-mail: qycai@hainanu.edu.cn (corresponding author)

quantum trajectory theory [18,19], the quantum potential of the universe describes the quantum effects of the universe. The change of the scale factor with time is determined by the guidance relation, which relates to the phase part of the universe wavefunction. By applying de Broglie–Bohm quantum trajectory approach to WDWE, we investigate the effects of the operator ordering factor and massless scalar fields of the dynamical universe, respectively.

This paper is organized as follows. In Sect. 2, de Broglie–Bohm quantum trajectory theory is applied to the WDWE in the minisuperspace model, and the expression of quantum potential and guidance relations are obtained. In Sect. 3, the evolution law of the vacuum universe is obtained with operator ordering factor. In Sect. 4, we get the evolution law of the universe with a massless scalar field, which is exactly the same as that given by operator ordering factor. In Sect. 5, we show that the equations determining the evolution of the universe with an operator ordering factor and a massless scalar field are the same under any boundary conditions both in the minisuperspace model and in the Kantowski–Sachs model. Finally, we will discuss and conclude in Sect. 6.

2 WDWE and quantum trajectory of the universe

The action integral for the scalar field, minimally coupled to gravity and having potential function $V(\phi)$, is given by

$$S_{EH} = \frac{1}{16\pi}\int\left[\mathcal{R} + \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right]\sqrt{-g}d^4x. \tag{1}$$

Here $\mathcal{R}$ is the Ricci scalar, and $V(\phi)$ is the potential of the scalar field ϕ. For a massless scalar field, one has $V(\phi) = 0$. Assumed to be homogeneous and isotropic, the universe can be described by a minisuperspace model [20,21], and the metric of the universe is thus given by

$$ds^2 = \sigma^2\left[-N^2(t)c^2dt^2 + a^2(t)d\Omega_3^2\right]. \tag{2}$$

Here, $d\Omega_3^2 = dr^2/(1-kr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ is the metric on a unit three-sphere, and $k = 1, 0, -1$ are for spatially closed, flat and open universe, respectively. $N(t)$ is an arbitrary lapse function, and $\sigma^2 = 2/3\pi$ is just a normalization factor for simplifying the formula [7]. Note that r is dimensionless and the scale factor $a(t)$ has dimensions of length. The WDWE is found to be [2,3,22],

$$\left[\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} - U(a,\phi)\right]\psi(a,\phi) = 0, \tag{3}$$

where $U(a,\phi) = a^2[k - a^2V(\phi)]$ is the classical potential and $\psi(a,\phi)$ is the wavefunction of the universe. The factor p represents the uncertainty in the choice of operator ordering.

Inserting $\psi(a,\phi) = R(a,\phi)e^{iS(a,\phi)}$ into Eq. (3) and separating the equation into real and imaginary parts, one obtains two equations:

$$S_{aa} + \frac{2R_aS_a}{R} + \frac{pS_a}{a} - \frac{S_{\phi\phi}}{a^2} - \frac{2R_\phi S_\phi}{a^2R} = 0, \tag{4}$$

$$(S_a)^2 - a^{-2}{S_\phi}^2 + U(a,\phi) + Q(a,\phi) = 0, \tag{5}$$

where X_a denotes the derivative of X with respect to a, etc. The quantum potential $Q(a,\phi)$ is

$$Q(a,\phi) = -\left(\frac{R_{aa}}{R} + \frac{p}{a}\frac{R_a}{R}\right) + \frac{R_{\phi\phi}}{a^2R}. \tag{6}$$

Quantum Hamilton–Jacobi theory gives the guidance relations [20]

$$\partial_aS(a,\phi) = -a\dot{a}, \tag{7}$$

$$\partial_\phi S(a,\phi) = a^3\dot{\phi}. \tag{8}$$

The evolution of the scale factor a and scalar field ϕ are determined by this guidance relations. If there is no scalar field, the scale factor a is completely determined by the first equation of guidance relations Eq. (7). Inserting Eqs. (7) and (8) into Eq. (5), we can obtain

$$H^2 = \dot{\phi}^2 - \frac{U(a,\phi) + Q(a,\phi)}{a^4}. \tag{9}$$

This shows that the Hubble parameter relates to the classical potential $U(a,\phi)$, quantum potential $Q(a,\phi)$ and $\dot{\phi}$.

3 The evolution of the vacuum universe with an operator ordering factor

In this section, we will study the change of quantum potential and the evolution of the vacuum universe in the minisuperspace model, where WDWE can be written as

$$\left(\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - ka^2\right)\psi(a) = 0, \tag{10}$$

the corresponding classical potential is $U(a) = -ka^2$. One can obtain general solutions of Eq. (10) for different type of the universe, for closed universe, $k = 1$

$$\psi_c(a) = a^{\frac{1-p}{2}}\left[ic_1I_\nu\left(\frac{a^2}{2}\right) + c_2K_\nu\left(\frac{a^2}{2}\right)\right], \tag{11}$$

for open universe, $k = -1$

$$\psi_o(a) = a^{\frac{1-p}{2}}\left[ic_1J_\nu\left(\frac{a^2}{2}\right) + c_2Y_\nu\left(\frac{a^2}{2}\right)\right], \tag{12}$$

for flat universe, $k = 0$

$$\psi_f(a) = ic_1 a^{\pm 4\nu} - c2. \tag{13}$$

Here, $J_\alpha(x)'s$, $I_\alpha(x)'s$ are Bessel functions and modified Bessel functions of the first kind, $Y_\alpha(x)'s$, $K_\alpha(x)'s$ are Bessel functions and modified Bessel functions of the second kind, respectively. The parameter $\nu = |(1-p)|/4$ is determined by the operator ordering factor p.

The constants c_1 and c_2 should be determined by boundary conditions. When $c_1 = c_2$, one can obtain Vilenkin's Tunneling wavefunction [7,8], and when c_1 or c_2 equals zero, one get the Hartle-Hawking's no boundary wavefunction. When $c_1 \neq c_2$, Hubble parameters will oscillate with the expansion of the universe, and the oscillation frequencies will increase as the universe growing up, which seems quite unreasonable [23]. In the following, we will set $c_1 = c_2$ for reasonable solutions.

For simplicity, we will only study the quantum potential and the evolution of closed universe $k = 1$ [1]. First, we study the evolution of the early universe ($a \ll l_p$). From the wavefunction Eq. (11), we can obtain

$$R = a^{\frac{1-p}{2}} \sqrt{I_\nu^2\left(\frac{a^2}{2}\right) + K_\nu^2\left(\frac{a^2}{2}\right)}. \tag{14}$$

And the phase part of the universe wavefunction is

$$S = \arctan\left[-\frac{I_\nu^2\left(\frac{a^2}{2}\right)}{K_\nu^2\left(\frac{a^2}{2}\right)}\right]. \tag{15}$$

Inserting Eq. (15) into guidance relation (7), the scale factor of the early universe can be obtained as [10]

$$a(t) = \begin{cases} 2\left[\frac{3-4\nu}{\Gamma^2(\nu)}(t+t_0)\right]^{\frac{1}{3-4\nu}}, & \nu \neq 0, \frac{3}{4} \\ e^{H(t+t_0)}, & \nu = \frac{3}{4}, \end{cases} \tag{16}$$

where $\Gamma(\nu)$ is Gamma function. It is clear that only the ordering factor takes the value $p = -2$ (or $p = 4$ for equivalence), i.e., $\nu = 3/4$, has the scale factor $a(t)$ of an exponential behavior. Since $H > 0$ corresponds to an expansionary bubble, therefore, we can draw the conclusion that, for a closed true vacuum bubble, it can expand exponentially, and then the early universe appears irreversible.

Inserting Eq. (14) into Eq. (6), and Expanding $Q(a)$ in series at $a = 0$, the quantum potential of the early universe can be obtained as

$$Q(a) = \begin{cases} -a^2 - \frac{2^{6-8\nu}a^{8\nu-2}}{\Gamma(\nu)^4}, & \nu \neq 0. \\ -\frac{1}{4a^2\ln^4 a}, & \nu = 0. \end{cases} \tag{17}$$

The quantum mechanism of spontaneous creation of the early universe can be seen from the quantum potential. For the case of $p = -2$ (or 4), the quantum potential of the small universe is

$$Q(a \to 0) = -a^2 - \frac{a^4}{\Gamma^4(3/4)}. \tag{18}$$

Here, we find that the first term in quantum potential $Q(a \to 0)$ exactly cancels the classical potential $V(a) = a^2$. The effect of the second term $-a^4/\Gamma^4(3/4)$ is quite similar to that of the scalar field potential in or the cosmological constant in [24] for inflation. We can see that the quantum potential of the small true vacuum bubble plays the role of the cosmological constant and provides the power for the exponential expansion.

When the universe is big, the evolution law of the universe will reduce to the classical solution which satisfies the Friedmann equation[23], and the operator ordering factor does not contribute in a significant way in this case [9].

4 The evolution of the universe with a massless scalar field

The WDWE of the minisuperspace model coupled with a massless scalar field can be written as

$$\left(\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} - ka^2\right)\Psi(a,\phi) = 0. \tag{19}$$

This binary differential equation above can be solved by the separation of variables. We set $p = 0$ to focus on the effect of the massless scalar field on the evolution of the universe. Inserting $\Psi(a,\phi) = \psi(a)\Phi(\phi)$ into Eq. (19), one obtains two equations

$$\left(\frac{\partial^2}{\partial a^2} + \frac{\varepsilon}{a^2} - ka^2\right)\psi(a) = 0, \tag{20}$$

$$\left(\frac{\partial^2}{\partial\phi^2} + \varepsilon\right)\Phi(\phi) = 0. \tag{21}$$

[1] For $k = 0, -1$, we can obtain similar results [10].

Here, ε is a constant. The solutions of Eqs. (20) and (21) can be obtained as

$$\psi_c(a) = a^{\frac{1}{2}}\left[ic_1 I_{\nu'}\left(\frac{a^2}{2}\right) + c_2 K_{\nu'}\left(\frac{a^2}{2}\right)\right], \tag{22}$$

$$\psi_o(a) = a^{\frac{1}{2}}\left[ic_1 J_{\nu'}\left(\frac{a^2}{2}\right) + c_2 Y_{\nu'}\left(\frac{a^2}{2}\right)\right], \tag{23}$$

$$\psi_f(a) = a^{-2\nu'}\left(ic_1 a^{4\nu'} - c_2\right), \tag{24}$$

$$\Phi(\phi) = c_3 e^{i\sqrt{\varepsilon}\phi} + c_4 e^{-i\sqrt{\varepsilon}\phi}, \quad \varepsilon \geq 0, \tag{25}$$

$$\Phi(\phi) = c_3 e^{\sqrt{-\varepsilon}\phi} + c_4 e^{-\sqrt{-\varepsilon}\phi}, \quad \varepsilon < 0. \tag{26}$$

where $\psi_c(a)$, $\psi_o(a)$, $\psi_f(a)$ are wave functions for closed, open and flat universes, respectively. And $\nu' = \sqrt{1-4\varepsilon}/4$. It is very interesting that the wavefunctions in Eqs. (22)–(24) are quite similar to the vacuum universe wavefunctions in Eqs. (11)–(13).

Following the line in the previous section, we can investigate the evolution of the closed ($k = 1$) universe containing a massless scalar field. The wavefunction of the closed universe $\Psi(a,\phi) = \psi_c(a)\Phi(\phi)$ can be written as

$$\Psi(a,\phi) = a^{\frac{1}{2}}\left[iI_{\nu'}\left(\tfrac{a^2}{2}\right) + K_{\nu'}\left(\tfrac{a^2}{2}\right)\right] \times \left(c_3 e^{i\sqrt{\varepsilon}\phi} + c_4 e^{-i\sqrt{\varepsilon}\phi}\right).$$

Here, $c_1 = c_2 = 1$ was chosen as that in the previous section, and we will first study the wavefunction related to ϕ for $\varepsilon \geq 0$. Assuming $\Psi(a,\phi) = R(a,\phi)e^{iS(a,\phi)}$, we can get

$$R(a,\phi) = a^{\frac{1}{2}}\sqrt{I^2_{\nu'}\left(\frac{a^2}{2}\right) + K^2_{\nu'}\left(\frac{a^2}{2}\right)} \times \sqrt{c_3^2 + c_4^2 + 2c_3c_4\cos(2\sqrt{\varepsilon}\phi)}, \tag{27}$$

$$S(a,\phi) = \arctan\left[-\frac{I_{\nu'}\left(\frac{a^2}{2}\right)}{K_{\nu'}\left(\frac{a^2}{2}\right)}\right] + \arctan\left[\frac{c_4 - c_3}{c_3 + c_4}\tan(\sqrt{\varepsilon}\phi)\right]. \tag{28}$$

Here we find that $S(a,\phi) = S(a) + S(\phi)$, where $S(a)$ is the first term of Eq. (28) which is independent of the scalar field ϕ, and $S(\phi)$ is the second term of Eq. (28) which is independent of a. Using the guidance relation (7), the evolution law of the early universe can be obtained as

$$a(t) = \begin{cases} 2\left[\dfrac{3-4\nu'}{\Gamma^2(\nu')}(t+t_0)\right]^{\frac{1}{3-4\nu'}}, & \nu' \neq 0, \dfrac{3}{4} \\ e^{H(t+t_0)}, & \nu' = \dfrac{3}{4}. \end{cases} \tag{29}$$

The influence of the massless scalar field on the evolution of the universe is determined by the parameter ν'. It is definite that the result in Eq. (29) is exactly the same as that in Eq. (16) of the vacuum universe without a scalar field. The quantum potential of the early universe ($a \ll l_p$) can be obtained as

$$Q(a,\phi) = \begin{cases} \dfrac{\varepsilon}{a^2} - a^2 - \dfrac{2^{6-8\nu'}a^{8\nu'-2}}{\Gamma(\nu')^4} + Q(\phi), \\ \quad \nu' \neq 0. \\ -\dfrac{1}{4a^2} - \dfrac{1}{4a^2\ln^4 a} + Q(\phi), \\ \quad \nu' = 0. \end{cases} \tag{30}$$

Here $Q(\phi)$ is the quantum potential relate to the massless scalar field,

$$Q(\phi) = \frac{\partial_{\phi,\phi}R(a,\phi)}{a^2 R(a,\phi)} \tag{31}$$

It is obvious that the quantum potential $Q(a,\phi)$ in Eq. (30) is different from the quantum potential $Q(a)$ in Eq. (17) without the scalar field. However, it is interesting that the evolution law of the small universe is identical for the vacuum universe with an operator ordering factor and the universe with a massless scalar field. As we know, quantum potential and classical potential together determine the evolution of the universe. At the first glance, there seems to be something wrong, since different quantum potentials lead to the same evolution law. Applying Eq. (28) to Eq. (8), one can finally obtain a concise equation

$$a^4\dot{\phi}^2 - Q(\phi) - \frac{\varepsilon}{a^2} = 0. \tag{32}$$

Considering the relations $Q(a,\phi) = Q(a) + Q(\phi)$ and $U(a,\phi) = U(a) + \varepsilon/a^2$, we can obtain the identity

$$\frac{Q(a) + U(a)}{a^4} \equiv -\dot{\phi}^2 + \frac{Q(a,\phi) + U(a,\phi)}{a^4}. \tag{33}$$

According to Eq. (9), we get Hubble parameter $H_p(a) \equiv H_\phi(a)$. Since Eq. (33) is always true whether the universe is small or big, we proved that the universe has identical evolution law when $\nu = \nu'$, with $\nu =\mid 1-p \mid /4$ determined by operator ordering and $\nu' = \sqrt{1-4\varepsilon}/4$ originated from the massless scalar field. It is surprising that two different effects of the operator ordering factor and the massless scalar field can give the same evolution law of the universe.

5 Operator ordering factors duel with massless scalar fields

In the previous sections, we have shown that the effect of operator ordering factor is the same to that of a massless scalar field in a minisuperspace model with some specific parameters. This makes us suspect that the duality between operator ordering factor and massless scalar field may be a coincidence in some special cases. In this section, we show that the duality between operator ordering factor and massless scalar field also exists in general cases.

5.1 Operator ordering factor duel with massless scalar field in minisuperspace model

In the Sects. 3 and 4, the special boundary condition ($c_1 = c_2$) was selected, and the wavefunction of such boundary condition corresponds to the tunneling wavefunction of the Vilenkin's proposal [23]. The different values of c_1 and c_2 will lead to different wavefunction, for example, $c_1 = 0, c_2 = 1$ lead to the Hartle-Hawking wavefunction. Next, we set c_1 and c_2 as arbitrary constants to study the relationship of operator ordering factor and massless scalar field in the case of arbitrary boundary conditions.

We can write the wavefunction of the minisuperspace model with an operator ordering factor which satisfies Eq. (10),

$$\psi(a) = a^{\frac{1-p}{2}}\alpha(a). \tag{34}$$

According to the guidance relation (7), the evolution law of the universe is completely determined by the phase part of the wavefunction

$$S_p(a) = \arctan\frac{\mathrm{Im}[\alpha(a)]}{\mathrm{Re}[\alpha(a)]}, \tag{35}$$

where $\mathrm{Im}[\alpha(a)]$ and $\mathrm{Re}[\alpha(a)]$ represent the imaginary and real parts of $\alpha(a)$, respectively. $S_p(a)$ is independent of the previous part of $\alpha(a)$ (i.e., $a^{(1-p)/2}$), which definitely shows that the evolution of the universe is entirely determined by $\alpha(a)$. Inserting $\psi(a)$ into Eq. (10), we can find that $\alpha(a)$ satisfy the equation

$$\alpha''(a) + \frac{\alpha'(a)}{a} - \left[ka^2 + \frac{4\nu^2}{a^2}\right]\alpha(a) = 0 \tag{36}$$

As a comparison, let us study the evolution of the universe with a massless scalar field. The wavefunction $\Psi(a,\phi)$ satisfies the Eq. (3) and can be rewritten as

$$\Psi(a,\phi) = a^{\frac{1}{2}}\Phi(\phi)\beta(a). \tag{37}$$

According to the guidance relation (7), the evolution of the scale factor a is completely determined by the phase part of the wavefunction $S(a,\phi)$, since the variables a and ϕ in Eq. (19) are separable, which means $S(a,\phi) = S_s(a) + S_s(\phi)$. The scale factor a is only dependent of the phase part of the wavefunction $S_s(a)$, and

$$S_s(a) = \arctan\frac{\mathrm{Im}[\beta(a)]}{\mathrm{Re}[\beta(a)]}. \tag{38}$$

Since $S_s(a)$ is determined by $\beta(a)$, the evolution of the universe will entirely be determined by $\beta(a)$ in this case. Inserting $\Psi(a,\phi)$ into Eq. (19), we have

$$\frac{a^2\beta''(a)}{\beta(a)} + \frac{a\beta'(a)}{\beta(a)} - \left(ka^4 + \frac{1}{4}\right) = \frac{\partial_{\phi,\phi}\Phi(\phi)}{\Phi(\phi)} = -\varepsilon. \tag{39}$$

we then obtain that

$$\beta''(a) + \frac{\beta'(a)}{a} - \left[ka^2 + \frac{4\nu'^2}{a^2}\right]\beta(a) = 0. \tag{40}$$

Equations (36) and (40) show that $\alpha(a)$ and $\beta(a)$ satisfy the same equation. In this way, it has been proved that the operator ordering factor always duel with a massless scalar field in the minisuperspace model no matter what boundary conditions are chosen. It can also be verified that our conclusion still holds when the WDWEs (10) and (19) contain terms of classical matter.

5.2 Operator ordering factors duel with massless scalar fields in Kantowski–Sachs models

The action integral for the scalar field, minimally coupled to gravity and having potential function $V(\phi)$, is given by

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\mathcal{R} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right]. \tag{41}$$

For the massless scalar field, one has $V(\phi) = 0$. The Kantowski–Sachs universe is one of the simplest anisotropic cosmological models and the Kantowski–Sachs line element is [25]

$$ds^2 = -N^2dt^2 + a^2(t)dr^2 + b^2(t)d\Omega_2^2, \tag{42}$$

where $a(t)$ and $b(t)$ are scale factors. In order to eliminate the cross term of the scale factor in the WDWE, one can redefine the coordinate $b(t) = a^{-1}(t)\sigma^{-1}(t)$. Taken the Lapse function $N = 1$, the Lagrangian of the universe can be written as

$$\mathcal{L} = \frac{\dot{a}^2}{2a^3\sigma^2} - \frac{\dot{\sigma}^2}{2a\sigma^4} - \frac{\dot{\phi}^2}{8a\sigma^2} + \frac{a}{2} - \frac{V(\phi)}{4a\sigma^2}. \tag{43}$$

The corresponding WDWE with a massless scalar field can be written as [26]

$$\left[-\frac{\partial^2}{\partial a^2} + 2\frac{\sigma^2}{a^2}\frac{\partial^2}{\partial\sigma^2} - \frac{4}{a^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{a^2\sigma^2}\right]\Psi(a,\sigma,\phi) = 0.$$

(44)

Following the de Broglie–Bohm quantum trajectory approach, the wavefunction can be represented as $\psi(a,\sigma,\phi) = R(a,\sigma,\phi)e^{iS(a,\sigma,\phi)}$, and the quantum Hamilton–Jacobi theory gives the guidance relations

$$\partial_a S(a,\sigma,\phi) = \frac{\partial\mathcal{L}}{\partial\dot{a}} = -\frac{\dot{a}}{a^3\sigma^2}, \tag{45}$$

$$\partial_\sigma S(a,\sigma,\phi) = \frac{\partial\mathcal{L}}{\partial\dot{\sigma}} = -\frac{\dot{\sigma}}{a\sigma^4}, \tag{46}$$

$$\partial_\phi S(a,\sigma,\phi) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -\frac{\dot{\phi}}{4a\sigma^2}. \tag{47}$$

The evolution of the scale factors a and σ is entirely determined by the first two equations of the above guidance relations.

Inserting $\Psi(a,\sigma,\phi) = a^{1/2}\Phi(\phi)\beta(a,\sigma)$ into Eq. (44), we can get that

$$\left[-\partial_{a,a} - \frac{1}{a}\partial_a + \frac{\sigma^2}{a^2}\partial_{\sigma,\sigma} + \left(\frac{\varepsilon}{a^2} + \frac{4+\sigma^2}{4a^2\sigma^2}\right)\right]\beta(a,\sigma) = 0. \tag{48}$$

Since the variables a, σ and ϕ are separable, the phase part of the wavefunction can be written as $S(a,\sigma,\phi) = S(a,\sigma) + S(\phi)$. Here, $S(a,\sigma)$ is determined by $\beta(a,\sigma) = R(a,\sigma)e^{iS(a,\sigma)}$, and $S(\phi)$ is determined by $\Phi(\phi) = R(\phi)e^{iS(\phi)}$. According to the guidance relations (45) and (46), the evolution law of the universe determined by $S(a,\sigma)$, i.e., the evolution of the universe entirely depends on $\beta(a,\sigma)$ that satisfies Eq. (48).

The WDWE with operator ordering factor for a vacuum universe (without scalar field ϕ) in Kantowski–Sachs model is [26]

$$\left[-\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} + 2\frac{\sigma^2}{a^2}\frac{\partial^2}{\partial\sigma^2} + \frac{1}{a^2\sigma^2}\right]\psi(a,\sigma) = 0. \tag{49}$$

Inserting $\psi(a,\sigma) = a^{(1-p)/2}\alpha(a,\sigma)$ into Eq. (49), we have

$$\left[-\partial_{a,a} - \frac{1}{a}\partial_a + \frac{\sigma^2}{a^2}\partial_{\sigma,\sigma} + \left(\frac{p^2-2p}{4a^2} + \frac{4+\sigma^2}{4a^2\sigma^2}\right)\right]\alpha(a,\sigma) = 0. \tag{50}$$

In this case, the evolution law of the universe can be calculated by the first two equations of the guidance relations (45) and (46) without scalar field ϕ, i.e., the evolution of the universe is totally determined by $\alpha(a,\sigma)$.

It is clear that $\alpha(a,\sigma)$ and $\beta(a,\sigma)$ satisfy the identical differential equations when the separation constant $\varepsilon = (2p-p^2)/4$, so the evolution laws of the universe in these two models are identical. In this way, we proved the duality of operator order factor and massless scalar field in the Kantowski–Sachs universe model.

6 Discussion

By applying de Broglie–Bohm quantum trajectory approach to WDWE, we have proved the duality of operator order factor and massless scalar field both in Kantowski–Sachs universe model and in minisuperspace model. The equivalence of WDWE with operator ordering factor and with massless scalar fields presents an avenue towards studying whether massless scalar fields can play the role of dark energy for current universe. Because of the equivalence, one can either choose operator order factor or choose massless scalar fields to study the quantum effects of the universe.

After the discovery of the accelerated expansion of the universe, it has been widely believed that dark energy may be quantum effects of the universe. The most possible candidate for dark energy is vacuum energy or called zero-point energy, while a simple calculation shows vacuum energy density is divergent and a Planck energy cut-off gives 120 orders of magnitude higher than what is observed. Another possible candidate for dark energy is massless scalar field, but it is not easy to draw a conclusion because of the complexity of WDWE coupled with a scalar field. Many studies have shown that the operator ordering factor may play an important role in the early universe, and the evolution of the universe is independent of the value of the ordering factor when the universe becomes large enough [1,7,9,10,23,27], i.e., the operator ordering factor can't play the role of dark energy for the current universe. Therefore, with the equivalence of operator ordering and massless scalar field, we can conclude that, in the minisuperspace model, the massless scalar field doesn't contribute significantly to the accelerating expansion of our current universe.

In non-isotropic quantum cosmology, there are more than two variables in the WDWE, which makes the WDWE too complicated to be resolved, e.g., in the simple Kantowski–Sachs universe model, the wave function can be written as $\Psi(a,b,\phi)$ [25]. Due to the equivalence of the operator ordering factor and massless scalar field, one can drop the variable ϕ, and thus the wavefunction can be simplified as $\Psi(a,b)$. Then the question of whether there exists an accelerated expansion solution of the universe in the Kantowski–Sachs model can be simplified and may be resolved, which can be studied in the future.

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References

1. J.B. Hartle, S.W. Hawking, Wave function of the universe. Phys. Rev. D **28**, 2960 (1983)
2. S.W. Hawking, The quantum state of the universe. Nucl. Phys. B **239**, 257 (1984)
3. A. Vilenkin, Boundary conditions in quantum cosmology. Phys. Rev. D **33**, 3560 (1986)
4. B.S. DeWitt, Quantum theory of gravity. I. The canonical theory. Phys. Rev. **160**, 1113 (1967)
5. S.W. Hawking, D. Page, Operator ordering and the flatness of the universe. Nucl. Phys. B **264**, 185–196 (1986)
6. N. Kontoleon, D.L. Wiltshire, Operator ordering and consistency of the wave function of the universe. Phys. Rev. D **59**, 063513 (1999)
7. A. Vilenkin, Quantum cosmology and the initial state of the Universe. Phys. Rev. D **37**, 888 (1988)
8. D. He, D. Gao, Q.-Y. Cai, Dynamical interpretation of the wavefunction of the universe. Phys. Lett. B **748**, 361–365 (2015)
9. H.S. Vieira, V.B. Bezerra, C.R. Muniz, M.S. Cunha, H.R. Christiansen, Some exact results on quantum relativistic cosmology: dynamical interpretation and tunneling phase. Phys. Lett. B **809**, 135712 (2020)
10. D. He, D. Gao, Q.-Y. Cai, Spontaneous creation of the universe from nothing. Phys. Rev. D **89**, 083510 (2014)
11. D. He, Q.-Y. Cai, Inflation of small true vacuum bubble by quantization of Einstein–Hilbert action. Sci. China Phys. Mech. Astron. **58**, 079801 (2015)
12. R. Šteigl, F. Hinterleitner, Factor ordering in standard quantum cosmology. Class. Quantum Gravity **23**(11), 3879–3893 (2006)
13. N. Pinto-Neto, E.S. Santini, The accelerated expansion of the Universe as a quantum cosmological effect. Phys. Lett. A **315**, 36–50 (2003)
14. N. Pinto-Neto, J.C. Fabris, Quantum cosmology from the de Broglie–Bohm perspective. Class. Quantum Gravity **30**, 143001 (2013)
15. A.T. Mithani, A. Vilenkin, Tunneling decay rate in quantum cosmology. Phys. Rev. D **91**, 123511 (2015)
16. L. Perlov, Wheeler–DeWitt equation for 4D supermetric and ADM with massless scalar field as internal time. Phys. Lett. B **743**, 143–146 (2015)
17. N. Pinto-neto, E.S. Santini, The accelerated expansion of the Universe as a quantum cosmological effect. Phys. Lett. A **315**, 36–50 (2003)
18. D. Bohm, A suggested interpretation of the quantum theory in terms of "hidden" variables. I. Phys. Rev. **85**, 166 (1952)
19. P.R. Holland, *The quantum theory of motion* (Cambridge University Press, Cambridge, 1993)
20. N. Pinto-Neto, F.T. Falciano, R. Pereira, E.S. Santini, Wheeler–DeWitt quantization can solve the singularity problem. Phys. Rev. D **86**, 063504 (2012)
21. S.P. Kim, Quantum potential and cosmological singularities. Phys. Lett. A **236**, 11 (1997)
22. D.L. Wiltshire, Wave functions for arbitrary operator ordering in the de Sitter minisuperspace approximation. Gen. Relativ. Gravit. **32**, 515 (2000)
23. D.S. He, Q.-Y. Cai, Wheeler–DeWitt equation rejects quantum effects of grown-up universes as a candidate for dark energy. Phys. Lett. B **809**, 135747 (2020)
24. D.H. Coule, Quantum cosmological models. Class. Quantum Gravity **22**, R125 (2005)
25. S. Dutta, M. Lakshmanan, S. Chakraborty, Non-minimally coupled scalar field in Kantowski–Sachs model and symmetry analysis. Ann. Phys. **393**, 254–263 (2018)
26. M. Aguero, J.A.S. Aguilar, C. Ortiz, M. Sabido, J. Socorro, Noncommutative Bianchi Type II quantum cosmology. Int. J. Theor. Phys. **46**, 2928–2934 (2007)
27. R. Garattini, M. Faizal, Cosmological constant from a deformation of the Wheeler–DeWitt equation. Nucl. Phys. B **905**, 313–326 (2016)