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Light-front description of infinite spin fields in six-dimensional Minkowski space

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Abstract We present a new $6D$ infinite spin field theory in the light-front formulation. The Lorentz-covariant counterparts of these fields depend on 6-vector coordinates and additional spinor variables. Casimir operators in this realization are found. We obtain infinite-spin fields in the light-cone frame which depend on two sets of the $\mathrm{SU}(2)$-harmonic variables. The generators of the $6D$ Poincaré group and the infinite spin field action in the light-front formulation are presented.

1 Introduction

The study of various aspects of classical and quantum field theory in higher dimensions attracts attention basically due to connections with the low-energy limit of superstring theory and miraculous cancelations of some divergences in supersymmetric field models. One of such aspects is a description of the massless representations of the Poincaré group in multi-dimensional spaces (see e.g. the recent works [1–5]).[1] In this paper, we continue our study of field irreducible massless representations of the six-dimensional Poincaré group [3–5] focusing on the infinite spin representations and their Lagrangian formulation.

The study of the infinite spin representations of the Poincaré group [8–10], their field realizations and dynamical description aroused considerable interest, which led to the formation of a certain research branch mainly in the context of the theory of higher spin fields (see e.g. the review [11] and earlier references therein, and recent papers [12–28]) where aspects of interactions and supersymmetry of infinite spin fields have been examined).Since field realizations of the Poincaré group representations in each concrete dimension have specific features, infinite spin fields in higher dimensions deserve a separate study.

To construct infinite spin fields in six-dimensional Minkowski space, we should describe a possible spectrum of states corresponding to these fields and, first of all, to clarify the spin structure. It can be achieved by considering massless representations of the $6D$ Poincaré group in state space in terms of the canonically conjugate position and momentum operators, as well as the canonically conjugate pair of spinor operators. Irreducible representations are formulated in terms of second-, fourth- and sixth-order Casimir operators, respectively. The corresponding eigenvalues for these Casimir operators in the irreducible infinite spin representation are $0\,, -\mu^2\,, -\mu^2 s(s+1)$ respectively, where μ is a nonzero real parameter and s is a non-negative integer or half-integer number (see the details in [3,4]).

To describe the spin structure of the $6D$ infinite spin fields, it is natural to refer to the light-cone frame for massless fields, where the eigenvalues of the energy-momentum operator are $p^0 = p^5 = k$, $p^{\hat a} = 0$, $\hat a = 1,2,3,4$ with some nonzero real parameter k. Here a remarkable result was unexpectedly discovered that any infinite spin field in this frame is necessarily a function on bi-harmonic space with the harmonics $u^{\pm}\,, v^{\pm}$ which were earlier essentially used to construct the

[1] For earlier activity in this direction see e.g. [6,7] and the references therein.

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unconstrained superfield formulation of $4D$, $\mathcal{N}=2$ supersymmetric field theories [29,30]. Taking into account this result, it is natural then to go to the light-front coordinate system $x^{\pm}, x^{\hat{a}}$, $a=1,2,3,4$, which inherits the properties of the light-cone frame [31]. Thus, we arrive at the function of both $x^{\pm}, x^{\hat{a}}$ and harmonics $u^{\pm}$, $v^{\pm}$ which is considered as the infinite spin field in the light-front coordinate system. The field dynamics in the light-front coordinates can be constructed following the generic scheme [31] (see also [32–37]).

The paper is organized as follows. In Sect. 2, we discuss the description of irreducible infinite spin representations of the $6D$ Poincaré group in state space formulated in terms of the position and momentum operators and spin operators. These operators are $6D$ vectors and a pair of SU(2) Majorana–Weyl spinors. We find expressions of the fourth- and sixth-orders Casimir operators for the system under consideration and discuss the conditions leading to fixing the eigenvalues of these operators on physical states. In Sect. 3, we derive infinite spin fields in the light-cone frame. Here we show that these fields are the function on bi-harmonic space with two sets of SU(2) harmonics $v_i^{\pm}$ and $v_{\hat{i}}^{\pm}$. The harmonics obtained here describe the coset space $[\mathrm{SU}(2)\otimes\mathrm{SU}(2)]/\mathrm{U}(1)$. Such a harmonic field possesses a harmonic charge which is determined by the eigenvalue of the sixth-order Casimir operator. We describe the general structure of the harmonic field in the light-cone frame and show that it is given by an infinite expansion in the harmonics. Using these results, in Sect. 4, we develop the light-front dynamical formulation of an infinite spin field. We find the generators of the $6D$ Poincaré group for the fields under consideration and propose the corresponding action. An important point in this approach is the use of harmonics as additional coordinates, which greatly simplifies the field analysis. In Sect. 5, we summarize the results obtained. Appendix A is devoted to the calculation of the sixth-order Casimir operator for the system considered. In Appendix B, we find the spinor part of the $6D$ Lorentz algebra generators.

2 Irreducible massless representation of the $D6$ Poincaré group

In this section, we discuss the construction of a massless irreducible representation of the six-dimensional Poincaré group emphasizing the specific use of spinor operators.

We consider the representations in the space of states described by vectors $|\Psi\rangle$. The basic operators acting in this space are

$$x^a, \quad p_a; \quad \xi^I_\alpha, \quad \rho^{\alpha I}. \tag{2.1}$$

Here the Hermitian coordinate $x^a=(x^a)^\dagger$ and momentum $p_a=(p_a)^\dagger$ operators are components of the six-vectors, $a=0,1,\ldots,5$ and they obey the standard commutation relations

$$[x^a, p_b]=i\delta^a_b. \tag{2.2}$$

The operators $\xi^I_\alpha, \rho^{\alpha I}$ are the SU(2) Majorana-Weyl spinors[2], where $\alpha=1,2,3,4$ and $I=1,2$ are, respectively, the spinorial SU*(4) and internal SU(2) indices. The Hermitian conjugation for these operators is defined as follows:

$$(\xi^I_\alpha)^\dagger=\epsilon_{IJ}B_{\dot{\alpha}}{}^{\beta}\xi^J_\beta, \quad (\rho^{\alpha I})^\dagger=\epsilon_{IJ}\rho^{\beta J}(B^{-1})_\beta{}^{\dot{\alpha}}, \tag{2.3}$$

where $B_{\dot{\alpha}}{}^{\beta}$ is the matrix related to complex conjugation, and the antisymmetric tensors $\epsilon_{IJ}, \epsilon^{IJ}$ have the components $\epsilon_{12}=\epsilon^{21}=1$ (see [5] for details). Nonzero commutation relations for the operators ξ^I_α, $\rho^{\alpha I}$ have the form

$$\left[\xi^I_\alpha, \rho^\beta_J\right]=i\delta^\beta_\alpha\delta^I{}_J. \tag{2.4}$$

The operators ξ^I_α, $\rho^{\alpha I}$ are to describe the spin degrees of freedom. The state space will be specified in the next section.

We assume that the operators p_a generate space-time translations. In this case, the generators $\{P_a, M_{ab}\}$ of the algebra $\mathfrak{iso}(1,5)$ of the Poincaré group are realized as

$$P_a=p_a, \tag{2.5}$$

$$M_{ab}=p_a x_b-p_b x_a+S_{ab}, \tag{2.6}$$

where the spin part of the Lorentz group generators looks like

$$S_{ab}=\xi^I_\alpha(\tilde{\sigma}_{ab})^\alpha{}_\beta\rho^\beta_I=-\rho^\alpha_I(\sigma_{ab})_\alpha{}^\beta\xi^I_\beta. \tag{2.7}$$

We consider the massless representations where the quadratic Casimir operator $C_2=P^2=P^aP_a$ of the algebra $\mathfrak{iso}(1,5)$ has zero eigenvalues

$$p^a p_a|\Psi\rangle=0. \tag{2.8}$$

In this case, the projection of the fourth-order Casimir operator in subspace (2.8) has the form [3,4]

$$C_4=\Pi^a\Pi_a, \tag{2.9}$$

where

$$\Pi_a=P^bM_{ba}. \tag{2.10}$$

As a result, we can see that in the representation (2.5), (2.6) and under the condition (2.8) the operator (2.9) takes the following form:

$$C_4=-\tilde{\ell}\,\ell, \tag{2.11}$$

where the scalar operators ℓ, $\tilde{\ell}$ are defined by the relations

$$\ell:=\frac{1}{2}\rho^\alpha_I(p_a\sigma^a)_{\alpha\beta}\rho^{\beta I}, \quad \tilde{\ell}:=\frac{1}{2}\xi^I_\alpha(p_a\tilde{\sigma}^a)^{\alpha\beta}\xi_{\beta I}. \tag{2.12}$$

[2] We use the spinor conventions of the works [3–5].

When deriving expression (2.11), we used the relation (A.3) for the $6D$ σ-matrices. The algebra of operators (2.12) is written in the form

$$[\tilde{\ell}, \ell] = N\, p^a p_a, \tag{2.13}$$

where the operator N is defined by the anticommutator

$$N := \frac{i}{2}\, \{\xi^I_\alpha, \rho^\alpha_I\}. \tag{2.14}$$

Besides, the operators (2.12) are the Poincaré group invariants and hence they commute with the generators (2.5), (2.6)

$$[P_a, \ell] = [P_a, \tilde{\ell}] = 0 \quad [M_{ab}, \ell] = [M_{ab}, \tilde{\ell}] = 0. \tag{2.15}$$

The infinite spin representation is characterized by the condition that the fourth-order Casimir operator has nonzero negative eigenvalue

$$C_4\, |\Psi\rangle = -\,\mu^2\, |\Psi\rangle, \tag{2.16}$$

where $\mu \neq 0$ is the dimensional real parameter which can be taken positive $\mu \in \mathbb{R}_{>0}$ without loss of generality. Using relations (2.16) and (2.11), we can see that it is sufficient to define infinite spin states by the constraints

$$\ell\, |\Psi\rangle = \mu\, |\Psi\rangle, \quad \tilde{\ell}\, |\Psi\rangle \ = \ \mu\, |\Psi\rangle. \tag{2.17}$$

For massless representations (2.8), the sixth-order Casimir operator has the form [3,4]

$$C_6 = -\,\Pi^b M_{ba}\, \Pi_c M^{ca} + \frac{1}{2}\left(M^{ab} M_{ab} - 8\right) \Pi^a \Pi_a, \tag{2.18}$$

where the operator Π_a is defined in (2.10). In the representation (2.5), (2.6) and under the conditions (2.8), (2.16), we obtain[3]

$$C_6\, |\Psi\rangle = -\,\mu^2\, J_{\rm i} J_{\rm i}\, |\Psi\rangle, \tag{2.19}$$

where the operators $J_{\rm i}$ ($\rm i = 1, 2, 3$) are defined as follows:

$$J_{\rm i} := \frac{i}{2}\, \xi^I_\alpha (\sigma_{\rm i})_I{}^J \rho^\alpha_J. \tag{2.20}$$

Here $\sigma_{\rm i}$ are the Pauli matrices. The operators $J_{\rm i}$ form the $\mathfrak{su}(2)$ algebra

$$[J_{\rm i}, J_{\rm j}] = i\epsilon_{\rm ijk} J_{\rm k}. \tag{2.21}$$

Expression (2.19) for the operator C_6 is the same as in [3,4] but the realization of the generators $J_{\rm i} \in \mathfrak{su}(2)$ in [3,4] is different.

As it was shown in [3,4], the space V of irreducible infinite spin representation is induced from the space of finite dimensional representation of (2.20) and the operator C_6 acts as follows:

$$C_6\, |\Psi\rangle = -\,\mu^2\, s(s+1)\, |\Psi\rangle, \tag{2.22}$$

where s is a nonzero integer or half-integer number, $s \in \mathbb{Z}_{\geq 0}/2$. Therefore, the states corresponding to the infinite spin irreducible representation obey the constraints

$$J_{\rm i} J_{\rm i}\, |\Psi\rangle = s(s+1)\, |\Psi\rangle, \tag{2.23}$$

where the operators $J_{\rm i}$ are defined in (2.20).

Note that the $\mathfrak{su}(2)$ algebra generators (2.20) commute with the SU(2) scalar operators (2.12):

$$[J_{\rm i}, \ell] = [J_{\rm i}, \tilde{\ell}] = 0. \tag{2.24}$$

Besides, the operators (2.20) commute with generators of six-dimensional translations (2.5) and with the Lorentz algebra $\mathfrak{so}(1,5)$ generators (2.6), (2.7):

$$[P_a, J_{\rm i}] = 0, \quad [M_{ab}, J_{\rm i}] = [S_{ab}, J_{\rm i}] = 0. \tag{2.25}$$

It is worth noting that the algebra $\mathfrak{so}(1,5) = \mathfrak{su}^*(4)$ generated by (2.7) is dual to the algebra $\mathfrak{su}(2)$ with the generators (2.20) in the sense of Howe duality [38,39].

It was shown earlier [6,7,12] that in the vector approach an infinite spin representation of $\mathfrak{so}(1,5)$ requires the use of the 6-dimensional Heisenberg algebra (2.2) generated by the operators of position x^a and momentum p_a and two additional 6-dimensional Heisenberg algebras with the coordinate operators y^a_1, y^a_2 and their momentum operators $p^{(y)}_{1a}$, $p^{(y)}_{2a}$. On the other hand, we proved in [5] that, in the twistor formulation, the $\mathfrak{iso}(1,5)$ representations of infinite spin are necessarily described in the bi-twistor space, which is defined by two pairs of canonically-conjugated SU(2) Majorana–Weyl spinors of type (2.4) (but having nonzero mass dimensions). Here we have shown that $6D$ infinite spin representations can be described in the space defined by only one 6-dimensional Heisenberg algebra (2.2) and one pair of canonically conjugated SU(2) Majorana–Weyl spinors (2.4), as it was indicated in (2.1).

In the next section, we will describe the infinite spin vectors $|\Psi\rangle$ in terms of appropriate fields.

3 Infinite spin fields in the light-cone frame

We consider a structure of $D6$ infinite spin fields in the light-cone frame. This frame is defined by the following conditions on eigenvalues of the energy-momentum operator:

$$p^0 = p^5 = k, \quad p^{\hat{a}} = 0\,, \qquad \hat{a} = 1, 2, 3, 4, \tag{3.1}$$

where k is a nonzero real parameter of mass dimension. Using the light-cone coordinates $p^{\pm} = \left(p^0 \pm p^5\right)/\sqrt{2}$, one gets for the same frame

$$p^+ = \sqrt{2}k, \quad p^- = p^{\hat{a}} = 0. \tag{3.2}$$

Consider the SU*(4) spinors ξ^I_α, $\rho^{\alpha I}$ with a four-component spinor index α and present them as objects with

[3] See the details in Appendix A.

two-component indices as follows:

$$\xi^I_\alpha = (\xi^I_i, \xi^I_{\underline{i}}), \quad \rho^\alpha_I = (\rho^i_I, \rho^{\underline{i}}_I), \tag{3.3}$$

where the two-component indices take the values $i = 1, 2$ and $\underline{i} = 1, 2$, i.e. $i = \alpha$ for $\alpha = 1, 2$, and $\underline{i} = \alpha - 2$ for $\alpha = 3, 4$.

In the light-cone frame (3.1) the operators (2.12) take the form

$$\tilde{\ell} = k\, \epsilon^{ij} \epsilon_{IJ}\, \xi^I_i \xi^J_j, \quad \ell = k\, \epsilon_{\underline{i}\underline{j}} \epsilon^{IJ}\, \rho^{\underline{i}}_I \rho^{\underline{j}}_J, \tag{3.4}$$

where the matrices $\tilde{\sigma}^-$ (B.5) and σ^- (B.4) were used. Then in this frame the constraints

$$\tilde{\ell} = \mu, \quad \ell = \mu \tag{3.5}$$

from (2.17) are written in the form

$$\epsilon_{IJ} u^I_i u^J_j = \epsilon_{ij}, \quad \epsilon_{IJ} v^I_{\underline{i}} v^J_{\underline{j}} = \epsilon_{\underline{i}\underline{j}}, \tag{3.6}$$

where we have used the spinor variables

$$u^I_i := \sqrt{2k/\mu}\, \xi^I_i; \quad v^I_{\underline{i}} := \sqrt{2k/\mu}\, \rho^I_{\underline{i}},$$
$$\rho^I_{\underline{i}} = \epsilon_{\underline{i}\underline{j}} \epsilon^{IJ} \rho^{\underline{j}}_J. \tag{3.7}$$

The conditions (2.3) in terms of the SU(2) spinors (3.7) look like

$$(u^I_i)^* = -\epsilon_{IJ} \epsilon^{ij} u^J_j, \quad (v^I_{\underline{i}})^* = -\epsilon_{IJ} \epsilon^{\underline{i}\underline{j}} v^J_{\underline{j}}. \tag{3.8}$$

Conditions (3.6), (3.8) are nothing but the ones of unimodularity, $\det u = 1$, $\det v = 1$, and unitarity, $u^\dagger u = 1$, $v^\dagger v = 1$, of the $2{\times}2$ matrices

$$u := \| u_i{}^I \|, \quad v := \| v_{\underline{i}}{}^I \| \,. \tag{3.9}$$

As a result, in the light-cone frame the variables u^I_i and $v^I_{\underline{i}}$ (3.7) are the elements of the SU(2) groups and parameterize the compact space. Further, analogously to [29,30], we will use the following notation:

$$u^1_i = u^+_i, \quad u^2_i = u^-_i, \quad v^1_{\underline{i}} = v^+_{\underline{i}}, \quad v^2_{\underline{i}} = v^-_{\underline{i}}. \tag{3.10}$$

In this notation, relations (3.6) are rewritten in the form [4]

$$u^+_i u^-_j - u^+_j u^-_i = \epsilon_{ij}, \qquad v^+_{\underline{i}} v^-_{\underline{j}} - v^+_{\underline{j}} v^-_{\underline{i}} = \epsilon_{\underline{i}\underline{j}} \tag{3.11}$$

or, in the equivalent form

$$u^{i+} u^-_i = 1, \quad v^{\underline{i}+} v^-_{\underline{i}} = 1, \tag{3.12}$$

where $u^{i\pm} = \epsilon^{ij} u^\pm_j$, $u^{\underline{i}\pm} = \epsilon^{\underline{i}\underline{j}} u^\pm_{\underline{j}}$. Note that both $u^\pm$ and $v^\pm$ have the same indices $\pm$, since they are obtained from the common SU(2)-index I for the SU(2) Majorana–Weyl spinors (3.3).

[4] Note that the U(1) charges $\pm$ in the variables $u^\pm$ and $v^\pm$ have a different meaning than the light-cone indices $\pm$ in quantities $p^\pm$. The later are the $SO(1,1)$ vector indices.

We emphasize that the variables $u^\pm$ and $v^\pm$ introduced in (3.7) completely determine the operators ℓ, $\tilde{\ell}$ in (3.4) and represent half of the different canonical pairs in the algebra (2.4). We treat the second half of the operators in (2.4) as differential operators. Thus, one considers a representation where the operators ρ^i_I and $\xi^I_{\underline{i}}$ in the algebra (2.4) are realized as differential operators

$$\rho^i_I = -i \frac{\partial}{\partial \xi^I_i} = -i\sqrt{2k/\mu}\, \frac{\partial}{\partial u^I_i},$$
$$\xi^I_{\underline{i}} = i \frac{\partial}{\partial \rho^{\underline{i}}_I} = i\sqrt{2k/\mu}\, \epsilon_{ij} \epsilon^{IJ} \frac{\partial}{\partial v^J_j}. \tag{3.13}$$

In the representation chosen, the operators ℓ and $\tilde{\ell}$ are realized by operators of multiplication by the functions of $u^\pm$, $v^\pm$.

In such a representation, the $\mathfrak{su}(2)$-generators $J_\pm := J_1 \pm i J_2$ and J_3, given by (2.20), are written as follows

$$J_\pm = D^{\pm\pm}_u + D^{\pm\pm}_v, \quad J_3 = \frac{1}{2}\left(D^0_u + D^0_v\right), \tag{3.14}$$

where

$$D^{\pm\pm}_u := u^\pm_i \frac{\partial}{\partial u^\mp_i}\,, \quad D^0_u := u^+_i \frac{\partial}{\partial u^+_i} - u^-_i \frac{\partial}{\partial u^-_i}, \tag{3.15}$$

$$D^{\pm\pm}_v := v^\pm_{\underline{i}} \frac{\partial}{\partial v^\mp_{\underline{i}}}\,, \quad D^0_v := v^+_{\underline{i}} \frac{\partial}{\partial v^+_{\underline{i}}} - v^-_{\underline{i}} \frac{\partial}{\partial v^-_{\underline{i}}} \tag{3.16}$$

coincide with the harmonic derivatives in the notation [29, 30].

Relations (2.21) are written in the form $[J_+, J_-] = 2J_3$, $[J_3, J_\pm] = \pm J_\pm$, and the Casimir operator of the $\mathfrak{su}(2)$ algebra has the standard expression

$$J_{\rm i} J_{\rm i} = J_3(J_3 + 1) + J_- J_+ = -J_3(-J_3 + 1) + J_+ J_-.$$

As a solution to the irreducibility condition (2.19), (2.22) for $6D$ representations we take the highest weight vector $|\Psi^{(2s)}\rangle$ which is defined by the equations

$$J_+ |\Psi^{(2s)}\rangle = 0, \tag{3.17}$$
$$(J_3 - s)|\Psi^{(2s)}\rangle = 0, \tag{3.18}$$

where the operators J_+ and J_3 are expressed via harmonic derivatives (3.15), (3.16) in (3.14). Recall that the vector $|\Psi^{(2s)}\rangle$ also obeys the conditions (2.17):

$$\ell\, |\Psi^{(2s)}\rangle = \mu\, |\Psi^{(2s)}\rangle, \quad \tilde{\ell}\, |\Psi^{(2s)}\rangle = \mu\, |\Psi^{(2s)}\rangle. \tag{3.19}$$

Now, we show that the vectors of the states $|\Psi^{(2s)}\rangle$ are realized as fields. In the representation (3.13) the corresponding fields in the light-cone frame are the functions $\Psi^{(2s)}(u^\pm, v^\pm)$ of four SU(2) spinors $u^\pm_i$, $v^\pm_{\underline{i}}$. Then, it is natural to present the solution of equations (3.19) by using δ-functions

$$\Psi^{(2s)}(u^\pm, v^\pm) = \delta(\ell - \mu)\delta(\tilde{\ell} - \mu)\Phi^{(2s)}(u^\pm, v^\pm), \tag{3.20}$$

where the arguments of the field $\Phi^{(2s)}(u^\pm, v^\pm)$ satisfy (3.11), (3.12), and it means that the field $\Psi^{(2s)}(u^\pm, v^\pm)$ is a function on the $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ group.

Using the relations (3.14), one rewrites the remaining conditions (3.17) and (3.18) in the form

$$\left(D_u^{++} + D_v^{++}\right)\Phi^{(2s)}(u^\pm, v^\pm) = 0, \tag{3.21}$$

$$\left(D_u^{0} + D_v^{0} - 2s\right)\Phi^{(2s)}(u^\pm, v^\pm) = 0. \tag{3.22}$$

Equation (3.22) means the U(1) covariance of the field $\Phi^{(2s)}(u^\pm, v^\pm)$:

$$\Phi^{(2s)}(e^{\pm i\varphi}u^\pm, e^{\pm i\alpha}v^\pm) = e^{2si\alpha}\Phi^{(2s)}(u^\pm, v^\pm). \tag{3.23}$$

The charge $(2s)$ in the notation of the field $\Phi^{(2s)}$ reflects the property (3.23) of this field. Transformations of the arguments of the field $\Phi^{(2s)}(u^\pm, v^\pm)$ in (3.23) are obtained from the right action on the matrices (3.9) by the diagonal unitary matrix h:

$$u_i{}^J \rightarrow u_i{}^K h_K{}^J, \quad v_{\underline{i}}{}^J \rightarrow v_{\underline{i}}{}^K h_K{}^J,$$
$$h = \| h_K{}^J \| := \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha}\end{pmatrix}, \tag{3.24}$$

where $K, J = (+,-)$. In the standard stereographic parametrization of the $SU(2)$ matrices

$$\begin{pmatrix} u_1^+ & u_1^- \\ u_2^+ & u_2^-\end{pmatrix} = \frac{1}{\sqrt{1+t_1\bar{t}_1}}\begin{pmatrix} 1 & -\bar{t}_1 \\ t_1 & 1\end{pmatrix}\begin{pmatrix} e^{i(\psi+\varphi)} & 0 \\ 0 & e^{-i(\psi+\varphi)}\end{pmatrix},$$
$$\begin{pmatrix} v_{\underline{1}}^+ & v_{\underline{1}}^- \\ v_{\underline{2}}^+ & v_{\underline{2}}^-\end{pmatrix} = \frac{1}{\sqrt{1+t_2\bar{t}_2}}\begin{pmatrix} 1 & -\bar{t}_2 \\ t_2 & 1\end{pmatrix}\begin{pmatrix} e^{i(\psi-\varphi)} & 0 \\ 0 & e^{-i(\psi-\varphi)}\end{pmatrix}, \tag{3.25}$$

the transformation (3.24) is represented by the phase shift $\psi \rightarrow \psi + \alpha$. Moreover, due to the fact that the field $\Phi^{(2s)}(u^\pm, v^\pm)$ has a fixed U(1)-charge equal to $2s$, its dependence on the phase variable ψ is factorized:

$$\Phi^{(2s)}(u^\pm, v^\pm) = e^{2si\psi}\,\hat{\Phi}(t_1, t_2, \bar{t}_1, \bar{t}_2, \varphi)\,. \tag{3.26}$$

The field $\hat{\Phi}(t_1, t_2, \bar{t}_1, \bar{t}_2, \varphi)$ on the right-hand side of equality (3.26) is the function on the coset $[\mathrm{SU}(2)\otimes\mathrm{SU}(2)]/\mathrm{U}(1)$ where the variable ψ is the coordinate of the stability subgroup U(1). Thus, the field $\Phi^{(2s)}(u^\pm, v^\pm)$ having a fixed U(1)-charge is in a one-to-one correspondence with the function on the coset space $[\mathrm{SU}(2)\otimes\mathrm{SU}(2)]/\mathrm{U}(1)$ [29,30].[5] For this reason, we may refer to the variables $u_i^\pm$, $v_{\underline{i}}^\pm$ used here as the $[\mathrm{SU}(2)\otimes\mathrm{SU}(2)]/\mathrm{U}(1)$ harmonics. Since the variables $u_i^\pm$, $v_{\underline{i}}^\pm$ consist of twice the number of harmonics used in [29,30], the space parameterized by these four SU(2) spinors can be called the bi-harmonic space.

Note that a slightly different type of the bi-harmonic space was previously used in the study of various supersymmetric models. For example, two types of harmonics were employed in [41,42] for constructing an off-shell superfield formulation of the $2D$, $(4,4)$ sigma-model. In those papers, the harmonics were used to parameterize the coset space $\mathrm{SU}_L(2)/\mathrm{U}_L(1)\otimes\mathrm{SU}(2)_R/\mathrm{U}_R(1)$, where the spaces $\mathrm{SU}_L(2)/\mathrm{U}_L(1)$ and $\mathrm{SU}(2)_R/\mathrm{U}_R(1)$ were associated with the harmonics $u^{\pm,0}$ and $v^{0,\pm}$, respectively, having the charges of different U(1) groups. As a result, the fields on this harmonic coset have two U(1) charges, which are defined as eigenvalues of the operators D_u^0 and D_v^0. On the other hand, in the case considered here, the field is defined by Eq. (3.22), where the only U(1) charge $2s$ of the field $\Phi^{(2s)}(u^\pm, v^\pm)$ is given as the eigenvalue of the U(1) generator $D^0 = D_u^0 + D_v^0$. Besides, the U(1) charges ($\pm$) of the two pairs of harmonics $u^\pm$, $v^\pm$ coincide unlike the harmonics in [41,42]. As discussed above, this means that the field $\Phi^{(2s)}(u^\pm, v^\pm)$ of a special type is defined on bi-harmonic space where the coordinates $u^\pm$ and $v^\pm$ parameterize the coset $[\mathrm{SU}(2)\otimes\mathrm{SU}(2)]/\mathrm{U}(1)$, as shown in (3.23). Another type of bi-harmonics was used, e.g. in [43], to describe the effective actions $\mathcal{N}=4$ SYM theory (see the details in [43] and the references therein).

Now we describe the general solution to equations (3.21) and (3.22).

First, we note that any function of the variables

$$y_{i\underline{j}} := u_i^+ v_{\underline{j}}^- - u_i^- v_{\underline{j}}^+ \tag{3.27}$$

satisfies Eqs. (3.21), (3.22) for $s = 0$.

Then, in general, the field $\Phi^{(2s)}(u^\pm, v^\pm)$, obeying equations (3.21), (3.22) for $2s \in \mathbb{Z}{\geq}0$, is written in the form

$$\Phi^{(2s)}(u^\pm, v^\pm) = \sum_{r=0}^{\infty}\Phi^{(2s)}_{k(r)\,\underline{l}(r)}(u^+, v^+)\; y^{k(r)\,\underline{l}(r)}, \tag{3.28}$$

where

$$\Phi^{(2s)}_{k(r)\,\underline{l}(r)}(u^+, v^+) = \sum_{\substack{p,q=0,\\ p+q=2s}}^{2s}\phi^{\,i(p)\,\underline{j}(q)}_{k(r)\,\underline{l}(r)}\, u^+_{i(p)} v^+_{\underline{j}(q)}. \tag{3.29}$$

Expressions (3.28) and (3.29) use the following concise notation for the monomials:

$$u^+_{i(r)} := u^+_{i_1}\ldots u^+_{i_r}, \quad v^+_{\underline{i}(r)} := v^+_{\underline{i}_1}\ldots v^+_{\underline{i}_r},$$
$$y^{i(r)\underline{j}(r)} := y^{i_1\underline{j}_1}\ldots y^{i_r\underline{j}_r}, \tag{3.30}$$

and we use the standard convention $y^{i\underline{j}} = \epsilon^{ik}\epsilon^{\underline{j}\underline{l}}y_{k\underline{l}}$ for raising and lowering the SU(2) indices.

The field $\Phi^{(2s)}(u^\pm, v^\pm)$ in (3.28) that satisfies (3.21) and (3.22) is a linear combination with the constant coefficients $\phi^{\,i(p)\,\underline{j}(q)}_{k(r)\,\underline{l}(r)}$ of an infinite number of basis states $u^+_{i(p)}v^+_{\underline{j}(q)}y_{k(r)\,\underline{l}(r)}$. The corresponding combinations of these basis vectors allow us to define the space for the irreducible infinite spin $\mathfrak{iso}(1,5)$ representation in the light-cone frame.

As a solution of the irreducibility condition (2.19), we chose one of the $2s{+}1$ possible vectors in the space of the

[5] Various aspects of functions on such a coset are discussed in [40].

$\mathfrak{su}(2)$ irreps with spin s, namely, we took the higher weight vector $|\Psi^{(2s)}\rangle$ defined by conditions (3.17), (3.18). This choice does not lead to loss of generality. The remaining $2s$ vectors are obtained from this vector $|\Psi^{(2s)}\rangle$ by acting of the operator $(J_-)^k$ at $k = 1, \dots, 2s$. In the representation (3.14), (3.21) and (3.22) the fields $\Phi^{(2s-2k)}$ are obtained by the action of the operator $(D^{--})^k$ on the field $\Phi^{(2s)}$: $\Phi^{(2s-2k)} = (D^{--})^k \Phi^{(2s)}$. Note that the action of $(D^{--})^k$ decreases the degree of the polynomial $\Phi^{(2s)}_{k(r)\underline{l}(r)}(u^+, v^+)$ (3.29) in the variables (u^+, v^+) and increases it in the variables (u^-, v^-). Choosing any other fields $\Phi^{(2s-2k)}$, $k = 1, \dots, 2s$ leads to the equivalent infinite spin representations of $\mathfrak{iso}(1,5)$.

4 Field theory in the light-front coordinates

In the previous section, we have developed a description of the irreducible $6D$ infinite spin representations in the light-cone frame and shown that this description is formulated in terms of fields in bi-harmonic space. Now we extend this analysis to the light-front coordinate system and construct the corresponding field theory.

The formulation of the field theory on the light-front was proposed by Dirac [31], its further development and applications were considered by many authors (see e.g. [32,33,36,37] and the references therein). The light-front is defined as the surface $x^+ = const$ in the six-dimensional Minkowski space $\mathbb{R}^{1,5}$. It means that the coordinate x^+ is interpreted as a "time" evolution parameter. Therefore, the role of the Hamiltonian in the case under consideration is played by the operator

$$H = P^-. \tag{4.1}$$

To define an infinite spin field in the light-front coordinates, we will use the results of the previous section, where the corresponding field is given by (3.28), (3.29). Note that the light-cone coordinate system is obtained from the light-front coordinate system by vanishing the coordinates $x^{\hat a}$ and fixing the coordinates $x^\pm$. Therefore, it is natural to assume that the infinite spin field in the light-front coordinates should have the form (3.28), (3.29), where, however, the coefficients $\phi^{i(p)\underline{j}(q)}_{k(r)\underline{l}(r)}$ are functions of $x^\pm$ and $x^{\hat a}$. As a result, the irreducible infinite spin field depending on the light-front coordinates is defined as

$$\Phi^{(2s)}(x^\pm, x^{\hat a}, u^\pm, v^\pm) = \sum_{\substack{p,q=0,\\ p+q=2s}}^{2s} \sum_{r=0}^{\infty} \phi^{i(p)\underline{j}(q)}_{k(r)\underline{l}(r)}(x^\pm, x^{\hat a})\, u^+_{i(p)} v^+_{\underline{j}(q)}\, y^{k(r)\underline{l}(r)}. \tag{4.2}$$

Taking into account a general principle of the light-cone dynamics [31], one concludes that the equation of motion for the field (4.2) is the Schrödinger-type equation

$$\left(-i\frac{\partial}{\partial x^+} - H\right)\Phi^{(2s)}(x^\pm, x^{\hat a}, u^\pm, v^\pm) = 0, \tag{4.3}$$

where the coordinate x^+ plays the role of time.

As usual, the generators P_a and M_{ab} of $\mathfrak{iso}(1,5)$ in the the light-front formulation are divided into kinematic and dynamic generators. One can show that these divisions in the field realization (4.2) have the form

- Kinematic generators

$$P^+ = p^+, \quad P^{\hat a} = p^{\hat a}, \tag{4.4}$$

$$M^{\hat a\hat b} = x^{\hat b} p^{\hat a} - x^{\hat a} p^{\hat b} + S^{\hat a\hat b}, \quad M^{+\hat a} = x^{\hat a} p^+ + S^{+\hat a},$$
$$M^{+-} = x^- p^+ + S^{+-}; \tag{4.5}$$

- Dynamic generators

$$P^- = \frac{p^{\hat a} p^{\hat a}}{2p^+} = H, \tag{4.6}$$

$$M^{-\hat a} = x^{\hat a} H - x^- p^{\hat a} + S^{-\hat a}, \tag{4.7}$$

where

$$p^{\hat a} = i\frac{\partial}{\partial x^{\hat a}}, \quad p^+ = -i\frac{\partial}{\partial x^-} \tag{4.8}$$

and all spin parts of the Lorentz rotation generators $S^{ab} = (S^{\hat a\hat b}, S^{\pm\hat a}, S^{+-})$ depend on the spinors $u_i^\pm$ and $v_{\underline{i}}^\pm$ in the same way as the operators (B.10), (B.14), (B.17), (B.18) depend on the spinors ξ_i^I and $\rho_{\underline{i}}^I$

$$S^{\hat a\hat b} = \frac{1}{2}\eta^{\mathrm{i}}_{\hat a\hat b}\left[\frac{\partial}{\partial u_i^+}(\tau_{\mathrm{i}})_i{}^j u_j^+ + \frac{\partial}{\partial u_i^-}(\tau_{\mathrm{i}})_i{}^j u_j^-\right]$$
$$+\frac{1}{2}\bar\eta^{\mathrm{i}'}_{\hat a\hat b}\left[\frac{\partial}{\partial v_{\underline{i}}^+}(\tau_{\mathrm{i}'})_{\underline{i}}{}^{\underline{j}} v_{\underline{j}}^+ + \frac{\partial}{\partial v_{\underline{i}}^-}(\tau_{\mathrm{i}'})_{\underline{i}}{}^{\underline{j}} v_{\underline{j}}^-\right], \tag{4.9}$$

$$S^{+\hat a} = \frac{i}{\sqrt{2}}\left[\frac{\partial}{\partial u_i^+}(\tau_{\hat a})_i{}^{\underline{j}}\epsilon_{\underline{jk}}\frac{\partial}{\partial v_{\underline{k}}^-} - \frac{\partial}{\partial u_i^-}(\tau_{\hat a})_i{}^{\underline{j}}\epsilon_{\underline{jk}}\frac{\partial}{\partial v_{\underline{k}}^+}\right],$$
$$\text{at } \hat a = 1,2,3, \tag{4.10}$$

$$S^{-\hat a} = \frac{i}{\sqrt{2}}\left[v_{\underline{k}}^-\epsilon^{\underline{k}i}(\tau_{\hat a})_{\underline{i}}{}^j u_j^+ - v_{\underline{k}}^+\epsilon^{\underline{k}i}(\tau_{\hat a})_{\underline{i}}{}^j u_j^-\right],$$
$$\text{at } \hat a = 1,2,3, \tag{4.11}$$

$$S^{+4} = \frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial u_i^+}\epsilon_{i\underline{k}}\frac{\partial}{\partial v_{\underline{k}}^-} - \frac{\partial}{\partial u_i^-}\epsilon_{i\underline{k}}\frac{\partial}{\partial v_{\underline{k}}^+}\right], \tag{4.12}$$

$$S^{-4} = \frac{1}{\sqrt{2}}\left[v_{\underline{k}}^-\epsilon^{\underline{k}i} u_i^+ - v_{\underline{k}}^+\epsilon^{\underline{k}i} u_i^-\right], \tag{4.13}$$

$$S^{+-} = \frac{1}{2}\left[u_i^+\frac{\partial}{\partial u_i^+} + u_i^-\frac{\partial}{\partial u_i^-} + v_{\underline{i}}^+\frac{\partial}{\partial v_{\underline{i}}^+} + v_{\underline{i}}^-\frac{\partial}{\partial v_{\underline{i}}^-} + 4\right]. \tag{4.14}$$

Turn attention that all the generators $S^{ab} = (S^{\hat{a}\hat{b}}, S^{\pm\hat{a}}, S^{+-})$ defined in (4.9)–(4.14) have zero U(1)-charge.

After acting by the operator p^+ on Eq. (4.3), this equation takes the form[6]

$$\Box\, \Phi^{(2s)}(x^\pm, x^{\hat{a}}, u^\pm, v^\pm) = 0, \tag{4.15}$$

where $\Box$ is the d'Alambertian operator in the six-dimensional Minkowski space in the light-front coordinates

$$\Box \ := \ 2\frac{\partial}{\partial x^+}\frac{\partial}{\partial x^-} - \frac{\partial}{\partial x^{\hat{a}}}\frac{\partial}{\partial x^{\hat{a}}}. \tag{4.16}$$

Equation (4.15) is the equation of motion corresponding to the action

$$S \ = \ \int d^6x\, du\, dv\ \bar{\Phi}^{(-2s)} \Box\, \Phi^{(2s)}, \tag{4.17}$$

where $d^6x = dx^+dx^-d^4x$ is the $6D$ Minkowski space measure and $dudv$ is the bi-harmonic space measure [29,30]. The function $\bar{\Phi}^{(-2s)}$ is obtained by complex conjugation of the function $\Phi^{(2s)}$:

$$\bar{\Phi}^{(-2s)} = (\Phi^{(2s)})^*. \tag{4.18}$$

Integration over harmonics is defined by simple rules (see [29,30] for details). The integral is is a linear operation and it does not vanish only for SU(2)-scalars with the following normalization condition:

$$\int du = 1, \quad \int dv = 1. \tag{4.19}$$

For all other harmonic monomials, the harmonic integral is equal to zero:

$$\begin{aligned} &\int du\, u^+_{(i_1} \ldots u^+_{i_m} u^-_{j_1} \ldots u^-_{j_n)} = 0, \\ &\int dv\, v^+_{(\underline{i}_1} \ldots v^+_{\underline{i}_m} v^-_{\underline{j}_1} \ldots v^-_{\underline{j}_n)} = 0, \end{aligned} \tag{4.20}$$

at arbitrary integers m and n which are not equal to zero simultaneously.

Reality conditions (3.8) are now written as follows:

$$(u^{i\pm})^* = \pm u^\mp_i, \quad (v^{i\pm})^* = \pm v^\mp_i. \tag{4.21}$$

Therefore, at complex conjugation the charge $2s$ of the harmonic field $\Phi^{(2s)}$ changes to $-2s$ in accordance with (4.18). As a result, the integrand in (4.17) has a zero harmonic charge as it should be for the non-vanishing harmonic integral.[7]

In expansion of the harmonic field (4.2) the indices i and $\underline{i}$ of the component fields $\phi^{(i(m)j(n))(\underline{i}(k)\underline{j}(n))}(x)$ are half of the $6D$ SU$^*(4)$-indices α. It means, for half-integer s, the harmonic field $\Phi^{(2s)}(x, u^\pm, v^\pm)$ is an odd order polynomial in $u^\pm$, $v^\pm$ and describes half-integer spin fields with an odd number of indices. Therefore, the fermionic fields should be endowed by the corresponding odd statistics. Besides, in the fermionic case, the natural Lagrangian is the one of the first order in space derivatives. This type of Lagrangian in the light-front formalism is obtained from the Lagrangian (4.17) by replacement

$$\Psi^{(2s)} = \sqrt{p^+}\, \Phi^{(2s)}. \tag{4.22}$$

Then, for the field $\Psi^{(2s)}$ expression (4.17) leads to the following Lagrangian:[8]

$$\bar{\Psi}^{(-2s)} \left(p^- - H\right) \Psi^{(2s)}. \tag{4.23}$$

Thus, the action (4.17) determines the field dynamics of infinite spin fields on the light front. A specific feature of the obtained theory is its formulation in terms of harmonic variables.

5 Summary

We have developed the $6D$ Minkowski space infinite spin free Lagrangian field theory in the light-cone formalism. First, we have studied this theory in the light-cone frame and unexpectedly found that the corresponding infinite spin field is a function on a special bi-harmonic space associated with the coset $[\mathrm{SU}(2)\otimes\mathrm{SU}(2)]/\mathrm{U}(1)$. Second, the result obtained was generalized to the light-front coordinate system, where the infinite spin field is described by the function $\Phi^{(2s)}(x^\pm, x^{\hat{a}}, u^\pm, v^\pm)$ (4.2) depending on the light-front coordinates and harmonics. Representations of all the $6D$ Poincaré group generators in this coordinate system are constructed. The field equation of motion in the light-front coordinate system has the form of Schrödinger-type Eq. (4.3) with the Hamiltonian (4.1). The corresponding action is given by (4.17).

The harmonic light-front approach formulated in this paper opens a possibility to construct an interacting theory for $6D$ infinite spin fields. One can expect that introducing an interaction will lead to a modification of the dynamic generators (4.6), (4.7) by the interaction terms (see the description of interactions in the light-front formalism, e.g., in [37] and the references therein). In particular, the Hamiltonian (4.6) should go to

$$H \quad \to \quad H + H_{\text{int}}. \tag{5.1}$$

The harmonic formalism allows one from the very beginning to make some simple predictions on the structure of the interacting Hamiltonian H_{int}. To preserve zero harmonic

6 For a discussion of the operator p^+ invertibility, see e.g. [34,35].

7 Note that in the harmonic superspace approach to $\mathcal{N}=2$ supersymmetric field theories [29,30] another rule of conjugation was used that combines complex conjugation with an antipodal map. However, in the case under consideration, the ordinary complex conjugation is totally appropriate.

8 For discussion of the light-front describing the fields with different statistics see e.g. [34,35].

charge of the action, this Hamiltonian should have zero harmonic charge as well. It immediately means that an arbitrary order self-interaction of the same harmonic fields $\Phi^{(2s)}$ is possible only for $s = 0$ if other charged harmonic quantities in the action are absent. Self-interaction of charged fields $\Phi^{(2s)}$, $s \neq 0$ can only be of an even order, such as $\sim \bar{\Phi}^{(-2s)}\bar{\Phi}^{(-2s)}\Phi^{(2s)}\Phi^{(2s)}$. Although for fields with different charges there is an additional choice in the structure of the interaction Lagrangian. For example, the following interacting terms $\sim \bar{\Phi}_1^{(-2s)}\left(\Phi_2^{(0)} + \bar{\Phi}_2^{(0)}\right)\Phi_1^{(2s)}$ or $\sim\left(\Phi_1^{(q_1)}\Phi_2^{(q_2)}\Phi_3^{(q_3)} + c.c.\right)$ at $q_1+q_2+q_3 = 0$ are allowed in the action. In general, the requirement of zero charge of interacting contributions to the action controls both charges of interacting fields and their number. We plan to construct interacting infinite spin $6D$ theories in the forthcoming works.

We also think that the appearance of bi-harmonic space in the infinite spin representations of $\mathfrak{iso}(1,5)$ can indicate the existence of manifest $\mathcal{N}=(1,0)$ supersymmetrization of the theory (4.17), (5.1).[9]

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Appendix A: Calculation of the Casimir operator C_6

The six-order Casimir operator in the $6D$ massless theory is given by (2.18). The derivation of this operator in [3,4] was based on the relations:

$$C_6 = \frac{1}{64}\,\Upsilon^a\Upsilon_a, \quad \Upsilon_a = \varepsilon_{abcdef}\,P^bM^{cd}M^{ef} \tag{A.1}$$

These expressions show that only the spin part S_{ab} (2.7) of the Lorentz group generators M_{ab} (2.6) contributes to the Casimir operator (A.1). However, if we substitute S_{ab} for M_{ab} into (2.18), an incorrect result is obtained, since when passing from (A.1) to (2.18), one has to rearrange the operators P_c and M_{ab} using commutators. At the replacement $M_{ab} \to S_{ab}$, a correct expression C_6 can be obtained only after preliminary "untangling" of the generators P_c и M_{ab} in expression (2.18).

We will act in the following way. First, we rearrange with the help of commutation relations all the operators P_c to the right on all the operators M_{ab} in expression (2.18). Second, after such an ordering is done, we replace the operator M_{ab} by the operator S_{ab} in the obtained expression.

Using the commutator $[M_{ab}, \Pi_c] = i\,(\eta_{ac}\Pi_b - \eta_{bc}\Pi_a)$, we rearrange the operators Π_a to the right in the first term of expression (2.18). All the terms proportional to $2\Pi_{[a}\Pi_{b]} = [\Pi_a, \Pi_b] = -i\,M_{ab}\,P^2$ can be omitted for a massless representation where $P^2 = 0$. Besides, since we consider irreducible infinite spin for which the condition (2.16) holds, we replace the operator C_4 by its eigenvalue $-\mu^2$ in the second term of (2.18). Now all operators M_{ab} to the left of the operators Π_a are replaced by the operators S_{ab}. As a result, one obtains

$$C_6 = -\,S^{(b}{}_a\,S^{c)a}\,\Pi_b\Pi_c \,-\, \frac{1}{2}\,\mu^2\left(S^{bc}S_{bc} - 8\right). \tag{A.2}$$

Using the relations for the σ-matrices from [5], the identity

$$\begin{aligned}(\tilde{\sigma}^b{}_a)^\alpha{}_\beta(\tilde{\sigma}^{ca})^\gamma{}_\delta =& \frac{1}{4}\,\eta^{bc}\left(\delta^\alpha_\beta\delta^\gamma_\delta - 2\delta^\alpha_\delta\delta^\gamma_\beta\right)\\ &+\frac{1}{2}(\tilde{\sigma}^{(b})^{\alpha\gamma}(\sigma^{c)})_{\beta\delta} + \frac{1}{2}\,\delta^\alpha_\delta\,(\tilde{\sigma}^{bc})^\gamma{}_\beta\\ &-\frac{1}{2}\,\delta^\gamma_\beta\,(\tilde{\sigma}^{bc})^\alpha{}_\delta,\end{aligned} \tag{A.3}$$

the commutation relations (2.4) and the realization (2.7) for the operators S_{ab}, one gets the equality

$$\begin{aligned}S^{(b}{}_a S^{c)a} =& \,\eta^{bc}\left[\frac{1}{2}\,(\xi_{(I}\rho_{K)})(\xi^I\rho^K) + i\,(\xi^I\rho_I)\right]\\ &+\frac{1}{2}\,\xi^I_\alpha\rho^\beta_I\,\xi^J_\gamma\,\rho^\delta_J\,(\tilde{\sigma}^{(b})^{\alpha\gamma}(\sigma^{c)})_{\beta\delta},\end{aligned} \tag{A.4}$$

which leads to

$$S^{bc}S_{bc} = -\frac{1}{2}\,(\xi^I\rho_I)^2 + 4i\,(\xi^I\rho_I) + 2(\xi_{(I}\rho_{K)})(\xi^I\rho^K), \tag{A.5}$$

where the notation $(\xi^I\rho_K) := \xi^I_\alpha\rho^\alpha_K$ has been used. After substituting (A.4) and (A.5) into (A.2), one gets

$$C_6 = \mu^2\left[\frac{1}{4}\,(\xi^I\rho_I)^2 - \frac{1}{2}\,(\xi_{(I}\rho_{J)})(\xi^I\rho^J) - i\,(\xi^I\rho_I) + 4\right]$$

[9] See, e.g., harmonic superfield formulation of the six-dimensional $\mathcal{N}=(1,0)$ and $\mathcal{N}=(1,1)$ supersymmetric theories in [44] and references therein.

$$-\frac{1}{2}\,\xi^I_\alpha\rho^\beta_I\xi^J_\gamma\rho^\delta_J(\tilde\sigma^b)^{\alpha\gamma}(\sigma^c)_{\beta\delta}\Pi_b\Pi_c. \qquad (A.6)$$

The last term in this expression is represented in the following form:

$$-\frac{1}{2}\,\xi^I_\alpha\rho^\beta_I\xi^J_\gamma\rho^\delta_J(\tilde\sigma^b)^{\alpha\gamma}(\sigma^c)_{\beta\delta}\Pi_b\Pi_c = -\frac{i}{2}\,\mu^2(\xi^I\rho_I) + \frac{1}{4}\,(\xi^I\tilde\sigma^b\xi_I)(\rho_J\sigma^c\rho^J)\Pi_b\Pi_c\,, \qquad (A.7)$$

where $(\xi^I\tilde\sigma^b\xi_I) := \xi^I_\alpha(\tilde\sigma^b)^{\alpha\beta}\xi_{\beta I}$, $(\rho_J\sigma^c\rho^J) := \rho^\alpha_J(\sigma^c)_{\alpha\beta}\rho^{\beta J}$.

The last step in deriving the expression for C_6 is to move the operators P_m to the right in expression (A.7). Using the equality $\Pi_a = M_{ba}P^b - 5i\,P_a$, we write the expression $\Pi_b\Pi_c$ in the form:

$$\Pi_b\Pi_c = M_{eb}M_{fc}P^eP^f - 6i\,M_{eb}P^eP_c - 5i\,M_{ec}P^eP_b - 30P_bP_c. \qquad (A.8)$$

Now that all operators M_{ab} on the right side of (A.8) are to the left of all operators P_c, we replace the operators M_{ab} with their spin parts S_{ab}.

After such a replacement $M_{ab} \to S_{ab}$, where S_{ab} are defined in (2.7), the substitution (A.8) into (A.7) and (A.6) and using the equalities

$$\frac{1}{4}\,(\xi^I\tilde\sigma^b\xi_I)(\rho_J\sigma^c\rho^J)S_{eb}S_{fc}P^eP^f = \mu^2\left[-\frac{1}{4}\,(\xi^I\rho_I)^2 + 2i(\xi^I\rho_I) - 6\right], \qquad (A.9)$$

$$-\frac{3i}{2}\,(\xi^I\tilde\sigma^b\xi_I)(\rho_J\sigma^c\rho^J)S_{eb}P^eP_c = \mu^2\left[-3i(\xi^I\rho_I) + 12\right], \qquad (A.10)$$

$$-\frac{5i}{4}\,(\xi^I\tilde\sigma^b\xi_I)(\rho_J\sigma^c\rho^J)S_{ec}P^eP_b = \mu^2\left[\frac{5i}{2}(\xi^I\rho_I) + 20\right], \qquad (A.11)$$

$$-\frac{15}{2}\,(\xi^I\tilde\sigma^b\xi_I)(\rho_J\sigma^c\rho^J)P_bP_c = -30\mu^2, \qquad (A.12)$$

which are valid at $P^2 = 0$ and $(\xi^I\tilde\sigma^b\xi_I)(\rho_J\sigma^c\rho^J)P_bP_c = 4\mu^2$ (second equality is the condition (2.16) for the fourth-order Casimir operator), one obtains

$$C_6 = -\frac{1}{2}\,\mu^2(\xi_{(I}\rho_{J)})(\xi^I\rho^J). \qquad (A.13)$$

When using the operators (2.20), this final expression (A.13) is represented as

$$C_6 = -\mu^2 J_{\rm i}J_{\rm i}, \qquad (A.14)$$

which is the same as (2.19).

Appendix B: Spinor part of the $\mathfrak{so}(1,5)$-generators

We consider a representation where the (4×4) σ-matrices [5,45,46] $\sigma^a = \|(\sigma^a)_{\alpha\dot\beta}\|$, $\tilde\sigma^a = \|(\tilde\sigma^a)^{\dot\alpha\beta}\|$ with the $6D$ vector index $a = 0,1,\ldots,5$ and spinor indices $\alpha,\dot\alpha = 1\ldots,4$ are realized in the form of the following matrices:

$$\sigma^a = (\sigma^0,\sigma^{\hat a},\sigma^5), \quad \tilde\sigma^a = (\sigma^0,-\sigma^{\hat a},-\sigma^5), \quad \hat a = 1,\ldots,4, \qquad (B.1)$$

where

$$\sigma^0 = 1_4, \quad \sigma^{\hat a} = \tau_2\otimes\tau_{\hat a} \ \text{ at } \ \hat a = 1,2,3, \quad \sigma^4 = -\tau_1\otimes 1_2, \quad \sigma^5 = \tau_3\otimes 1_2 \qquad (B.2)$$

and $\tau_{1,2,3}$ are the Pauli matrices.[10] The antisymmetric σ-matrices with non-dotted spinor subscripts and superscripts are defined as follows:

$$(\sigma^a)_{\alpha\beta} = (\sigma^a)_{\alpha\dot\gamma}(B^{-1})_\beta{}^{\dot\gamma}, \quad (\tilde\sigma^a)^{\alpha\beta} = B_{\dot\gamma}{}^\alpha(\tilde\sigma^a)^{\dot\gamma\beta}, \qquad (B.3)$$

where $B = \|B_{\dot\alpha}{}^\beta\| = 1_2\otimes i\tau_2 = \begin{pmatrix} i\tau_2 & 0 \\ 0 & i\tau_2\end{pmatrix}$ is the matrix defining complex conjugation of the $6D$ Weyl spinors [5,45,46]. In particular, the matrices $\sigma^\pm = (\sigma^0\pm\sigma^5)/\sqrt{2}$, $\tilde\sigma^\pm = (\tilde\sigma^0\pm\tilde\sigma^5)/\sqrt{2}$ with undotted indices have the form:

$$(\sigma^+)_{\alpha\beta} = \sqrt{2}\begin{pmatrix} i\tau_2 & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} \sqrt{2}\epsilon_{ij} & 0 \\ 0 & 0\end{pmatrix}, \quad (\sigma^-)_{\alpha\beta} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & i\tau_2\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\epsilon_{\underline{i}\underline{j}}\end{pmatrix}, \qquad (B.4)$$

$$(\tilde\sigma^+)^{\alpha\beta} = \sqrt{2}\begin{pmatrix} 0 & 0 \\ 0 & -i\tau_2\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2}\epsilon^{\underline{i}\underline{j}}\end{pmatrix}, \quad (\tilde\sigma^-)^{\alpha\beta} = \sqrt{2}\begin{pmatrix} -i\tau_2 & 0 \\ 0 & 0\end{pmatrix} = \begin{pmatrix} \sqrt{2}\epsilon^{ij} & 0 \\ 0 & 0\end{pmatrix}. \qquad (B.5)$$

Also we use the standard representation for the 't Hooft symbols $\eta^{\rm i}_{\hat a\hat b} = -\eta^{\rm i}_{\hat b\hat a}$, i $= 1,2,3$ and $\bar\eta^{\rm i'}_{\hat a\hat b} = -\bar\eta^{\rm i'}_{\hat b\hat a}$, i$' = 1,2,3$ (see e.g. [3,4,47,48])

$$\eta^{\rm i}_{\hat a\hat b} = \begin{cases} \epsilon_{{\rm i}\hat a\hat b} & \hat a,\hat b = 1,2,3, \\ \delta_{{\rm i}\hat a} & \hat b = 4, \end{cases} \quad \bar\eta^{\rm i'}_{\hat a\hat b} = \begin{cases} \epsilon_{{\rm i'}\hat a\hat b} & \hat a,\hat b = 1,2,3, \\ -\delta_{{\rm i'}\hat a} & \hat b = 4. \end{cases} \qquad (B.6)$$

First, we will consider the $\mathfrak{so}(4)$-part of the generators (2.7), i.e. the operators

$$S_{\hat a\hat b} = \xi^I_\alpha(\tilde\sigma_{\hat a\hat b})^\alpha{}_\beta\rho^\beta_I, \quad \hat a = 1,2,3,4. \qquad (B.7)$$

[10] In [5,45] the representation $\sigma^1 = \tau_1\otimes 1_2$, $\sigma^{\hat a} = \tau_2\otimes\tau_{\hat a-1}$ at $\hat a = 2,3,4$ and σ^0,σ^5 was used as in (B.2). That is, distinction between the representations [5,45] and (B.2) lies in the difference in the notation of the four space coordinates labeled by $\hat a$. However, the representation (B.2) is more convenient when using the standard realizations (B.6) for the 't Hooft symbols.

These six generators $S_{\hat{a}\hat{b}}$ are written as the sum

$$S_{\hat{a}\hat{b}} = S^{(+)}_{\hat{a}\hat{b}} + S^{(-)}_{\hat{a}\hat{b}}, \tag{B.8}$$

where the SO(4)-(anti-)self-dual parts $S^{(\pm)}_{\hat{a}\hat{b}} = \pm\frac{1}{2}\,\epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}S^{(\pm)}_{\hat{c}\hat{d}}$ are expressed in terms of the SO(3)-vectors $S^{(+)}_{\rm i}$, $S^{(-)}_{\rm i'}$:

$$S^{(+)}_{\hat{a}\hat{b}} = -\eta^{\rm i}_{\hat{a}\hat{b}}S^{(+)}_{\rm i}, \quad S^{(-)}_{\hat{a}\hat{b}} = -\bar{\eta}^{\rm i'}_{\hat{a}\hat{b}}S^{(-)}_{\rm i'}, \tag{B.9}$$

if we use the 't Hooft symbols (B.6). Thus, the generator (B.7) has the expansion

$$S_{\hat{a}\hat{b}} = -\eta^{\rm i}_{\hat{a}\hat{b}}S^{(+)}_{\rm i} - \bar{\eta}^{\rm i'}_{\hat{a}\hat{b}}S^{(-)}_{\rm i'}, \tag{B.10}$$

where the operators $S^{(+)}_{\rm i}$ и $S^{(-)}_{\rm i'}$ form two $\mathfrak{su}(2)$ algebras:

$$[S^{(+)}_{\rm i}, S^{(+)}_{\rm j}] = i\epsilon_{\rm ijk}S^{(+)}_{\rm k}, \quad [S^{(-)}_{\rm i'}, S^{(-)}_{\rm j'}] = i\epsilon_{\rm i'j'k'}S^{(-)}_{\rm k'},$$
$$[S^{(+)}_{\rm i}, S^{(-)}_{\rm j'}] = 0. \tag{B.11}$$

Using the equalities $\eta^{\rm i}_{ab}\eta^{\rm j}_{ab} = 4\delta^{\rm ij}$, $\bar{\eta}^{\rm i'}_{ab}\bar{\eta}^{\rm j'}_{ab} = 4\delta^{\rm i'j'}$ and $\eta^{\rm i}_{ab}\bar{\eta}^{\rm j'}_{ab} = 0$, one finds the inverse to (B.9) relations

$$S^{(+)}_{\rm i} = -\frac{1}{4}\,\eta^{\rm i}_{\hat{a}\hat{b}}S_{\hat{a}\hat{b}}, \quad S^{(-)}_{\rm i'} = -\frac{1}{4}\,\bar{\eta}^{\rm i'}_{\hat{a}\hat{b}}S_{\hat{a}\hat{b}}. \tag{B.12}$$

However, using (B.1), (B.2), (B.3) and (B.6) we obtain that the matrices present in the definition of the generators (B.12) have only one diagonal 2×2 matrix block:

$$-\frac{1}{4}\,\eta^{\rm i}_{\hat{a}\hat{b}}(\tilde{\sigma}_{\hat{a}\hat{b}})^{\alpha}{}_{\beta} = \frac{i}{2}\begin{pmatrix} -\tau^T_{\rm i} & 0 \\ 0 & 0 \end{pmatrix},$$
$$-\frac{1}{4}\,\bar{\eta}^{\rm i'}_{\hat{a}\hat{b}}(\tilde{\sigma}_{\hat{a}\hat{b}})^{\alpha}{}_{\beta} = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & -\tau^T_{\rm i'} \end{pmatrix}. \tag{B.13}$$

Substituting (B.7) and (B.13) into (B.12), one finds

$$S^{(+)}_{\rm i} = -\frac{i}{2}\,\rho^i_I(\tau_{\rm i})_i{}^j\xi^I_j, \quad S^{(-)}_{\rm i'} = -\frac{i}{2}\,\rho^{\underline{i}}_I(\tau_{\rm i'})_{\underline{i}}{}^{\underline{j}}\xi^I_{\underline{j}}. \tag{B.14}$$

Thus, the generators $S^{(+)}_{\rm i}$ are built using the canonical pairs (ξ^I_i, ρ^j_J) from (3.3), while the generators $S^{(-)}_{\rm i'}$ are built using the other canonical pairs $(\xi^I_{\underline{i}}, \rho^{\underline{j}}_J)$.

Now using the matrix expressions

$$(\tilde{\sigma}^{+\hat{a}})^{\alpha}{}_{\beta} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -\tau^T_{\hat{a}} & 0 \end{pmatrix}, \quad (\tilde{\sigma}^{-\hat{a}})^{\alpha}{}_{\beta} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & \tau^T_{\hat{a}} \\ 0 & 0 \end{pmatrix}$$
$$\text{at } \hat{a} = 1, 2, 3, \tag{B.15}$$
$$(\tilde{\sigma}^{+4})^{\alpha}{}_{\beta} = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix}, \quad (\tilde{\sigma}^{-4})^{\alpha}{}_{\beta} = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & 1_2 \\ 0 & 0 \end{pmatrix},$$
$$(\tilde{\sigma}^{+-})^{\alpha}{}_{\beta} = \frac{1}{2}\begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \tag{B.16}$$

obtained from (B.1), (B.2), (B.3), and expansion (3.3), we find the spin part of the remaining Lorentz group generators (2.7):

$$S^{+\hat{a}} = -\frac{i}{\sqrt{2}}\,\rho^i_I(\tau_{\hat{a}})_i{}^{\underline{j}}\xi^I_{\underline{j}}, \quad S^{-\hat{a}} = \frac{i}{\sqrt{2}}\,\rho^{\underline{i}}_I(\tau_{\hat{a}})_{\underline{i}}{}^j\xi^I_j,$$
$$\text{at } \hat{a} = 1, 2, 3, \tag{B.17}$$
$$S^{+4} = -\frac{1}{\sqrt{2}}\,\rho^i_I\delta_i{}^{\underline{j}}\xi^I_{\underline{j}}, \quad S^{-4} = -\frac{1}{\sqrt{2}}\,\rho^{\underline{i}}_I\delta_{\underline{i}}{}^j\xi^I_j,$$
$$S^{+-} = -\frac{1}{2}\left(\xi^I_i\rho^i_I - \xi^I_{\underline{i}}\rho^{\underline{i}}_I\right). \tag{B.18}$$

The found expressions (B.10), (B.14), (B.17), (B.18) are used in Sect. 4 to construct the spin part of the Lorentz algebra generators in the bi-harmonic space.

References

1. S. Weinberg, Massless particles in higher dimensions. Phys. Rev. D **102**, 095022 (2020). arXiv:2010.05823 [hep-th]
2. S.M. Kuzenko, A.E. Pindur, Massless particles in five and higher dimensions. Phys. Lett. B **812** (2021). arXiv:2010.07124 [hep-th]
3. I.L. Buchbinder, S.A. Fedoruk, A.P. Isaev, M.A. Podoinitsyn, Massless finite and infinite spin representations of Poincaré group in six dimensions. Phys. Lett. B **813**, 136064 (2021). arXiv:2011.14725 [hep-th]
4. I.L. Buchbinder, S.A. Fedoruk, A.P. Isaev, M.A. Podoinitsyn, Massless representations of the ISO(1, 5) group. Phys. Part. Nucl. Lett. **18**, 721 (2021)
5. I.L. Buchbinder, S.A. Fedoruk, A.P. Isaev, Twistor formulation of massless 6D infinite spin fields. Nucl. Phys. B **973**, 115576 (2021). arXiv:2108.04716 [hep-th]
6. X. Bekaert, N. Boulanger, The unitary representations of the Poincaré group in any spacetime dimension, in *Lectures presented at 2nd Modave Summer School in Theoretical Physics*, 6–12 Aug 2006, Belgium. arXiv:hep-th/0611263
7. X. Bekaert, N. Boulanger, Tensor gauge fields in arbitrary representations of $GL(D, R)$. Commun. Math. Phys. **271** (2007). arXiv:hep-th/0606198
8. E.P. Wigner, On unitary representations of the inhomogeneous Lorentz group. Ann. Math. **40**, 149 (1939)
9. E.P. Wigner, Relativistische Wellengleichungen. Z. Physik **124**, 665 (1947)
10. V. Bargmann, E.P. Wigner, Group theoretical discussion of relativistic wave equations. Proc. Natl. Acad. Sci. USA **34**, 211 (1948)
11. X. Bekaert, E.D. Skvortsov, Elementary particles with continuous spin. Int. J. Mod. Phys. A **32**, 1730019 (2017). arXiv:1708.01030 [hep-th]
12. X. Bekaert, J. Mourad, The continuous spin limit of higher spin field equations. JHEP **0601**, 115 (2006). arXiv:hep-th/0509092
13. X. Bekaert, J. Mourad, M. Najafizadeh, Continuous-spin field propagator and interaction with matter. JHEP **1711**, 113 (2017). arXiv:1710.05788 [hep-th]
14. M. Najafizadeh, Modified Wigner equations and continuous spin gauge field. Phys. Rev. D **97**, 065009 (2018). arXiv:1708.00827 [hep-th]
15. M.V. Khabarov, Y.M. Zinoviev, Infinite (continuous) spin fields in the frame-like formalism. Nucl. Phys. B **928**, 182 (2018). arXiv:1711.08223 [hep-th]
16. K.B. Alkalaev, M.A. Grigoriev, Continuous spin fields of mixed-symmetry type. JHEP **1803**, 030 (2018). arXiv:1712.02317 [hep-th]
17. R.R. Metsaev, BRST-BV approach to continuous-spin field. Phys. Lett. B **781**, 568 (2018). arXiv:1803.08421 [hep-th]

18. I.L. Buchbinder, S. Fedoruk, A.P. Isaev, A. Rusnak, Model of massless relativistic particle with continuous spin and its twistorial description. JHEP **1807**, 031 (2018). arXiv:1805.09706 [hep-th]
19. I.L. Buchbinder, V.A. Krykhtin, H. Takata, BRST approach to Lagrangian construction for bosonic continuous spin field. Phys. Lett. B **785**, 315 (2018). arXiv:1806.01640 [hep-th]
20. I.L. Buchbinder, S. Fedoruk, A.P. Isaev, V.A. Krykhtin, Towards Lagrangian construction for infinite half-integer spin field. Nucl. Phys. B **958**, 115114 (2020). arXiv:2005.07085 [hep-th]
21. K. Alkalaev, A. Chekmenev, M. Grigoriev, Unified formulation for helicity and continuous spin fermionic fields. JHEP **1811**, 050 (2018). arXiv:1808.09385 [hep-th]
22. R.R. Metsaev, Cubic interaction vertices for massive/massless continuous-spin fields and arbitrary spin fields. JHEP **1812**, 055 (2018). arXiv:1809.09075 [hep-th]
23. I.L. Buchbinder, S. Fedoruk, A.P. Isaev, Twistorial and space-time descriptions of massless infinite spin (super)particles and fields. Nucl. Phys. B **945**, 114660 (2019). arXiv:1903.07947 [hep-th]
24. R.R. Metsaev, Light-cone continuous-spin field in AdS space. Phys. Lett. B **793**, 134 (2019). arXiv:1903.10495 [hep-th]
25. I.L. Buchbinder, M.V. Khabarov, T.V. Snegirev, Y.M. Zinoviev, Lagrangian formulation for the infinite spin $N = 1$ supermultiplets in $d = 4$. Nucl. Phys. B **946**, 114717 (2019). arXiv:1904.05580 [hep-th]
26. M. Najafizadeh, Supersymmetric Continuous Spin Gauge Theory. JHEP **2003**, 027 (2020). arXiv:1912.12310 [hep-th]
27. M. Najafizadeh, Off-shell supersymmetric continuous spin Gauge theory. JHEP **02**, 038 (2022). arXiv:2112.10178 [hep-th]
28. I.L. Buchbinder, S.A. Fedoruk, A.P. Isaev, V.A. Krykhtin, On the off-shell superfield Lagrangian formulation of 4D, N=1 supersymmetric infinite spin theory. Phys. Lett. B **829**, 137139 (2022). arXiv:2203.12904 [hep-th]
29. A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, Unconstrained N=2 matter, Yang–Mills and supergravity theories in harmonic superspace. Class. Quantum Gravity **1**, 469 (1984)
30. A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky, E.S. Sokatchev, *Harmonic superspace* (Cambridge Univ. Press, 2001), p. 306
31. P.A.M. Dirac, Forms of relativistic dynamics. Rev. Mod. Phys. **21**, 392 (1949)
32. A.K.H. Bengtsson, I. Bengtsson, L. Brink, Cubic interaction terms for arbitrary spin. Nucl. Phys. B **227**, 31 (1983)
33. A.K.H. Bengtsson, I. Bengtsson, N. Linden, Interacting higher-spin gauge fields on the light front. Class. Quantum Gravity **4**, 1333 (1987)
34. W. Siegel, Introduction to string field theory. Adv. Ser. Math. Phys. **8**, 1 (1988). arXiv:hep-th/0107094
35. W. Siegel, Fields. arXiv:hep-th/9912205
36. R.R. Metsaev, Cubic interaction vertices for massive and massless higher spin fields. Nucl. Phys. B **759**, 147 (2006). arXiv:hep-th/0512342
37. D. Ponomarev, E.D. Skvortsov, Light-front higher-spin theories in flat space. J. Phys. A **50**, 095401 (2017). arXiv:1609.04655 [hep-th]
38. R. Howe, Transcending classical invariant theory. J. Am. Math. Soc. **2**, 535 (1989)
39. R. Howe, Remarks on classical invariant theory. Trans. Am. Math. Soc. **313**, 539 (1989)
40. S.M. Kuzenko, Projective superspace as a double punctured harmonic superspace. Int. J. Mod. Phys. A **14**, 1737 (1999). arXiv:hep-th/9806147
41. E. Ivanov, A. Sutulin, Sigma models in (4, 4) harmonic superspace. Nucl. Phys. B **432**, 246 (1994). arXiv:hep-th/9404098
42. E. Ivanov, A. Sutulin, Diversity of off-shell twisted (4, 4) multiplets in $SU(2) \times SU(2)$ harmonic superspace. Phys. Rev. D **70**, 045022 (2004). arXiv:hep-th/0403130
43. I.L. Buchbinder, E.A. Ivanov, V.A. Ivanovskiy, New bi-harmonic superspace formulation of $4D$, $N = 4$ SYM theory. JHEP **04**, 010 (2021). arXiv:2012.09669 [hep-th]
44. G. Bossard, E. Ivanov, A. Smilga, Ultraviolet behavior of 6D supersymmetric Yang–Mills theories and harmonic superspace. JHEP **12**, 085 (2015). arXiv:1509.08027 [hep-th]
45. T. Kugo, P.K. Townsend, Supersymmetry and the division algebras. Nucl. Phys. B **221**, 357 (1983)
46. A.P. Isaev, V.A. Rubakov, *Theory of Groups and Symmetries II. Representations of Groups and Lie Algebras, Applications* (World Scientific, Singapore, 2021), p. 600
47. A.P. Isaev, V.A. Rubakov, *Theory of groups and symmetries (I): finite groups, lie groups and lie algebras* (World Scientific, Singapore, 2019)
48. G. 't Hooft, Computation of the quantum effects due to a four-dimensional pseudoparticle. Phys. Rev. D **14**, 3432 (1976)