Higher-derivative field redefinitions in the presence of boundary

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Abstract Recently it has been proposed that the consistency with T-duality requires the effective action of string theory at order α'^n to satisfy the least action principle provided that the values of the massless fields and their derivatives up to order n are known on the boundary. In this paper we speculate that this boundary condition constrains the field redefinitions and the corrections to the T-duality transformations in the presence of boundary, e.g., at order α', the metric does not change, and all other massless fields should change to include only the first derivative of the massless fields. Using the above restricted field redefinitions, we write all gauge invariant bulk and boundary couplings in the bosonic string theory at order α' in a minimal scheme. Then using the assumption that the effective action of string theory at the critical dimension is background independent, we fix the coefficients of the tree-level gauge invariant couplings by imposing $O(1, 1)$ symmetry when the background has a circle and by imposing $O(d, d)$ symmetry when the background has T^d. These constraints fix the bulk action up to an overall factor, and the boundary action up to two parameters. By requiring the gravity couplings in the boundary action to be consistent with those in the Chern-Simons gravity, the two boundary parameters are also fixed. Up to a restricted field redefinition, the bulk and boundary couplings are exactly those in the K.A. Meissner action and its corresponding boundary action.

1 Introduction

string theory is expected to be a background independent theory of quantum gravity. The spectrum of the free string at the critical dimension which is 26 for the bosonic string theory and 10 for the superstring and heterotic string theories, includes a finite number of massless and a tower of infinite number of massive excitations. In the interacting theory and

at low energies, however, the massive modes are integrated out to produce an effective action which includes only the massless fields. The effective action of string theory on an arbitrary background has a double expansions: the genus-expansion and the stringy-expansion. The genus-expansion includes the classical sphere-level and a tower of quantum loop-level corrections. The stringy-expansion is an expansion in terms of higher derivative couplings at each loop-level. The classical effective action of string theory on an arbitrary open manifold has both bulk and boundary couplings, i.e., $\mathbf{S}_{\text{eff}} + \partial\mathbf{S}_{\text{eff}}$. At the critical dimension, it has the following higher-derivative or α'-expansion:

$$\mathbf{S}_{\text{eff}} = \sum_{m=0}^{\infty} \alpha'^m \mathbf{S}_m = \mathbf{S}_0 + \alpha'\mathbf{S}_1 + \alpha'^2\mathbf{S}_2 + \alpha'^3\mathbf{S}_3 + \cdots$$

$$\partial\mathbf{S}_{\text{eff}} = \sum_{m=0}^{\infty} \alpha'^m \partial\mathbf{S}_m = \partial\mathbf{S}_0 + \alpha'\partial\mathbf{S}_1 + \alpha'^2\partial\mathbf{S}_2$$
$$+ \alpha'^3\partial\mathbf{S}_3 + \cdots$$

$$(1)$$

The leading order bulk action $\mathbf{S}_0$ includes the Hilbert-Einstein term in the critical dimension and the boundary action $\partial\mathbf{S}_0$ includes the corresponding Hawking-Gibbons term [1,2]. These actions and their appropriate higher derivative extensions may be found by requiring the effective actions to be invariant under the gauge symmetries corresponding to the massless fields and by imposing various constraints from the global symmetries/dualities of string theory.

The Einstein action is background independent in the sense that only the gauge symmetry corresponding to the metric is required to specify the theory. We expect that string theory effective action at the critical dimension which is a higher-derivative extension of the Einstein term, to be background independent too. Unlike the Einstein action which has only one coupling, however, there are many gauge invariant couplings in the effective action of string theory at each order of α'. The background independence requires the coef-

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ficients of all gauge invariant couplings to be independent of the geometry of the spacetime. At the higher orders of α', there is also the complication that the effective action has the freedom of the higher-derivative field redefinitions [3]. If one could fix the parameters, up to field redefinitions, for a specific compact geometry in which the lower-dimensional action enjoys a specific global symmetry, then those parameters would be valid for any other geometry in which the effective action may have no compact sub-manifold and no symmetry.

One of the most exciting discoveries in the perturbative string theory is T-duality [4,5] which appears when one compactifies the theory on a torus, e.g., the spectrum of the free string on torus T^d is invariant under $O(d, d, Z)$ transformations. In the interacting theory and after integrating out the massive modes, T-duality should appear as symmetry in the dimensional reduction of the effective actions. For the closed spacetime manifolds, it has been shown in [6,7] that the dimensional reduction of the classical effective actions of the bosonic and heterotic string theories on a torus T^d are in fact invariant under $O(d, d, R)$ transformations. We expect there should be such symmetry even when the string lives in an open spacetime manifold which has boundary. Using the background independent assumption, then one may consider the background which has the torus T^n for $1 \leq n \leq d$, to study the allowed gauge invariant couplings in the effective action at the critical dimension. That is, the requirement that the dimensional reduction of the classical effective action to have the symmetry $O(n, n, R)$, strongly constrains the parameters of the gauge invariant couplings in both bulk and boundary actions.

For the closed spacetime manifolds, if one considers a background which includes only one circle $S^{(1)}$, then the requirement that the dimensional reduction of the classical effective action of the bosonic string theory to have the $O(1, 1)$ symmetry, fixes all couplings in the critical dimension up to, at most, one parameter at each order of α'. In fact, the constraints from the Z_2-subgroup of $O(1, 1)$ have been used in [8,9] to fix the classical effective actions of the bosonic string theory at orders α', α'^2 up to one parameter, i.e., couplings at orders α' and α'^2 have the overall factors a_1 and a_1^2, respectively. The Z_2-constraints also fix the NS-NS couplings of the type II superstring theories at order α'^3 up to another parameter $a_2 \sim \zeta(3)$ [10–13]. The background independence then indicates that the couplings found in this way are valid for any other geometry. In fact, it has been observed in [12,13] that the NS-NS couplings are fully consistent with the sphere-level S-matrix element of four NS-NS vertex operators in flat spacetime. Moreover, it has been shown in [14,15] that the above couplings are fully consistent with the $O(d, d)$ symmetry when the background has torus T^d.

The reason that the $O(1, 1)$ symmetry can not fully fix all couplings in the classical effective action of the bosonic string theory up to one overall factor is that the full T-duality transformations are the Buscher rules [16,17] plus derivative corrections at all orders of α' [18]. Even though the $O(1, 1)$ symmetry can fix all couplings at orders α', α'^2 up to one parameter, however, it can not fix the couplings at order α'^3, up to the same parameter. In fact, if one extends the calculations in [9] to the order α'^3, one would find that the couplings at orders α'^0, α' and α'^2 are related to some of the couplings at order α'^3 by the T-duality transformations at orders α'^3, α'^2 and α', respectively. These couplings carry the same parameter a_1 which appears in the couplings at orders α', α'^2. In other words, they all belong to one T-dual multiplet. However, there are couplings at this order that are not connected to the couplings at order α', α'^2 by the T-duality transformations. They are connected only to the couplings at order α'^0 by the T-duality transformations at order α'^3. These couplings belong to another T-dual multiplet with a different parameter a_2.

In other words, if one tries to find the classical effective action of the bosonic string theory at order α'^3 by the S-matrix method, instead of the T-duality method, one would find it has two factors a_1, a_2, i.e., $\mathbf{S}_3 = a_1^3 \mathbf{S}_3^1 + a_2 \mathbf{S}_3^2$. One factor which is resulted from the expansion of the tachyonic pole in the amplitude, should be the same as the one appearing in the couplings at order α', α'^2, and another one which is resulted from the expansion of the massive poles in the amplitude, is proportional to $\zeta(3)$ [19]. Hence, there are two T-dual multiplets in the bosonic string theory at order α'^3: one with coefficient a_1^3 and another one with coefficient $a_2 \sim \zeta(3)$. The multiplet with coefficient $\zeta(3)$ appears also in the type II superstring theory. At order α'^4, there are nonzero couplings in the bosonic string theory. However, there are no couplings at this order in the superstring theory [20]. Hence, there are still two T-dual multiplets at this order: one with coefficient a_1^4 and another one with coefficient $a_1 \zeta(3)$. There is no new parameter at order α'^4 in this case. At order α'^5, there are couplings R^6 in the superstring theory with coefficient $\zeta(5)$ [20], hence, there are three T-dual multiplets at order α'^5 in the bosonic string: one with coefficient a_1^5, one with coefficient $a_1^2 \zeta(3)$ and another one with coefficient $\zeta(5)$. This means that the $O(1, 1)$ symmetry can fix the couplings at order α'^5 up to three parameters. Two of them already appeared at the lower orders of α' and one new parameter appears at order α'^5.

In general, there is no coupling in the superstring theory with coefficient $\zeta(2k)$ [20] which can be seen by studying the α'-expansion of the sphere-level four point functions. As a result, the T-dual multiplets in the effective action of the bosonic string theory should have an expansion in terms of powers of $(a_1^{m_1} \zeta(2k + 1)^{m_2} \zeta(2k + 3)^{m_3} \cdots) \alpha'^{m_1 + m_2(2k+1) + m_3(2k+3) + \cdots}$ where $k = 1, 2, \ldots$ and

$m_1, m_2, \cdots = 0, 1, 2, \ldots$ For $m_2 = m_3 = \cdots = 0$ there is one T-dual multiplet with parameter a_1. For $m_1 = m_3 = \cdots = 0$ and $k = 1$, there is one T-dual multiplet with parameter proportional to $\zeta(3)$. Similarly for other values of k, m_i. Schematically, the bulk action (1) has the following expansion in terms of the T-dual multiplets:

$$\mathbf{S}_{\text{eff}} = \sum_{n=0}^{\infty} T_n = T_0 + T_1 + T_2 + T_3 + T_4 + T_5 \cdots \quad (2)$$

The T-dual multiplets in the bosonic string theory on a closed manifold have the following structures:

$$
\begin{aligned}
T_0 &= \mathbf{S}_0 \\
T_1 &= a_1\alpha'\mathbf{S}_1 + a_1^2\alpha'^2\mathbf{S}_2 + a_1^3\alpha'^3\mathbf{S}_3^1 + a_1^4\alpha'^4\mathbf{S}_4^1 + a_1^5\alpha'^5\mathbf{S}_5^1 \\
&\quad + \cdots \\
T_2 &= \zeta(3)\alpha'^3\mathbf{S}_3^2 + \zeta(3)^2\alpha'^6\mathbf{S}_6^2 + \zeta(3)^3\alpha'^9\mathbf{S}_9^2 \\
&\quad + \zeta(3)^4\alpha'^{12}\mathbf{S}_{12}^2 \\
&\quad + \cdots \\
T_3 &= a_1\zeta(3)\alpha'^4\mathbf{S}_4^3 + a_1^2\zeta(3)^2\alpha'^8\mathbf{S}_8^3 + a_1^3\zeta(3)^3\alpha'^{12}\mathbf{S}_{12}^3 \\
&\quad + a_1^4\zeta(3)^4\alpha'^{16}\mathbf{S}_{16}^3 + \cdots \\
T_4 &= a_1^2\zeta(3)\alpha'^5\mathbf{S}_5^4 + a_1^4\zeta(3)^2\alpha'^{10}\mathbf{S}_{10}^4 + a_1^6\zeta(3)^3\alpha'^{15}\mathbf{S}_{15}^4 \\
&\quad + a_1^8\zeta(3)^4\alpha'^{20}\mathbf{S}_{20}^4 + \cdots \\
T_5 &= \zeta(5)\alpha'^5\mathbf{S}_5^5 + \zeta(5)^2\alpha'^{10}\mathbf{S}_{10}^5 + \zeta(5)^3\alpha'^{15}\mathbf{S}_{15}^5 \\
&\quad + \zeta(5)^4\alpha'^{20}\mathbf{S}_{20}^5 + \cdots \\
&\vdots
\end{aligned}
\quad (3)
$$

where $a_1 \sim 1$. In the type II superstring theory, $a_1 = 0$ and there are T-dual multiplets which include the R-R couplings. If the background has no boundary in which the total derivative terms can be ignored, then each of the above multiplets should be invariant under the full T-duality transformations. We expect that the $O(1, 1)$-constraint to fix all couplings in the above T-dual multiplets, up to the overall factors $a_1, \zeta(3), a_1\zeta(3), a_1^2\zeta(3), \zeta(5), \ldots$ The couplings in the multiplet T_1 at orders α' and α'^2 in a particular minimal scheme have been found in [8,9]. The couplings in the multiplet T_2 at order α'^3 in a particular minimal scheme have been also found in [12,13]. At the present time, the higher derivative terms in the full T-duality transformations are not known. They depend on the scheme of the gauge invariant couplings [8]. They can be found at each order of α', up to overall factors, by imposing the effective action at that order to be invariant under the $O(1, 1)$ symmetry. Hence, the T-duality constraint can not predict the overall factors $a_1, \zeta(3), a_1\zeta(3), a_1^2\zeta(3), \zeta(5), \ldots$ which appear in the full T-duality transformations and in the above T-dual multiplets. Since each multiplet includes four-field couplings, however, these factors may be found from the α'-expansion of the sphere-level S-matrix element of four graviton vertex operators.

When the background has boundary, one should keep the total derivative terms before and after reduction and use the Stokes's theorem to transfer them to the boundary. They dictate that the invariance under the T-duality transformations requires some couplings on the boundary as well [21]. Hence, the bulk T-dual multiplets (3) should be accompanied with appropriate boundary couplings to be fully invariant under the T-duality. Schematically, the boundary action (1) should have the following expansion in terms of the boundary T-dual multiplets:

$$
\begin{aligned}
\partial\mathbf{S}_{\text{eff}} = \sum_{n=0}^{\infty} \partial T_n &= \partial T_0 + \partial T_1 + \partial T_2 + \partial T_3 + \partial T_4 \\
&\quad + \partial T_5 + \cdots
\end{aligned}
\quad (4)
$$

The boundary mutiplets corresponding to the bulk multiplets (3) have the following structures:

$$
\begin{aligned}
\partial T_0 &= \partial\mathbf{S}_0 \\
\partial T_1 &= a_1\alpha'\partial\mathbf{S}_1 + a_1^2\alpha'^2\partial\mathbf{S}_2 + a_1^3\alpha'^3\partial\mathbf{S}_3^1 + a_1^4\alpha'^4\partial\mathbf{S}_4^1 \\
&\quad + a_1^5\alpha'^5\partial\mathbf{S}_5^1 \\
&\quad + \cdots \\
\partial T_2 &= \zeta(3)\alpha'^3\partial\mathbf{S}_3^2 + \zeta(3)^2\alpha'^6\partial\mathbf{S}_6^2 + \zeta(3)^3\alpha'^9\partial\mathbf{S}_9^2 \\
&\quad + \zeta(3)^4\alpha'^{12}\partial\mathbf{S}_{12}^2 + \cdots \\
\partial T_3 &= a_1\zeta(3)\alpha'^4\partial\mathbf{S}_4^3 + a_1^2\zeta(3)^2\alpha'^8\partial\mathbf{S}_8^3 + a_1^3\zeta(3)^3\alpha'^{12}\partial\mathbf{S}_{12}^3 \\
&\quad + a_1^4\zeta(3)^4\alpha'^{16}\partial\mathbf{S}_{16}^3 + \cdots \\
\partial T_4 &= a_1^2\zeta(3)\alpha'^5\partial\mathbf{S}_5^4 + a_1^4\zeta(3)^2\alpha'^{10}\partial\mathbf{S}_{10}^4 + a_1^6\zeta(3)^3\alpha'^{15}\partial\mathbf{S}_{15}^4 \\
&\quad + a_1^8\zeta(3)^4\alpha'^{20}\partial\mathbf{S}_{20}^4 + \cdots \\
\partial T_5 &= \zeta(5)\alpha'^5\partial\mathbf{S}_5^5 + \zeta(5)^2\alpha'^{10}\partial\mathbf{S}_{10}^5 + \zeta(5)^3\alpha'^{15}\partial\mathbf{S}_{15}^5 \\
&\quad + \zeta(5)^4\alpha'^{20}\partial\mathbf{S}_{20}^5 + \cdots \\
&\vdots
\end{aligned}
\quad (5)
$$

The combination of the bulk and the boundary multiplets, i.e., $T_i + \partial T_i$, are then invariant under the T-duality transformations. In other words, neither the bulk multiplets nor the boundary multiplets are invariant separately under the T-duality transformations. Their anomalies cancel each other in the T-dual multiplets $T_i + \partial T_i$. This constrains the parameters in the general gauge invariant couplings in the boundary action (5). However, there are boundary couplings in (5) which are invariant under the T-duality transformations with no anomaly. They form T-dual boundary multiplets which have no bulk partner. To constrain these multiplets, one may use the background which has torus T^d for which the cosmological reduction of the effective action should have the $O(d, d)$ symmetry [6,7].

To impose the $O(1, 1)$-constraint on the gauge invariant couplings in the closed spacetime manifold, one should first use the most general field redefinitions to find the independent couplings in a minimal scheme [3] in which the number of independent gauge invariant couplings is minimum, and

then should use the most general corrections to the Buscher rules [16,17] to fix the T-dual multiplets in (3) at order α'^n up to the overall factors at each order of α'. It turns out that in the presence of boundary, however, if one uses the most general field redefinitions and the most general corrections to the T-duality transformations, then the couplings in the minimal scheme are not invariant under the T-duality transformations! It may indicate that in the present of boundary, one is not allowed to use the most general field redefinitions and the most general corrections to the Buscher rules.

It has been observed in [22] that only the gravity couplings in the Euler characters satisfy the least action principle with the usual boundary condition that the values of the metric are known on the boundary. On the other hand, it has been shown in [23] that the Euler character at order α' is not consistent with $O(1, 1)$ nor with $O(d, d)$ symmetries. Hence, the T-duality dictates that in the least action principle, not only the values of the massless fields but also the values of some derivatives of the massless field must be known on the boundary. In fact, as it has been argued in [22], the least action principle in the bosonic string field theory produces the correct string field equations of motion with the usual boundary condition that the values of the string field are known on the boundary. The string field includes the massless fields and infinite tower of the massive fields. Hence, in the least action the values of the massless fields an all massive fields are known on the boundary. When the massive fields are integrated out to produce the effective action, the values of the massive fields on the boundary should appear in the effective action as the values of the derivatives of the massless fields on the boundary. It has been proposed in [23] that to extremize the effective action at order α'^n for $n > 0$, not only the massless fields Ψ but also their derivatives up to order n, i.e., $\nabla\Psi, \ldots, \nabla^n\Psi$ must be known on the boundary. It means that in the least action principle $\delta(\mathbf{S}_{eff} + \partial\mathbf{S}_{eff}) = 0$, the variation of the massless fields and the variation of the derivatives of the massless fields up to order n must be zero on the boundary, i.e., $\delta\Psi = \delta\nabla\Psi = \cdots = \delta\nabla^n\Psi = 0$ on the boundary. However, the variation of the derivatives of the massless fields at the higher orders are not zero on the boundary, i.e., $\delta\nabla^{n+1}\Psi \neq 0, \delta\nabla^{n+2}\Psi \neq 0$, and so on. These boundary values do not constrain the couplings in the bulk effective action. They, however, constrain the couplings in the boundary actions. In fact, using the Stokes's theorem to remove the total derivative terms in the bulk action to the boundary and using the appropriate Bianchi identities, one can show that the gauge invariant bulk couplings at order α'^n have no derivative terms $\nabla^{n+2}\Psi$ and higher. Then in extremizing the bulk action, using the Stokes's theorem, one finds no term with variation $\delta\nabla^{n+1}\Psi$ and higher. Hence, the extremization $\delta(\mathbf{S}_{eff}) = 0$ produces equations of motion with no constraint on the parameters in the bulk effective action. However, in extremizing the boundary action $\delta(\partial\mathbf{S}_{eff}) = 0$, the variations

$\delta\nabla^{n+1}\Psi, \delta\nabla^{n+2}\Psi$ and higher may appear on the boundary which are not zero. The extremization $\delta(\partial\mathbf{S}_{eff}) = 0$ then constrains the parameters in the boundary action (5).

Therefore, when the background has boundary, because of the above boundary conditions, one is allowed to use only a restricted field redefinitions in which the values of the massless fields and their derivatives to order α'^n on the boundary remain invariant under the field redefinitions. The field redefinitions should not change the known values to the unknown values. In other words, the least action principle forces the variations of fields and their derivatives up to order n to be zero on the boundary, i.e., $\delta\Psi = \delta\nabla\Psi = \cdots = \delta\nabla^n\Psi = 0$ and $\delta\nabla^{n+1}\Psi \neq 0, \delta\nabla^{n+2}\Psi \neq 0, \ldots$, on the boundary. The field redefinitions should not change $\delta\Psi, \delta\nabla\Psi, \ldots, \delta\nabla^n\Psi$ which are zero, to $\delta\nabla^{n+1}\Psi, \delta\nabla^{n+2}\Psi, \cdots$ which are non-zero on the boundary. For example, for the effective action at order α', the values of the massless fields and their first derivatives are known on the boundary and the values of their higher derivatives are not known. The field redefinitions at order α' are applied on the effective action at order α'^0 to produce couplings at order α'. At order α'^0, only the values of the massless fields are known on the boundary. Using the fact that the first derivative of dilaton and B-field appear in the leading order effective action, one realizes that in the least action, using the Stokes's theorem, only the variation of the dilaton and B-field appears on the boundary. These variations are zero on the boundary at the leading order. The field redefinition of these fields which includes only the first derivative of the massless fields, are then allowed because they produce the variations for the first derivative of the massless fields. Such variations are allowed for the effective action at order α' because they are zero at this order. On the other hand, the second derivative of metric appears in the leading order effective action. In the variation of the action, the variation of the second derivative of the metric appears. Then upon the use of the Stokes's theorem, it produces the variation of the first derivative of metric on the boundary. If one uses a field redefinition for the metric which includes the first derivative of the massless fields, then it would produce variation of the second derivative of the massless field on the boundary which are not allowed because the variation of the second derivative of the massless fields are not zero on the boundary for the effective action at order α'. Hence, the correct field redefinitions at order α' do not allow to change the metric and allow to change the dilaton and B-field to include only the first derivative of the massless fields. Similarly for the field redefinition for the effective actions at higher orders of α'. Using the above restricted field redefinitions, we will find that there are 17 independent bulk couplings at order α'.

If the background has boundary and a compact submanifold independent of the boundary, then the dimensionally reduced action at order α'^n should satisfy the least action principle in the base space with the boundary conditions that

the values of the base space massless fields and their derivatives up to order n are known on the boundary. Hence, the higher-derivative field redefinitions in the base space should be restricted as those specified in the previous paragraph, i.e., they should not change the data on the boundary. If the compact manifold is a circle, then the T-duality transformations in the presence of boundary should be the Buscher rules [16,17] plus some restricted higher derivative corrections, e.g., at order α', the base space metric receives no corrections, and all other massless fields, except the torsion, should receive corrections which include only the first derivative of the massless fields. The torsion 3-form in the base space however is not the exterior derivative of a 2-form. Hence its variation is not the derivative of the variation of a 2-form. As a result, the corrections to the torsion 3-form may involve the second derivative of the massless fields without changing the boundary conditions on the massless fields and their first derivative. Similarly for the corrections at the higher orders of α'. Hence, in applying the $O(1, 1)$-constraint on the independent gauge invariant couplings in the presence of the boundary, one should use the above restricted T-duality transformations.

When one uses the cosmological reduction on the classical bulk effective action (1), the resulting one-dimensional effective action has $O(d, d)$ symmetry provided that the $O(d, d)$ transformations receive higher derivative corrections [24]. Using the most general corrections for the $O(d, d)$-transformations including corrections for the lapse function, and using the integration by parts, it has been shown in [25,26] that the cosmological reduction of the bulk action (1) at order α' and higher, can be written in a scheme in which only the first time-derivative of the generalized metric $\mathcal{S}$ appears, i.e., the *nonlocal* cosmological action becomes *local* in the new variables. It has been shown in [12,13] that the cosmological reduction of the couplings at order α'^2, α'^3 that have been found by the $O(1, 1)$-constraint, can be written in terms of only traces of $\dot{\mathcal{S}}$. In the presence of boundary, however, the lapse function is the unite vector orthogonal to the boundary. Hence, one should not use corrections for the laps function any more. Moreover, in the presence of the boundary the higher derivative corrections to the $O(d, d)$-transformations should be restricted to those which do not change the data on the boundary, e.g., the corrections at order α' should include only the first time-derivative of the massless fields.

In studying the $O(d, d)$ symmetry of the cosmological reduction of the boundary action in (1), one has to take into account the one-dimensional total derivative terms that are needed to write the cosmological bulk couplings in $O(d, d)$-invariant form. The total derivative terms, in general, are not invariant under the $O(d, d)$ transformations, hence, they put some constraints on the parameters in the cosmological reduction of the boundary couplings. In other words,

the cosmological reduction of the boundary couplings are anomalous under the $O(d, d)$-transformations such that their anomalies are canceled with the anomalies of the total derivative terms. In this case also there are boundary couplings that their cosmological reduction are invariant under the $O(d, d)$ symmetry with no anomaly. It has been speculated in [23] that in the scheme that the $O(d, d)$-invariant couplings are in terms of the first derivative of the generalized metric and dilaton, the cosmological boundary action should be zero, i.e.,

$$\partial \mathbf{S}_k^c = 0 \tag{6}$$

The above discussion is valid not only for the cosmological reduction whose boundary is spacelike, i.e., $n^2 = -1$, but also for any one-dimensional reduction whose boundary is timelike, i.e., $n^2 = 1$. At the leading order which has no field redefinitions freedom, the explicit calculation confirms the above constraint. The above equation constrains the parameters in the gauge invariant couplings in the boundary action $\partial \mathbf{S}_k$ in (5).

The remainder of the paper is as follows: in Sect. 2, using the restricted field redefinitions, removing the total derivative terms from the bulk to the boundary and using the Bianchi identities, we find that there are at least 17 independent gauge invariant couplings at order α' in the bulk. We write them in a specific minimal scheme which includes 17 bulk couplings, and write the 38 independent gauge invariant boundary couplings at order α' which have been found in [23]. In Sect. 3, using the background independence assumption, we consider the background which has a boundary and one circle, and use the dimensional reduction to find the corresponding couplings in the base space. We then impose the $O(1, 1)$ symmetry with the above restricted T-duality transformations, on the reduced actions to constrain the parameters in the actions. In Sect. 4, we consider the background which has a boundary and the torus T^d, and use the cosmological/one-dimensional reduction to find the one-dimensional bulk action and the zero-dimensional boundary action. We then impose the $O(d, d)$ symmetry with the above restricted corrections to the $O(d, d)$ transformations, on the resulting bulk action and the constraint (6) on the boundary action to further constrain the remaining parameters. The above constraints fix the bulk action up to an overall factor, and the boundary action up to two parameters. The gravity couplings in the bulk action are those in the Gauss–Bonnet gravity. By requiring the gravity couplings on the boundary action to be consistent with those in the Chern–Simons form, the two boundary parameters are also fixed. In Sect. 6, we compare the bulk and boundary actions that we have found in this paper with the Meissner action and its corresponding boundary action that has been recently found by the similar constraints. We show that they are related by a restricted field redefinitions. In Sect. 7, we briefly discuss our results.

2 Gauge invariance couplings at order α'

The classical effective action of the bosonic string theory on an open manifold has both bulk and boundary actions. At the leading order these actions in the string frame are

$$
\mathbf{S}_0 + \partial\mathbf{S}_0 = -\frac{2}{\kappa^2}\Bigg[\int_M d^{26}x\sqrt{-G}e^{-2\Phi}\,(R
$$
$$
+4\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{12}H^2) + 2\int_{\partial M} d^{25}\sigma\sqrt{|g|}e^{-2\Phi}K \Bigg]
\tag{7}
$$

where G is determinant of the bulk metric $G_{\mu\nu}$ and boundary is specified by the functions $x^\mu = x^\mu(\sigma^{\tilde\mu})$. In the boundary term, g is determinant of the induced metric on the boundary

$$
g_{\tilde\mu\tilde\nu} = \frac{\partial x^\mu}{\partial\sigma^{\tilde\mu}}\frac{\partial x^\nu}{\partial\sigma^{\tilde\nu}}G_{\mu\nu}
\tag{8}
$$

and K is the trace of the extrinsic curvature. The normal vector to the boundary is n^μ. It is outward-pointing (inward-pointing) if the boundary is spacelike (timelike).

At order α' these actions in terms of their Lagrangians are

$$
\mathbf{S}_1 = -\frac{2}{\kappa^2}\int_M d^{26}x\sqrt{-G}e^{-2\Phi}\mathcal{L}_1;
$$
$$
\partial\mathbf{S}_1 = -\frac{2}{\kappa^2}\int_{\partial M} d^{25}\sigma\sqrt{|g|}e^{-2\Phi}\partial\mathcal{L}_1
\tag{9}
$$

The Lagrangians must be invariant under the coordinate transformations and under the B-field gauge transformations. Using the package "xAct" [27], one finds there are 41 couplings in the bulk Lagrangian. As we have clarified in the Introduction section, one is free to use the restricted field redefinitions, i.e.,

$$
g_{\mu\nu} \rightarrow g_{\mu\nu}
$$
$$
B_{\mu\nu} \rightarrow B_{\mu\nu} + \alpha'\delta B^{(1)}_{\mu\nu}
$$
$$
\Phi \rightarrow \Phi + \alpha'\delta\Phi^{(1)}
\tag{10}
$$

where the tensors $\delta B^{(1)}_{\mu\nu}$ and $\delta\Phi^{(1)}$ include all possible odd- and even-parity, respectively, gauge invariant terms at two-derivative level which involve only the first derivative of the massless fields, i.e., ,

$$
\delta B^{(1)}_{\mu\nu} = \alpha_1 H_{\mu\nu\alpha}\nabla^\alpha\Phi
$$
$$
\delta\Phi^{(1)} = \alpha_2 H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} + \alpha_3\nabla_\alpha\Phi\nabla^\alpha\Phi
\tag{11}
$$

The coefficients $\alpha_1, \alpha_2, \alpha_3$ are arbitrary parameters. When the field variables in $\mathbf{S}_0$ are changed according to the above field redefinitions, the following couplings at order α' are produced:

$$
\delta\mathbf{S}_0 = -\frac{2}{\kappa^2}\int d^{26}x\sqrt{-G}e^{-2\Phi}\Big[-\frac{1}{2}H^{\alpha\beta\gamma}\nabla_\gamma\delta B^{(1)}_{\alpha\beta}
$$
$$
+8\nabla^\alpha\Phi\nabla_\alpha\delta\Phi^{(1)} - 2(R + 4\nabla_\alpha\Phi\nabla^\alpha\Phi
$$

$$
-\frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma})\delta\Phi^{(1)}\Big]
\tag{12}
$$

No integration by parts has been used in finding the above equation.

The couplings in the bulk action $\mathbf{S}_1$ which are total derivative terms can be transferred to the boundary action $\partial\mathbf{S}_1$ by using the Stokes's theorem. Moreover, the independent couplings should not be related to each other by the Bianchi identities

$$
R_{\alpha[\beta\gamma\delta]} = 0
$$
$$
\nabla_{[\mu}R_{\alpha\beta]\gamma\delta} = 0
$$
$$
\nabla_{[\mu}H_{\alpha\beta\gamma]} = 0
$$
$$
[\nabla, \nabla]\mathcal{O} - R\mathcal{O} = 0
\tag{13}
$$

Removing the above freedoms from the most general gauge invariant couplings in the bulk, one finds that there are 17 independent couplings. In fact, if one does not use the field redefinition, it has been shown in [3] that there are 20 independent couplings. There are three parameters in the restricted field redefinition (11), which reduces these couplings to 17 independent couplings. These couplings in a specific scheme are

$$
\mathcal{L}_1 = a_1 H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^\varepsilon H_{\gamma\epsilon\varepsilon} + a_2 H_{\alpha\beta}{}^\delta H^{\alpha\beta\gamma}H_\gamma{}^{\epsilon\varepsilon}H_{\delta\epsilon\varepsilon}
$$
$$
+a_3 H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}R^{\alpha\beta} + a_4 R_{\alpha\beta}R^{\alpha\beta}
$$
$$
+a_5 H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}R + a_6 R^2 + a_7 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}
$$
$$
+a_8 H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta\gamma\delta\epsilon} + a_9 R\nabla_\alpha\Phi\nabla^\alpha\Phi
$$
$$
+a_{10}R^{\alpha\beta}\nabla_\beta\nabla_\alpha\Phi + a_{11}R_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi
$$
$$
+a_{12}\nabla_\alpha\Phi\nabla^\alpha\Phi\nabla_\beta\Phi\nabla^\beta\Phi + a_{13}\nabla^\alpha\Phi\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\Phi
$$
$$
+a_{14}\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\nabla^\alpha\Phi + a_{15}\nabla_\alpha H^{\alpha\beta\gamma}\nabla_\delta H_{\beta\gamma}{}^\delta
$$
$$
+a_{16}H_\alpha{}^{\beta\gamma}\nabla^\alpha\Phi\nabla_\delta H_{\beta\gamma}{}^\delta + a_{17}\nabla_\delta H_{\alpha\beta\gamma}\nabla^\delta H^{\alpha\beta\gamma}
\tag{14}
$$

where $a_1, \ldots, a_{17}$ are 17 parameters. The background independent assumption dictates that these parameters are independent of the geometry of the spacetime. They may be fixed by considering the background which has a circle or a torus T^{25}. Then the dimensional reduction of the above couplings should have the symmetry $O(1, 1)$ or $O(25, 25)$, respectively. These symmetries constraint the parameters in the above action. Note that as we have anticipated in the Introduction section, the independent couplings have no term with three derivatives. Hence, the least action principle does not constrain the bulk parameters $a_1, \ldots, a_{17}$.

The boundary of the spacetime has a unite normal vector n^μ, hence, the boundary Lagrangian $\partial\mathcal{L}_1$ should include this vector and its derivatives as well as the other tensors. Since the field redefinition freedom has been already used in the bulk action, one is not allowed to use any field redefinition in the boundary action. Removing the total derivative terms from the most general gauge invariant boundary couplings,

and using the Bianchi identities and the identities corresponding to the unit vector, it has been shown in [23] that there are 38 independent couplings in the boundary action. They are

$$
\begin{aligned}
\partial\mathcal{L}_1 =\ & b_1 H_{\beta\gamma\delta} H^{\beta\gamma\delta} K^\alpha{}_\alpha + b_2 H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} K^{\alpha\beta} \\
& + b_3 K_\alpha{}^\gamma K^{\alpha\beta} K_{\beta\gamma} + b_4 K^\alpha{}_\alpha K_{\beta\gamma} K^{\beta\gamma} \\
& + b_5 K^\alpha{}_\alpha K^\beta{}_\beta K^\gamma{}_\gamma + b_6 H_\alpha{}^{\delta\epsilon} H_{\beta\delta\epsilon} K^\gamma{}_\gamma n^\alpha n^\beta \\
& + b_7 H_{\alpha\gamma}{}^\epsilon H_{\beta\delta\epsilon} K^{\gamma\delta} n^\alpha n^\beta + b_8 K^{\alpha\beta} R_{\alpha\beta} \\
& + b_9 K^\gamma{}_\gamma n^\alpha n^\beta R_{\alpha\beta} + b_{10} K^\alpha{}_\alpha R \\
& + b_{11} K^{\gamma\delta} n^\alpha n^\beta R_{\alpha\gamma\beta\delta} \\
& + b_{12} H^{\beta\gamma\delta} n^\alpha \nabla_\alpha H_{\beta\gamma\delta} \\
& + b_{13} K^{\beta\gamma} n^\alpha \nabla_\alpha K_{\beta\gamma} + b_{14} K^\beta{}_\beta n^\alpha \nabla_\alpha K^\gamma{}_\gamma \\
& + b_{15} n^\alpha \nabla_\alpha R + b_{16} H_{\beta\gamma\delta} H^{\beta\gamma\delta} n^\alpha \nabla_\alpha \Phi \\
& + b_{17} K_{\beta\gamma} K^{\beta\gamma} n^\alpha \nabla_\alpha \Phi \\
& + b_{18} K^\beta{}_\beta K^\gamma{}_\gamma n^\alpha \nabla_\alpha \Phi \\
& + b_{19} H_\beta{}^{\delta\epsilon} H_{\gamma\delta\epsilon} n^\alpha n^\beta n^\gamma \nabla_\alpha \Phi \\
& + b_{20} n^\alpha n^\beta n^\gamma R_{\beta\gamma} \nabla_\alpha \Phi + b_{21} n^\alpha R \nabla_\alpha \Phi \\
& + b_{22} K^\beta{}_\beta \nabla_\alpha \Phi \nabla^\alpha \Phi + b_{23} n^\alpha n^\beta \nabla_\alpha \Phi \nabla_\beta K^\gamma{}_\gamma \\
& + b_{24} K^\gamma{}_\gamma n^\alpha n^\beta \nabla_\alpha \Phi \nabla_\beta \Phi + b_{25} n^\alpha n^\beta \nabla_\beta \nabla_\alpha K^\gamma{}_\gamma \\
& + b_{26} K^{\alpha\beta} \nabla_\beta \nabla_\alpha \Phi \\
& + b_{27} K^\gamma{}_\gamma n^\alpha n^\beta \nabla_\beta \nabla_\alpha \Phi + b_{28} H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} n^\alpha \nabla^\beta \Phi \\
& + b_{29} n^\alpha R_{\alpha\beta} \nabla^\beta \Phi + b_{30} K_{\alpha\beta} \nabla^\alpha \Phi \nabla^\beta \Phi \\
& + b_{31} n^\alpha \nabla_\alpha \Phi \nabla_\beta \Phi \nabla^\beta \Phi + b_{32} n^\alpha \nabla_\beta \nabla_\alpha \Phi \nabla^\beta \Phi \\
& + b_{33} H_\alpha{}^{\delta\epsilon} n^\alpha n^\beta n^\gamma \nabla_\gamma H_{\beta\delta\epsilon} \\
& + b_{34} n^\alpha n^\beta n^\gamma \nabla_\alpha \Phi \nabla_\beta \Phi \nabla_\gamma \Phi \\
& + b_{35} n^\alpha n^\beta n^\gamma \nabla_\alpha \Phi \nabla_\gamma \nabla_\beta \Phi + b_{36} n^\alpha n^\beta n^\gamma \nabla_\gamma \nabla_\beta \nabla_\alpha \Phi \\
& + b_{37} n^\alpha n^\beta \nabla_\beta K_{\alpha\gamma} \nabla^\gamma \Phi + b_{38} n^\alpha n^\beta n^\gamma n^\delta \nabla_\delta \nabla_\gamma K_{\alpha\beta}
\end{aligned}
$$
(15)

where $b_1, \ldots, b_{38}$ are 38 background independent parameters. In the above couplings, the first derivative of the unit vector appears in the extrinsic curvature of boundary as

$$
K_{\mu\nu} = \nabla_\mu n_\nu \mp n_\mu n^\rho \nabla_\rho n_\nu
$$
(16)

where the minus (plus) sign is for timelike (spacelike) boundary in which $n^\mu n_\mu = 1$ ($n^\mu n_\mu = -1$). It is symmetric and satisfies $n^\mu K_{\mu\nu} = 0$ and $n^\mu n^\nu \nabla_\alpha K_{\mu\nu} = 0$ which can easily be seen by writing the unit normal vector of the boundary as

$$
n^\mu = \pm(|\nabla_\alpha f \nabla^\alpha f|)^{-1/2} \nabla^\mu f
$$
(17)

where plus (minus) sign is for timelike (spacelike) boundary and the function f specifies the boundary. Note that the metric in the curvatures in the boundary couplings (15) is the bulk metric $G_{\mu\nu}$ and the metric that rises the indices is the inverse bulk metric $G^{\mu\nu}$.

The parameters in the boundary action (15) may be fixed by imposing the $O(1,1)$ and $O(25,25)$ symmetries when the background has a circle and a torus T^{25}, respectively,

and by imposing the least action principle. Note that, unlike the bulk couplings, the above boundary couplings do have terms with two and three derivatives, hence, in extremizing the above Lagrangian one encounters with the variation of the second and third derivatives of massless fields on the boundary which are non-zero. As a result, the least action principle constrains the parameters $b_1, \ldots, b_{38}$. However, as we will see there would be no constraint on top of the T-duality constraints.

3 Constraint from $O(1,1)$ symmetry

We now try to fix the parameters in the actions (14) and (15). The assumption that the effective action at the critical dimension is background independent, means that the parameters in these actions are independent of the background. Hence, to fix them we consider a specific background which has a circle. That is, the open manifold has the structure $M^{(26)} = M^{(25)} \times S^{(1)}$, $\partial M^{(26)} = \partial M^{(25)} \times S^{(1)}$. The manifold $M^{(26)}$ has coordinates $x^\mu = (x^a, y)$ and its boundary $\partial M^{(26)}$ has coordinates $\sigma^{\tilde\mu} = (\sigma^{\tilde a}, y)$ where y is the coordinate of the circle $S^{(1)}$. The dimensionally reduced action then should have the $O(1,1)$ symmetry. To simplify the calculation, we consider the Z_2-subgroup of the $O(1,1)$-group.

The reduction of the effective actions on the circle $S^{(1)}$ should then be invariant under the Z_2-transformations [21], i.e.,

$$
S_{\text{eff}}(\psi) + \partial S_{\text{eff}}(\psi) = S_{\text{eff}}(\psi') + \partial S_{\text{eff}}(\psi')
$$
(18)

where S_{eff} and ∂S_{eff} are the reductions of the bulk action $\mathbf{S}_{eff}$ and boundary action $\partial\mathbf{S}_{\text{eff}}$, respectively. In above equation ψ represents all the massless fields in the base space which are defined in the following reductions:

$$
G_{\mu\nu} = \begin{pmatrix} \bar{g}_{ab} + e^\varphi g_a g_b & e^\varphi g_a \\ e^\varphi g_b & e^\varphi \end{pmatrix},
$$

$$
B_{\mu\nu} = \begin{pmatrix} \bar{b}_{ab} + \frac{1}{2}b_a g_b - \frac{1}{2}b_b g_a & b_a \\ -b_b & 0 \end{pmatrix},
$$

$$
\Phi = \bar\phi + \varphi/4, \quad n^\mu = (n^a, 0)
$$
(19)

and ψ' represents its transformation under the Z_2-transformations or the T-duality transformations.

In [23], it has been shown that the constraint (18) can be written as two separate constraints. One for the bulk couplings and the other one for the boundary couplings. These constraints for the couplings at order α' are [23]

$$
S_1(\psi) - S_1(\psi'_0) - \Delta S_0 - \frac{2}{\kappa^2} \int d^{25}x \sqrt{-\bar{g}} \nabla_a (A_1^a e^{-2\bar\phi})
$$
$$
= 0
$$
$$
\partial S_1(\psi) - \partial S_1(\psi'_0) - \Delta \partial S_0 + T_1(\psi)
$$

$$+\frac{2}{\kappa^2}\int d^{24}\sigma\sqrt{|\tilde{g}|}n_a A_1^a e^{-2\bar{\phi}}=0 \tag{20}$$

where $\bar{g}$ is the determinant of the base space metric $\bar{g}_{ab}$ and $\tilde{g}$ is the determinant of the induced base space metric on its boundary, i.e.,

$$\tilde{g}_{\tilde{a}\tilde{b}}=\frac{\partial x^a}{\partial\sigma^{\tilde{a}}}\frac{\partial x^b}{\partial\sigma^{\tilde{b}}}\bar{g}_{ab} \tag{21}$$

In Eq. (20), ψ_0' is the transformation of the base space field ψ under the Buscher rules, A_1^a is a vector made of the massless fields in the base space at order α' with arbitrary coefficients, and $T_1(\psi)$ is the most general total derivative terms in the boundary, i.e.,

$$T_1(\psi)=-\frac{2}{\kappa^2}\int_{\partial M}d^{D-2}\sigma\sqrt{|\tilde{g}|}n_a\partial_b(e^{-2\bar{\phi}}F_1^{ab}) \tag{22}$$

where F_1^{ab} is an antisymmetric tensor constructed from the massless fields in the base space at order α' with arbitrary coefficients. In the Eq. (20), ΔS_0, $\Delta\partial S_0$ are defined in the following α'-expansions of the reduction of the leading order actions:

$$S_0(\psi_0'+\alpha'\psi_1')-S_0(\psi_0')=\alpha'\Delta S_0+\cdots$$
$$\partial S_0(\psi_0'+\alpha'\psi_1')-\partial S_0(\psi_0')=\alpha'\Delta\partial S_0+\cdots \tag{23}$$

where dots represent some terms at higher orders of α' in which we are not interested in this paper. In the above equation, S_0 and ∂S_0 are the reduction of the leading order actions (7). The reduction of these actions are [18,23]

$$S_0(\psi)=-\frac{2}{\kappa^2}\int d^{D-1}x e^{-2\bar{\phi}}\sqrt{-\bar{g}}\Big[\bar{R}-\nabla^a\nabla_a\varphi$$
$$-\frac{1}{4}\nabla_a\varphi\nabla^a\varphi-\frac{1}{4}(e^\varphi V^2+e^{-\varphi}W^2)$$
$$+4\nabla_a\bar{\phi}\nabla^a\bar{\phi}+2\nabla_a\bar{\phi}\nabla^a\varphi-\frac{1}{12}\bar{H}_{abc}\bar{H}^{abc}\Big]$$
$$\partial S_0(\psi)=-\frac{4}{\kappa^2}\int d^{D-2}\sigma\, e^{-2\bar{\phi}}\sqrt{|\tilde{g}|}\Big[\bar{g}^{ab}\bar{K}_{ab}+\frac{1}{2}n^a\nabla_a\varphi\Big] \tag{24}$$

In the first equation, V_{ab} is field strength of the $U(1)$ gauge field g_a, i.e., $V_{ab}=\partial_a g_b-\partial_b g_a$, and $W_{\mu\nu}$ is field strength of the $U(1)$ gauge field b_a, i.e., $W_{ab}=\partial_a b_v-\partial_b b_a$. The three-form $\bar{H}$ is defined as $\bar{H}_{abc}=\hat{H}_{abc}-\frac{3}{2}g_{[a}W_{bc]}-\frac{3}{2}b_{[a}V_{bc]}$ where the three-form $\hat{H}$ is field strength of the two-form $\bar{b}_{ab}$ in (19). Our notation for making the antisymmetry is such that e.g., $g_{[a}W_{bc]}=\frac{1}{3}(g_a W_{bc}-g_b W_{ac}-g_c W_{ba})$. Since $\bar{H}$ is not exterior derivative of a two-form, it satisfies anomalous Bianchi identity, whereas the W, V satisfy the ordinary Bianchi identity, i.e.,

$$\partial_{[a}\bar{H}_{bcd]}=-\frac{3}{2}V_{[ab}W_{cd]}$$
$$\partial_{[a}W_{bc]}=0$$
$$\partial_{[a}V_{bc]}=0 \tag{25}$$

In the second equation in (24), the extrinsic curvature $\bar{K}_{ab}$ is made of the covariant derivative of the base space normal vector n^a.

In (23), ψ_1' represents the corrections to the Buscher rules at order α', i.e.,

$$\varphi'=-\varphi+\alpha'\Delta\varphi\ ,\ g_a'=b_a+\alpha'e^{\varphi/2}\Delta g_a\ ,\ b_a'=g_a$$
$$+\alpha'e^{-\varphi/2}\Delta b_a\ ,$$
$$\bar{g}_{ab}'=\bar{g}_{ab}\ ,\ \bar{H}_{abc}'=\bar{H}_{abc}+\alpha'\Delta\bar{H}_{abc}\ ,\ \bar{\phi}'=\bar{\phi}$$
$$+\alpha'\Delta\bar{\phi}\ ,\ n_a'=n_a \tag{26}$$

As we have clarified in the Introduction section, the metric has no correction at order α', and the corrections $\Delta\varphi$, Δb_a, Δg_a, $\Delta\bar{\phi}$ contain all contractions of the massless fields in the base space at order α' which involve only the first derivative of the massless fields. The correction $\Delta\bar{H}_{abc}$ is related to the corrections Δg_a, Δb_a through the following relation which is resulted from the Bianchi identity (25):

$$\Delta\bar{H}_{abc}=\tilde{H}_{abc}-3e^{-\varphi/2}W_{[ab}\Delta b_{c]}-3e^{\varphi/2}\Delta g_{[a}V_{bc]} \tag{27}$$

where $\tilde{H}_{abc}$ is a $U(1)\times U(1)$ gauge invariant closed 3-form at order α' which is odd under parity. It has the following terms:

$$\tilde{H}_{abc}=e_1\partial_{[a}W_b{}^d V_{c]d}+e_2\partial_{[a}\bar{H}_{bc]d}\nabla^d\varphi \tag{28}$$

where e_1, e_2 and the coefficients in the corrections $\Delta\varphi$, Δb_a, Δg_a, $\Delta\bar{\phi}$ are parameters that the Z_2-symmetry of the effective action should fix them. The above transformations should also form the Z_2-group [8].

It is important to note that the torsion in the base space is $\bar{H}_{abc}$ which is not a field strength of a two-form. Hence, in the least action principle, this field and its first derivative are known on the boundary. In the variation of the leading order bulk action (24), the variation $\delta\bar{H}_{abc}$ appears. The field redefinitions of $\bar{H}_{abc}$ that involve the second derivatives of the massless fields, produce the variation of the second derivative of the massless fields. Under the use of the Stokes's theorem, it produces the variation of the first derivative of the massless fields on the boundary which is zero. Hence, the corrections (28) which involve the second derivative of the massless fields b_a are consistent with our proposal for the restricted corrections to the Buscher rules which should not change the data on the boundary in the base space.

Using the reduction (24), then one can calculate ΔS_0 and $\Delta\partial S_0$ from the expansion (23) in terms of ψ_1', i.e.,

$$\Delta S_0=-\frac{2}{\kappa^2}\int d^{25}x e^{-2\bar{\phi}}\sqrt{-\bar{g}}\Big[-4\Big(\frac{1}{2}\bar{R}+2\partial_c\bar{\phi}\partial^c\bar{\phi}$$
$$-\frac{1}{8}\partial_c\varphi\partial^c\varphi-\frac{1}{24}\bar{H}^2-\frac{1}{8}e^\varphi V^2$$
$$-\frac{1}{8}e^{-\varphi}W^2+\frac{1}{2}\nabla_c\nabla^c\varphi-\partial_c\bar{\phi}\partial^c\varphi\Big)\Delta\bar{\phi}$$
$$+\frac{1}{4}\Big(e^\varphi V^2-e^{-\varphi}W^2\Big)\Delta\varphi$$

$$+\frac{1}{2}e^{-\varphi/2}\partial_b\varphi\,W^{ab}\Delta g_a - \frac{1}{2}e^{\varphi/2}\partial_b\varphi\,V^{ab}\Delta b_a$$

$$-\frac{1}{6}\bar{H}^{abc}\Delta\bar{H}_{abc}$$

$$+\frac{1}{2}(\partial_a\varphi + 4\partial_a\bar{\phi})\nabla^a(\Delta\varphi) - \nabla_a\nabla^a(\Delta\varphi)$$

$$-2(\partial_a\varphi - 4\partial_a\bar{\phi})\nabla^a(\Delta\bar{\phi})$$

$$+e^{-\varphi/2}W_{ab}\nabla^b(\Delta g^a) + e^{\varphi/2}V_{ab}\nabla^b(\Delta b^a)\Big]$$

$$\Delta\partial S_0 = -\frac{2}{\kappa^2}\int d^{24}\sigma\, e^{-2\bar{\phi}}\sqrt{|\tilde{g}|}\Big[n^a\nabla_a(\Delta\varphi) - 4(\nabla_a n^a$$
$$-\frac{1}{2}n^a\nabla_a\varphi)\Delta\bar{\phi}\Big] \tag{29}$$

where no integration by parts has been used. Note that if one would like to find the equations of motion in the base space, the variation of the leading order action is the same as the above equation in which Δ should be replaced by δ. In particular, the variation of the bulk action against $\delta\bar{H}_{abc}$ is given by the last term in the third line above. As we have pointed out in the previous paragraph, unlike the other massless fields, it does not involve derivative of $\delta\bar{H}_{abc}$.

Following the same steps as those in [23], one finds that the Z_2-symmetry fixes the bulk Lagrangian (14) as

$$\mathcal{L}_1 = a_1 H_\alpha{}^{\delta\epsilon} H^{\alpha\beta\gamma} H_{\beta\delta}{}^\varepsilon H_{\gamma\epsilon\varepsilon} + \Big(3a_1 + \frac{1}{64}a_{10}$$

$$+\frac{1}{64}a_{11}\Big)H_{\alpha\beta}{}^\delta H^{\alpha\beta\gamma} H_\gamma{}^{\epsilon\varepsilon} H_{\delta\epsilon\varepsilon}$$

$$-\frac{1}{16}a_{11}H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} R^{\alpha\beta}$$

$$+\Big(\frac{1}{4}a_{10} + \frac{1}{4}a_{11}\Big)R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{192}a_{11}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}R$$

$$-\frac{1}{16}a_{11}R^2 + 24a_1 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$$

$$+(-36a_1 - \frac{1}{8}a_{10} - \frac{1}{16}a_{11})H_\alpha{}^{\delta\epsilon} H^{\alpha\beta\gamma} R_{\beta\gamma\delta\epsilon}$$

$$-\frac{1}{4}a_{11}R\nabla_\alpha\Phi\nabla^\alpha\Phi + a_{10}R^{\alpha\beta}\nabla_\beta\nabla_\alpha\Phi$$

$$+a_{11}R_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi + a_{10}\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\nabla^\alpha\Phi$$

$$-\frac{1}{16}a_{10}\nabla_\alpha H^{\alpha\beta\gamma}\nabla_\delta H_{\beta\gamma}{}^\delta$$

$$+\frac{1}{8}a_{10}H_\alpha{}^{\beta\gamma}\nabla^\alpha\Phi\nabla_\delta H_{\beta\gamma}{}^\delta + \Big(8a_1 + \frac{1}{24}a_{10}$$

$$+\frac{1}{48}a_{11}\Big)\nabla_\delta H_{\alpha\beta\gamma}\nabla^\delta H^{\alpha\beta\gamma} \tag{30}$$

and the boundary Lagrangian (15) for timelike boundary as

$$\partial\mathcal{L}_1 = b_1 H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + \frac{1}{16}(-2a_{10} - a_{11})H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}K^{\alpha\beta}$$

$$+b_{11}K_\alpha{}^\gamma K^{\alpha\beta}K_{\beta\gamma}$$

$$+\frac{1}{4}(a_{11} + 96b_1 - 2b_{17})K^\alpha{}_\alpha K_{\beta\gamma}K^{\beta\gamma} + \Big(-\frac{1}{12}a_{11}$$

$$-8b_1 - \frac{1}{6}b_{18}\Big)K^\alpha{}_\alpha K^\beta{}_\beta K^\gamma{}_\gamma$$

$$-\frac{1}{2}b_{19}H_\alpha{}^{\delta\epsilon} H_{\beta\delta\epsilon}K^\gamma{}_\gamma n^\alpha n^\beta + \Big(a_{10} + \frac{1}{2}a_{11}$$

$$+12b_{12}\Big)K^{\alpha\beta}R_{\alpha\beta}$$

$$+\frac{1}{2}(-a_{10} + 48b_1 - 24b_{12} - b_{17} + 4b_{19})$$
$$\times K^\gamma{}_\gamma n^\alpha n^\beta R_{\alpha\beta}$$

$$+\Big(-\frac{1}{8}a_{11} - 12b_1\Big)K^\alpha{}_\alpha R$$

$$+b_{11}K^{\gamma\delta}n^\alpha n^\beta R_{\alpha\gamma\beta\delta} + b_{12}H^{\beta\gamma\delta}n^\alpha\nabla_\alpha H_{\beta\gamma\delta}$$

$$+\Big(\frac{1}{48}a_{11} - 2b_1\Big)H_{\beta\gamma\delta}H^{\beta\gamma\delta}n^\alpha\nabla_\alpha\Phi$$

$$+b_{17}K_{\beta\gamma}K^{\beta\gamma}n^\alpha\nabla_\alpha\Phi + b_{18}K^\beta{}_\beta K^\gamma{}_\gamma n^\alpha\nabla_\alpha\Phi$$

$$+b_{19}H_\beta{}^{\delta\epsilon} H_{\gamma\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi$$

$$+\Big(a_{10} + \frac{1}{2}a_{11} + 24b_{12} + b_{17} - 4b_{19}\Big)$$
$$\times n^\alpha n^\beta n^\gamma R_{\beta\gamma}\nabla_\alpha\Phi$$

$$+\Big(-\frac{1}{4}a_{11} + 24b_1\Big)n^\alpha R\nabla_\alpha\Phi$$

$$-48b_1 K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi + (96b_1 - 2b_{18})$$
$$\times K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\alpha\Phi\nabla_\beta\Phi$$

$$+\frac{1}{2}\big(4a_{10} + a_{11} + 48(-2b_1 + b_{12})\big)K^{\alpha\beta}\nabla_\beta\nabla_\alpha\Phi$$

$$+\Big(-a_{10} - \frac{1}{2}a_{11} - 24b_{12} - b_{17} + 4b_{19}\Big)$$
$$\times K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\beta\nabla_\alpha\Phi$$

$$+\frac{1}{8}a_{10}H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta}n^\alpha\nabla^\beta\Phi$$

$$+\frac{1}{2}(a_{11} - 96b_1)n^\alpha R_{\alpha\beta}\nabla^\beta\Phi + a_{11}K_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi$$

$$+(-a_{11} + 96b_1)n^\alpha\nabla_\alpha\Phi\nabla_\beta\Phi\nabla^\beta\Phi$$

$$+(a_{11} - 96b_1)n^\alpha\nabla_\beta\nabla_\alpha\Phi\nabla^\beta\Phi$$

$$+\frac{1}{8}\big(-2a_{10} - a_{11} - 2(b_{11} + 24b_{12})\big)$$
$$\times H_\alpha{}^{\delta\epsilon} n^\alpha n^\beta n^\gamma\nabla_\gamma H_{\beta\delta\epsilon}$$

$$+\frac{2}{3}\big(a_{11} + 2(-96b_1 + b_{18})\big)n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\beta\Phi\nabla_\gamma\Phi$$

$$+2(a_{10} + 48b_1 + 24b_{12} + b_{17} - 4b_{19})$$
$$\times n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\gamma\nabla_\beta\Phi$$

$$+b_{38}n^\alpha n^\beta n^\gamma n^\delta\nabla_\delta\nabla_\gamma K_{\alpha\beta} \tag{31}$$

The bulk Lagrangian has three parameters a_1, a_{10}, a_{11} and the boundary Lagrangian has two bulk parameters a_{10}, a_{11} and 7 boundary parameters $b_1, b_{11}, b_{12}, b_{17}, b_{18}, b_{19}, b_{38}$. Note that the parameter b_{38} appears in only one term, hence, its corresponding couplings is invariant under the Z_2-

transformations. Note also that the Z_2-symmetry does not constrain the gravity couplings in the bulk Lagrangian to be the Gauss–Bonnet combination.

The corrections to the T-duality transformation (26) corresponding to the above Lagrangians are

$$
\begin{aligned}
\Delta\bar{\phi} &= -\frac{1}{128}a_{11}e^{\varphi}V_{ab}V^{ab} + \frac{1}{128}a_{11}e^{-\varphi}W_{ab}W^{ab} \\
&\quad -\frac{1}{8}a_{11}\nabla_a\varphi\nabla^a\bar{\phi} \\
\Delta\varphi &= 24a_1e^{\varphi}V_{ab}V^{ab} + 24a_1e^{-\varphi}W_{ab}W^{ab} \\
&\quad +48a_1\nabla_a\varphi\nabla^a\varphi \\
\Delta g_a &= 24a_1e^{\varphi/2}\bar{H}_{abc}V^{bc} + \frac{1}{4}(a_{10}+a_{11})e^{-\varphi/2}W_{ab}\nabla^b\bar{\phi} \\
&\quad +\frac{1}{16}(768a_1+a_{10}+a_{11})e^{-\varphi/2}W_{ab}\nabla^b\varphi \\
\Delta b_a &= -24a_1e^{-\varphi/2}\bar{H}_{abc}W^{bc} \\
&\quad -\frac{1}{4}(a_{10}+a_{11})e^{\varphi/2}V_{ab}\nabla^b\bar{\phi} \\
&\quad +\frac{1}{16}(768a_1+a_{10}+a_{11})e^{\varphi/2}V_{ab}\nabla^b\varphi \\
\Delta\bar{H}_{abc} &= -144a_1\partial_{[a}W_b{}^dV_{c]d} + \left(-\frac{3}{8}a_{10} - \frac{3}{8}a_{11}\right) \\
&\quad \times\partial_{[a}\bar{H}_{bc]d}\nabla^d\varphi \\
&\quad -3e^{\varphi/2}V_{[ab}\Delta g_{c]} - 3e^{-\varphi/2}W_{[ab}\Delta b_{c]}
\end{aligned}
\tag{32}
$$

which involve the three bulk parameters a_1, a_{10}, a_{11}. In [23], the field redefinition (11) has not been used to write the bulk couplings in the minimal scheme, and a particular correction to the Buscher rules has been used that does not include $\Delta\bar{\phi}$, as in [18]. Note that if one sets a_{11} to zero, then as we will see in the next section, the bulk couplings would not be consistent with the $O(25, 25)$ symmetry whereas the couplings in [23] are consistent with the $O(25, 25)$ symmetry. Hence, as expected, the corrections to the Buscher rules depend on the scheme that one uses for the gauge invariant couplings.

4 Constraint from $O(25, 25)$ symmetry

In the previous section we have considered the background which has a boundary and a circle independent of it. For this background the circle reduction of the effective actions should have the symmetry $O(1, 1)$. For the simplicity of the calculations we have considered only the Z_2-subgroup of $O(1, 1)$. The constraint from this Z_2-symmetry, fixes the 17 bulk parameters and 38 boundary parameters in terms of three bulk and 7 boundary parameters. One may expect if one imposes the full $O(1, 1)$-symmetry, then the remaining parameters would be fixed. Or one may consider the background which has a boundary and a torus T^2, then the dimensional reduction of the effective actions should have the symmetry $O(2, 2)$. Imposing this symmetry, one may be able to

fix all parameters in (14), (15) in terms of one overall factor. We leave this calculation for the future works, however, in this section we consider a simpler calculation. We consider the background which depends only on one coordinate ζ, i.e., all other coordinates are assumed to be the torus T^{25}. If $\zeta = t$ in which the boundary is spacelike, then all circles in $T^{(25)}$ are spacial coordinates, and if $\zeta = x$ in which the boundary is timelike, then one of the circle is along the time direction. Then the cosmological/one-dimensional reduction of the bulk and boundary couplings in (30), (31) should have the symmetry $O(25, 25)$. This symmetry should constrain the parameters in (30), (31). Furthermore, in this case the lower-dimensional boundary action should be zero. This further constrains the parameters in the boundary action.

When fields depend only on one coordinate ζ, using the gauge symmetries it is possible to write the metric, B-field and dilaton as

$$
G_{\mu\nu} = \begin{pmatrix} \mp n^2(\zeta) & 0 \\ 0 & G_{ij}(\zeta) \end{pmatrix}, \ B_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ij}(\zeta) \end{pmatrix}, \ 2\Phi
$$

$$
= \phi + \frac{1}{2}\log|\det(G_{ij})|
\tag{33}
$$

where minus (plus) sign is when $\zeta = t$ ($\zeta = x$), and the lapse function $n(\zeta)$ can also be fixed to $n = 1$ ($n = -1$) when $\zeta = t$ ($\zeta = x$). This function at the boundary is the unit vector orthogonal to the boundary. Using the above reduction, then the cosmological reduction of the bulk action in (7) in terms of the generalized metric $\mathcal{S}$ which is defined as

$$
\mathcal{S} \equiv \eta \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}
\tag{34}
$$

where η is the metric of the $O(25, 25)$ group, i.e.,

$$
\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\tag{35}
$$

becomes [28,29]

$$
\mathbf{S}_0^c = -\frac{2}{\kappa^2 n}\int d\zeta\, e^{-\phi}\left[-\dot{\phi}^2 - \frac{1}{8}\mathrm{tr}(\dot{\mathcal{S}}^2)\right]
\tag{36}
$$

where dot represents the ζ-derivative. The above action is invariant under the global $O(25, 25)$ transformations because the one-dimensional dilaton is invariant and the generalized metric transforms as

$$
\mathcal{S} \to \Omega^T \mathcal{S}\Omega
\tag{37}
$$

where Ω belong to the $O(25, 25)$ group, i.e., $\Omega^T\eta\Omega = \eta$. The cosmological/one-dimensional reduction of the boundary action in (7) becomes zero [23], i.e., $\partial\mathbf{S}_0^c = 0$.

Using the reductions (33), one finds the following cosmological/one-dimensional reduction for the bulk action (30):

$$\mathbf{S}_1^c = -\frac{2}{\kappa^2 n}\int d\zeta\, e^{-\phi}\bigg[\frac{1}{16}(240a_1 + a_{10} + a_{11})$$
$$\times \dot{B}_i{}^k \dot{B}^{ij}\dot{B}_j{}^l\dot{B}_{kl}$$
$$+\frac{1}{64}(192a_1 + a_{10} + a_{11})\dot{B}_{ij}\dot{B}^{ij}\dot{B}_{kl}\dot{B}^{kl}$$
$$+\left(30a_1 + \frac{1}{16}a_{11}\right)\dot{B}^{ij}\dot{B}^{kl}\dot{G}_{ik}\dot{G}_{jl}$$
$$+\left(-36a_1 - \frac{1}{4}a_{10} - \frac{1}{8}a_{11}\right)\dot{B}_i{}^k \dot{B}^{ij}\dot{G}_j{}^l\dot{G}_{kl}$$
$$+\frac{1}{16}(48a_1 + a_{10} + a_{11})\dot{G}_i{}^k\dot{G}^{ij}\dot{G}_j{}^l\dot{G}_{kl}$$
$$-\frac{1}{16}a_{11}\dot{G}^i{}_i\dot{G}_j{}^l\dot{G}^{jk}\dot{G}_{kl}$$
$$+\left(6a_1 + \frac{1}{256}(8a_{10} + 3a_{11})\right)\dot{B}_{ij}\dot{B}^{ij}\dot{G}_{kl}\dot{G}^{kl}$$
$$+\left(3a_1 + \frac{1}{256}(4a_{10} - 5a_{11})\right)\dot{G}_{ij}\dot{G}^{ij}\dot{G}_{kl}\dot{G}^{kl}$$
$$+\frac{11}{256}a_{11}\dot{G}^i{}_i\dot{G}^j{}_j\dot{G}_{kl}\dot{G}^{kl}$$
$$+\frac{1}{32}(2a_{10} + a_{11})\dot{B}_i{}^k\dot{B}^{ij}\dot{G}_{jk}\dot{G}^l{}_l$$
$$-\frac{1}{256}a_{11}\dot{B}_{ij}\dot{B}^{ij}\dot{G}^k{}_k\dot{G}^l{}_l$$
$$-\frac{1}{8}a_{10}\dot{B}_i{}^k\dot{B}^{ij}\dot{G}_{jk}\dot{\phi} + \frac{1}{8}a_{10}\dot{G}_i{}^k\dot{G}^{ij}\dot{G}_{jk}\dot{\phi}$$
$$+\frac{1}{64}a_{11}\dot{G}^i{}_i\dot{G}_{jk}\dot{G}^{jk}\dot{\phi} + \frac{1}{32}a_{10}\dot{B}_{ij}\dot{B}^{ij}\dot{G}^k{}_k\dot{\phi}$$
$$+\frac{1}{64}a_{11}\dot{G}^i{}_i\dot{G}^j{}_j\dot{G}^k{}_k\dot{\phi} + \frac{1}{64}(4a_{10} + a_{11})\dot{G}_{ij}\dot{G}^{ij}\dot{\phi}^2$$
$$+\frac{1}{64}a_{11}\dot{G}^i{}_i\dot{G}^j{}_j\dot{\phi}^2 - \frac{1}{32}a_{10}\dot{B}^{ij}\dot{G}^k{}_k\ddot{B}_{ij}$$
$$+\frac{1}{16}a_{10}\dot{B}^{ij}\dot{\phi}\ddot{B}_{ij} + \frac{1}{16}(384a_1 + a_{10} + a_{11})\ddot{B}_{ij}\ddot{B}^{ij}$$
$$+\left(48a_1 + \frac{1}{8}a_{11}\right)\dot{B}^{ij}\dot{G}_i{}^k\ddot{B}_{jk} - \frac{1}{8}a_{10}\dot{G}^{ij}\dot{\phi}\ddot{G}_{ij}$$
$$-\frac{1}{16}a_{11}\dot{\phi}^2\ddot{G}^i{}_i + \frac{1}{16}(384a_1 + a_{10} + a_{11})\ddot{G}_{ij}\ddot{G}^{ij}$$
$$+\left(72a_1 + \frac{1}{4}a_{10} + \frac{3}{16}a_{11}\right)\dot{B}_i{}^k\dot{B}^{ij}\ddot{G}_{jk}$$
$$-\left(24a_1 + \frac{1}{8}a_{10} + \frac{1}{8}a_{11}\right)\dot{G}_i{}^k\dot{G}^{ij}\ddot{G}_{jk}$$
$$+\frac{1}{16}a_{11}\dot{G}^i{}_i\dot{G}^{jk}\ddot{G}_{jk} - \frac{1}{16}a_{11}\dot{G}^i{}_i\dot{\phi}\ddot{G}^j{}_j$$
$$+\frac{1}{64}a_{11}\dot{B}_{ij}\dot{B}^{ij}\ddot{G}^k{}_k + \frac{1}{32}a_{11}\dot{G}_{ij}\dot{G}^{ij}\ddot{G}^k{}_k$$
$$-\frac{3}{64}a_{11}\dot{G}^i{}_i\dot{G}^j{}_j\ddot{G}^k{}_k - \frac{1}{8}a_{10}\dot{G}_{ij}\dot{G}^{ij}\ddot{\phi} + \frac{1}{4}a_{10}\ddot{\phi}^2\bigg] \tag{38}$$

This action is not invariant under $O(25, 25)$ transformations. Some of the non-invariant terms are total derivative terms

which should be transferred to the boundary by using the Stokes's theorem. Moreover, the action in terms of the variables G_{ij}, B_{ij}, Φ is not invariant. As we have clarified in the Introduction section, it should be invariant in terms of some other variables which involve the first derivatives of G_{ij}, B_{ij}, Φ.

To find the total derivatives terms in (38), we add all total derivative terms at order α' with arbitrary coefficients to (38). We add the following total derivative terms:

$$-\frac{2}{\kappa^2 n}\int d\zeta\,\frac{d}{d\zeta}(e^{-\phi}\mathcal{I}_1) = -\frac{2}{\kappa^2 n}e^{-\phi}\mathcal{I}_1 \tag{39}$$

where $\mathcal{I}_1$ is all possible terms at three-derivative level with even parity which are constructed from the derivatives of ϕ, B_{ij}, G_{ij}. Using the package "xAct", one finds there are 18 such terms. One can also change the field variables in the bulk action (38) as

$$G_{ij} \to G_{ij} + \alpha'\delta G_{ij}^{(1)}$$
$$B_{ij} \to B_{ij} + \alpha'\delta B_{ij}^{(1)}$$
$$\phi \to \phi + \alpha'\delta\phi^{(1)} \tag{40}$$

where the matrices $\delta G_{ij}^{(1)}$, $\delta B_{ij}^{(1)}$ and $\delta\phi^{(1)}$ are all possible terms at 2-derivative level constructed from $\dot{\phi}$, $\dot{B}$, $\dot{G}$. The perturbations $\delta G_{ij}^{(1)}$, $\delta\phi^{(1)}$ contain even-parity terms and $\delta B_{ij}^{(1)}$ contains odd-parity terms.

When the field variables are changed according to the above restricted field redefinitions, then $\mathbf{S}_0^c$ produces some couplings at order α' and higher. In this section we are interested in the resulting couplings at order α', i.e.,

$$\delta\mathbf{S}_0^c = -\frac{2}{\kappa^2 n}\int d\zeta\, e^{-\phi}\bigg[\delta\phi^{(1)}\left(-\frac{1}{4}\dot{B}_{ij}\dot{B}^{ij} - \frac{1}{4}\dot{G}_{ij}\dot{G}^{ij} + \dot{\phi}^2\right)$$
$$-2\dot{\phi}\frac{d}{d\zeta}\delta\phi^{(1)}$$
$$+\delta G_{ij}^{(1)}\left(-\frac{1}{2}\dot{B}_k{}^j\dot{B}^{ki} - \frac{1}{2}\dot{G}_k{}^j\dot{G}^{ki}\right) + \frac{1}{2}\dot{G}^{ij}\frac{d}{d\zeta}\delta G_{ij}^{(1)}$$
$$+\frac{1}{2}\dot{B}^{ij}\frac{d}{d\zeta}\delta B_{ij}^{(1)}\bigg] \tag{41}$$

We use the field redefinitions and the total derivative terms to remove all terms in the bulk action except the couplings which have only the first derivative of G_{ij}, B_{ij}, ϕ.

Using the following field redefinitions:

$$\delta\phi^{(1)} = -6a_1\dot{B}_{ij}\dot{B}^{ij} + 6a_1\dot{G}^i{}_i\dot{G}^j{}_j + 24a_1\dot{G}^i{}_i\dot{\phi} + 24a_1\dot{\phi}^2$$
$$\delta G_{ij}^{(1)} = -24a_1\dot{B}_i{}^k\dot{B}_{jk} - 24a_1\dot{G}_i{}^k\dot{G}_{jk}$$
$$\delta B_{ij}^{(1)} = 24a_1\dot{B}_j{}^k\dot{G}_{ik} - 24a_1\dot{B}_i{}^k\dot{G}_{jk} + 24a_1\dot{B}_{ij}\dot{G}^k{}_k$$
$$\qquad + 48a_1\dot{B}_{ij}\dot{\phi} \tag{42}$$

and the following total derivative terms:

$$\mathcal{I}_1 = 24a_1\dot{B}_i{}^k\dot{B}^{ij}\dot{G}_{jk} + 12a_1\dot{G}^i{}_i\dot{G}_{jk}\dot{G}^{jk} - 6a_1\dot{B}_{ij}\dot{B}^{ij}\dot{G}^k{}_k$$

$$-6a_1 \dot{G}^i{}_i \dot{G}^j{}_j \dot{G}^k{}_k - 24a_1 \dot{B}_{ij} \dot{B}^{ij} \dot{\phi} + 24a_1 \dot{G}^i{}_i \dot{\phi}^2$$
$$+32a_1 \dot{\phi}^3 \tag{43}$$

one finds the cosmological action (38) becomes invariant under the $O(25, 25)$-transformations when there is the following relation between the bulk parameters a_1, a_{10}, a_{11}:

$$a_{11} = -384a_1, \quad a_{10} = 0 \tag{44}$$

Note that a_{11} is not zero. Hence, in the minimal scheme that we have chosen for the bulk couplings (14), the correction $\Delta\bar{\phi}$ to the Buscher rules, i.e., (32), can not be zero.

The $O(25, 25)$-invariant form of the action is

$$S_1^c = \mathbf{S}_1^c + \delta\mathbf{S}_0^c = -\frac{2}{\kappa^2 n} 24a_1 \int dt e^{-\phi} \left[\frac{1}{16} \mathrm{tr}(\dot{\mathcal{S}}^4) \right.$$
$$\left. -\frac{1}{64} (\mathrm{tr}(\dot{\mathcal{S}}^2))^2 + \frac{1}{2} \mathrm{tr}(\dot{\mathcal{S}}^2) \dot{\phi}^2 - \frac{1}{3} \dot{\phi}^4 \right] \tag{45}$$

where

$$\mathrm{tr}(\dot{\mathcal{S}}^2) = 2\dot{B}_{ij} \dot{B}^{ij} + 2\dot{G}_{ij} \dot{G}^{ij}$$
$$\mathrm{tr}(\dot{\mathcal{S}}^4) = 2\dot{B}_i{}^k \dot{B}^{ij} \dot{B}_j{}^l \dot{B}_{kl} - 4\dot{B}^{ij} \dot{B}^{kl} \dot{G}_{ik} \dot{G}_{jl}$$
$$+8\dot{B}_i{}^k \dot{B}^{ij} \dot{G}_j{}^l \dot{G}_{kl}$$
$$+2\dot{G}_i{}^k \dot{G}^{ij} \dot{G}_j{}^l \dot{G}_{kl} \tag{46}$$

The cosmological boundary action (45) is the one considered in [24].

Since the lower-dimensional boundary action at the leading order is zero, the field redefinitions (42) produce no term at order α'. However, the total derivative terms (43) appear in the cosmological/one-dimensional reduction of the boundary terms at order α'. Note that most of the terms in the total derivative terms are not consistent with the $O(25, 25)$ symmetry. They must be cancelled with the anomalous terms in the cosmological reduction of the boundary couplings.

The one-dimensional reduction of the timelike boundary couplings (31) is the following:

$$\partial\mathbf{S}_1^c = -\frac{2}{\kappa^2 n} e^{-\phi} \left[\frac{1}{4} \left(96a_1 - (b_{11} + 12b_{12}) \right) \dot{B}_i{}^k \dot{B}^{ij} \dot{G}_{jk} \right.$$
$$+\frac{1}{4} \left(192a_1 - (b_{11} + 12b_{12}) \right) \dot{G}_i{}^k \dot{G}^{ij} \dot{G}_{jk}$$
$$-12a_1 \dot{G}^i{}_i \dot{G}_{jk} \dot{G}^{jk} + 6a_1 \dot{B}_{ij} \dot{B}^{ij} \dot{G}^k{}_k$$
$$+6a_1 \dot{G}^i{}_i \dot{G}^j{}_j \dot{G}^k{}_k$$
$$+\frac{1}{2} (24a_1 + 6b_1 - b_{19}) \dot{B}_{ij} \dot{B}^{ij} \dot{\phi} + \frac{1}{2} (-24a_1$$
$$+6b_1 - b_{19}) \dot{G}_{ij} \dot{G}^{ij} \dot{\phi}$$
$$-24a_1 \dot{G}^i{}_i \dot{\phi}^2 + \frac{1}{6} (-96a_1 + 24b_1 - b_{18}) \dot{\phi}^3$$
$$+\frac{1}{4} (-192a_1 + b_{11} + 12b_{12}) \dot{B}^{ij} \ddot{B}_{ij}$$
$$+\frac{1}{4} (-192a_1 + b_{11} + 12b_{12}) \dot{G}^{ij} \ddot{G}_{ij}$$

$$+\frac{1}{2} (192a_1 - 24b_{12} - b_{17} + 4b_{19}) \dot{\phi}\ddot{\phi} \right] \tag{47}$$

where we have also used the relations (44). The above action is not invariant under the $O(25, 25)$ transformations. Note that the boundary parameter b_{38} does not appear in the above action which indicates that its corresponding couplings is invariant under the $O(25, 25)$ transformations. This coupling is also invariant under the $O(1, 1)$ transformations. The boundary couplings in (31) which have the coefficient b_1 are also invariant under the $O(25, 25)$ because this parameter does not appear in the above cosmological action. If one adds to the above action the total derivative term (39) in which $\mathcal{I}_1$ is given in (43), one can choose the boundary parameters such that the result becomes invariant. For the following relations between the parameters:

$$b_{11} = 96a_1 - 24b_1, \quad b_{12} = 8a_1 + 2b_1, \quad b_{19} = 12b_1 \frac{1}{4} b_{17} \tag{48}$$

The boundary action becomes $O(25, 25)$-invariant which involves the first derivative of the dilaton. The assumption that the boundary action must be zero, i.e., (6), also implies the following relations for the parameters:

$$b_{17} = -96a_1 - 24b_1, \quad b_{18} = 96a_1 + 24b_1 \tag{49}$$

That is, for the above relations one has $\partial\mathbf{S}_1^c = 0$. The above relations, reduce the 7 boundary parameters in (31) to 2 parameters b_1, b_{38}. In the next section we show that the couplings satisfy the least action principle with no further constraint on the parameters, however, the condition that the boundary couplings should include the Chern–Simons form fully constrains the parameters.

5 Constraint from the Chern–Simons form

For the spacetime manifolds which have boundary, both the bulk and boundary actions should satisfy the least action principle, i.e., $\delta(\mathbf{S}_1 + \partial\mathbf{S}_1) = 0$ with the appropriate boundary condition on the massless fields. For the effective action at order α', the massless fields and their first derivative must be known on the boundary [23]. The couplings in the bulk action (30) contain terms which have at most two derivative on the massless fields. The variation of the bulk action then produces at most the variation of second derivative of the massless fields in the bulk. Using the Stokes's theorem, they appear on the boundary as the variation of the first derivative of the massless fields which are zero on the boundary. Hence, the least action does not constrain the couplings in (30). The boundary couplings (31), however, contain terms which have second derivative of the massless fields. In the

least action, they produce the variation of the second derivative of the massless fields on the boundary which are not zero. To have no variation of the second derivative of the massless fields, the parameters in the boundary action may satisfy some constraints.

Inserting the relations (44), (48) and (49) into the boundary action (31), one finds that the variation of the resulting boundary action against the metric variation produces the following terms in the local frame in which the first partial derivative of metric is zero:

$$
\begin{aligned}
&24(4a_1 + b_1)\partial^\alpha \Phi f1^\beta P^{\gamma\delta}\partial_\alpha \partial_\beta \delta G_{\gamma\delta} \\
&24(4a_1 + b_1)\partial^\alpha \Phi f1_\alpha f1^\beta f1^\gamma P^{\delta\epsilon}\partial_\beta \partial_\gamma \delta G_{\delta\epsilon}
\end{aligned}
\tag{50}
$$

where $f1^\alpha = \partial^\alpha f$. We have used the assumption that the variation of metric and its first derivative, and their tangent derivatives are zero, i.e., $\delta G_{\alpha\beta} = \partial_\mu \delta G_{\alpha\beta} = 0$ and $P^{\mu\nu}\partial_\mu \partial_\gamma \delta G_{\alpha\beta} = 0$. The above variations must be zero, up to some total derivative terms on the boundary. One finds that , up to some total derivative terms, the above variation becomes zero with no relation between b_1, a_1. We also find that the variation of the boundary action against the dilaton and the B-field variations produces zero result, up to some total derivative terms. Hence, the least action principle produces no constraint on the parameters on top of the constraints that are found by the T-duality.

Inserting the relations (44) into the bulk action (30), one finds the bulk action is fixed up to one overall parameter, i.e.,

$$
\begin{aligned}
\mathcal{L}_1 = a_1 \Big[&H_\alpha{}^{\delta\epsilon} H^{\alpha\beta\gamma} H_{\beta\delta}{}^\varepsilon H_{\gamma\epsilon\varepsilon} - 3H_{\alpha\beta}{}^\delta H^{\alpha\beta\gamma} H_\gamma{}^{\epsilon\varepsilon} H_{\delta\epsilon\varepsilon} \\
&+24H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} R^{\alpha\beta} \\
&-96R_{\alpha\beta}R^{\alpha\beta} - 2H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}R + 24R^2 \\
&+24R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \\
&-12H_\alpha{}^{\delta\epsilon} H^{\alpha\beta\gamma} R_{\beta\gamma\delta\epsilon} + 96R\nabla_\alpha \Phi \nabla^\alpha \Phi \\
&-384R_{\alpha\beta}\nabla^\alpha \Phi \nabla^\beta \Phi \Big]
\end{aligned}
\tag{51}
$$

The gravity part of the above couplings is the Gauss–Bonnet gravity. However, inserting the relations (44), (48) and (49) into the boundary action, one finds that the boundary action is not fixed up to the bulk parameter a_1, i.e., it has two boundary parameters b_1, b_{38} which should be fixed by some other constraints. One may consider other backgrounds to fix the remaining parameters. For example, one may consider the background which has a boundary and a torus T^2. Then the dimensional reduction of the effective actions on this torus should have the symmetry $O(2, 2)$. The bulk couplings (51) should satisfy this symmetry automatically because there is only one overall parameter in the bulk action, however, the boundary couplings may satisfy this symmetry for some specific relations between the parameters a_1, b_1, b_{38}. We leave this calculation for the future works.

Here we fix the remaining parameters in the boundary action by noting that the boundary couplings include the structures as those in the Chern–Simons form. Hence, we fix the remaining parameters in the boundary action such that the gravity couplings in the boundary include the Chern–Simons form. The Chern–Simons form has the following gravity couplings for timelike boundary [22]:

$$
\begin{aligned}
Q_2 = 4\Big[&K^\mu{}_\mu \tilde{R} - 2K^{\mu\nu}\tilde{R}_{\mu\nu} + \frac{1}{3}(3K^\alpha{}_\alpha K_{\mu\nu}K^{\mu\nu} \\
&-K^\mu{}_\mu K^\nu{}_\nu K^\alpha{}_\alpha - 2K_\mu{}^\nu K_{\nu\alpha}K^{\alpha\mu}) \Big]
\end{aligned}
\tag{52}
$$

where $\tilde{R}_{\mu\nu}$ and $\tilde{R}$ are curvatures that are constructed from the induced metric (8). In terms of the spacetime metric and curvature, it is

$$
\begin{aligned}
Q_2 = 4\Big[&K^\mu{}_\mu R - 2K^{\mu\nu}R_{\mu\nu} - 2K_\alpha{}^\alpha n^\mu n^\nu R_{\mu\nu} \\
&+2K^{\mu\nu}n^\alpha n^\beta R_{\alpha\mu\beta\nu} \\
&-\frac{1}{3}(6K^\alpha{}_\alpha K_{\mu\nu}K^{\mu\nu} - 2K^\mu{}_\mu K^\nu{}_\nu K^\alpha{}_\alpha \\
&-4K_\mu{}^\nu K_{\nu\alpha}K^{\alpha\mu}) \Big]
\end{aligned}
\tag{53}
$$

Using the following identity [23]:

$$
n^\alpha n^\beta n^\gamma n^\delta \nabla_\delta \nabla_\gamma K_{\alpha\beta} = -2K_\alpha{}^\gamma K^{\alpha\beta}K_{\beta\gamma} + n^\alpha n^\beta \nabla_\gamma \nabla^\gamma K_{\alpha\beta}
\tag{54}
$$

one finds that the gravity couplings in the boundary action become the same as the couplings in Q_2 for the following relations:

$$
b_1 = -4a_1, \ b_{38} = 32a_1
\tag{55}
$$

In fact, inserting the relations (44), (48), (49) and (55) into the boundary action (31), one finds

$$
\begin{aligned}
\partial \mathcal{L}_1 = a_1 \Big[&24Q_2 + 32n^\alpha n^\beta \nabla_\gamma \nabla^\gamma K_{\alpha\beta} - 4H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha \\
&+24H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} K^{\alpha\beta} \\
&+24H_\alpha{}^{\delta\epsilon} H_{\beta\delta\epsilon} K^\gamma{}_\gamma n^\alpha n^\beta \\
&-48H_\beta{}^{\delta\epsilon} H_{\gamma\delta\epsilon} n^\alpha n^\beta n^\gamma \nabla_\alpha \Phi \\
&+192K^\beta{}_\beta \nabla_\alpha \Phi \nabla^\alpha \Phi \\
&-384K^\gamma{}_\gamma n^\alpha n^\beta \nabla_\alpha \Phi \nabla_\beta \Phi - 384K_{\alpha\beta}\nabla^\alpha \Phi \nabla^\beta \Phi \\
&+256n^\alpha n^\beta n^\gamma \nabla_\alpha \Phi \nabla_\beta \Phi \nabla_\gamma \Phi \Big]
\end{aligned}
\tag{56}
$$

Hence, the boundary couplings are also fixed up to the overall bulk parameter a_1.

6 Comparing with Meissner action

We have used the restricted field redefinitions to find the independent bulk couplings in the minimal scheme (14). Then using the assumption that the effective action is background independent, we considered the backgrounds in which the effective actions have symmetries $O(1, 1)$ and $O(25, 25)$ to fix the parameters. We have also used the assumption that the gravity couplings on the boundary should include the Chern–Simons form. Then we have found the bulk and boundary couplings up to an overall factor. Similar calculations have been done in [23] in which no field redefinition has been used and a particular T-duality transformations has been used which produces the bulk action to be the Meissner action [24]. The Meissner bulk action and its corresponding timelike boundary actions are

$$
\begin{aligned}
\mathbf{S}_1^M = -\frac{48a_1}{\kappa^2} \int_M d^{26}x \sqrt{-G} e^{-2\Phi} \Big[& R_{GB}^2 \\
& + \frac{1}{24} H_\alpha{}^{\delta\epsilon} H^{\alpha\beta\gamma} H_{\beta\delta}{}^\varepsilon H_{\gamma\epsilon\varepsilon} \\
& - \frac{1}{8} H_{\alpha\beta}{}^\delta H^{\alpha\beta\gamma} H_\gamma{}^{\epsilon\varepsilon} H_{\delta\epsilon\varepsilon} \\
& + \frac{1}{144} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} H_{\delta\epsilon\varepsilon} H^{\delta\epsilon\varepsilon} + H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} R^{\alpha\beta} \\
& - \frac{1}{6} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} R - \frac{1}{2} H_\alpha{}^{\delta\epsilon} H^{\alpha\beta\gamma} R_{\beta\gamma\delta\epsilon} \\
& - \frac{2}{3} H_{\beta\gamma\delta} H^{\beta\gamma\delta} \nabla_\alpha \nabla^\alpha \Phi + \frac{2}{3} H_{\beta\gamma\delta} H^{\beta\gamma\delta} \nabla_\alpha \Phi \nabla^\alpha \Phi \\
& + 8R \nabla_\alpha \Phi \nabla^\alpha \Phi + 16 \nabla_\alpha \Phi \nabla^\alpha \Phi \nabla_\beta \nabla^\beta \Phi \\
& - 16 R_{\alpha\beta} \nabla^\alpha \Phi \nabla^\beta \Phi - 16 \nabla_\alpha \Phi \nabla^\alpha \Phi \nabla_\beta \Phi \nabla^\beta \Phi \\
& + 2 H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} \nabla^\beta \nabla^\alpha \Phi \Big]
\end{aligned}
$$

$$
\begin{aligned}
\partial \mathbf{S}_1^M = -\frac{48a_1}{\kappa^2} \int_{\partial M} d^{25}\sigma \sqrt{-g} e^{-2\Phi} \Big[& Q_2 \\
& + \frac{4}{3} n^\alpha n^\beta \nabla_\gamma \nabla^\gamma K_{\alpha\beta} \\
& + \frac{2}{3} H_{\beta\gamma\delta} H^{\beta\gamma\delta} n^\alpha \nabla_\alpha \Phi \\
& - 2 H_\beta{}^{\delta\epsilon} H_{\gamma\delta\epsilon} n^\alpha n^\beta n^\gamma \nabla_\alpha \Phi - \frac{1}{3} H_{\beta\gamma\delta} H^{\beta\gamma\delta} K^\alpha{}_\alpha \\
& + H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} K^{\alpha\beta} + H_\alpha{}^{\delta\epsilon} H_{\beta\delta\epsilon} K^\gamma{}_\gamma n^\alpha n^\beta \\
& - 16 K^\gamma{}_\gamma n^\alpha n^\beta \nabla_\alpha \Phi \nabla_\beta \Phi \\
& + 16 K^\beta{}_\beta \nabla_\alpha \Phi \nabla^\alpha \Phi - 16 K_{\alpha\beta} \nabla^\alpha \Phi \nabla^\beta \Phi \\
& - 16 n^\alpha \nabla_\alpha \Phi \nabla_\beta \Phi \nabla^\beta \Phi \\
& + \frac{32}{3} n^\alpha n^\beta n^\gamma \nabla_\alpha \Phi \nabla_\beta \Phi \nabla_\gamma \Phi \Big]
\end{aligned}
$$

where R_{GB}^2 is the Gauss–Bonnet bulk couplings and Q_2 is the Chern-Simons boundary couplings (53).

In this section we are going to show that the above couplings and the couplings (51) and (56) are identical up to a

particular restricted field redefinition. In general, under the field redefinition $\Psi \to \Psi'$, the form of actions are changes. However, up to some total derivative terms, the sum of the bulk and boundary actions in terms of variable Ψ should be the same as the new actions in terms of new field Ψ', i.e.,

$$
\mathbf{S}_{\text{eff}}(\Psi) + \partial\mathbf{S}_{\text{eff}}(\Psi) = \mathbf{S}'_{\text{eff}}(\Psi') + \partial\mathbf{S}'_{\text{eff}}(\Psi') \tag{58}
$$

Using the α'-expansion (1) for the couplings and the α'-expansion for the new field variable Ψ', i.e.,

$$
\Psi' = \sum_{m=0}^\infty \alpha'^m \Psi'_m \tag{59}
$$

where $\Psi'_0 = \Psi$, one finds the following relations for the bulk and boundary couplings at order α' which are similar to the relations (20) in the base space:

$$
\begin{aligned}
& \mathbf{S}_1(\Psi) - \mathbf{S}'_1(\Psi) - \delta\mathbf{S}_0 - \frac{2}{\kappa^2} \int d^{26}x \sqrt{-G} \nabla_\alpha (J_1^\alpha e^{-2\Phi}) \\
& = 0 \\
& \partial\mathbf{S}_1(\Psi) - \partial\mathbf{S}'_1(\Psi) - \delta\partial\mathbf{S}_0 + \mathcal{T}_1(\Psi) \\
& \quad + \frac{2}{\kappa^2} \int d^{25}\sigma \sqrt{|g|} n_\alpha J_1^\alpha e^{-2\Phi} = 0
\end{aligned} \tag{60}
$$

where $\delta\mathbf{S}_0$ and $\delta\partial\mathbf{S}_0$ are the variations of the leading order bulk and boundary actions, respectively. $\mathcal{T}_1$ is arbitrary total derivative terms on the boundary at order α' and J_1^α is arbitrary vector at order α' in the bulk.

The difference between the bulk couplings in the Meissner action (57) and the bulk couplings (51) are the following terms:

$$
\begin{aligned}
a_1 \Big[& \frac{1}{6} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} H_{\delta\epsilon\varepsilon} H^{\delta\epsilon\varepsilon} - 2 H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} R \\
& - 16 H_{\beta\gamma\delta} H^{\beta\gamma\delta} \nabla_\alpha \nabla^\alpha \Phi + 16 H_{\beta\gamma\delta} H^{\beta\gamma\delta} \nabla_\alpha \Phi \nabla^\alpha \Phi \\
& + 96 R \nabla_\alpha \Phi \nabla^\alpha \Phi + 384 \nabla_\alpha \Phi \nabla^\alpha \Phi \nabla_\beta \nabla^\beta \Phi \\
& - 384 \nabla_\alpha \Phi \nabla^\alpha \Phi \nabla_\beta \Phi \nabla^\beta \Phi + 48 H_\alpha{}^{\gamma\delta} H_{\beta\gamma\delta} \nabla^\beta \nabla^\alpha \Phi \Big]
\end{aligned}
$$

Using the variation (12), one finds that the above terms are removable by the following field redefinitions:

$$
\delta\Phi^{(1)} = a_1 \Big[H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - 48 \nabla_\alpha \Phi \nabla^\alpha \Phi \Big]
$$

$$
\delta B_{\alpha\beta}^{(1)} = -96 a_1 H_{\alpha\beta\gamma} \nabla^\gamma \Phi \tag{61}
$$

and by the total derivative terms with the following vector:

$$
J_1^\alpha = a_1 \Big[16 H_{\beta\gamma\delta} H^{\beta\gamma\delta} \nabla^\alpha \Phi - 384 \nabla^\alpha \Phi \nabla_\beta \Phi \nabla^\beta \Phi \Big] \tag{62}
$$

When the field redefinition (61) is imposed on the boundary coupling (7) at the leading order, it produces the following boundary couplings order α':

$$
\delta\partial\mathbf{S}_0 = -\frac{2}{\kappa^2} \int d^{25}\sigma \sqrt{-g} e^{-2\Phi} (-4a_1) K^\mu{}_\mu \Big[H_{\alpha\beta\gamma} H^{\alpha\beta\gamma}
$$

$$
-48\nabla_\alpha\Phi\nabla^\alpha\Phi\Bigg] \tag{63}
$$

One the other hand, the difference between the boundary couplings in (57) and the boundary couplings (56) are

$$
a_1\Bigg[-4H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + 16H_{\beta\gamma\delta}H^{\beta\gamma\delta}n^\alpha\nabla_\alpha\Phi
$$

$$
+192K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi - 384n^\alpha\nabla_\alpha\Phi\nabla_\beta\Phi\nabla^\beta\Phi\Bigg]
$$

which are removed by the variation (63) and the vector (62). No total derivative term on the boundary, i.e., $\mathcal{T}_1(\Psi)$, is needed to satisfy the second equation in (60). Therefore, the bulk and boundary action that we have found in this paper are the same as the couplings in (57) up to the field redefinition (61)

7 Discussion

It is known that the classical effective actions in string theory have the gauge invariant couplings at all orders of α'. Hence, one is free to use the higher-derivative field redefinitions. When there is no boundary, the field redefinition is to write the massless fields in terms of the derivatives of the massless fields at all orders of α' with arbitrary coefficients [3]. In this paper we have proposed that in the presence of boundary, the higher-derivative field redefinitions should be restricted to those which do not change the data on the boundary. It has been proposed in [23] that in the least action principle in the presence of the boundary, the massless fields and their derivatives up to order n must be known on the boundary for the effective action at order α'^n. The field redefinitions should not change this information on the boundary. For example, for the effective action at order α', the massless fields and their first derivatives are known on the boundary. The field redefinition that does not change these data, requires that the metric does not change, and all other massless fields include only the first derivative of the massless fields. Using this restricted field redefinition for the effective action of the bosonic string theory at order α', we find there are 17 independent bulk couplings, i.e., (14). There are also 38 independent boundary couplings.

We then use the assumption that the effective action of string theory at each order of α' and at the critical dimension is background independent, i.e., the assumption that coefficients of the above 55 independent couplings are constant. We choose the background which has a boundary and a circle. The dimensional reduction of the effective action should then have the $O(1, 1)$ symmetry. We constrain the 55 constants such that the couplings become invariant under the Z_2-subgroup of the $O(1, 1)$-group. The Z_2-transformations are the Buscher rules plus some corrections that involve

only the first derivative of the massless fields in the base space. This constraint does not fully fix all the constants. We then choose another background in which all fields depend only on one coordinate. In this case the cosmological/one-dimensional reduction of the effective action should have $O(25, 25)$ symmetry. We constrain the remaining parameters such that the one-dimensional actions become invariant under the $O(25, 25)$-transformations after using some one-dimensional field redefinitions which involve only the first derivative of the massless fields in the one-dimensional base space. We then impose the constraint that the boundary action is zero in the scheme in which the one-dimensional bulk action involves only the first derivative terms, to further fix the coefficients of the couplings. We find that these three set of constraints fixed the 17 bulk parameters up to one overall factor. However, 2 of the boundary parameters remain unfixed. When B-field and dilaton are zero, the resulting bulk couplings are exactly the bulk couplings in the Gauss-Bonnet gravity. Imposing the constraint that the gravity couplings in the boundary action should be consistent with the Chern-Simons form, the 2 boundary parameters are also fixed in terms of the overall factor of the bulk couplings. We have found the following effective actions at order α' for the spacetime with timelike boundary:

$$
\mathbf{S}_1 = -\frac{48a_1}{\kappa^2}\int_M d^{26}x\sqrt{-G}e^{-2\Phi}\Bigg[R_{\mathrm{GB}}^2
$$

$$
+\frac{1}{24}H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}H_{\beta\delta}{}^\epsilon H_{\gamma\epsilon\epsilon} - \frac{1}{8}H_{\alpha\beta}{}^\delta H^{\alpha\beta\gamma}H_\gamma{}^{\epsilon\epsilon}H_{\delta\epsilon\epsilon}
$$

$$
+R^{\alpha\beta}H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta} - \frac{1}{12}RH_{\alpha\beta\gamma}H^{\alpha\beta\gamma}
$$

$$
-\frac{1}{2}H_\alpha{}^{\delta\epsilon}H^{\alpha\beta\gamma}R_{\beta\gamma\delta\epsilon}
$$

$$
+4R\nabla_\alpha\Phi\nabla^\alpha\Phi - 16R^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi\Bigg]
$$

$$
\partial\mathbf{S}_1 = -\frac{48a_1}{\kappa^2}\int_{\partial M} d^{25}\sigma\sqrt{-g}e^{-2\Phi}\Bigg[Q_2
$$

$$
+\frac{4}{3}n^\alpha n^\beta\nabla_\gamma\nabla^\gamma K_{\alpha\beta}
$$

$$
-\frac{1}{6}H_{\beta\gamma\delta}H^{\beta\gamma\delta}K^\alpha{}_\alpha + H_\alpha{}^{\gamma\delta}H_{\beta\gamma\delta}K^{\alpha\beta}
$$

$$
+H_\alpha{}^{\delta\epsilon}H_{\beta\delta\epsilon}K^\gamma{}_\gamma n^\alpha n^\beta - 2H_\beta{}^{\delta\epsilon}H_{\gamma\delta\epsilon}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi
$$

$$
+8K^\beta{}_\beta\nabla_\alpha\Phi\nabla^\alpha\Phi
$$

$$
-16K^\gamma{}_\gamma n^\alpha n^\beta\nabla_\alpha\Phi\nabla_\beta\Phi - 16K_{\alpha\beta}\nabla^\alpha\Phi\nabla^\beta\Phi
$$

$$
+\frac{32}{3}n^\alpha n^\beta n^\gamma\nabla_\alpha\Phi\nabla_\beta\Phi\nabla_\gamma\Phi\Bigg] \tag{64}
$$

where R_{GB}^2 is the Gauss–Bonnet gravity couplings and Q_2 is the Chern–Simons couplings (53). We have shown that the above couplings are the same as the Meissner action and its corresponding timelike boundary couplings (57) up to

the restricted field redefinition (61). Unlike in the Meissner action, in the bulk action (64) all fields except the metric carry only the first derivative.

It has been observed in [23] that the consistency of the effective action at order α' with the T-duality requires not only the values of the massless fields but also the values of their first derivatives must be known on the boundary. This makes us to use the restricted field redefinition which involves only the first derivative of the massless fields. On the other hand, if one uses the field redefinitions at order α' which involve the second derivatives of the massless fields, then one would find 8 independent couplings in (14). We have checked it explicitly that the bulk and boundary couplings are not consistent with the T-duality in this case. Hence, the T-duality does not allow the values of the second derivatives of the massless fields on the boundary to be known for the effective action at order α'.

The cosmological action (38) in terms of the functions ϕ, B_{ij}, G_{ij} which appear in the cosmological reduction of Φ, B_μ, $G_{\mu\nu}$ have second derivatives of the functions ϕ, B_{ij}, G_{ij}, i.e., the cosmological action (38) is *nonlocal*. Then in the variation of the action, the first derivative of $\delta\phi$, δB_{ij}, δG_{ij} appear on the boundary which are zero according to the boundary condition proposed in [23] for the nonlocal effective action at order α' that the massless fields and their first derivative should be known on the boundary. However, after the field redefinition (40), (42), the action (45) involves only the first derivatives of the generalized metric $\mathcal{S}$ and dilaton ϕ, i.e., the cosmological action (45) becomes *local* in the new variables. Then in the least action, only variation of these new variables appear on the boundary. Hence, in terms of the new variables, the boundary condition should be the usual boundary condition for the local actions that only the massless fields are known on the boundary. In this case, then there should be no cosmological boundary action, i.e., (6). In other words, the usual boundary condition for the local cosmological actions in terms of the variables $\mathcal{S}$, ϕ requires the constraint (6).

We have found the effective actions (64) for timelike boundary by requiring that the dimensional reduction of the effective action on a circle and on a torus T^{25} to be invariant under the Z_2-transformations, and under the $O(25, 25)$ transformations, respectively. Moreover, we have imposed the constraint from requiring the boundary action to have the Chern–Simons form. The assumption that effective actions at the critical dimensions are background independent then requires that if one considers the background which has a boundary and a compact sub-manifold T^2, then the dimensional reduction of the couplings in (64) should have the $O(2, 2)$ symmetry, up to some restricted field redefinitions. It would be interesting to perform this calculations to check if the couplings (64) are consistent with the $O(2, 2)$ symmetry. The dimensional reduction of the bulk actions at orders

α'^0, α' on torus T^2 have been studied in [30,31]. It would be also interesting to find the boundary couplings for the spacelike boundary for which the Chern–Simons form is given by (52) with different sign for the first two terms [22]. It would be also interesting to extend the calculations in this paper to the order α'^2 to find the boundary couplings at this order, i.e., $\partial\mathbf{S}_2$ in (5).

References

1. J.W. York Jr., Phys. Rev. Lett. **28**, 1082 (1972). https://doi.org/10.1103/PhysRevLett.28.1082
2. G.W. Gibbons, S.W. Hawking, Phys. Rev. D **15**, 2752 (1977). https://doi.org/10.1103/PhysRevD.15.2752
3. R.R. Metsaev, A.A. Tseytlin, Nucl. Phys. B **293**, 385 (1987). https://doi.org/10.1016/0550-3213(87)90077-0
4. A. Giveon, M. Porrati, E. Rabinovici, Phys. Rep. **244**, 77 (1994). https://doi.org/10.1016/0370-1573(94)90070-1. arXiv:hep-th/9401139
5. E. Alvarez, L. Alvarez-Gaume, Y. Lozano, Nucl. Phys. Proc. Suppl. **41**, 1 (1995). https://doi.org/10.1016/0920-5632(95)00429-D. arXiv:hep-th/9410237
6. A. Sen, Phys. Lett. B **271**, 295–300 (1991). https://doi.org/10.1016/0370-2693(91)90090-D
7. O. Hohm, A. Sen, B. Zwiebach, JHEP **02**, 079 (2015). https://doi.org/10.1007/JHEP02(2015)079. arXiv:1411.5696 [hep-th]
8. M.R. Garousi, Phys. Rev. D **99**(12), 126005 (2019). https://doi.org/10.1103/PhysRevD.99.126005. arXiv:1904.11282 [hep-th]
9. M.R. Garousi, Eur. Phys. J. C **79**(10), 827 (2019). https://doi.org/10.1140/epjc/s10052-019-7357-4. arXiv:1907.06500 [hep-th]
10. H. Razaghian, M.R. Garousi, Phys. Rev. D **97**, 106013 (2018). https://doi.org/10.1103/PhysRevD.97.106013. arXiv:1801.06834 [hep-th]
11. M.R. Garousi, Eur. Phys. J. C **80**(11), 1086. https://doi.org/10.1140/epjc/s10052-020-08662-9. arXiv:2006.09193 [hep-th]
12. M.R. Garousi, JHEP **02**, 157 (2021). https://doi.org/10.1007/JHEP02(2021)157. arXiv:2011.02753 [hep-th]
13. M.R. Garousi, Nucl. Phys. B **971**, 115510 (2021). https://doi.org/10.1016/j.nuclphysb.2021.115510. arXiv:2012.15091 [hep-th]
14. M.R. Garousi, Phys. Rev. D **104**(6), 066013 (2021). https://doi.org/10.1103/PhysRevD.104.066013. arXiv:2105.07598

15. M.R. Garousi, Eur. Phys. J. C **81**(8), 711 (2021). https://doi.org/10.1140/epjc/s10052-021-09518-6. arXiv:2107.04796 [hep-th]
16. T.H. Buscher, Phys. Lett. B **194**, 59 (1987). https://doi.org/10.1016/0370-2693(87)90769-6
17. T.H. Buscher, Phys. Lett. B **201**, 466 (1988). https://doi.org/10.1016/0370-2693(88)90602-8
18. N. Kaloper, K.A. Meissner, Phys. Rev. D **56**, 7940 (1997). https://doi.org/10.1103/PhysRevD.56.7940. arXiv:hep-th/9705193
19. I. Jack, D.R.T. Jones, N. Mohammedi, Nucl. Phys. B **322**, 431–470 (1989). https://doi.org/10.1016/0550-3213(89)90422-7
20. S. Stieberger, Phys. Rev. Lett. **106**, 111601 (2011). https://doi.org/10.1103/PhysRevLett.106.111601. arXiv:0910.0180 [hep-th]
21. M.R. Garousi, Phys. Lett. B **809**, 135733 (2020). https://doi.org/10.1016/j.physletb.2020.135733. arXiv:1907.09168 [hep-th]
22. R.C. Myers, Phys. Rev. D **36**, 392 (1987). https://doi.org/10.1103/PhysRevD.36.392
23. M.R. Garousi, Effective action of string theory at order α' in the presence of boundary. arXiv:2103.13682 [hep-th]
24. K.A. Meissner, Phys. Lett. B **392**, 298–304 (1997). https://doi.org/10.1016/S0370-2693(96)01556-0. arXiv:hep-th/9610131
25. O. Hohm, B. Zwiebach, JHEP **04**, 101 (2016). https://doi.org/10.1007/JHEP04(2016)101. arXiv:1510.00005 [hep-th]
26. O. Hohm, B. Zwiebach, Phys. Rev. D **100**(12), 126011 (2019). https://doi.org/10.1103/PhysRevD.100.126011. arXiv:1905.06963 [hep-th]
27. T. Nutma, Comput. Phys. Commun. **185**, 1719 (2014). https://doi.org/10.1016/j.cpc.2014.02.006. arXiv:1308.3493 [cs.SC]
28. G. Veneziano, Phys. Lett. B **265**, 287–294 (1991). https://doi.org/10.1016/0370-2693(91)90055-U
29. K.A. Meissner, G. Veneziano, Phys. Lett. B **267**, 33–36 (1991). https://doi.org/10.1016/0370-2693(91)90520-Z
30. J. Maharana, J.H. Schwarz, Nucl. Phys. B **390**, 3–32 (1993). https://doi.org/10.1016/0550-3213(93)90387-5. arXiv:hep-th/9207016
31. C. Eloy, O. Hohm, H. Samtleben, Phys. Rev. D **101**(12), 126018 (2020). https://doi.org/10.1103/PhysRevD.101.126018. arXiv:2004.13140 [hep-th]