Eur. Phys. J. C (2022) 82:615
https://doi.org/10.1140/epjc/s10052-022-10583-8

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

The exclusive vision of rare K and B decays and of the quark mixing in the standard model

Andrzej J. Buras[1,2,a], **Elena Venturini**[2]

[1] TUM Institute for Advanced Study, Lichtenbergstr. 2a, 85747 Garching, Germany
[2] Physik Department, TU München, James-Franck-Straße, 85748 Garching, Germany

Received: 20 April 2022 / Accepted: 2 July 2022 / Published online: 15 July 2022

Abstract The most common predictions for rare K and B decay branching ratios in the Standard Model in the literature are based on the CKM elements $|V_{cb}|$ and $|V_{ub}|$ resulting from global fits, that are in the ballpark of their inclusive and exclusive determinations, respectively. In the present paper we follow another route, which to our knowledge has not been explored for $\Delta M_{s,d}$ and rare K and B decays by anybody to date. We assume, in contrast to the prevailing *inclusive* expectations for $|V_{cb}|$, that the future true values of $|V_{cb}|$ and $|V_{ub}|$ will be both from *exclusive* determinations; in practice we use the most recent averages from FLAG. With the precisely known $|V_{us}|$ the resulting rare decay branching ratios, ε_K, ΔM_d, ΔM_s and $S_{\psi K_S}$ depend then only on the angles β and γ in the unitarity triangle that moreover are correlated through the CKM unitarity. An unusual pattern of SM predictions results from this study with some existing tensions being dwarfed and new tensions being born. In particular using HPQCD $B^0_{s,d}-\bar{B}^0_{s,d}$ hadronic matrix elements a 3.1σ tension in ΔM_s independently of γ is found. For $60^\circ \le \gamma \le 75^\circ$ the tension in ΔM_d between 4.0σ and 1.1σ is found and in the case of ε_K between 5.2σ and 2.1σ. Moreover, the room for new physics in $K^+\to\pi^+\nu\bar\nu$, $K_L\to\pi^0\nu\bar\nu$ and $B\to K(K^*)\nu\bar\nu$ decays is significantly increased. We compare the results in this EXCLUSIVE scenario with the HYBRID one in which $|V_{cb}|$ in the former scenario is replaced by the most recent inclusive $|V_{cb}|$ and present the dependence of all observables considered by us in both scenarios as functions of γ. As a byproduct we compare the determination of $|V_{cb}|$ from ΔM_s, ΔM_d, ε_K and $S_{\psi K_S}$ using $B^0_{s,d}-\bar{B}^0_{s,d}$ hadronic matrix elements from LQCD with $2+1+1$ flavours, $2+1$ flavours and their average. Only for the $2+1+1$ case values for β and γ exist for which the same value of $|V_{cb}|$ is found: $|V_{cb}| = 42.6(4)\times 10^{-3}$, $\gamma = 64.6(16)^\circ$ and $\beta = 22.2(7)^\circ$. This in turn implies a 2.7σ anomaly in $B_s\to\mu^+\mu^-$.

Contents

1 Introduction

The rare K and B decays and the quark mixing being GIM [1] suppressed in the Standard Model (SM) and simultaneously being often theoretically clean are very powerful tools for the search of New Physics (NP) [2]. Unfortunately the persistent tension between inclusive and exclusive determinations of $|V_{cb}|$ (see e.g. [3–6]) weakens this power significantly. As recently reemphasized by us [7] this is in particular the case of the branching ratios for rare K-meson decays and the parameter ε_K that exhibit stronger $|V_{cb}|$ dependences than rare B decay branching ratios and the $\Delta M_{s,d}$ mass differences. Also similar tensions in the determination of $|V_{ub}|$ [8] matter.

One possible solution to cope with this difficulty is to consider within the SM suitable ratios of two properly chosen observables so that the dependences on $|V_{cb}|$ and $|V_{ub}|$ are eliminated [7,9,10]. While in [9,10] B physics observables were considered, the analysis in [7] was dominated by the K system and its correlation with rare B decays and $B^0_{s,d}-\bar{B}^0_{s,d}$ mixing. In this manner we could construct 16 $|V_{cb}|$-independent ratios that were either independent of the CKM parameters or only dependent on the angles β and γ, that can be determined in tree-level processes. Having one day precise experimental values for the ratios in question

---

[a] e-mail: aburas@ph.tum.de (corresponding author)

and also precise values on β and γ will hopefully allow one to identify particular pattern of deviations from SM expectations independently of $|V_{cb}|$ pointing towards a particular extension of the SM.

But these ratios, even if useful in the context of the tensions in question, are not as interesting as the observables themselves. Therefore, assuming in addition no NP in ε_K, ΔM_d and ΔM_s and in the mixing induced CP-asymmetry $S_{\psi K_S}$, these ratios allowed to obtain $|V_{cb}|$-independent SM predictions for a number of branching ratios [7]. As these four quark mixing observables are very precisely measured and theoretically rather clean, the resulting SM predictions obtained in this manner turned out to be the most precise to date. A brief summary of the results of this analysis just appeared [11].

Another insight in this problematic has been provided recently by the authors of [12] who made a determination of $|V_{cb}|$ and $|V_{ub}|$ from loop processes, rare decays and quark mixing, by assuming no NP contributions to these observables. To this end they could use only well measured observables in the B system and ε_K. This strategy has already been explored in [13] but there only ε_K, ΔM_d and ΔM_s and $S_{\psi K_S}$ have been considered.

There is no question about that the analyses in [7,9,10] will help us to identify possible departures from SM predictions for the $|V_{cb}|$-independent ratios and possible pattern of $|V_{cb}|$ determinations from various loop processes as analysed in [12,13], but also to some extent in [7]. See in particular Figs. 12 and 14 of the latter paper. Yet, eventually the most obvious procedure to look for NP is to determine all CKM parameters in tree-level processes under the assumption that NP contributions to these decays are negligible. This assumption is more likely to be correct than assuming no NP contributions in loop induced decays. Subsequently the resulting values of the CKM parameters inserted into SM amplitudes for loop induced processes would allow for definite predictions for GIM suppressed observables.

In this spirit in the present paper we follow a more direct but a novel route, which to our knowledge has not been explored by anybody to date, at least as far as SM predictions for theoretically clean observables like rare K and B decays, $\Delta M_{s,d}$ and the mixing induced CP asymmetry $S_{\psi K_S}$ are concerned. Instead of using the values of $|V_{cb}|$ and $|V_{ub}|$ resulting from the global fits of the CKM matrix [14,15], we assume, in contrast to the prevailing *inclusive* expectations in the case of $|V_{cb}|$, that the future values of both $|V_{cb}|$ and $|V_{ub}|$ will be determined from *exclusive* tree-level decays. Therefore we use FLAG [6] averages,[1] which are based on a number of LQCD calculations that are listed after (2). With the precisely known $|V_{us}|$ the resulting rare decay branching ratios, ε_K, ΔM_d, ΔM_s and $S_{\psi K_S}$ depend than only on the angles β and γ in the unitarity triangle that are moreover correlated through the CKM unitarity. An unusual pattern of SM predictions results from this study. Some present tensions are dwarfed and new tensions are born. In some cases their sizes depend sensitively on the value of γ which enhances the importance of precise measurements of this parameter, stressed in particular in [7,16].

The view that exclusive decays will eventually lead to the best determination of $|V_{cb}|$ is rather unusual but has been already expressed by the first author in the past [2]. The point is that precise measurements of formfactors by Lattice QCD (LQCD) accompanied by improved measurements of the relevant branching ratios should allow eventually a better control over theoretical uncertainties than it is possible in inclusive decays and consequently determinations of $|V_{cb}|$ and $|V_{ub}|$ that do not rely on quark-hadron duality. Yet, to be on the safe side, in view of the important progress in the inclusive determination of $|V_{cb}|$ [3–6], we compare at all stages the results in this EXCLUSIVE scenario with the HYBRID one in which $|V_{cb}|$ in the former scenario is replaced by the most recent inclusive $|V_{cb}|$ from [4].

To our knowledge in the literature only the authors of [17] performed a similar study, but only for ε_K, finding, similar to us, a significant deviation of the SM prediction from the data. However, their analysis differs from ours in that for the CKM parameters they used the values obtained from global fits of the UT which can be questioned because in fact these SM global analyses used already ε_K in their fits. Moreover, until now they did not incorporate the theoretical advances in ε_K from [18] which have been taken by us into account in [7] and also in the present analysis.

The outline of our paper is as follows. In Sect. 2 we set up our strategy as far as CKM parameters are concerned. We also list the input parameters used in our numerical analysis. However, we refrain from the expressions for the observables which have been studied already by us in [7] and are collected there and in [2]. The numerical analysis is presented in Sect. 3 with the SM predictions for many observables resulting from the EXCLUSIVE LQCD scenario and from the HYBRID scenario. In Sect. 4 we calculate the impact of the hadronic matrix elements with 2+1+1 flavours from the HPQCD collaboration [19] on our results for rare B decays in [7], where the averages of HPQCD results and 2 + 1 results from Fermilab Lattice and MILC Collaborations (FNAL/MILC) [20] calculated in [10] have been used. We also illustrate how the determination of $|V_{cb}|$ from ΔM_s, ΔM_d and ε_K depends on the number of flavours used in LQCD calculations of the relevant hadronic matrix elements. We conclude in Sect. 5. Two short appendices list LQCD results for F_{B_q} and $F_{B_q}\sqrt{\hat{B}_{B_q}}$ for $N_f = 2+1$ and $N_f = 2+1+1$.

---

[1] In fact we will use for $|V_{cb}|$ its preliminary value that should appear in the 2022 FLAG's edition.

2 Strategy

The CKM parameters entering our analysis will be

$$\boxed{\lambda = |V_{us}|, \quad |V_{cb}|, \quad |V_{ub}|, \quad \gamma} \tag{1}$$

with γ one of the angles in the UT, shown in Fig. 1. It is equal, within an excellent accuracy, to the single phase in the standard parametrization of the CKM matrix [21,22].

As the input parameters we will use $\lambda = 0.225$ and the FLAG values for $|V_{cb}|$ and $|V_{ub}|$ extracted from tree-level decays. Now these values, as given in the latest FLAG's report, read [6]

$$|V_{cb}| = 39.48(68) \times 10^{-3}, \quad |V_{ub}| = 3.63(14) \times 10^{-3}, \quad (\text{FLAG} - 2021). \tag{2}$$

These results are based on a number of different LQCD analyses as summarized in Fig. 38 of [6]. These are from FNAL/MILC [23–26], HPQCD [27–29] and RBC/UKQCD [30] with further details given in the original papers and [6].

However, the value for $|V_{cb}|$ in (2) does not include the most recent one from Fermilab/MILC [31] that is significantly lower $38.40(74) \times 10^{-3}$. Fortunately, we were able to obtain from FLAG a preliminary result for $|V_{cb}|$[2] that includes the latter result. Our basic values for $|V_{cb}|$ and $|V_{ub}|$ obtained from the overall 2022 FLAG's ($|V_{cb}|$,$|V_{ub}|$) fit will be then as follows[3]

$$\boxed{|V_{cb}| = 39.21(62) \times 10^{-3}, |V_{ub}| = 3.61(13) \times 10^{-3}, (\text{FLAG} - 2022),} \tag{3}$$

with $|V_{ub}|$ practically unchanged. Larger values for $|V_{cb}|$ from exclusive decays using LQCD, in the ballpark of 41.0×10^{-3}, have been reported in [32–35] and we are looking forward to the 2023 FLAG report incorporating these results.

We will also compare our EXCLUSIVE scenario with the HYBRID one in which the value for $|V_{cb}|$ is the inclusive one from [4] and the exclusive one for $|V_{ub}|$ as above:

$$\boxed{|V_{cb}| = 42.16(50) \times 10^{-3}, |V_{ub}| = 3.61(13) \times 10^{-3}, (\text{HYBRID}).} \tag{4}$$

For γ we will use a broad range $60^\circ \leq \gamma \leq 75^\circ$. Using CKM unitarity the angle β in the UT can then be determined through the correlation of β and γ

$$\boxed{\cot\beta = \frac{1 - R_b \cos\gamma}{R_b \sin\gamma}, \quad R_b = \left(1 - \frac{\lambda^2}{2}\right)\frac{1}{\lambda}\left|\frac{V_{ub}}{V_{cb}}\right|.} \tag{5}$$

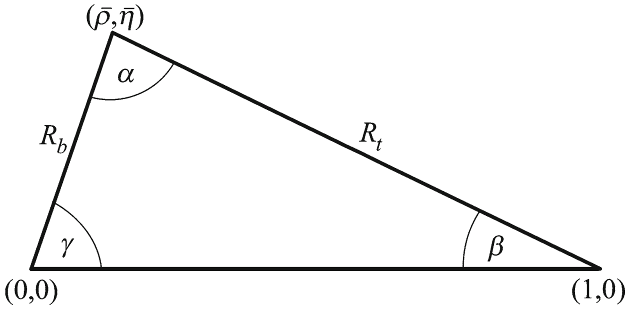


Fig. 1 The unitarity triangle

On the other hand the sides of the UT, R_t and R_b, can be solely expressed in terms of the angles β and γ, as follows [36]

$$R_t = \frac{\sin\gamma}{\sin(\beta+\gamma)} \approx \sin\gamma, \quad R_b = \frac{\sin\beta}{\sin(\beta+\gamma)} \approx \sin\beta. \tag{6}$$

We observe that R_t depends dominantly on γ, while R_b on β. These approximations follow from the experimental fact that $\beta + \gamma \approx 90^\circ$ and it is an excellent approximation to set $\sin(\beta + \gamma) = 1$ in the formulae below although we will not do it in the numerical evaluations.

The values of $|V_{cb}|$ and $|V_{ub}|$ in (3) imply then

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.0921 \pm 0.0036, \quad R_b = 0.399 \pm 0.016, \quad (\text{EXCLUSIVE}) \tag{7}$$

and using [37]

$$\gamma = (65.4^{+3.8}_{-4.2})^\circ. \tag{8}$$

$$\beta = (23.50 \pm 0.93)^\circ, \; S_{\psi K_S} = \sin(2\beta) = 0.731 \pm 0.024, \quad (\text{EXCLUSIVE}) \tag{9}$$

that both differ mildly from the measured values [22]

$$\beta = (22.2 \pm 0.7)^\circ, \quad S_{\psi K_S} = 0.699(17), \quad (\text{PDG}). \tag{10}$$

On the other hand in the HYBRID scenario we find

$$\frac{|V_{ub}|}{|V_{cb}|} = 0.0856 \pm 0.0032, \quad R_b = 0.371 \pm 0.014, \quad (\text{HYBRID}) \tag{11}$$

and

$$\beta = (21.74 \pm 0.84)^\circ, \quad S_{\psi K_S} = \sin(2\beta) = 0.688 \pm 0.022, \quad (\text{HYBRID}) \tag{12}$$

in perfect agreement with the experimental measurements in (10).

This brief exercise is an overture to the new tensions emerging from the exclusive strategy. In Fig. 2 we show β as a function of γ for the values of $|V_{ub}|/|V_{cb}|$ in two scenarios in question and compare them to the one-sigma range for β

---

[2] We thank Enrico Lunghi for providing this number prior to the official new FLAG's update.

[3] The value for $|V_{cb}|$ should be considered as preliminary.

in (10). We observe that in the case of the EXCLUSIVE strategy there is indeed a mild tension. To remove this tension a negative NP phase has to be added to β in the formula for $S_{\psi K_S}$ in (9). See (23). This negative phase originates in a NP phase in $B_d^0-\bar{B}_d^0$ mixing. Significantly larger tensions will be found in most observables analyzed by us.

In this context we would like to mention the analysis in [38] in which the ratio $|V_{ub}|/|V_{cb}|$ was proposed as a useful test of the SM because of reduced hadronic uncertainties combined with the fact that this ratio is almost the same for the exclusive and inclusive determinations of $|V_{cb}|$ and $|V_{ub}|$.

Finally, useful are also the following expressions ($\lambda_t=V_{td}V_{ts}^*$)

$$|V_{td}|=\lambda|V_{cb}|\sin\gamma=(8.82\pm0.14)\,\sin\gamma\times10^{-3},\quad\text{(EXCLUSIVE)},\tag{13}$$

$$\mathrm{Im}\lambda_t=|V_{ub}||V_{cb}|\sin\gamma=(1.42\pm0.06)\,\sin\gamma\times10^{-4},\quad\text{(EXCLUSIVE)},\tag{14}$$

where the values in (3) have been used. For the HYBRID scenario we have

$$|V_{td}|=\lambda|V_{cb}|\sin\gamma=(9.49\pm0.12)\,\sin\gamma\times10^{-3},\quad\text{(HYBRID)},\tag{15}$$

$$\mathrm{Im}\lambda_t=|V_{ub}||V_{cb}|\sin\gamma=(1.52\pm0.06)\,\sin\gamma\times10^{-4},\quad\text{(HYBRID)}.\tag{16}$$

Moreover

$$|V_{ts}|=[1+\frac{\lambda^2}{2}(1-2\sin\gamma\cos\beta)]|V_{cb}|\approx0.983|V_{cb}|.\tag{17}$$

As $|V_{td}|$ and $\mathrm{Im}\lambda_t$ play an important role in rare K and B decays we show in Fig. 3 their dependence on γ in both scenarios.

A recent review of tree-level determinations of β and γ can be found in Chapter 8 of [2]. See also [39,40]. But here we will use γ as a free parameter and β as an output by means of (5).

3 Numerical analysis

Our numerical analysis uses the formulae for various branching ratios that we have collected in [7]. The parameters, other than the CKM ones, entering the formulae in [2,7] are collected in the Table 1. Except for the values of $F_{B_s}\sqrt{\hat{B}_s}$ and $F_{B_d}\sqrt{\hat{B}_d}$ that are taken this time from the HPQCD collaboration [19],[4] other parameters are unchanged. These two inputs from $N_f=2+1+1$ LQCD calculations are only slightly lower than the ones used in [2,7] but are significantly lower than the ones from $N_f=2+1$ LQCD average given by FLAG in [6]. These differences are summarized in Appendix B. Their impact on the determination of $|V_{cb}|$ from ΔM_s and ΔM_d will be analysed in Sect. 4.

Here we only recall that the dependence of the observables considered by us on $|V_{us}|$ is negligible. As far as β and γ are concerned, the angle β is already known from the mixing induced CP-asymmetry $S_{\psi K_S}$ with respectable precision as given in (10) and there is a significant progress by the LHCb collaboration on the determination of γ from tree-level strategies [37] so that we have presently from tree-level decays the value given in (8). Moreover, in the coming years the determination of γ by the LHCb and Belle II collaborations should be significantly improved so that precision tests of the SM using the strategy in [7] and the one presented here will be possible.

However we emphasize that we do not use the value of γ above as an input parameter. Our strategy will be to treat γ as a free parameter in the rather broad range $60^\circ\le\gamma\le75^\circ$ and in view of the future measurements of γ by LHCb and Belle II to exhibit the γ dependence of the observables considered by us. We recall that the angle β is the output by means of the unitarity relation (5) as given in Fig. 2.

In Table 2 we show SM predictions for a number of rare K and B branching ratios and $\Delta F=2$ observables resulting from the EXCLUSIVE input in (3) setting $\gamma=65.4^\circ$, the central LHCb value. The uncertainties appearing therein, thus, do not include any error on the γ determination. We also show our results in the HYBRID scenario defined in (4). The latter are not far from the ones obtained in [7] where the absence of NP contributions to $\Delta F=2$ observables was assumed. We do not consider the decays like $B\to K(K^*)\ell^+\ell^-$ that have larger theoretical uncertainties than the observables considered by us. Their $|V_{cb}|$ dependence has been investigated recently in [12].

The decay $B_s\to X_s\gamma$ was not considered in [7]. The result for $B_s\to X_s\gamma$ in both scenarios is obtained here from [48] that effectively corresponds to the inclusive $|V_{cb}|=42.0\times10^{-3}$. We just rescaled it using the exclusive and inclusive values of $|V_{cb}|$ for EXCLUSIVE and HYBRID scenarios, respectively.

In Figs. 4, 5 and 6 we show the γ dependence of the following observables

$$\mathcal{B}(K^+\to\pi^+\nu\bar\nu),\quad\mathcal{B}(K_L\to\pi^0\nu\bar\nu),\quad\Delta M_d,\quad\mathcal{B}(B_d\to\mu^+\mu^-),\quad\varepsilon_K,\quad S_{\psi K_S}\tag{18}$$

in both scenarios for $|V_{cb}|$ and $|V_{ub}|$. $\mathcal{B}(K_S\to\mu^+\mu^-)$ has the same γ dependence as $\mathcal{B}(K_L\to\pi^0\nu\bar\nu)$ and the remaining observables do not depend or only weakly depend on γ.

---

[4] These latest LQCD results are in good agreement with the ones from HQET sum rules [41].

Fig. 2 The UT angle β as functions of γ, in the EXCLUSIVE and HYBRID scenarios. The bands represent the uncertainties related to $|V_{cb}|$, $|V_{ub}|$ and $|V_{us}|$. The one-sigma range for β *in* (10) is shown as a red band

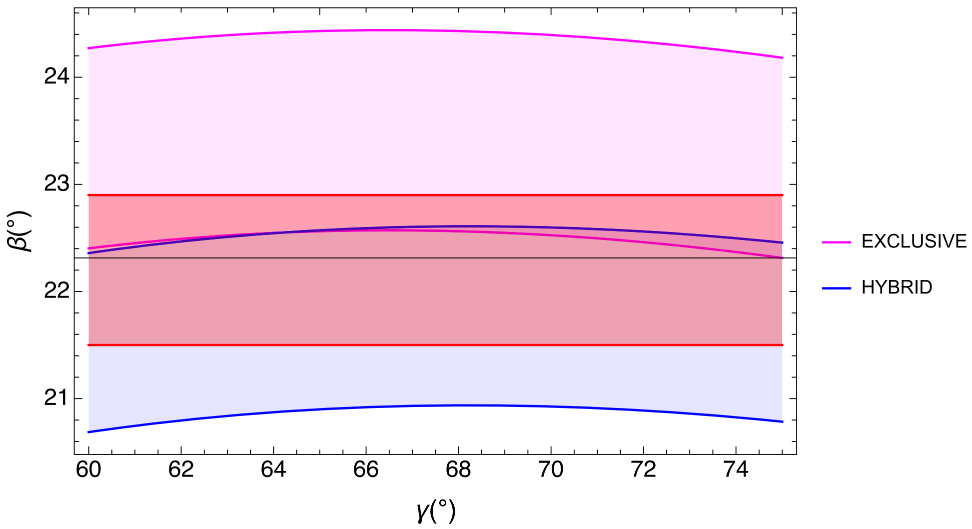
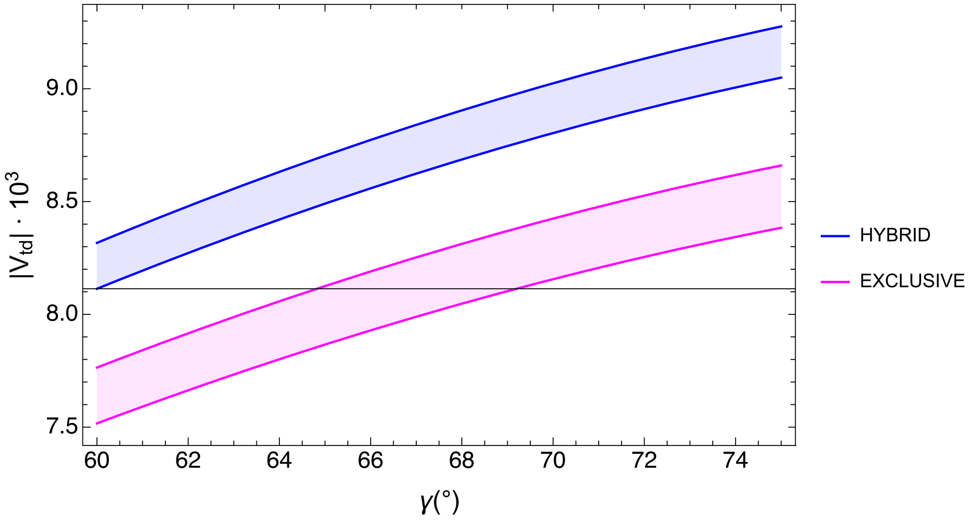


Fig. 3 $|V_{td}|$ and Imλ_t as functions of γ in the EXCLUSIVE and HYBRID scenarios



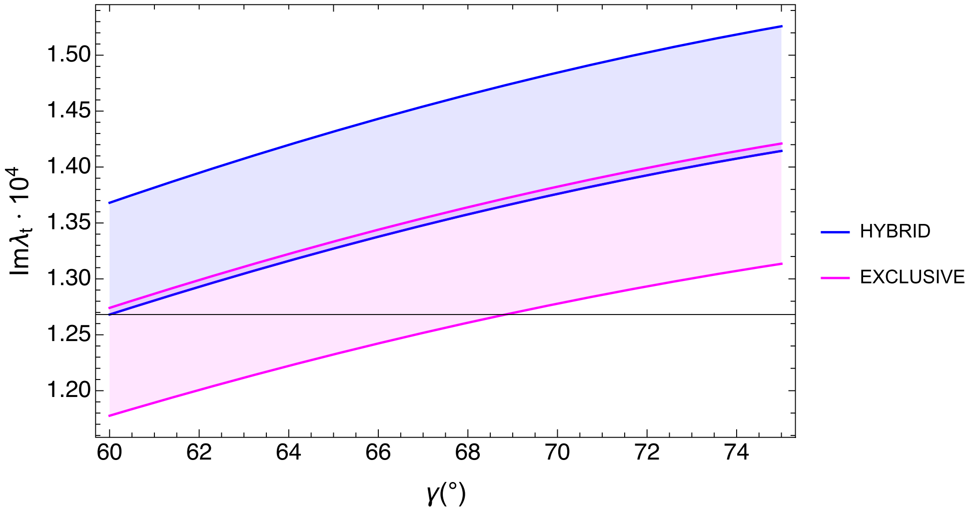


Table 1 Values of the experimental and theoretical quantities used as input parameters. For future updates see FLAG [6], *PDG* [22] *and HFLAV* [42]

$m_{B_s} = 5366.8(2)\text{MeV}$ [22]	$m_{B_d} = 5279.58(17)\text{MeV}$ [22]
$\Delta M_s = 17.749(20)\,\text{ps}^{-1}$ [22]	$\Delta M_d = 0.5065(19)\,\text{ps}^{-1}$ [22]
$\Delta M_K = 0.005292(9)\,\text{ps}^{-1}$ [22]	$m_{K^0} = 497.61(1)\text{MeV}$ [22]
$S_{\psi K_S} = 0.699(17)$ [22]	$F_K = 155.7(3)\text{MeV}$ [42]
$\lvert V_{us}\rvert = 0.2253(8)$ [22]	$\lvert\epsilon_K\rvert = 2.228(11)\cdot 10^{-3}$ [22]
$F_{B_s} = 230.3(1.3)\text{MeV}$ [6]	$F_{B_d} = 190.0(1.3)\text{MeV}$ [6]
$F_{B_s}\sqrt{\hat{B}_s} = 256.1(5.7)\text{MeV}$ [19]	$F_{B_d}\sqrt{\hat{B}_d} = 210.6(5.5)\text{MeV}$ [19]
$\hat{B}_s = 1.232(53)$ [19]	$\hat{B}_d = 1.222(61)$ [19]
$m_t(m_t) = 162.83(67)\,\text{GeV}$ [43]	$m_c(m_c) = 1.279(13)\,\text{GeV}$
$S_{tt}(x_t) = 2.303$	$S_{ut}(x_c, x_t) = -1.983\times 10^{-3}$
$\eta_{tt} = 0.55(2)$ [18]	$\eta_{ut} = 0.402(5)$ [18]
$\kappa_\varepsilon = 0.94(2)$ [44]	$\eta_B = 0.55(1)$ [45,46]
$\tau_{B_s} = 1.515(4)\,\text{ps}$ [47]	$\tau_{B_d} = 1.519(4)\,\text{ps}$ [47]

Table 2 Predictions (second column) for various observables within the SM using the EXCLUSIVE strategy for $|V_{cb}|$ and $|V_{ub}|$ and $\gamma = 65.4°$. In the third column we show the results for the HYBRID choice of $|V_{cb}|$ and $|V_{ub}|$ as given in (4) and in the fourth the experimental data

Decay	EXCLUSIVE	HYBRID	DATA
$\mathcal{B}(K^+\to\pi^+\nu\bar\nu)\times 10^{11}$	6.88(38)	8.44(41)	10.9(38) [49]
$\mathcal{B}(K_L\to\pi^0\nu\bar\nu)\times 10^{11}$	2.37(15)	2.74(14)	< 300 [50]
$\mathcal{B}(K_S\to\mu^+\mu^-)\times 10^{13}$	1.49(10)	1.72(8)	10^4 [51]
$\overline{\mathcal{B}}(B_s\to\mu^+\mu^-)\times 10^{9}$	3.18(12)	3.67(12)	2.86(33) [52–54]
$\mathcal{B}(B_d\to\mu^+\mu^-)\times 10^{10}$	0.864(34)	0.999(34)	< 2.05 [52]
$\mathcal{B}(B^+\to K^+\nu\bar\nu)\times 10^{6}$	3.83(53)	4.42(60)	11 ± 4 [55]
$\mathcal{B}(B^0\to K^{0*}\nu\bar\nu)\times 10^{6}$	8.32(82)	9.61(93)	< 18 [56]
$\mathcal{B}(B\to X_s\gamma)\times 10^{4}$	2.93(20)	3.39(23)	3.32(15) [22]
$\lvert\varepsilon_K\rvert\times 10^3$	1.78(11)	2.14(12)	2.228(11) [22]
$S_{\psi K_S}$	0.731(24)	0.688(22)	0.699(17) [22]
$\Delta M_s\,\text{ps}^{-1}$	15.02(87)	17.35(94)	17.749(20) [22]
$\Delta M_d\,\text{ps}^{-1}$	0.434(28)	0.502(31)	0.5065(19) [22]

Concentrating first on the EXCLUSIVE scenario we observe:

- As seen in Table 2 for $\gamma = 65.4°$, the largest tensions are found for ε_K (4.1σ), ΔM_s (3.1σ) and ΔM_d (2.6σ).
- As ΔM_s is practically independent of γ this tension remains for other values of γ.
- As seen in Fig. 4, the branching ratios for $K^+\to\pi^+\nu\bar\nu$ and $K_L\to\pi^0\nu\bar\nu$ are significantly suppressed below the values found in the literature that are in the ballpark of 8.5×10^{-11} and 3.0×10^{-11}, respectively [7]. But this suppression decreases with increasing γ.
- As seen in Fig. 5, the tension in ΔM_d decreases to $0.6\,\sigma$ for $\gamma = 75°$ but is as large as 4σ for $\gamma = 60°$. The branching ratio for $B_d\to\mu^+\mu^-$ shows a similar behaviour because its ratio to ΔM_d is CKM parameters independent. The uncertainty in ΔM_d is a bit larger because of the additional hadronic uncertainty in the parameter $\hat{B}_d$. The red band in the upper panel represents the experimental value for ΔM_d, with its 1σ uncertainty.
- As seen in Fig. 6, the tension for ε_K (with the experimental measurement shown in red) is practically linear in γ and in the range of γ considered varies from 2.0σ for $\gamma = 75°$ to 5.2σ for $\gamma = 60°$. While significant tension in ε_K in the EXCLUSIVE scenario has been already identified in [17], our analysis differs in several respects from that paper as we already stated at the beginning of this writing. On the other hand the tension for $S_{\psi K_S}$ is practically independent of γ and in the ballpark of 1.0σ so that in this case one really cannot talk about an anomaly.
- The tension in $B_s\to\mu^+\mu^-$ basically disappears.

On the other hand in the HYBRID scenario all these tensions disappear but the one in $B_s\to\mu^+\mu^-$ is independently of γ in the ballpark of 2.1σ [10].

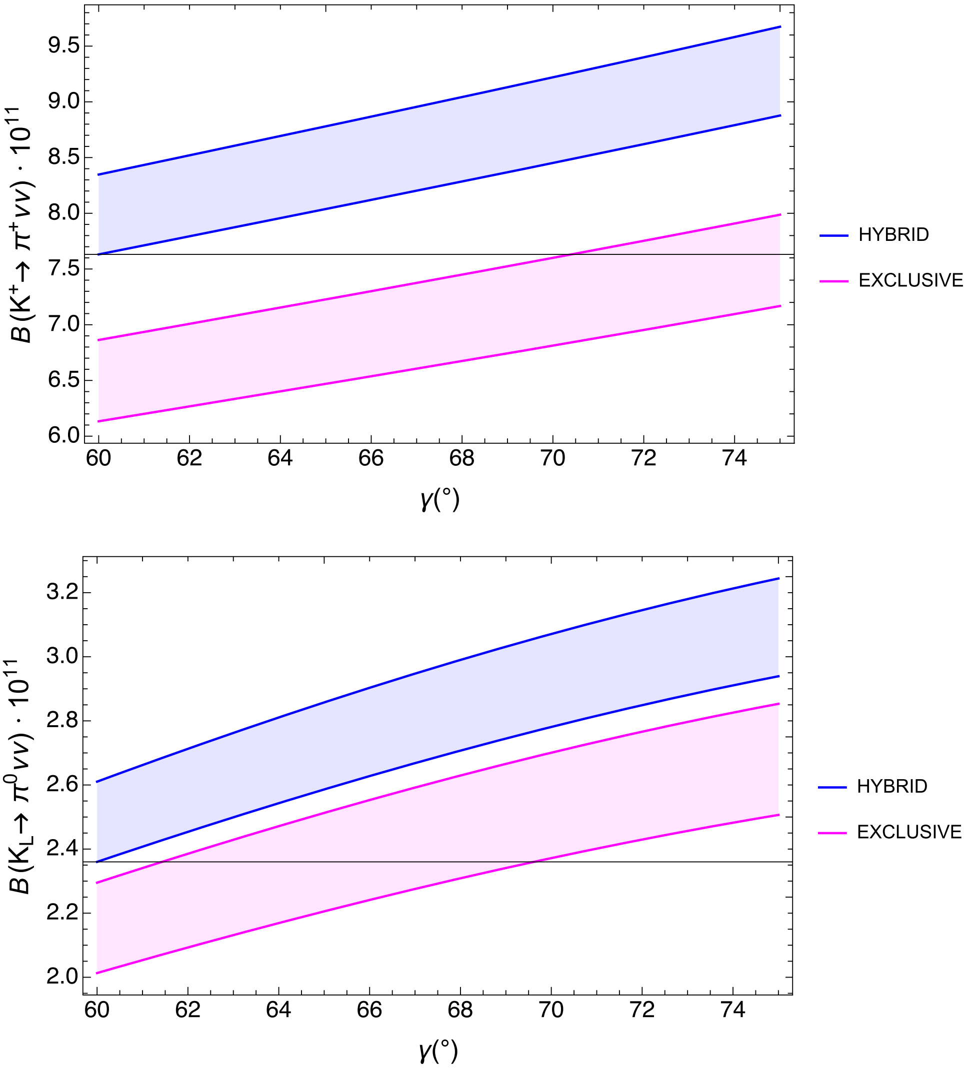


Fig. 4 The branching ratios $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$ and $\mathcal{B}(K_L \to \pi^0\nu\bar{\nu})$ *as functions of* γ, in the EXCLUSIVE and HYBRID scenarios. The bands represent the uncertainties related to $|V_{cb}|$, $|V_{ub}|$, $|V_{us}|$ and to the non-CKM parameters

4 The impact of the HPQCD results

It is of interest to see how the use of 2 + 1 + 1 hadronic matrix elements from the HPQCD collaboration [19] used in the present paper, instead of the ones used in [7] (the average of 2 + 1 and 2 + 1 + 1 results), would modify our results for rare B decays of the latter paper in which no NP in ΔM_s and ΔM_d has been assumed. We make this comparison in Table 3. For completeness we list there also results for rare K decays which remain unchanged. This also allows the comparison with the results obtained in EXCLUSIVE and HYBRID scenarios in Table 2.

We observe that all B decays branching ratios in Table 3 are larger than our results obtained in [7] that were rather close to the ones in the HYBRID scenario. In view of still sizable experimental errors this impact of the HPQCD results cannot be fully appreciated with the exception of $B_s \to \mu^+\mu^-$. With the branching ratio for this decay in Table 3 and the experimental data in Table 2, assuming no NP in ΔM_s, the anomaly in $B_s \to \mu^+\mu^-$ of 2.1σ found in [10] is raised to 2.7σ.

It should be emphasized at this point that all the correlations found in [7] that do not involve ΔM_s and ΔM_d remain unchanged but the predictions for B decay branching ratios change as we have just seen. This then implies a different SM region in the correlation between $K^+ \to \pi^+\nu\bar{\nu}$ and $B_s \to \mu^+\mu^-$ that we illustrate in Fig. 7. There the result using HPQCD $2+1+1$ input (left panel) is compared with the one of [7] (right panel) where the average of $2+1+1$ and $2+1$ matrix elements from [10] has been used. This difference shows the importance of charm contribution in LQCD calculations. Note that the result for $K^+ \to \pi^+\nu\bar{\nu}$ did not change relative to [7].

In fact our results for $B_{s,d} \to \mu^+\mu^-$ are rather consistent with the ones obtained by the HPQCD collaboration [19] in 2019. But since then the experimental accuracy of $B_{s,d} \to \mu^+\mu^-$ significantly increased [52–54] allowing a better estimate of the anomaly in question.

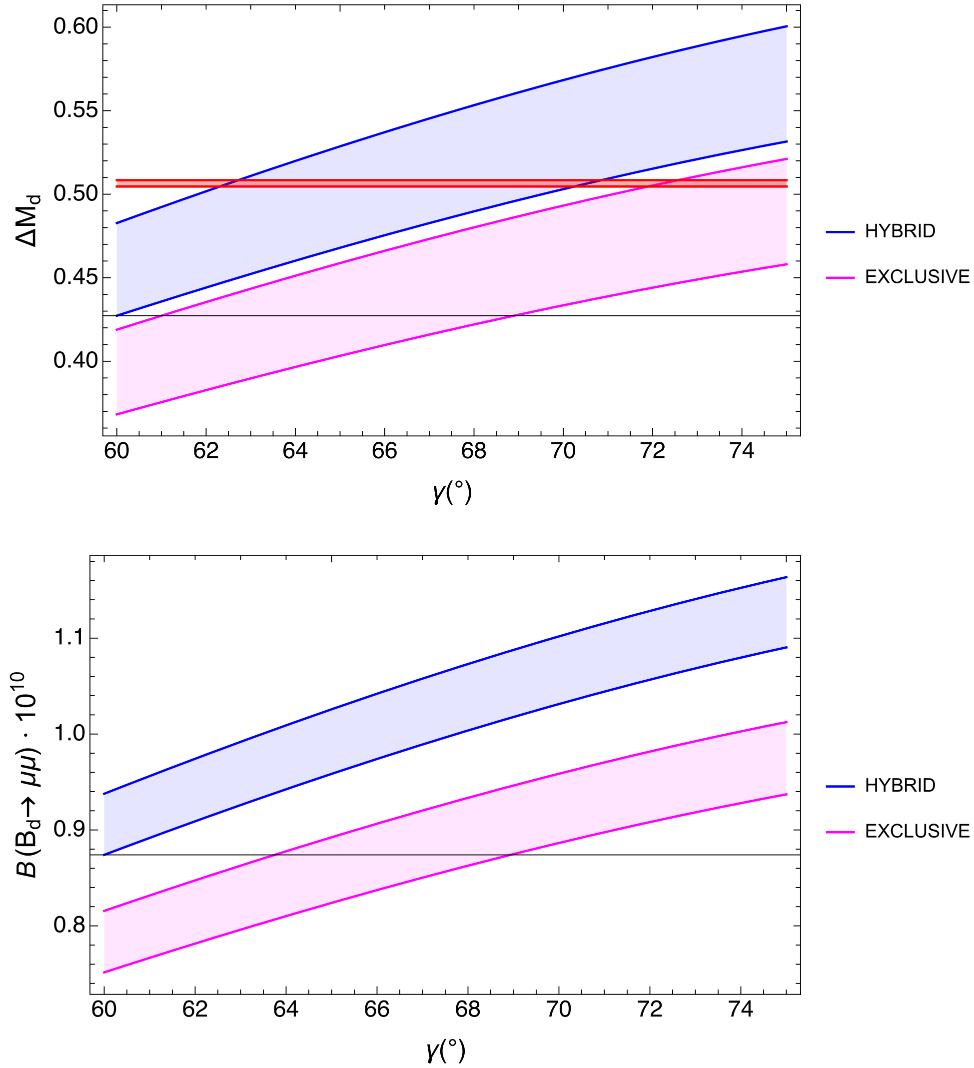


Fig. 5 ΔM_d and the branching ratio $\mathcal{B}(B_d \to \mu^+\mu^-)$ as functions of γ, in the EXCLUSIVE and HYBRID scenarios. The bands represent the uncertainties related to $|V_{cb}|$, $|V_{ub}|$, $|V_{us}|$ and to the non-CKM parameters. The red band in the upper panel represents the experimental value for ΔM_d, with its 1σ uncertainty

Finally, we would like to address another important issue. In [7], using the averages of $B^0_{s,d} - \bar{B}^0_{s,d}$ hadronic matrix elements from LQCD calculations with $2+1$ and $2+1+1$ flavours, as given in (32), we found that there are no values of β and γ for which the same value for $|V_{cb}|$ can be obtained from ε_K, ΔM_d and ΔM_s when imposing the experimental constraint from $S_{\psi K_S}$. It is then of interest to investigate what happens when this analysis is repeated separately for the $2+1$ and $2+1+1$ matrix elements as given in (26) and (29), respectively.

The result of this exercise is shown in Fig. 8. We observe that only in the case of $2+1+1$ flavours consistent result for $|V_{cb}|$ from all observables considered by us is obtained which in turn provides unique values of $|V_{cb}|$ and γ. The determination of γ and $|V_{cb}|$ can be further improved by considering first the $|V_{cb}|$-independent ratio $\Delta M_d/\Delta M_s$ from which one derives an accurate formula for $\sin\gamma$

$$\sin\gamma = \frac{0.983(1)}{\lambda}\sqrt{\frac{m_{B_s}}{m_{B_d}}}\xi\sqrt{\frac{\Delta M_d}{\Delta M_s}},$$

$$\xi = \frac{F_{B_s}\sqrt{\hat{B}_{B_s}}}{F_{B_d}\sqrt{\hat{B}_{B_d}}} = 1.216(16), \tag{19}$$

with the value for ξ from HPQCD [19] and where (17) has been used.. The advantage of using this ratio for the determination of γ over studying ΔM_s and ΔM_d separately is the reduced error on ξ from LQCD relative to the individual errors of hadronic parameters in ΔM_s and ΔM_d. See Appendix B.

Combining then ε_K, ΔM_d, ΔM_s, $S_{\psi K_S}$ and using (19) we obtain finally the following values of the CKM parameters

$$\boxed{|V_{cb}| = 42.6(4)\times 10^{-3}, \quad \gamma = 64.6(16)^\circ, \quad \beta = 22.2(7)^\circ.} \tag{20}$$

The value of $|V_{cb}|$ is somewhat larger than in the HYBRID scenario in (4) but consistent with it. It should be noted that the determination of γ in this manner is more accurate than its present determination from tree-level decays in (8). The

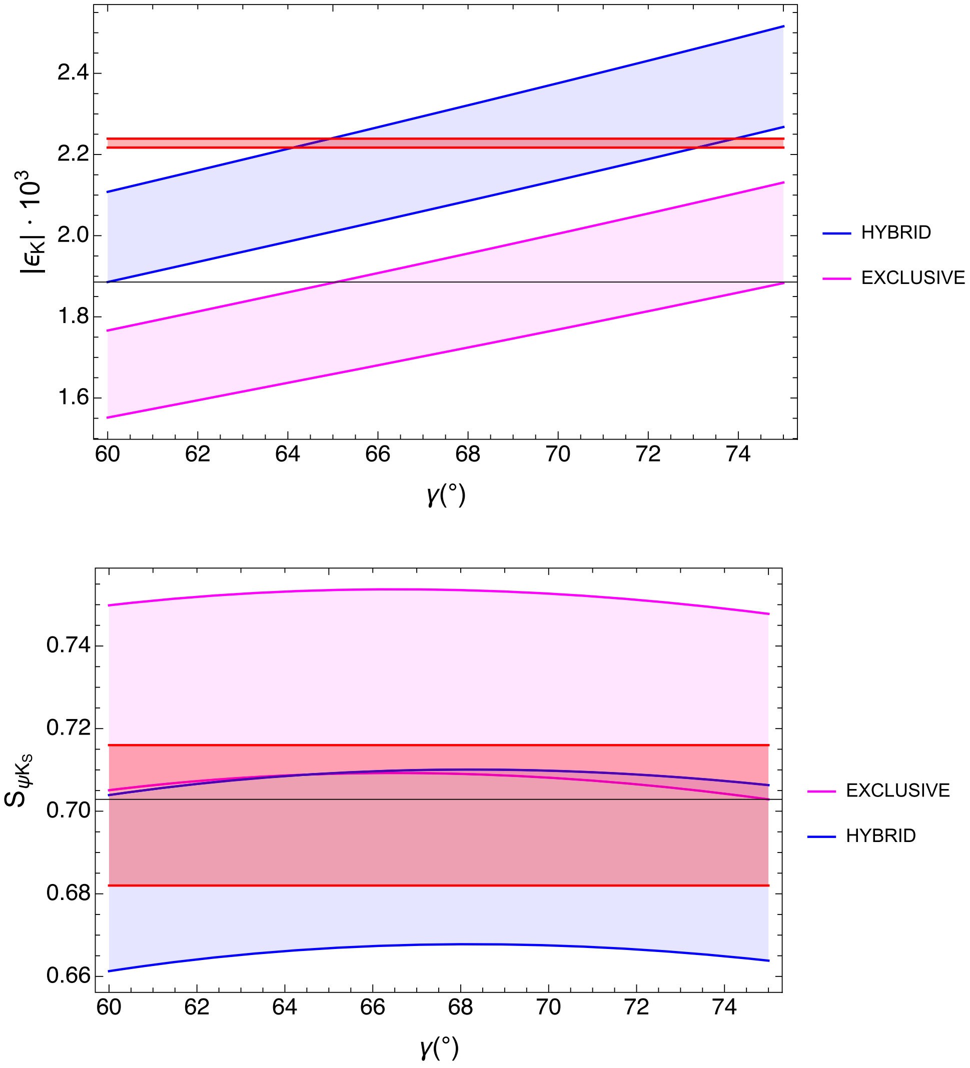


Fig. 6 ε_K and the CP asymmetry $S_{\psi K_S}$ as functions of γ, *in the* EXCLUSIVE and HYBRID scenarios. The bands represent the uncertainties related to $|V_{cb}|$, $|V_{ub}|$, $|V_{us}|$ and to the non-CKM parameters. The red band in the upper panel represents the experimental value for ε_K, with its 1σ uncertainty. The same for $S_{\psi K_S}$

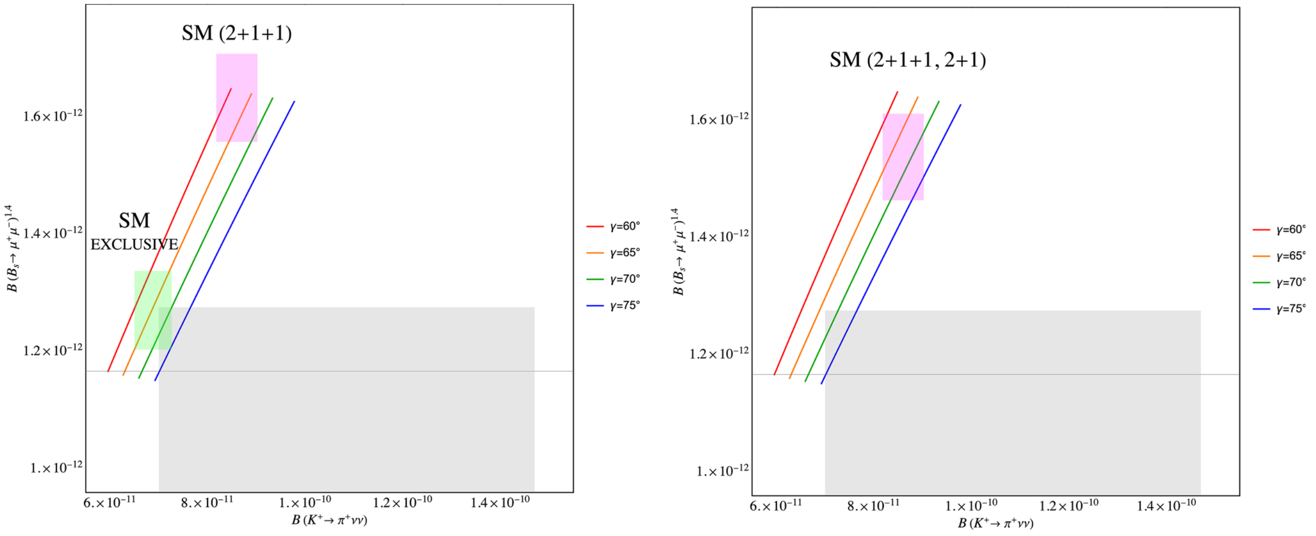


Fig. 7 The correlation of $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$ with $\overline{\mathcal{B}}(B_s \to \mu^+\mu^-)^{1.4}$of [7] for different values of γ within the SM. The SM area corresponds to the HPQCD $2+1+1$ input (left panel) and to the $2+1+1$ *and* $2+1$ average (right panel) used in [7]. The green area represents the EXCLUSIVE scenario and the gray area the present experimental situation

corresponding value of $|V_{ub}|$ is

$$\boxed{|V_{ub}| = 3.72(11) \times 10^{-3},} \qquad (21)$$

which is slightly larger than the FLAG determination but consistent with it. We observe that ε_K dominates this determination of $|V_{cb}|$. This can be traced back to its larger senitivity to $|V_{cb}|$ than it is the case for $\Delta M_{s,d}$. While $\Delta M_{s,d}$ are proportional to $|V_{cb}|^2$, $|\varepsilon_K|$ exhibits approximately $|V_{cb}|^{3.4}$ dependence [7]. Reducing the error on β, represented by the green band, and decreasing the error on γ from its tree-level measurements will provide a determination of $|V_{cb}|$ with an error below 1%.

The 2 + 1 case demonstrates significant inconsistencies between $|V_{cb}|$ values from $\Delta M_{d,s}$ and ε_K. The average in (32) considered by us in [7] is in a better shape but also various tensions are identified that we discussed in detail in the latter paper. The message from this exercise is clear. The inclusion of charm in the evaluation of $B^0_{s,d} - \bar{B}^0_{s,d}$ hadronic matrix elements by LQCD is mandatory and it is important that in addition to HPQCD [19] a second LQCD collaboration includes charm in the evaluation of these matrix elements.

Assuming that the HPQCD values will be confirmed by another LQCD group, the SM predictions in the left panel of Fig. 7 will be favoured implying

$$\overline{\mathcal{B}}(B_s \to \mu^+\mu^-) = (3.78^{+0.15}_{-0.10}) \times 10^{-9},$$
$$\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu}) = (8.60 \pm 0.42) \times 10^{-11}. \qquad (22)$$

But it should be remembered that in contrast to the EXCLUSIVE scenario discussed in previous sections these results assume that the SM predictions for ΔM_s and ε_K agree with the data. If the EXCLUSIVE scenario will turn out to be true, the predictions above will be invalid and will be replaced by the ones in Table 2 and the green area in Fig. 7.

5 Conclusions

The EXCLUSIVE vision of rare decays and quark mixing is still not excluded and could become reality in the coming years. The present paper shows, similarly to our analysis in [7], how important is the determination of $|V_{cb}|$ for rare decays, in particular for rare Kaon decays. A precise determination of the γ in tree-level decays in the coming years will shed additional light on the tensions identified by us.

As we have seen, an unusual pattern of SM predictions results from this study with some existing tensions disappearing or being dwarfed and new ones being born. In particular the $B_s \to \mu^+\mu^-$ tension disappears and instead the anomalies at the level of $(2-5)\sigma$ are present in ΔM_s, ΔM_d and in particular in ε_K. While the 3.1σ tension in ΔM_s is practically independent of γ, the one in ΔM_d increases from $0.6\,\sigma$ to 4σ when γ is decreased from 75° to 60°. In the case of ε_K the corresponding variation is from 2.0σ to 5.2σ.

Moreover, the room left for NP in $K^+ \to \pi^+\nu\bar{\nu}$, $K_L \to \pi^0\nu\bar{\nu}$ and $B \to K(K^*)\nu\bar{\nu}$ is significantly increased but as seen in Figs. 4 and 5 it depends sensitively on γ. The tension in $B \to X_s\gamma$ is also interesting.

It should be recalled that in 2018 with the values of $\gamma \approx 74°$ from the LHCb, with the inclusive $|V_{cb}|$ and the $N_f = 2+1$ hadronic $B^0_{s,d} - \bar{B}^0_{s,d}$ matrix elements, ΔM_d in the SM was found significantly above the data with a smaller enhancement in ΔM_s [16]. In 2022 with lower values for $\gamma \approx 65°$ from the LHCb [37] and $N_f = 2+1+1$ hadronic $B^0_{s,d} - \bar{B}^0_{s,d}$ matrix elements from HPQCD [19] the inclusive values of $|V_{cb}|$ imply good agreement of the SM with the data on $\Delta M_{s,d}$. But for the exclusive values of $|V_{cb}|$ used in the present paper ΔM_s is significantly below the experimental data. This also applies to ΔM_d unless γ is chosen above 70° that is not yet excluded by experiments.

In this context we should emphasize that the $R(K)$ and $R(K^*)$ anomalies being independent of $|V_{cb}|$ remain. On the other hand, as analysed in [12], lowering the value of $|V_{cb}|$ decreases the anomalies in $B \to K\mu^+\mu^-$, $B \to K^*\mu^+\mu^-$ and $B_s \to \phi\mu^+\mu^-$ decay branching ratios but to remove them completely values of $|V_{cb}|$ significantly lower than the exclusive ones are required.

It is premature to make a detailed analysis of possible BSM scenarios that could remove the anomalies in the EXCLUSIVE scenario considered by us. Despite of this let us close our paper with a few observations.

As in the EXCLUSIVE scenario NP is required to enhance ΔM_s, ΔM_d and ε_K, a natural scenario would be at first sight the constrained Minimal Flavour Violation scenario [57] because, as pointed out in [58], in this scenario the $\Delta F = 2$ observables can only be enhanced. However, the fact that a new phase $\varphi_{\text{new}} \approx -1.3°$ is required to fit the data for $S_{\psi K_S}$, a more appropriate here would be the $\mathrm{U}(2)^3$ scenario [59–61]. As pointed out in [62], in the $\mathrm{U}(2)^3$ scenario the CP-asymmetry $S_{\psi K_S}$ is anti-correlated with the CP-asymmetry $S_{\psi\phi}$

$$S_{\psi K_S} = \sin(2\beta + 2\varphi_{\text{new}}), \quad S_{\psi\phi} = \sin(2|\beta_s| - 2\varphi_{\text{new}}), \quad (\mathrm{U}(2)^3), \qquad (23)$$

so that with $|\beta_s| \approx 1°$ an enhancement of the latter asymmetry from the SM prediction 0.0363 ± 0.0013 to 0.080 ± 0.020 would follow. Somewhat above the present data 0.054 ± 0.020 [22] but consistent with it.

As a byproduct we have investigated in Sect. 4 the impact of the hadronic matrix elements from the HPQCD collaboration [19] on our results for rare B decays in [7]. The most interesting result is the increase of the $B_s \to \mu^+\mu^-$ anomaly from 2.1σ to 2.7σ. Moreover we compared the determination of $|V_{cb}|$ from ΔM_s, ΔM_d, ε_K and $S_{\psi K_S}$ using $B^0_{s,d} - \bar{B}^0_{s,d}$

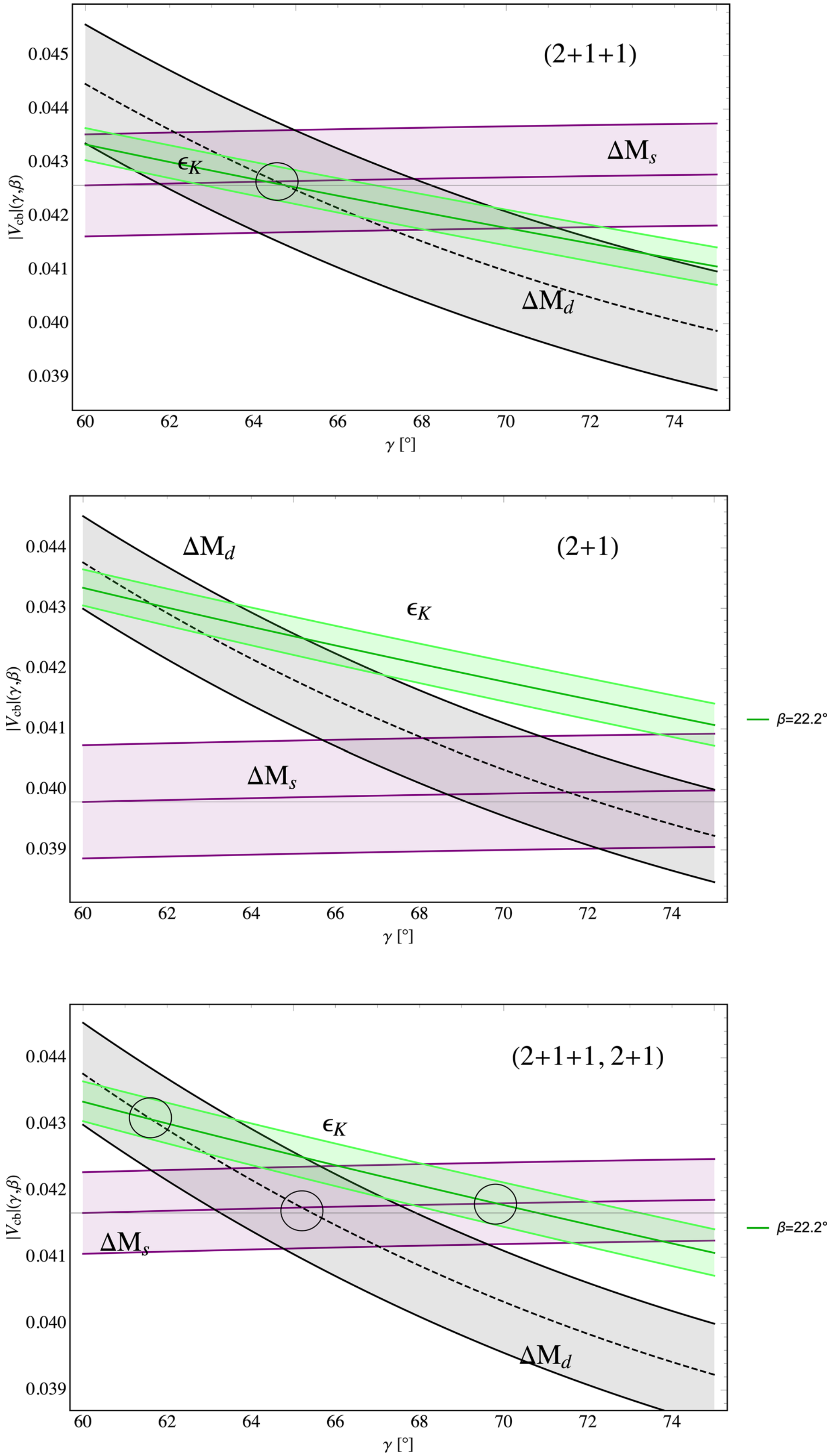


Fig. 8 The values of $|V_{cb}|$ extracted from ε_K, ΔM_d and ΔM_s as functions of γ. $2+1+1$ flavours (top), $2+1$ flavours (middle), average of $2+1+1$ and $2+1$ cases (bottom). The green band represents experimental $S_{\psi K_S}$ constraint on β

Table 3 Results for the rare B decay branching ratios using the strategy of [7] with the $2+1+1$ LQCD hadronic matrix elements of [19] (second column) compared with the ones obtained in [7] using the average of $2+1+1$ and $2+1$ LQCD data from [10] (third column). Results for rare K decays remain unchanged

Decay	Branching ratio with [7,19]	Branching ratio with [7,10]
$B_s \to \mu^+\mu^-$	$(3.78^{+0.15}_{-0.10}) \times 10^{-9}$	$(3.62^{+0.15}_{-0.10}) \times 10^{-9}$
$B_d \to \mu^+\mu^-$	$(1.02^{+0.05}_{-0.03}) \times 10^{-10}$	$(0.99^{+0.05}_{-0.03}) \times 10^{-10}$
$B^+ \to K^+\nu\bar\nu$	$(4.65 \pm 0.62) \times 10^{-6}$	$(4.45 \pm 0.62) \times 10^{-6}$
$B^0 \to K^{0*}\nu\bar\nu$	$(10.13 \pm 0.92) \times 10^{-6}$	$(9.70 \pm 0.92) \times 10^{-6}$
$K^+ \to \pi^+\nu\bar\nu$	$(8.60 \pm 0.42) \times 10^{-11}$	$(8.60 \pm 0.42) \times 10^{-11}$
$K_L \to \pi^0\nu\bar\nu$	$(2.94 \pm 0.15) \times 10^{-11}$	$(2.94 \pm 0.15) \times 10^{-11}$
$K_S \to \mu^+\mu^-$	$(1.85 \pm 0.10) \times 10^{-13}$	$(1.85 \pm 0.10) \times 10^{-13}$

hadronic matrix elements from LQCD with $2+1+1$ flavours, $2+1$ flavours and their average. As seen in Fig. 8 only for the $2+1+1$ case values for β and γ can be found for which the same value of $|V_{cb}|$ is found. The resulting $|V_{cb}|$, γ and β are given in (20) and $|V_{ub}|$ in (21).

In any case the coming years will hopefully reveal for us which scenario for $|V_{cb}|$ and $|V_{ub}|$ has been chosen by nature. The measurement of γ combined with the 16 $|V_{cb}|$-independent ratios constructed in [7] and with γ-dependence of various observables presented here will also play an important role in the search for NP. The importance of rare K decays in the search for NP has been recently summarized in [63] and the prospects for reducing hadronic uncertainties in K decays through intensive LQCD computations in the coming years are very good [64].

The EXCLUSIVE scenario appears to us to be more interesting than the HYBRID one because it implies more tensions between the SM predictions and the data. On the other the proponents of the inclusive determinations of $|V_{cb}|$ could consider the tensions found by us as an argument against exclusive determinations of $|V_{cb}|$.

Acknowledgements We would like to thank Steven Gottlieb and Andreas Kronfeld for the discussions on $|V_{cb}|$ from LQCD and in particular Enrico Lunghi for providing preliminary value of $|V_{cb}|$ from FLAG. Thanks go also to Fulvia De Fazio for a few numerical checks. A.J.B acknowledges financial support from the Excellence Cluster ORIGINS, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Excellence Strategy, EXC-2094, 390783311. E.V. has been partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy—EXC-2094—390783311, by the Collaborative Research Center SFB1258 and the BMBFgrant 05H18WOCA1 and thanks the Munich Institute for Astro- and Particle Physics (MIAPP) for hospitality.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: No data because it is theory paper.]

Funded by SCOAP^3. SCOAP^3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

Appendix A: Weak decay constants from LQCD

For $N_f = 2+1$ FLAG averages based on [65–70] read [6]

$$F_{B_d} = 192.0(4.3)\ \text{MeV}, \quad F_{B_s} = 228.4(3.7)\ \text{MeV}, \qquad \frac{F_{B_s}}{F_{B_d}} = 1.201(16), \tag{24}$$

while for $N_f = 2+1+1$ FLAG averages based on [71–74] are [6]

$$F_{B_d} = 190.0(1.3)\ \text{MeV}, \quad F_{B_s} = 230.3(1.3)\ \text{MeV}, \qquad \frac{F_{B_s}}{F_{B_d}} = 1.209(5). \tag{25}$$

While the central values in (24) and (25) are close to each other, the latter ones are much more accurate and we use them in our analysis.

Appendix B: Hadronic matrix elements from LQCD

For $N_f = 2+1$ the FLAG averages dominated by FNAL/MILC results [20] and including [68,75] are [6]

$$F_{B_d}\sqrt{\hat B_{B_d}} = 225(9)\ \text{MeV}\, F_{B_s}\sqrt{\hat B_{B_s}} = 274(8)\ \text{MeV} \tag{26}$$

$$\hat B_{B_d} = 1.30(10) \quad \hat B_{B_s} = 1.35(6) \tag{27}$$

$$\xi = 1.206(17) \quad \hat B_{B_s}/\hat B_{B_d} = 1.032(38). \tag{28}$$

For $N_f = 2+1+1$ one finds [19]

$$F_{B_d}\sqrt{\hat B_{B_d}} = 210.6(5.5)\ \text{MeV} \quad F_{B_s}\sqrt{\hat B_{B_s}} = 256.1(5.7)\ \text{MeV} \tag{29}$$

$$\hat{B}_{B_d} = 1.222(61) \quad \hat{B}_{B_s} = 1.232(53), \tag{30}$$

$$\xi = 1.216(16) \quad \hat{B}_{B_s}/\hat{B}_{B_d} = 1.008(25)\,. \tag{31}$$

In this case there are significant differences between the $N_f = 2+1$ and $N_f = 2+1+1$ results. Moreover, the latter ones are more accurate and we use them in the present paper. In [7] we have used the averages of both results given by [10]

$$F_{B_d}\sqrt{\hat{B}_{B_d}} = 214.0(39)\,\text{MeV},$$

$$F_{B_s}\sqrt{\hat{B}_{B_s}} = 261.7(38)\,\text{MeV}. \tag{32}$$

These are consistent with the ones from [19] alone but higher. However FLAG-2021 advices not to make such averages so that this time we use the $N_f = 2+1+1$ values.

References

1. S.L. Glashow, J. Iliopoulos, L. Maiani, Weak interactions with lepton-hadron symmetry. Phys. Rev. D **2**, 1285–1292 (1970)
2. A.J. Buras, *Gauge Theory of Weak Decays* (Cambridge University Press, Cambridge, 2020), p. 6
3. M. Bordone, N. Gubernari, D. van Dyk, M. Jung, Heavy-quark expansion for $\bar{B}_s \to D_s^{(*)}$ form factors and unitarity bounds beyond the $SU(3)_F$ limit. Eur. Phys. J. C **80**(4), 347 (2020). arXiv:1912.09335
4. M. Bordone, B. Capdevila, P. Gambino, Three loop calculations and inclusive Vcb. Phys. Lett. B **822**, 136679 (2021). arXiv:2107.00604
5. G. Ricciardi, Theory: semileptonic B decays and $|V_{xb}|$ update. PoS **BEAUTY2020**, 031 (2021). arXiv:2103.06099
6. Y. Aoki et al., FLAG Review 2021. arXiv:2111.09849
7. A.J. Buras, E. Venturini, Searching for new physics in rare K and B decays without $|V_{cb}|$ and $|V_{ub}|$ uncertainties. arXiv:2109.11032
8. D. Leljak, B. Melić, D. van Dyk, The $\overline{B}\pi$ form factors from QCD and their impact on $|V_{ub}|$. JHEP **07**, 036 (2021). arXiv:2102.07233
9. A.J. Buras, Relations between $\Delta M_{s,d}$ and $B_{s,d} \to \mu^+\mu^-$ in models with minimal flavour violation. Phys. Lett. B **566**, 115–119 (2003). arXiv:hep-ph/0303060
10. C. Bobeth, A.J. Buras, Searching for new physics with $\overline{\mathcal{B}}(B_{s,d} \to \mu\bar{\mu})/\Delta M_{s,d}$. Acta Phys. Polon. B **52**, 1189 (2021). arXiv:2104.09521
11. A.J. Buras, E. Venturini, Standard model predictions for rare K and B decays without $|V_{cb}|$ and $|V_{ub}|$ uncertainties, 3, 2022. arXiv:2203.10099
12. W. Altmannshofer, N. Lewis, Loop-induced determinations of V_{ub} and V_{cb}. Phys. Rev. D **105**(3), 033004 (2022). arXiv:2112.03437
13. A.J. Buras, D. Buttazzo, J. Girrbach-Noe, R. Knegjens, $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \pi^0\nu\bar{\nu}$ in the Standard Model: status and perspectives. JHEP **11**, 033 (2015). arXiv:1503.02693
14. UTfit Collaboration, M. Bona et al., Model-independent constraints on ΔF=2 operators and the scale of new physics. JHEP**0803** 049, (2008). arXiv:0707.0636. Updates available on https://urldefense.com/v3
15. CKMfitter Group Collaboration, J. Charles et al., CP violation and the CKM matrix: Assessing the impact of the asymmetric B factories. Eur. Phys. J. C**41** 1–131 (2005). arXiv:hep-ph/0406184. http://www.ckmfitter.in2p3.fr
16. M. Blanke, A.J. Buras, Emerging ΔM_d -anomaly from tree-level determinations of $|V_{cb}|$ and the angle γ. Eur. Phys. J. C **79**(2), 159 (2019). arXiv:1812.06963
17. S.W.M.E. Collaboration, J. Kim, Y.-C. Jang, S. Lee, W. Lee, J. Leem, C. Park, S. Park, 2021 update on ε_K with lattice QCD inputs. PoS **LATTICE2021**, 078 (2021). arXiv:2202.11473
18. J. Brod, M. Gorbahn, E. Stamou, Standard-Model prediction of ϵ_K with manifest quark-mixing unitarity. Phys. Rev. Lett. **125**(17), 171803 (2020). arXiv:1911.06822
19. R.J. Dowdall, C.T.H. Davies, R.R. Horgan, G.P. Lepage, C.J. Monahan, J. Shigemitsu, M. Wingate, Neutral B-meson mixing from full lattice QCD at the physical point. Phys. Rev. D **100**(9), 094508 (2019). arXiv:1907.01025
20. Fermilab Lattice, MILC Collaboration, A. Bazavov et al., $B^0_{(s)}$-mixing matrix elements from lattice QCD for the Standard Model and beyond. Phys. Rev. D **93**(11), 113016 (2016). arXiv:1602.03560
21. L.-L. Chau, W.-Y. Keung, Comments on the parametrization of the Kobayashi–Maskawa matrix. Phys. Rev. Lett. **53**, 1802 (1984)
22. Particle Data Group Collaboration, P.A. Zyla et al., Review of particle physics. PTEP **2020**(8), 083C01 (2020)
23. [FNAL/MILC 14] J.A. Bailey et al., Update of $|V_{cb}|$ from the $\bar{B} \to D^*\ell\bar{\nu}$ form factor at zero recoil with three-flavor lattice QCD. Phys. Rev. D **89**(11), 114504 (2014). arXiv:1403.0635
24. [FNAL/MILC 15] J.A. Bailey et al., $|V_{ub}|$ from $B \to \pi\ell\nu$ decays and (2+1)-flavor lattice QCD. Phys. Rev. D **92**(1), 014024 (2015). arXiv:1503.07839
25. [FNAL/MILC 15C] J.A. Bailey et al., $B \to D\ell\nu$ form factors at nonzero recoil and $|V_{cb}|$ from 2+1-flavor lattice QCD. Phys. Rev. D **92**(3), 034506 (2015). arXiv:1503.07237
26. [FNAL/MILC 19] A. Bazavov et al., $B_s \to K\ell\nu$ decay from lattice QCD. Phys. Rev. D **100**(3), 034501 (2019). arXiv:1901.02561
27. [HPQCD 15] H. Na, C.M. Bouchard, G.P. Lepage, C. Monahan, J. Shigemitsu, $B \to D\ell\nu$ form factors at nonzero recoil and extraction of $|V_{cb}|$. Phys. Rev. D **92**(5), 054510 (2015). arXiv:1505.03925
28. C.M. Bouchard, G.P. Lepage, C. Monahan, H. Na, J. Shigemitsu, $B_s \to K\ell\nu$ form factors from lattice QCD. Phys. Rev. D **90**, 054506 (2014). arXiv:1406.2279
29. [HPQCD 19] E. McLean, C.T.H. Davies, J. Koponen, A.T. Lytle, $B_s \to D_s\ell\nu$ Form Factors for the full q^2 range from Lattice QCD with non-perturbatively normalized currents. Phys. Rev. D **101**(7), 074513 (2020). arXiv:1906.00701
30. [RBC/UKQCD 15] J. M. Flynn, T. Izubuchi, T. Kawanai, C. Lehner, A. Soni, R.S. Van de Water, O. Witzel, $B \to \pi\ell\nu$ and $B_s \to K\ell\nu$ form factors and $|V_{ub}|$ from 2+1-flavor lattice QCD with domain-wall light quarks and relativistic heavy quarks. Phys. Rev. D **91**(7), 074510 (2015). arXiv:1501.05373
31. Fermilab Lattice, MILC Collaboration, A. Bazavov et al., Semileptonic form factors for $B \to D^*\ell\nu$ at nonzero recoil from 2 + 1-flavor lattice QCD. arXiv:2105.14019
32. G. Martinelli, S. Simula, L. Vittorio, $|V_{cb}|$ and $R(D^{(*)})$ using lattice QCD and unitarity. Phys. Rev. D **105**(3), 034503 (2022). arXiv:2105.08674
33. G. Martinelli, S. Simula, L. Vittorio, Exclusive determinations of $|V_{cb}|$ and $R(D^*)$ through unitarity. arXiv:2109.15248
34. G. Martinelli, S. Simula, L. Vittorio, Non-perturbative bounds for $B \to D^{(*)}\ell\nu_\ell$ decays and phenomenological applications, in *38th International Symposium on Lattice Field Theory*, 11, 2021. arXiv:2111.10582
35. G. Martinelli, M. Naviglio, S. Simula, L. Vittorio, $|V_{cb}|$, Lepton flavour universality and $SU(3)_F$ symmetry breaking in semileptonic $B_s \to D_s^{(*)}\ell\nu_\ell$ decays through unitarity and lattice QCD. arXiv:2204.05925

36. A.J. Buras, F. Parodi, A. Stocchi, The CKM matrix and the unitarity triangle: another look. JHEP **0301**, 029 (2003). arXiv:hep-ph/0207101
37. LHCb Collaboration, R. Aaij et al., Simultaneous determination of CKM angle γ and charm mixing parameters. arXiv:2110.02350
38. A. Bansal, N. Mahajan, D. Mishra, $\frac{|V_{ub}|}{|V_{cb}|}$ and quest for new physics. arXiv:2112.00363
39. S. Descotes-Genon, P. Koppenburg, The CKM parameters. Annu. Rev. Nucl. Part. Sci. **67**, 97–127 (2017). arXiv:1702.08834
40. A. Cerri, V.V. Gligorov, S. Malvezzi, J. Martin Camalich, J. Zupan, Opportunities in flavour physics at the HL-LHC and HE-LHC. arXiv:1812.07638
41. D. King, A. Lenz, T. Rauh, B_s mixing observables and $|V_{td}/V_{ts}|$ from sum rules. JHEP **05**, 034 (2019). arXiv:1904.00940
42. Flavour Lattice Averaging Group Collaboration, S. Aoki et al., FLAG Review 2019: Flavour Lattice Averaging Group (FLAG). Eur. Phys. J. C **80**(2), 113 (2020). arXiv:1902.08191
43. J. Brod, M. Gorbahn, and E. Stamou, Updated Standard Model prediction for $K \to \pi\nu\bar{\nu}$ and ϵ_K, in *19th International Conference on B-Physics at Frontier Machines, 5, 2021*. arXiv:2105.02868
44. A.J. Buras, D. Guadagnoli, G. Isidori, On ϵ_K beyond lowest order in the Operator Product Expansion. Phys. Lett. B **688**, 309–313 (2010). arXiv:1002.3612
45. A.J. Buras, M. Jamin, P.H. Weisz, Leading and next-to-leading QCD corrections to ε parameter and $B^0-\bar{B}^0$ mixing in the presence of a heavy top quark. Nucl. Phys. B **347**, 491–536 (1990)
46. J. Urban, F. Krauss, U. Jentschura, G. Soff, Next-to-leading order QCD corrections for the $B^0 - \bar{B}^0$ mixing with an extended Higgs sector. Nucl. Phys. B **523**, 40–58 (1998). arXiv:hep-ph/9710245
47. Heavy Flavor Averaging Group (HFAG) Collaboration, Y. Amhis et al., Averages of *b*-hadron, *c*-hadron, and τ-lepton properties as of summer 2016. arXiv:1612.07233
48. M. Misiak et al., Updated NNLO QCD predictions for the weak radiative B-meson decays. Phys. Rev. Lett. **114**(22), 221801 (2015). arXiv:1503.01789
49. NA62 Collaboration, E. Cortina Gil et al., An investigation of the very rare $K^+ \to \pi^+\nu\bar{\nu}$ decay. arXiv:2007.08218
50. KOTO Collaboration, J. Ahn et al., Search for the $K_L \to \pi^0\nu\bar{\nu}$ and $K_L \to \pi^0 X^0$ decays at the J-PARC KOTO experiment. Phys. Rev. Lett.**122**(2), 021802 (2019). arXiv:1810.09655
51. LHCb Collaboration, R. Aaij et al., Improved limit on the branching fraction of the rare decay $K^0_S \to \mu^+\mu^-$. Eur. Phys. J. C **77**(10), 678 (2017). arXiv:1706.00758
52. LHCb Collaboration, R. Aaij et al., Measurement of the $B^0_s \to \mu^+\mu^-$ decay properties and search for the $B^0 \to \mu^+\mu^-$ and $B^0_s \to \mu^+\mu^-\gamma$ decays. arXiv:2108.09283
53. CMS Collaboration, Combination of the ATLAS, CMS and LHCb results on the $B^0_{(s)} \to \mu^+\mu^-$ decays, CMS-PAS-BPH-20-003
54. ATLAS Collaboration, Combination of the ATLAS, CMS and LHCb results on the $B^0_{(s)} \to \mu^+\mu^-$ decays. ATLAS-CONF-2020-049
55. T.E. Browder, N.G. Deshpande, R. Mandal, R. Sinha, Impact of $BK\nu\nu$ measurements on beyond the Standard Model theories. Phys. Rev. D **104**(5), 053007 (2021). arXiv:2107.01080
56. Belle Collaboration, J. Grygier et al., Search for $B \to h\nu\bar{\nu}$ decays with semileptonic tagging at Belle. Phys. Rev. D **96**(9), 091101 (2017). arXiv:1702.03224 [Addendum: Phys. Rev.D97,no.9,099902(2018)]
57. A.J. Buras, P. Gambino, M. Gorbahn, S. Jäger, L. Silvestrini, Universal unitarity triangle and physics beyond the standard model. Phys. Lett. B **500**, 161–167 (2001). (**[hep-ph/0007085]**)
58. M. Blanke, A.J. Buras, Lower bounds on $\Delta M_{s,d}$ from constrained minimal flavour violation. JHEP **0705**, 061 (2007). (**[hep-ph/0610037]**)
59. R. Barbieri, G. Isidori, J. Jones-Perez, P. Lodone, D.M. Straub, U(2) and minimal flavour violation in supersymmetry. Eur. Phys. J. C **71**, 1725 (2011). arXiv:1105.2296
60. R. Barbieri, P. Campli, G. Isidori, F. Sala, D.M. Straub, B-decay CP-asymmetries in SUSY with a $U(2)^3$ flavour symmetry. Eur. Phys. J. C **71**, 1812 (2011). arXiv:1108.5125
61. A. Crivellin, L. Hofer, U. Nierste, The MSSM with a softly broken $U(2)^3$ flavor symmetry. PoS **EPS-HEP2011**, 145 (2011). arXiv:1111.0246
62. A.J. Buras, J. Girrbach, On the correlations between flavour observables in minimal $U(2)^3$ models. JHEP **1301**, 007 (2013). arXiv:1206.3878
63. J. Aebischer, A.J. Buras, J. Kumar, On the importance of rare kaon decays: a snowmass 2021 white paper, 3, 2022. arXiv:2203.09524
64. T. Blum et al., Discovering new physics in rare kaon decays, 3, 2022. arXiv:2203.10998
65. [HPQCD 11A] C. McNeile, C.T.H. Davies, E. Follana, K. Hornbostel, G.P. Lepage, High-precision f_{B_s} and HQET from relativistic lattice QCD. Phys. Rev. D **85**, 031503 (2012). arXiv:1110.4510
66. [FNAL/MILC 11] A. Bazavov et al., *B*- and *D*-meson decay constants from three-flavor lattice QCD. Phys. Rev. D **85**, 114506 (2012). arXiv:1112.3051
67. H. Na, C.J. Monahan, C.T. Davies, R. Horgan, G.P. Lepage et al., The B and B_s meson decay constants from lattice QCD. Phys. Rev. D **86**, 034506 (2012). arXiv:1202.4914
68. [RBC/UKQCD 14A] Y. Aoki, T. Ishikawa, T. Izubuchi, C. Lehner, A. Soni, Neutral B meson mixings and B meson decay constants with static heavy and domain-wall light quarks. Phys. Rev. D **91**(11), 114505 (2015). arXiv:1406.6192
69. [RBC/UKQCD 14] N.H. Christ, J.M. Flynn, T. Izubuchi, T. Kawanai, C. Lehner, et al., B-meson decay constants from 2+1-flavor lattice QCD with domain-wall light quarks and relativistic heavy quarks. Phys. Rev. D **91**(5), 054502 (2015). arXiv:1404.4670
70. [RBC/UKQCD 18A] P.A. Boyle, L. Del Debbio, N. Garron, A. Juttner, A. Soni, J.T. Tsang, O. Witzel, SU(3)-breaking ratios for $D_{(s)}$ and $B_{(s)}$ mesons. arXiv:1812.08791
71. R.J. Dowdall, C. Davies, R. Horgan, C. Monahan, J. Shigemitsu, B-meson decay constants from improved lattice NRQCD and physical u, d, s and c sea quarks. Phys. Rev. Lett. **110**, 222003 (2013). arXiv:1302.2644
72. [ETM 16B] A. Bussone et al., Mass of the b quark and B -meson decay constants from N$_f$=2+1+1 twisted-mass lattice QCD. Phys. Rev. D **93**(11), 114505 (2016). arXiv:1603.04306
73. [HPQCD 17A] C. Hughes, C.T.H. Davies, C.J. Monahan, New methods for B meson decay constants and form factors from lattice NRQCD. Phys. Rev. D **97**(5), 054509 (2018). arXiv:1711.09981
74. [FNAL/MILC 17] A. Bazavov et al., B- and D-meson leptonic decay constants from four-flavor lattice QCD. Phys. Rev. D **98**(7), 074512 (2018). arXiv:1712.09262
75. HPQCD Collaboration, E. Gamiz, C. T. Davies, G. P. Lepage, J. Shigemitsu, M. Wingate, Neutral B meson mixing in unquenched lattice QCD. Phys. Rev. D **80**, 014503 (2009). arXiv:0902.1815