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A deformed supersymmetric $w_{1+\infty}$ symmetry in the celestial conformal field theory

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Abstract By using the K-free complex bosons and the K-free complex fermions, we construct the $\mathcal{N}=2$ supersymmetric $W_\infty^{K,K}$ algebra which is the matrix generalization of previous $\mathcal{N}=2$ supersymmetric W_∞ algebra. By twisting this $\mathcal{N}=2$ supersymmetric $W_\infty^{K,K}$ algebra, we obtain the $\mathcal{N}=1$ supersymmetric W_∞^K algebra which is the matrix generalization of known $\mathcal{N}=1$ supersymmetric topological W_∞ algebra. From this two-dimensional symmetry algebra, we propose the operator product expansion (OPE) between the soft graviton and gravitino (as a first example), at nonzero deformation parameter, in the supersymmetric Einstein–Yang–Mills theory explicitly. Other six OPEs between the graviton, gravitino, gluon and gluino can be determined completely. At vanishing deformation parameter, we reproduce the known result of Fotopoulos, Stieberger, Taylor and Zhu on the above OPEs via celestial holography.

Contents

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1 Introduction

Recently, the celestial holography has been proposed by the fact that there exists a duality between the gravitational scattering in asymptotically flat spacetimes and the conformal field theory which lives on the celestial sphere. See the review papers [1–4] on the celestial holography. By using the low energy scattering problems, the symmetry algebra of the conformal field theory for flat space (a celestial conformal field theory) has been studied in [5]. Moreover, in [6], the group of symmetries on the celestial sphere satisfies the wedge subalgebra of $w_{1+\infty}$ algebra [7]. This implies that we should understand the structures behind these findings thoroughly in order to check the above duality. In [8], the supersymmetric $w_{1+\infty}$ algebra has been identified with the corresponding soft current algebra in the supersymmetric Einstein–Yang–Mills theory. The relevant works on the celestial holography in the various directions can be found in [9–35]. See [1] for more complete literatures.

The higher-spin extension of the Virasoro algebra has been found by Zamolodchikov [36]. The so-called W_3 algebra consists of the spin-2 stress energy tensor and the spin-3 current. Subsequently, this W_3 algebra is generalized to the W_N algebra [37,38] which is generated by the spin-2 stress energy tensor and the higher-spin currents of each spin, $s = 3, 4, \ldots, N$. It is also possible to construct a linear W_∞ algebra [39,40] generated by the currents with spins $s = 2, 3, 4, \ldots, \infty$. A simple contraction of the W_∞ algebra leads to the w_∞ algebra [7]. Moreover, the $W_{1+\infty}$ algebra [41] contains all spins $s = 1, 2, 3, \ldots, \infty$. The $\mathcal{N} = 2$ supersymmetric W_∞ algebra [42] whose bosonic sector is given by W_∞ and $W_{1+\infty}$ is obtained.

Bakas and Kiritsis [43] have found the W^K_∞ algebra which is an $U(K)$-matrix generalization of W_∞ algebra. For each current of spin s, there are K^2 multicomponent generators. Odake and Sano [44] also have constructed the $W^L_{1+\infty}$ algebra which is an $U(L)$-matrix extension of $W_{1+\infty}$ algebra. There exist L^2 multicomponent generators for each current of spin s. Furthermore, the supersymmetric $W^{K,L}_\infty$ algebra, whose bosonic sector is given by W^K_∞ and $W^L_{1+\infty}$, has been studied in [45].

In this paper, by taking the condition $K = L$, we construct the $\mathcal{N}=2$ supersymmetric $W^{K,K}_\infty$ algebra with $U(K)\times U(K)$ symmetry. Due to the above condition $K = L$, we can multiply the generators in the fermionic currents. The $\mathcal{N}=2$ supersymmetry is reduced to the $\mathcal{N}=1$ supersymmetry by topological twisting [46–48]. Then the $\mathcal{N}=1$ supersymmetric W^K_∞ algebra with $U(K)$ symmetry is obtained. That is, we obtain the matrix generalization of [48]. The seven commutator relations between the bosonic and fermionic currents can be written in terms of the various structure constants and the deformation parameter. By considering the vanishing limit of this deformation parameter, we reproduce the previous result of [49]. We propose that the OPEs between the graviton, gravitino, gluon and gluino in the supersymmetric Einstein–Yang–Mills theory can be determined from the above two-dimensional symmetry algebra.[1]

For the nonvanishing deformation parameter, the commutators contain the possible terms in the right hand sides. In general, the structure constants depend on the two modes of the commutator and the weights. Among the weights, the weights h_1 and h_2 appearing in the left hand side of the commutator are given. On the other hand, the weight h appearing in the right hand side vary from its lowest value to the highest value depending on the previous weights h_1 and h_2. The weight h plays the role of dummy variable in the summation of the right hand side of the commutator. We would like to determine the OPEs between the above soft currents in the supersymmetric Einstein–Yang–Mills theory by looking at the two dimensional symmetry algebra characterized by seven commutators. Then the question is how we can determine the OPEs in the soft currents which will eventually lead to the commutators we have found in two dimensional conformal field theory after performing the appropriate contour integrals.

As mentioned before, the structure constants consists of mode dependent part and mode independent part and they do depend on the above three weights dependence. We should figure out how the mode dependent part can be read off from the relevant OPE between the soft currents because the mode independent part can be multiplied into this inside of the dummy variable weight h. From the experience of the various contour integrals [5], we expect that there should be h dependence in the OPE when we consider the case of the nonzero deformation parameter. Once we have obtained the correct OPEs which produces the mode dependent part of the commutators, then it is straightforward to determine the full OPEs by multiplying the weights dependence parts and summing over above dummy variable h within the possible range.

In Sect. 2, we obtain the $\mathcal{N}=2$ supersymmetric $W^{K,K}_\infty$ algebra after reviewing the supersymmetric $W^{K,L}_\infty$ algebra. In Sect. 3, we determine the $\mathcal{N} = 1$ supersymmetric W^K_∞ algebra. The free field realization is given. At the vanishing deformation parameter, the previous result [8] is reproduced.

[1] We are focusing on the soft currents where the celestial operators have the specific conformal dimensions for the bosonic and fermionic fields. For the former $\Delta = 1, 0, -1, \ldots$ and for the latter $\Delta = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \ldots$. See also [5,31].

We also present the seven commutator relations in terms of the corresponding OPEs. Finally, we propose its realization in the $\mathcal{N}=1$ supersymmetric Einstein–Yang–Mills theory. In Sect. 4, we summarize what we have obtained in this paper. In Appendices, we provide some details in Sects. 2 and 3.

2 The $\mathcal{N}=2$ supersymmetric $W_{\infty}^{K,K}$ algebra with $U(K)\times U(K)$ symmetry

2.1 The supersymmetric $W_{\infty}^{K,L}$ algebra with $U(K)\times U(L)$ symmetry: review

The nontrivial (anti)commutator relations of $W_{\infty}^{K,L}$ algebra [45] (see also [50]) are given by

$$
\begin{aligned}
\big[(W^{\bar{\alpha}\beta}_{\mathrm{F},h_1})_m,(W^{\bar{\gamma}\delta}_{\mathrm{F},h_2})_n\big] &= \sum_{h=-1}^{h_1+h_2-3}\frac{\lambda^h}{2}\,p_{\mathrm{F}}^{h_1,h_2,h}(m,n)\\
&\times\Big[\delta^{\bar{\gamma}\beta}\,(W^{\bar{\alpha}\delta}_{\mathrm{F},h_1+h_2-2-h})_{m+n}\\
&+(-1)^h\delta^{\bar{\alpha}\delta}(W^{\bar{\gamma}\beta}_{\mathrm{F},h_1+h_2-2-h})_{m+n}\Big]\\
&+c_{W_{\mathrm{F},h_1}}(m)\delta^{\bar{\alpha}\delta}\delta^{\beta\bar{\gamma}}\delta^{h_1h_2}\lambda^{2(h_1-2)}\delta_{m+n},\\
\big[(W^{\bar{a}b}_{\mathrm{B},h_1})_m,(W^{\bar{c}d}_{\mathrm{B},h_2})_n\big] &= \sum_{h=-1}^{h_1+h_2-4}\frac{\lambda^h}{2}\,p_{\mathrm{B}}^{h_1,h_2,h}(m,n)\\
&\times\Big[\delta^{\bar{c}b}\,(W^{\bar{a}d}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\\
&+(-1)^h\delta^{\bar{a}d}(W^{\bar{c}b}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\Big]\\
&+c_{W_{\mathrm{B},h_1}}(m)\delta^{\bar{a}d}\delta^{b\bar{c}}\delta^{h_1h_2}\lambda^{2(h_1-2)}\delta_{m+n},\\
\big[(W^{\bar{\alpha}\beta}_{\mathrm{F},h_1})_m,(Q^{\bar{a}\gamma}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3}\lambda^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\\
&\times(m,r)\,\delta^{\bar{\alpha}\gamma}\,(Q^{\bar{a}\beta}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W^{\bar{\alpha}\beta}_{\mathrm{F},h_1})_m,(\bar{Q}^{a\bar{\gamma}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}\\
&\times(m,r)\,\delta^{\beta\bar{\gamma}}\,(\bar{Q}^{a\bar{\alpha}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W^{\bar{a}b}_{\mathrm{B},h_1})_m,(Q^{\bar{c}\alpha}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3}\lambda^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\,\delta^{\bar{c}b}\\
&\times(Q^{\bar{a}\alpha}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W^{\bar{a}b}_{\mathrm{B},h_1})_m,(\bar{Q}^{c\bar{\alpha}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}\\
&\times(m,r)\,\delta^{\bar{a}c}\,(\bar{Q}^{b\bar{\alpha}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\{(Q^{\bar{a}\alpha}_{h_1+\frac{1}{2}})_r,(\bar{Q}^{b\bar{\beta}}_{h_2+\frac{1}{2}})_s\} &= \sum_{h=0}^{h_1+h_2-1}\lambda^h\,o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}\\
&\times(r,s)\,\delta^{\bar{a}b}\,(W^{\bar{\beta}\alpha}_{\mathrm{F},h_1+h_2-h})_{r+s}\\
&+\sum_{h=0}^{h_1+h_2-2}\lambda^h\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s)\,\delta^{\alpha\bar{\beta}}\,(W^{\bar{a}b}_{\mathrm{B},h_1+h_2-h})_{r+s}\\
&+c_{Q\bar{Q}_{h_1+\frac{1}{2}}}(r)\,\delta^{\bar{a}b}\,\delta^{\alpha\bar{\beta}}\,\delta^{h_1h_2}\,\lambda^{2(h_1+\frac{1}{2}-1)}\,\delta_{r+s}. \qquad (2.1)
\end{aligned}
$$

The bosonic W_{∞}^{K} subalgebra corresponding to the second equation of (2.1) is generated by the $U(K)$-adjoint $W^{\bar{a}b}_{\mathrm{B},h}$ with an integer weight $h=2,3,\ldots,\infty$. The subscript B stands for the bilinear of complex free bosons in next subsection. The fundamental index $a,b,\ldots$ of $U(K)$ runs over $a,b,\ldots=1,2,\ldots,K$ and the antifundamental index $\bar{a},\bar{b},\ldots$ of $U(K)$ runs over $\bar{a},\bar{b},\ldots=1,2,\ldots,K$. The bosonic $W^{L}_{1+\infty}$ subalgebra corresponding to the first equation of (2.1) is generated by the $U(L)$-adjoint $W^{\bar{\alpha}\beta}_{\mathrm{F},h}$ with an integer weight $h=1,2,\ldots,\infty$. Note the presence of weight-1 current. The subscript F stands for the bilinear of complex free fermions in next subsection. The fundamental index $\alpha,\beta,\ldots$ of $U(L)$ runs over $\alpha,\beta,\ldots=1,2,\ldots,L$ and the antifundamental index $\bar{\alpha},\bar{\beta},\ldots$ of $U(L)$ runs over $\bar{\alpha},\bar{\beta},\ldots=1,2,\ldots,L$. There are also the bifundamental $Q^{\bar{a}\alpha}_{h+\frac{1}{2}}$ and the bifundamental $\bar{Q}^{b\bar{\beta}}_{h+\frac{1}{2}}$ under the $U(K)\times U(L)$ with the half-integer weight $h+\frac{1}{2}=\frac{3}{2},\frac{5}{2},\ldots$ for the remaining (anti)commutator relations. Note that the lower and upper limits for the dummy variable h in (2.1) can be determined by the fact that (i) the maximum weight for the current in the right hand side is equal to the sum of two weights in the left hand side minus one and (ii) the minimum weight for the current in the right hand side is equal to 2, 1 or $\frac{3}{2}$ as above.

The λ is a deformation parameter[2] and the central terms in (2.1) except the λ-dependent factors are given by

$$
\begin{aligned}
c_{W_{\mathrm{F},h}}(m) &= N\,k\,\frac{2^{2(h-3)}\,(h-1)!)\,(h-1)!}{(2h-3)!!\,(2h-1)!!}\prod_{j=1-h}^{h-1}(m+j),\\
c_{W_{\mathrm{B},h}}(m) &= N\,k\,\frac{2^{2(h-3)}(h-2)!\,h!}{(2h-3)!!\,(2h-1)!!}\prod_{j=1-h}^{h-1}(m+j),\\
c_{Q\bar{Q}_h}(r) &= N\,k\,\frac{2^{2(h-\frac{3}{2})}(h-\frac{3}{2})!\,(h-\frac{1}{2})!}{(2h-2)!!\,(2h-2)!!}\\
&\times\prod_{j=\frac{1}{2}-h}^{h-\frac{3}{2}}\left(r+j+\frac{1}{2}\right). \qquad (2.2)
\end{aligned}
$$

There exists an overall factor N which is related to the number of free complex bosons (or fermions). The k is the

[2] This corresponds to the parameter q in [45] and is nothing to do with the one in the higher spin algebra in [50].

level of $\hat{SU}(L)$ and the corresponding weight-one current is given by $\frac{4\lambda}{\sqrt{LN}} W_{\mathrm{F},1}^{\bar{\alpha}\beta}\,\delta_{\beta\bar{\alpha}}$ with $c_{W_{\mathrm{F},1}}(m) = \frac{Nm}{16}$. By introducing the k copies of the free field realization, we construct the general level k realization [45] because $W_\infty^{K,L}$ algebra is linear. The Virasoro central charge is given by $c = Nk(2K+L)$ from $c_{W_{\mathrm{F},2}}(m) = \frac{1}{12}\,N\,k\,m(m^2-1)$ and $c_{W_{\mathrm{B},2}}(m) = \frac{1}{6}\,N\,k\,m(m^2-1)$ from (2.2) and the Sugawara stress energy tensor is given by $W_{\mathrm{B},2}^{\bar{a}b}\,\delta_{b\bar{a}} - W_{\mathrm{F},2}^{\bar{\alpha}\beta}\,\delta_{\beta\bar{\alpha}}$.

The mode-dependent structure constants appearing in (2.1) are described as follows:

$$
\begin{aligned}
p_{\mathrm{F}}^{h_1,h_2,h}(m,n) &\equiv \frac{1}{2(h+1)!}\,\phi_h^{h_1,h_2}(0,-\tfrac{1}{2})\,N_h^{h_1,h_2}(m,n),\\
p_{\mathrm{B}}^{h_1,h_2,h}(m,n) &\equiv \frac{1}{2(h+1)!}\,\phi_h^{h_1,h_2}(0,0)\,N_h^{h_1,h_2}(m,n),\\
q_{\mathrm{F}}^{h_1,h_2,h}(m,r) &\equiv \frac{(-1)^h}{4(h+2)!}\Big[(h_1-1)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,0)\\
&\quad -(h_1-h-3)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,-\tfrac{1}{2})\Big]\,N_h^{h_1,h_2}(m,r),\\
q_{\mathrm{B}}^{h_1,h_2,h}(m,r) &\equiv \frac{-1}{4(h+2)!}\Big[(h_1-h-2)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,0)\\
&\quad -(h_1)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,-\tfrac{1}{2})\Big]\,N_h^{h_1,h_2}(m,r),\\
o_{\mathrm{F}}^{h_1,h_2,h}(r,s) &\equiv \frac{4(-1)^h}{h!}\Big[(h_1+h_2-1-h)\,\phi_h^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\tfrac{1}{2},-\tfrac{1}{4})\\
&\quad -(h_1+h_2-\frac{3}{2}-h)\,\phi_{h+1}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\tfrac{1}{2},-\tfrac{1}{4})\Big]\,N_{h-1}^{h_1,h_2}(r,s),\\
o_{\mathrm{B}}^{h_1,h_2,h}(r,s) &\equiv -\frac{4}{h!}\Big[(h_1+h_2-2-h)\,\phi_h^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\tfrac{1}{2},-\tfrac{1}{4})\\
&\quad -(h_1+h_2-\frac{3}{2}-h)\,\phi_{h+1}^{h_1+\frac{1}{2},h_2+\frac{1}{2}}(\tfrac{1}{2},-\tfrac{1}{4})\Big]\,N_{h-1}^{h_1,h_2}(r,s).
\end{aligned}
\tag{2.3}
$$

The structure constants are polynomials in the modes. The modes $m, n, \ldots$ are integers and the modes $r, s, \ldots$ are half-integers. We introduce the following quantities

$$
\begin{aligned}
N_h^{h_1,h_2}(m,n) &\equiv \sum_{l=0}^{h+1}(-1)^l \binom{h+1}{l}\\
&\quad\times[h_1-1+m]_{h+1-l}[h_1-1-m]_l\\
&\quad\times[h_2-1+n]_l[h_2-1-n]_{h+1-l},\\
\phi_h^{h_1,h_2}&(x,y)\\
&\equiv {}_4F_3\Big[\begin{matrix}-\frac{1}{2}-x-2y,\ \frac{3}{2}-x+2y,\ -\frac{h+1}{2}+x,\ -\frac{h}{2}+x\\ -h_1+\frac{3}{2},\ -h_2+\frac{3}{2},\ h_1+h_2-h-\frac{3}{2}\end{matrix};1\Big].
\end{aligned}
\tag{2.4}
$$

We use the falling Pochhammer symbol $[a]_n \equiv a(a-1)\cdots(a-n+1)$ in (2.4) and we use the binomial coefficients for parentheses. Moreover, the generalized hypergeometric function, with four upper arguments a_i, three lower arguments b_i and variable z, is defined as the series

$$
{}_4F_3\Big[\begin{matrix}a_1,a_2,a_3,a_4\\ b_1,b_2,b_3\end{matrix};z\Big] = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n(a_4)_n}{(b_1)_n(b_2)_n(b_3)_n}\,\frac{z^n}{n!},
\tag{2.5}
$$

where the rising Pochhammer symbol $(a)_n \equiv a(a+1)\cdots(a+n-1)$ is used in (2.5).[3]

In this paper, by acting the generators of $U(K)$ (or $U(L)$) with the contractions of the (anti)fundamental indices on the (anti)commutator relations in (2.1), we will determine the $\mathcal{N}=2$ supersymmetric $W_\infty^{K,K}$ algebra and the $\mathcal{N}=1$ supersymmetric W_∞^K algebra. Then the fermionic currents will not have any (anti)fundamental indices.

2.2 Free field realization: review

The $W_\infty^{K,L}$ algebra with level $k=1$ is realized by K-free complex bosons of weight-1 ($\bar{\partial}\,\phi^{\bar{i},a}$ and $\bar{\partial}\,\bar{\phi}^{i,\bar{a}}$) and L-free complex fermions of weight-$\frac{1}{2}$ ($\psi^{\bar{i},\alpha}$ and $\bar{\psi}^{i,\bar{\alpha}}$). The index i is the fundamental index of $U(N)$ and the index $\bar{i}$ is the antifundamental index of $U(N)$. Their operator product expansions in the antiholomorphic sector are

$$
\begin{aligned}
\bar{\partial}\,\bar{\phi}^{i,\bar{a}}(\bar{z})\,\bar{\partial}\,\phi^{\bar{j},b}(\bar{w}) &= \frac{1}{(\bar{z}-\bar{w})^2}\,\delta^{i\bar{j}}\,\delta^{\bar{a}b}+\cdots,\\
\bar{\psi}^{i,\bar{\alpha}}(\bar{z})\,\psi^{\bar{j},\beta}(\bar{w}) &= \frac{1}{(\bar{z}-\bar{w})}\,\delta^{i\bar{j}}\,\delta^{\bar{\alpha}\beta}+\cdots.
\end{aligned}
\tag{2.6}
$$

The $U(N)$-singlet currents of $W_\infty^{K,L}$ algebra are described by the bilinears of these free fields as follows (See also [51]):

$$
\begin{aligned}
W_{\mathrm{F},h}^{\bar{\alpha}\beta} &= n_{W_{\mathrm{F},h}}\sum_{l=0}^{h-1}\sum_{i,\bar{i}=1}^{N}\delta_{i,\bar{i}}(-1)^l\binom{h-1}{l}^2\\
&\quad\times(\bar{\partial}^{h-l-1}\bar{\psi}^{i,\bar{\alpha}}\,\bar{\partial}^l\psi^{\bar{i},\beta}),\\
W_{\mathrm{B},h}^{\bar{a}b} &= n_{W_{\mathrm{B},h}}\sum_{l=0}^{h-2}\sum_{i,\bar{i}=1}^{N}\delta_{i,\bar{i}}\frac{(-1)^l}{(h-1)}\binom{h-1}{l}\binom{h-1}{l+1}\\
&\quad\times(\bar{\partial}^{h-l-1}\bar{\phi}^{i,\bar{a}}\,\bar{\partial}^{l+1}\phi^{\bar{i},b}),\\
Q_{h+\frac{1}{2}}^{\bar{a}\alpha} &= n_{Q_{h+\frac{1}{2}}}\sum_{l=0}^{h-1}\sum_{i,\bar{i}=1}^{N}\delta_{i,\bar{i}}(-1)^l\binom{h-1}{l}\binom{h}{l}\\
&\quad\times(\bar{\partial}^{h-l}\bar{\phi}^{i,\bar{a}}\,\bar{\partial}^l\psi^{\bar{i},\alpha}),\\
\bar{Q}_{h+\frac{1}{2}}^{a\bar{\alpha}} &= n_{\bar{Q}_{h+\frac{1}{2}}}\sum_{l=0}^{h-1}\sum_{i,\bar{i}=1}^{N}\delta_{i,\bar{i}}(-1)^{h-1+l}\binom{h-1}{l}\binom{h}{l}
\end{aligned}
$$

[3] In the right hand side of (2.4), the h_1 and h_2 play the role of $(i+2)$ (or $(i+\frac{3}{2})$) and $(j+2)$ (or $(j+\frac{3}{2})$) of [45]. The h corresponds to their r. Due to the typos in [42] (See also the footnote 2 of [48]), the structure constants in [45] are different from the ones in [42]. For example, the structure constant $a_l^{i\,\alpha}(m,r)$ of [42] is given by our $(-1)^h\,q_B^{h_1,h_2,h}(m,r)$ with the identification $h_1 = i+2$, $h_2 = \alpha+\frac{3}{2}$, $h = l-1$ and the structure constant $\tilde{a}_l^{i\,\alpha}(m,r)$ of [42] is given by our $(-1)^h\,q_F^{h_1,h_2,h}(m,r)$ with the identification $h_1 = i+2$, $h_2 = \alpha+\frac{3}{2}$, $h = l-1$.

$$\times\,(\,\bar{\partial}^{h-l}\phi^{\bar{i},a}\bar{\partial}^{l}\bar{\psi}^{i,\bar{\alpha}}\,). \tag{2.7}$$

The $l = 0$ cases of single summations correspond to the lowest weights, 1, 2, $\frac{3}{2}$ and $\frac{3}{2}$. The normalizations are given by

$$n_{W_{\mathrm{F},h}} = \frac{2^{h-3}(h-1)!}{(2h-3)!!}\,\lambda^{h-2}, \quad n_{W_{\mathrm{B},h}} = \frac{2^{h-3}\,h!}{(2h-3)!!}\,\lambda^{h-2},$$
$$n_{Q_{h+\frac{1}{2}}} = \frac{2^{h-\frac{1}{2}}h!}{(2h-1)!!}\,\lambda^{h-1} = n_{\bar{Q}_{h+\frac{1}{2}}}. \tag{2.8}$$

Then we can check that the modes of (2.7) satisfy the previous (anti)commutator relations (2.1) by using the mode expansion for the normal ordering between the free fields in the conformal field theory. After using the (anti)commutator relations corresponding to (2.6), the left hand sides of (2.1) contain the quadratic free fields having the explicit modes (where the coefficients depend on the weights, the modes and the dummy variable from the infinite sum) and the central terms. Similarly, the right hand sides of (2.1) contain the quadratic free fields and the central terms in the presence of the nontrivial structure constants (2.3). For several low values of the weights and the modes, we can check the several nontrivial identities. Alternatively, after using the Thielemans package [52] for low values of the weights, simplifying the right hand sides of the OPEs between the currents and rewriting down them in terms of the (anti)commutator relations with the help of the explicit formula in [53,54], the previous algebra (2.1) can be checked also explicitly.[4]

2.3 The $\mathcal{N}=2$ supersymmetric $W_{\infty}^{K,K}$ algebra with $U(K)\times U(K)$ symmetry

Let us consider the case of $K = L$. Then the number of (anti)fundamental indices is the same. From the decomposition of $U(K+K) = U(K)\oplus U(K)\oplus(\mathbf{K},\bar{\mathbf{K}})\oplus(\bar{\mathbf{K}},\mathbf{K})$, the generators consists of $t^{\hat{A}}_{\alpha\bar{\beta}}$, $t^{\hat{A}}_{a\bar{b}}$, $t^{\hat{A}}_{\alpha\bar{a}}$ and $t^{\hat{A}}_{a\bar{\alpha}}$ in addition to $\delta_{\alpha\bar{\beta}}$, $\delta_{a\bar{b}}$, $\delta_{\alpha\bar{a}}$ and $\delta_{a\bar{\alpha}}$ with $\hat{A} = 1, 2, \ldots, (K^2-1)$.

By multiplying the generators into the four kinds of currents, we obtain four kinds of singlets and adjoints of $U(K)$ as follows:

$$W_{\mathrm{F},h} \equiv W^{\bar{\alpha}\beta}_{\mathrm{F},h}\,\delta_{\beta\bar{\alpha}}, \quad W^{\hat{A}}_{\mathrm{F},h} \equiv W^{\bar{\alpha}\beta}_{\mathrm{F},h}\,t^{\hat{A}}_{\beta\bar{\alpha}},$$
$$W_{\mathrm{B},h} \equiv W^{\bar{a}b}_{\mathrm{B},h}\,\delta_{b\bar{a}}, \quad W^{\hat{A}}_{\mathrm{B},h} \equiv W^{\bar{a}b}_{\mathrm{B},h}\,t^{\hat{A}}_{b\bar{a}},$$
$$Q_{h+\frac{1}{2}} \equiv Q^{\bar{a}\alpha}_{h+\frac{1}{2}}\,\delta_{\alpha\bar{a}}, \quad Q^{\hat{A}}_{h+\frac{1}{2}} \equiv Q^{\bar{a}\alpha}_{h+\frac{1}{2}}\,t^{\hat{A}}_{\alpha\bar{a}},$$
$$\bar{Q}_{h+\frac{1}{2}} \equiv \bar{Q}^{a\bar{\alpha}}_{h+\frac{1}{2}}\,\delta_{a\bar{\alpha}}, \quad \bar{Q}^{\hat{A}}_{h+\frac{1}{2}} \equiv \bar{Q}^{a\bar{\alpha}}_{h+\frac{1}{2}}\,t^{\hat{A}}_{a\bar{\alpha}},$$
$$\hat{A} = 1, 2, \ldots, (K^2-1). \tag{2.9}$$

[4] This paragraph is based on the discussion with S. Odake some years ago.

Now we would like to rewrite down (2.1) in terms of (2.9) after multiplying the various generators [5].

2.3.1 The $W^{K}_{1+\infty}$ algebra

Now we can multiply the generators into the first equation of (2.1) and the three commutator relations can be obtained as follows:

$$\big[(W_{\mathrm{F},h_1})_m, (W_{\mathrm{F},h_2})_n\big] = \sum_{h=0,\mathrm{even}}^{h_1+h_2-3} \lambda^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n)$$
$$\times (W_{\mathrm{F},h_1+h_2-2-h})_{m+n} + K\, c_{W_{\mathrm{F},h_1}}\,\delta^{h_1 h_2}\,\lambda^{2(h_1-2)}\,\delta_{m+n},$$
$$\big[(W_{\mathrm{F},h_1})_m, (W^{\hat{A}}_{\mathrm{F},h_2})_n\big] = \sum_{h=0,\mathrm{even}}^{h_1+h_2-3} \lambda^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n)$$
$$\times (W^{\hat{A}}_{\mathrm{F},h_1+h_2-2-h})_{m+n},$$
$$\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m, (W^{\hat{B}}_{\mathrm{F},h_2})_n\big] = -\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-3} \lambda^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n)$$
$$\times \frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}}\,(W^{\hat{C}}_{\mathrm{F},h_1+h_2-2-h})_{m+n}$$
$$+ c_{W_{\mathrm{F},h_1}}\,\delta^{\hat{A}\hat{B}}\,\delta^{h_1h_2}\,\lambda^{2(h_1-2)}\,\delta_{m+n}$$
$$+ \sum_{h=0,\mathrm{even}}^{h_1+h_2-3} \lambda^h\, p_{\mathrm{F}}^{h_1,h_2,h}(m,n)\,\Big[\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}$$
$$\times (W^{\hat{C}}_{\mathrm{F},h_1+h_2-2-h})_{m+n} + \frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,(W_{\mathrm{F},h_1+h_2-2-h})_{m+n}\Big]. \tag{2.10}$$

This $W^{K}_{1+\infty}$ algebra (or $\hat{SU}(K)_k\ W_{1+\infty}$ algebra) was found in [44]. The first equation of (2.10) generated by the singlet current of $U(K)$ is $W_{1+\infty}$ algebra and its extension with the adjoint of $U(K)$ appears in the remaining equations. In the last equation of (2.10), the identity for the product of two generators $t^{\hat{A}}\,t^{\hat{B}} = \frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\mathbf{1}_K + \frac{1}{2}\,(i\,f + d)^{\hat{A}\hat{B}\hat{C}}\,t^{\hat{C}}$ is used. Note that the weights in the right hand side appear in even or odd integers. Of course, by taking the contractions of the currents with the vanishing λ limit, the first equation reduces to the $w_{1+\infty}$ algebra (which is the weight-1 extension of w_∞ algebra [7]) as shown in [8].[6]

[5] From now on, we do not have to distinguish the two (anti) fundamental indices.

[6] By redefining the currents of weights 2, 3, 4 nonlinearly, we can decouple the weight-1 current from other currents. That is, there are no singular OPEs between the weight-1 current and others. The OPEs between the above currents of weights 2, 3, 4 do not contain the weight-1 current at the poles in the right hand side. The lowest pole we are considering contains the weight-4 current. For the poles having the higher spin current of spins, 5, 6, ..., we need to find out the corresponding redefined currents step by step. We expect that this will be true for higher weights. See also the relevant paper [55].

2.3.2 The W_∞^K algebra

The second equation of (2.1) can be rewritten as

$$\begin{aligned}
\big[(W_{\mathrm{B},h_1})_m,(W_{\mathrm{B},h_2})_n\big] &= \sum_{h=0,\mathrm{even}}^{h_1+h_2-4} \lambda^h\, p_{\mathrm{B}}^{h_1,h_2,h}(m,n) \\
&\times (W_{\mathrm{B},h_1+h_2-2-h})_{m+n} + K\, c_{W_{\mathrm{B},h_1}}\, \delta^{h_1 h_2}\, \lambda^{2(h_1-2)}\, \delta_{m+n},\\
\big[(W_{\mathrm{B},h_1})_m,(W^{\hat{A}}_{\mathrm{B},h_2})_n\big] &= \sum_{h=0,\mathrm{even}}^{h_1+h_2-4} \lambda^h\, p_{\mathrm{B}}^{h_1,h_2,h}(m,n) \\
&\times (W^{\hat{A}}_{\mathrm{B},h_1+h_2-2-h})_{m+n},\\
\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m,(W^{\hat{B}}_{\mathrm{B},h_2})_n\big] &= -\sum_{h=-1,\mathrm{odd}}^{h_1+h_2-4} \lambda^h\, p_{\mathrm{B}}^{h_1,h_2,h}(m,n)\, \frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}} \\
&\times (W^{\hat{C}}_{\mathrm{B},h_1+h_2-2-h})_{m+n} + c_{W_{\mathrm{B},h_1}}\, \delta^{\hat{A}\hat{B}}\, \delta^{h_1 h_2}\, \lambda^{2(h_1-2)}\, \delta_{m+n}\\
&+ \sum_{h=0,\mathrm{even}}^{h_1+h_2-4} \lambda^h\, p_{\mathrm{B}}^{h_1,h_2,h}(m,n) \Big[\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, (W^{\hat{C}}_{\mathrm{B},h_1+h_2-2-h})_{m+n}\\
&+ \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, (W_{\mathrm{B},h_1+h_2-2-h})_{m+n}\Big]. \qquad (2.11)
\end{aligned}$$

This W_∞^K algebra was found in [43]. The first equation of (2.11) generated by the singlet current of $U(K)$ is W_∞ algebra and its extension with the adjoint of $U(K)$ appears in the remaining equations. The algebraic structure of (2.11) looks similar to the one of (2.10). The upper bound of dummy variable h and the structure constants are different from each other. By taking the contractions of the currents with the vanishing λ limit, the first equation reduces to the w_∞ algebra [7].

2.3.3 The commutators between the bosonic and fermionic currents

In this case, we have four commutator relations after multiplying the generators into the third equation of (2.1)

$$\begin{aligned}
\big[(W_{\mathrm{F},h_1})_m,(Q_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r) \\
&\times (Q_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W_{\mathrm{F},h_1})_m,(Q^{\hat{A}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r) \\
&\times (Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m,(Q_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r) \\
&\times (Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m,(Q^{\hat{B}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r)\, \frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}} \\
&\times (Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} + \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times \Big[\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, (Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, (Q_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big]. \qquad (2.12)
\end{aligned}$$

2.3.4 The commutators between the other bosonic and fermionic currents

From the fifth equation of (2.1), the following four commutator relations can be obtained by multiplying the generators

$$\begin{aligned}
\big[(W_{\mathrm{B},h_1})_m,(Q_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r) \\
&\times (Q_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W_{\mathrm{B},h_1})_m,(Q^{\hat{A}}_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r) \\
&\times (Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m,(Q_{h_2+\frac{1}{2}})_r\big] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r) \\
&\times (Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m,(Q^{\hat{B}}_{h_2+\frac{1}{2}})_r\big] &\\
= -\sum_{h=-1}^{h_1+h_2-3} \lambda^h\, &q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\, \frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}} \\
&\times (Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} + \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times \Big[\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, (Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+ \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, (Q_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big]. \qquad (2.13)
\end{aligned}$$

We present here (2.12) and (2.13) which are necessary to describe the discussion of next section and the remaining (anti)commutator relations are presented in Appendix A.

2.4 Free field realization

From (2.9) and (2.7) with (2.8), we obtain the following free field realization

$$\begin{aligned}
W_{\mathrm{F},h} &= n_{W_{\mathrm{F},h}} \sum_{l=0}^{h-1} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} (-1)^l \binom{h-1}{l}^2 \\
&\times (\,\bar{\partial}^{h-l-1} \bar{\psi}^{i,\bar{\alpha}}\, \delta_{\beta\bar{\alpha}}\, \bar{\partial}^l \psi^{\bar{i},\beta}\,),
\end{aligned}$$

$$W^{\hat{A}}_{\mathrm{F},h} = n_{W_{\mathrm{F},h}} \sum_{l=0}^{h-1} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} (-1)^l \binom{h-1}{l}^2 \times (\bar{\partial}^{h-l-1} \bar{\psi}^{i,\bar{\alpha}}\, t^{\hat{A}}_{\beta\bar{\alpha}}\, \bar{\partial}^l \psi^{\bar{i},\beta}),$$

$$W_{\mathrm{B},h} = n_{W_{\mathrm{B},h}} \sum_{l=0}^{h-2} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} \frac{(-1)^l}{(h-1)} \binom{h-1}{l} \binom{h-1}{l+1} \times (\bar{\partial}^{h-l-1} \bar{\phi}^{i,\bar{a}}\, \delta_{b\bar{a}}\, \bar{\partial}^{l+1} \phi^{\bar{i},b}),$$

$$W^{\hat{A}}_{\mathrm{B},h} = n_{W_{\mathrm{B},h}} \sum_{l=0}^{h-2} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} \frac{(-1)^l}{(h-1)} \binom{h-1}{l} \binom{h-1}{l+1} \times (\bar{\partial}^{h-l-1} \bar{\phi}^{i,\bar{a}}\, t^{\hat{A}}_{b\bar{a}}\, \bar{\partial}^{l+1} \phi^{\bar{i},b}),$$

$$Q_{h+\frac{1}{2}} = n_{Q_{h+\frac{1}{2}}} \sum_{l=0}^{h-1} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} (-1)^l \binom{h-1}{l} \binom{h}{l} \times (\bar{\partial}^{h-l} \bar{\phi}^{i,\bar{a}}\, \delta_{\alpha\bar{a}}\, \bar{\partial}^l \psi^{\bar{i},\alpha}),$$

$$Q^{\hat{A}}_{h+\frac{1}{2}} = n_{Q_{h+\frac{1}{2}}} \sum_{l=0}^{h-1} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} (-1)^l \binom{h-1}{l} \binom{h}{l} \times (\bar{\partial}^{h-l} \bar{\phi}^{i,\bar{a}}\, t^{\hat{A}}_{\alpha\bar{a}}\, \bar{\partial}^l \psi^{\bar{i},\alpha}),$$

$$\bar{Q}_{h+\frac{1}{2}} = n_{\bar{Q}_{h+\frac{1}{2}}} \sum_{l=0}^{h-1} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} (-1)^{h-1+l} \binom{h-1}{l} \binom{h}{l} \times (\bar{\partial}^{h-l} \phi^{\bar{i},a}\, \delta_{a\bar{\alpha}}\, \bar{\partial}^l \bar{\psi}^{i,\bar{\alpha}}),$$

$$\bar{Q}^{\hat{A}}_{h+\frac{1}{2}} = n_{\bar{Q}_{h+\frac{1}{2}}} \sum_{l=0}^{h-1} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}} (-1)^{h-1+l} \binom{h-1}{l} \binom{h}{l} \times (\bar{\partial}^{h-l} \phi^{\bar{i},a}\, t^{\hat{A}}_{a\bar{\alpha}}\, \bar{\partial}^l \bar{\psi}^{i,\bar{\alpha}}). \qquad (2.14)$$

The number N which is contracted in (2.14) does not play any role of our discussion. In this basis, the singlet and adjoint property of $U(K)$ is clear. It is evident that the above free field realizations satisfy (2.12) and (2.13) and Appendices (A.1), (A.3) and (A.5). Except the last two of (2.14), the remaining ones will be used in next section.

2.5 The existence of $\mathcal{N}=2$ supersymmetric $w^{K,K}_{\infty}$ algebra with $U(K)\times U(K)$ symmetry?

By taking the simple rescalings

$$W_h \rightarrow \lambda\, W_h, \quad W^{\hat{A}}_h \rightarrow \lambda\, W^{\hat{A}}_h,$$
$$Q_{h+\frac{1}{2}} \rightarrow \lambda\, Q_{h+\frac{1}{2}}, \quad Q^{\hat{A}}_{h+\frac{1}{2}} \rightarrow \lambda\, Q^{\hat{A}}_{h+\frac{1}{2}}, \qquad (2.15)$$

and putting the λ to vanish, we obtain the following commutator relations

$$\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m, (W^{\hat{B}}_{\mathrm{F},h_2})_n\big] = -\frac{i}{4} f^{\hat{A}\hat{B}\hat{C}} (W^{\hat{C}}_{\mathrm{F},h_1+h_2-1})_{m+n},$$
$$\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m, (W^{\hat{B}}_{\mathrm{B},h_2})_n\big] = -\frac{i}{4} f^{\hat{A}\hat{B}\hat{C}} (W^{\hat{C}}_{\mathrm{B},h_1+h_2-1})_{m+n},$$
$$\big[(W_{\mathrm{F},h_1})_m, (Q_{h_2+\frac{1}{2}})_r\big] = -\frac{1}{4} (Q_{h_1+h_2-\frac{1}{2}})_{m+r},$$
$$\big[(W_{\mathrm{F},h_1})_m, (Q^{\hat{A}}_{h_2+\frac{1}{2}})_r\big] = -\frac{1}{4} (Q^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},$$
$$\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m, (Q_{h_2+\frac{1}{2}})_r\big] = -\frac{1}{4} (Q^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},$$
$$\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m, (Q^{\hat{B}}_{h_2+\frac{1}{2}})_r\big] = -\frac{i}{8} f^{\hat{A}\hat{B}\hat{C}} (Q^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r} -\frac{1}{4}\Big[\frac{1}{2} d^{\hat{A}\hat{B}\hat{C}} (Q^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r} + \frac{1}{K}\delta^{\hat{A}\hat{B}} (Q_{h_1+h_2-\frac{1}{2}})_{m+r}\Big],$$
$$\big[(W_{\mathrm{B},h_1})_m, (Q_{h_2+\frac{1}{2}})_r\big] = \frac{1}{4} (Q_{h_1+h_2-\frac{1}{2}})_{m+r},$$
$$\big[(W_{\mathrm{B},h_1})_m, (Q^{\hat{A}}_{h_2+\frac{1}{2}})_r\big] = \frac{1}{4} (Q^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},$$
$$\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m, (Q_{h_2+\frac{1}{2}})_r\big] = \frac{1}{4} (Q^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},$$
$$\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m, (Q^{\hat{B}}_{h_2+\frac{1}{2}})_r\big] = -\frac{i}{8} f^{\hat{A}\hat{B}\hat{C}} (Q^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r} +\frac{1}{4}\Big[\frac{1}{2} d^{\hat{A}\hat{B}\hat{C}} (Q^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r} + \frac{1}{K}\delta^{\hat{A}\hat{B}} (Q_{h_1+h_2-\frac{1}{2}})_{m+r}\Big]. \qquad (2.16)$$

There are no mode dependent terms in the right hand side. In the OPE language, the $\frac{1}{\lambda}$ term in the first order pole in the original OPEs survives. Other reduced commutator relations similar to (2.16) appear in Appendix A. Once we keep the commutators in the bosonic singlet currents, then we have some $\frac{1}{\lambda}$ dependence in other commutators.

3 The $\mathcal{N}=1$ supersymmetric W^K_{∞} algebra with $U(K)$ symmetry

3.1 The $\mathcal{N}=1$ supersymmetric W^K_{∞} algebra with $U(K)$ symmetry

3.1.1 The commutators between the bosonic currents

In [48], the bosonic current of weight h is given by the linear combination of $W_{B,h}$, $W_{F,h}$, $\bar{\partial}\, W_{B,h-1}$ and $\bar{\partial}\, W_{F,h-1}$. In terms of their modes with correct deformation parameter λ (the power of λ should be equal to $(h-2)$), we obtain the following $U(K)$-singlet and $U(K)$-adjoint currents together with (2.9)

$$(W_h)_m \equiv (W_{\mathrm{B},h})_m + (W_{\mathrm{F},h})_m + \lambda\,\frac{2(h-2)(m+(h-2)+1)}{2(h-2)+1} (W_{\mathrm{B},h-1})_m - \lambda\,\frac{(2(h-2)+2)(m+(h-2)+1)}{2(h-2)+1} (W_{\mathrm{F},h-1})_m,$$
$$(W^{\hat{A}}_h)_m \equiv (W^{\hat{A}}_{\mathrm{B},h})_m + (W^{\hat{A}}_{\mathrm{F},h})_m + \lambda\,\frac{2(h-2)(m+(h-2)+1)}{2(h-2)+1} (W^{\hat{A}}_{\mathrm{B},h-1})_m - \lambda\,\frac{(2(h-2)+2)(m+(h-2)+1)}{2(h-2)+1} (W^{\hat{A}}_{\mathrm{F},h-1})_m. \qquad (3.1)$$

According to the construction of (3.1), the lowest value for the weight h of the bosonic current $W_{B,h}$ is given by $h = 2$. When $h = 2$, the third term of each expression vanishes due to the coefficient. The $U(K)$-adjoint current is new and is the matrix generalization of [48]. The mode m dependence in the coefficients appears in the third and fourth terms when we write down the mode of derivative of current in terms of mode of the current itself.[7]

Then we can determine their commutator relations explicitly by using both (2.10) and (2.11) as follows:

$$
\begin{aligned}
\big[(W_{h_1})_m, (W_{h_2})_n\big] &= \sum_{h=0}^{h_1+h_2-4} \lambda^h\, q^{h_1,h_2,h}(m,n) \\
&\quad \times (W_{h_1+h_2-2-h})_{m+n} + c_{W,h_1}(m), \\
\big[(W_{h_1})_m, (W^{\hat{A}}_{h_2})_n\big] &= \sum_{h=0}^{h_1+h_2-4} \lambda^h\, q^{h_1,h_2,h}(m,n) \\
&\quad \times (W^{\hat{A}}_{h_1+h_2-2-h})_{m+n}, \\
\big[(W^{\hat{A}}_{h_1})_m, (W^{\hat{B}}_{h_2})_n\big] &= -\sum_{h=-1}^{h_1+h_2-4} \lambda^h\, \tilde{q}^{h_1,h_2,h}(m,n) \\
&\quad \times \frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}}\, (W^{\hat{C}}_{h_1+h_2-2-h})_{m+n} \\
&\quad + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, c_{W,h_1}(m) + \sum_{h=0}^{h_1+h_2-4} \lambda^h\, q^{h_1,h_2,h}(m,n) \\
&\quad \times \bigg[\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, (W^{\hat{C}}_{h_1+h_2-2-h})_{m+n} \\
&\quad + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, (W_{h_1+h_2-2-h})_{m+n}\bigg].
\end{aligned}
\tag{3.2}
$$

Again, the last two relations in (3.2) are new. Compared to (2.11), the algebraic structure looks similar, but the structure constants are different from each other and the range for the dummy variable h in the right hand sides is different. It is claimed in [48] that the algebra in the first relation of (3.2) is isomorphic to the W_∞ algebra [39,40].[8] Here the central terms appearing in the first and the last equations of (3.2) are given by the following expression

$$
\begin{aligned}
c_{W,h_1}(m) = K\bigg[& c_{W_{B,h_1}}(m)\, \delta^{h_1 h_2}\, \lambda^{2(h_1-2)} \\
& +\lambda\, \frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\, c_{W_{B,h_1}} \\
& \times (m)\, \delta^{h_1,h_2-1}\, \lambda^{2(h_1-2)} \\
& +\lambda\, \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, c_{W_{B,h_1-1}} \\
& \times (m)\, \delta^{h_1-1,h_2}\, \lambda^{2(h_1-1-2)} \\
& +\lambda^2\, \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1} \\
& \times \frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1} \\
& \times c_{W_{B,h_1-1}}(m)\, \delta^{h_1-1,h_2-1}\, \lambda^{2(h_1-1-2)} \\
& + c_{W_{F,h_1}}(m)\, \delta^{h_1 h_2}\, \lambda^{2(h_1-2)} \\
& -\lambda\, \frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\, c_{W_{F,h_1}} \\
& \times (m)\, \delta^{h_1,h_2-1}\, \lambda^{2(h_1-2)} \\
& -\lambda\, \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, c_{W_{F,h_1-1}} \\
& \times (m)\, \delta^{h_1-1,h_2}\, \lambda^{2(h_1-1-2)} \\
& +\lambda^2\, \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1} \\
& \times \frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1} \\
& \times c_{W_{F,h_1-1}}(m)\, \delta^{h_1-1,h_2-1}\, \lambda^{2(h_1-1-2)} \bigg]\, \delta_{m+n}.
\end{aligned}
\tag{3.3}
$$

We can check that the above expression (3.3) vanishes (topological property) by using the Kronecker delta conditions properly. The second and the sixth, the third and the seventh, and the remaining ones can be combined as the independent terms. We introduce the following structure constants

$$
\begin{aligned}
q^{h_1,h_2,h}(m,n) &\equiv q_{\mathrm{B}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) \\
&\quad + \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\mathrm{B}}^{h_1-1,h_2-\frac{1}{2},h-1} \\
&\quad \times \left(m,n+\frac{1}{2}\right) + q_{\mathrm{F}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) \\
&\quad - \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\mathrm{F}}^{h_1-1,h_2-\frac{1}{2},h-1} \\
&\quad \times \left(m,n+\frac{1}{2}\right), \\
\tilde{q}^{h_1,h_2,h}(m,n) &\equiv q_{\mathrm{B}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) \\
&\quad + \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\mathrm{B}}^{h_1-1,h_2-\frac{1}{2},h-1} \\
&\quad \times \left(m,n+\frac{1}{2}\right) - q_{\mathrm{F}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) \\
&\quad + \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\mathrm{F}}^{h_1-1,h_2-\frac{1}{2},h-1} \\
&\quad \times \left(m,n+\frac{1}{2}\right).
\end{aligned}
\tag{3.4}
$$

[7] In principle [55], we can add the weight-1 currents by reversing the procedure in the footnote 6 and obtain the $W^K_{1+\infty}$ algebra and by contractions the corresponding $w^K_{1+\infty}$ algebra can be obtained).

[8] It is known in [48] that the diagonal W_∞ algebra from the current W_h is generated in W_∞ algebra (generated by $W_{B,h}$) and $W_{1+\infty}$ algebra (generated by $W_{F,h}$).

The first relation of (3.4) was found in [48] with our convention and is more natural in the commutator relation between the bosonic current and the fermionic current in next subsection. That is the reason why there are shifts in the weight h_2 and the mode n. The second and the fourth terms have the explicit mode m dependence due to the derivative terms as we explained before. Note that there are precise relations between the structure constants (p_B, p_F) which appear in the equations (2.10) and (2.11) and the structure constants (q_F, q_B) which appear in the equations (2.12) and (2.13) at each term in the commutator relations. Their relations will appear later.

3.1.2 *The commutators between the bosonic currents and the fermionic currents*

Now we obtain the commutator relations including the fermionic currents. By using (2.12), (2.13) and (2.9), the following relations satisfy

$$
\begin{aligned}
\left[(W_{h_1})_m, (Q_{h_2+\frac{1}{2}})_r\right] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q^{h_1,h_2+1,h}\left(m, r-\frac{1}{2}\right) \\
&\times \left(Q_{h_1+h_2-\frac{3}{2}-h}\right)_{m+r}, \\
\left[(W_{h_1})_m, (Q^{\hat{A}}_{h_2+\frac{1}{2}})_r\right] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q^{h_1,h_2+1,h}\left(m, r-\frac{1}{2}\right) \\
&\times \left(Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h}\right)_{m+r}, \\
\left[(W^{\hat{A}}_{h_1})_m, (Q_{h_2+\frac{1}{2}})_r\right] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, q^{h_1,h_2+1,h}\left(m, r-\frac{1}{2}\right) \\
&\times (Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}, \\
\left[(W^{\hat{A}}_{h_1})_m, (Q^{\hat{B}}_{h_2+\frac{1}{2}})_r\right] &= \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, \hat{q}^{h_1,h_2+\frac{1}{2},h}(m,r)\, \frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}} \\
&\times (Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} + \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, \check{q}^{h_1,h_2+\frac{1}{2},h}(m,r) \\
&\times \left[\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, (Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r} + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, (Q_{h_1+h_2-\frac{3}{2}-h})_{m+r}\right] \\
&+ \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, \dot{q}^{h_1,h_2+\frac{1}{2},h}(m,r)\, \frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}}\, (Q^{\hat{C}}_{h_1-1+h_2-\frac{3}{2}-h})_{m+r} \\
&+ \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, \bar{q}^{h_1,h_2+\frac{1}{2},h}(m,r) \left[\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, (Q^{\hat{C}}_{h_1-1+h_2-\frac{3}{2}-h})_{m+r}\right. \\
&\left. + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, (Q_{h_1-1+h_2-\frac{3}{2}-h})_{m+r}\right].
\end{aligned}
\tag{3.5}
$$

Again, the first relation of (3.5) was found in [48]. The remaining ones are the matrix generalization. In the last equation of (3.5), the following structure constants are introduced by collecting each contribution

$$
\begin{aligned}
\hat{q}^{h_1,h_2,h}(m,r) &\equiv -q_{\rm B}^{h_1,h_2,h}(m,r) + q_{\rm F}^{h_1,h_2,h}(m,r), \\
\check{q}^{h_1,h_2,h}(m,r) &\equiv q_{\rm B}^{h_1,h_2,h}(m,r) + q_{\rm F}^{h_1,h_2,h}(m,r), \\
\dot{q}^{h_1,h_2,h}(m,r) &\equiv -\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\rm B}^{h_1-1,h_2,h}(m,r) \\
&- \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\rm F}^{h_1-1,h_2,h}(m,r), \\
\bar{q}^{h_1,h_2,h}(m,r) &\equiv \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\rm B}^{h_1-1,h_2,h}(m,r) \\
&- \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_{\rm F}^{h_1-1,h_2,h}(m,r).
\end{aligned}
\tag{3.6}
$$

Although we introduce the weight h_2 in the structure constant in (3.6), their appearance in (3.5) takes the form of $(h_2+\frac{1}{2})$ which is the weight of the fermionic current in the left hand side. The shift in the weight h_1 of the structure constants (q_B, q_F) in the last two of (3.6) can be understood from the derivative of the bosonic current in the left hand side. Moreover, the last three terms in the last equation of (3.5) have the weight of the first three terms minus one. We can combine the last three terms by introducing a new dummy variable $(h+1)$ with the first three terms in addition to other term. In this way, we can simplify the last equation of (3.5) further.

Then one of our main results with a deformation parameter λ is summarized by (3.2) and (3.5) together with (3.4) and (3.6). We will present the realization of this algebra in the supersymmetric Einstein–Yang–Mills theory.

3.2 Free field realization

By combining (2.9) and (2.14), we can write down the free field realization for the singlet current and the adjoint current of $U(K)$ as follows:

$$
\begin{aligned}
W_h &= W_{B,h} + W_{F,h} - \lambda\, \frac{2(h-2)}{2(h-2)+1}\, \bar{\partial}\, W_{B,h-1} \\
&+ \lambda\, \frac{2(h-2)+2}{2(h-2)+1}\, \bar{\partial}\, W_{F,h-1}, \\
&= \frac{2^{h-2}(h-1)!}{(2(h-2)+1)!!}\, \lambda^{h-2} \sum_{l=0}^{h-2} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}}\, (-1)^l \\
&\times \binom{h-2}{l} \binom{h-1}{l} \\
&\times \left(\bar{\partial}^{h-l-1} \bar{\phi}^{i,\bar{a}}\, \delta_{b\bar{a}}\, \bar{\partial}^{l+1} \phi^{\bar{i},b} + \bar{\partial}^{h-l-1} \bar{\psi}^{i,\bar{\alpha}}\, \delta_{\beta\bar{\alpha}}\, \bar{\partial}^{l} \psi^{\bar{i},\beta}\right), \\
W^{\hat{A}}_h &= W^{\hat{A}}_{B,h} + W^{\hat{A}}_{F,h} - \lambda\, \frac{2(h-2)}{2(h-2)+1}\, \bar{\partial}\, W^{\hat{A}}_{B,h-1} \\
&+ \lambda\, \frac{2(h-2)+2}{2(h-2)+1}\, \bar{\partial}\, W^{\hat{A}}_{F,h-1} \\
&= \frac{2^{h-2}(h-1)!}{(2(h-2)+1)!!}\, \lambda^{h-2} \sum_{l=0}^{h-2} \sum_{i,\bar{i}=1}^{N} \delta_{i,\bar{i}}\, (-1)^l \\
&\times \binom{h-2}{l} \binom{h-1}{l}
\end{aligned}
$$

$$\times\left(\bar{\partial}^{h-l-1}\bar{\phi}^{i,\bar{a}}\,t^{\hat{A}}_{b\bar{a}}\,\bar{\partial}^{l+1}\phi^{\bar{i},b}+\bar{\partial}^{h-l-1}\bar{\psi}^{i,\bar{\alpha}}\,t^{\hat{A}}_{\beta\bar{\alpha}}\,\bar{\partial}^{l}\psi^{\bar{i},\beta}\right). \tag{3.7}$$

There is some difference in the sign when we compare with the result of [48] because this comes from the fact that we are using different normalization in the Footnote 3.

The remaining superpartner currents come from (2.14) as follows:

$$\begin{aligned}Q_{h+\frac{1}{2}}&=\frac{2^{h-\frac{1}{2}}h!}{(2h-1)!!}\lambda^{h-1}\sum_{l=0}^{h-1}\sum_{i,\bar{i}=1}^{N}\delta_{i,\bar{i}}(-1)^l\binom{h-1}{l}\binom{h}{l}\\&\quad\times(\bar{\partial}^{h-l}\bar{\phi}^{i,\bar{a}}\,\delta_{\alpha\bar{a}}\,\bar{\partial}^{l}\psi^{\bar{i},\alpha}),\\Q^{\hat{A}}_{h+\frac{1}{2}}&=\frac{2^{h-\frac{1}{2}}h!}{(2h-1)!!}\lambda^{h-1}\sum_{l=0}^{h-1}\sum_{i,\bar{i}=1}^{N}\delta_{i,\bar{i}}(-1)^l\binom{h-1}{l}\binom{h}{l}\\&\quad\times(\bar{\partial}^{h-l}\bar{\phi}^{i,\bar{a}}\,t^{\hat{A}}_{\alpha\bar{a}}\,\bar{\partial}^{l}\psi^{\bar{i},\alpha}).\end{aligned} \tag{3.8}$$

In principle, we can check the previous commutator relations by using these free field realizations, calculating the various OPEs and rewriting down these in terms of commutator relations. The power of deformation parameter is given by the weight minus 2 or $\frac{3}{2}$. As in the abstract, the results of (3.7) and (3.8) are the matrix generalization of previous work of [48].

3.3 The $\mathcal{N}=1$ supersymmetric w^K_∞ algebra with $U(K)$ symmetry

In order to keep the nonzero lowest power of the deformation parameter in each term of (3.2) and (3.5), we consider the scales for the currents with the deformation parameter whose power depends on the weights. According to the following transformations,

$$\begin{aligned}&W_h\to\lambda^{h-2}\,W_h,\quad W^{\hat{A}}_h\to\lambda^h\,W^{\hat{A}}_h,\\&Q_{h+\frac{1}{2}}\to\lambda^{h+\frac{1}{2}-2}\,Q_{h+\frac{1}{2}},\quad Q^{\hat{A}}_{h+\frac{1}{2}}\to\lambda^{h+\frac{1}{2}}\,Q^{\hat{A}}_{h+\frac{1}{2}},\end{aligned} \tag{3.9}$$

we obtain the following the $\mathcal{N}=1$ supersymmetric $w^K_{1+\infty}$ algebra with $U(K)$ symmetry after redefining as (3.9) and taking $\lambda\to 0$ limit with the help of (3.4) and (3.6)

$$\begin{aligned}&\left[(W_{h_1})_m,(W_{h_2})_n\right]\to q^{h_1,h_2,0}(m,n)\,(W_{h_1+h_2-2})_{m+n}\\&\quad=\left[m(h_2-1)-n(h_1-1)\right](W_{h_1+h_2-2})_{m+n},\\&\left[(W_{h_1})_m,(W^{\hat{A}}_{h_2})_n\right]\to q^{h_1,h_2,0}(m,n)\,(W^{\hat{A}}_{h_1+h_2-2})_{m+n}\\&\quad=\left[m(h_2-1)-n(h_1-1)\right](W^{\hat{A}}_{h_1+h_2-2})_{m+n},\\&\left[(W^{\hat{A}}_{h_1})_m,(W^{\hat{B}}_{h_2})_n\right]\to -\tilde{q}^{h_1,h_2,-1}(m,n)\,\frac{i}{2}\,f^{\hat{A}\hat{B}\hat{C}}\,(W^{\hat{C}}_{h_1+h_2-1})_{m+n}\\&\quad=-\frac{i}{4}\,f^{\hat{A}\hat{B}\hat{C}}\,(W^{\hat{C}}_{h_1+h_2-1})_{m+n},\\&\left[(W_{h_1})_m,(Q_{h_2+\frac{1}{2}})_r\right]\to q^{h_1,h_2+1,0}\left(m,r-\frac{1}{2}\right)(Q_{h_1+h_2-\frac{3}{2}})_{m+r}\\&\quad=\left[m(h_2+1-1)-\left(r-\frac{1}{2}\right)(h_1-1)\right](Q_{h_1+h_2-\frac{3}{2}})_{m+r},\\&\left[(W_{h_1})_m,(Q^{\hat{A}}_{h_2+\frac{1}{2}})_r\right]\to q^{h_1,h_2+1,0}\left(m,r-\frac{1}{2}\right)(Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}})_{m+r}\\&\quad=\left[m(h_2+1-1)-\left(r-\frac{1}{2}\right)(h_1-1)\right](Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}})_{m+r},\\&\left[(W^{\hat{A}}_{h_1})_m,(Q_{h_2+\frac{1}{2}})_r\right]\to q^{h_1,h_2+1,0}\left(m,r-\frac{1}{2}\right)(Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}})_{m+r}\\&\quad=\left[m(h_2+1-1)-\left(r-\frac{1}{2}\right)(h_1-1)\right](Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}})_{m+r},\\&\left[(W^{\hat{A}}_{h_1})_m,(Q^{\hat{B}}_{h_2+\frac{1}{2}})_r\right]\to\hat{q}^{h_1,h_2+\frac{1}{2},-1}(m,r)\,\frac{i}{2}\,f^{\hat{A}\hat{B}\hat{C}}\\&\quad\times(Q^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r}=-\frac{i}{4}\,f^{\hat{A}\hat{B}\hat{C}}\,(Q^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r}.\end{aligned} \tag{3.10}$$

Due to the weight $(h_2+\frac{1}{2})$ for the fermionic current rather than $(h_2+\frac{3}{2})$, the shift in h_2 in the right hand side appears. Note that the structure constant $q^{h_1,h_2+1,-1}(m,r-\frac{1}{2})$ vanishes. In other words, the lowest dummy variable h in the first three equations of (3.5) starts with $h=0$. Moreover, the structure constant $\check{q}^{h_1,h_2+\frac{1}{2},-1}(m,r)$ in the last equation of (3.5) vanishes. This algebra (3.10) with a rescaling of the structure constant $f^{\hat{A}\hat{B}\hat{C}}$ was found in [8] previously. The point here is that the present description is more transparent because the last three commutator relations of (3.10) in [8] are introduced abstractly but in this paper we prove that they can be obtained from the above $\mathcal{N}=1$ supersymmetric W^K_∞ algebra by taking the vanishing λ limit. Therefore, at least, the celestial holography between the above two-dimensional symmetry algebra and the OPEs [49] from the supersymmetric Einstein–Yang–Mills theory holds at vanishing λ limit.

3.4 The seven OPEs

In order to present the above commutator relations in terms of OPEs, we need to introduce the following quantity [40]

$$\begin{aligned}M^{h_1,h_2}_h(m,n)&\equiv\sum_{k=0}^{h+1}(-1)^k\binom{h+1}{k}(2h_1-h-2)_k\\&\quad\times[2h_2-2-k]_{h+1-k}\,m^{h+1-k}\,n^k.\end{aligned} \tag{3.11}$$

The degree of this polynomial is given by $(h+1)$. Then the above seven commutator relations can be written in terms of the OPEs (see also [50])

$$W_{h_1}(\bar{z})\, W_{h_2}(\bar{w}) = \sum_{h=0}^{h_1+h_2-4} \lambda^h\, (-1)^{h-1}\, f^{h_1,h_2,h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{W_{h_1+h_2-2-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots,$$

$$W_{h_1}(\bar{z})\, W^{\hat{A}}_{h_2}(\bar{w}) = \sum_{h=0}^{h_1+h_2-4} \lambda^h\, (-1)^{h-1}\, f^{h_1,h_2,h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{W^{\hat{A}}_{h_1+h_2-2-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots,$$

$$W^{\hat{A}}_{h_1}(\bar{z})\, W^{\hat{B}}_{h_2}(\bar{w}) = \sum_{h=-1}^{h_1+h_2-4} \lambda^h\, (-1)^{h-1}\, \tilde{f}^{h_1,h_2,h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{-\frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}}\, W^{\hat{C}}_{h_1+h_2-2-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots, + \sum_{h=0}^{h_1+h_2-4} \lambda^h\, (-1)^{h-1}\, f^{h_1,h_2,h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, W^{\hat{C}}_{h_1+h_2-2-h}(\bar{w}) + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, W_{h_1+h_2-2-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots,$$

$$W_{h_1}(\bar{z})\, Q_{h_2+\frac{1}{2}}(\bar{w}) = \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, (-1)^{h-1}\, f^{h_1,h_2+1,h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{Q_{h_1+h_2-\frac{3}{2}-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots,$$

$$W_{h_1}(\bar{z})\, Q^{\hat{A}}_{h_2+\frac{1}{2}}(\bar{w}) = \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, (-1)^{h-1}\, f^{h_1,h_2+1,h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots,$$

$$W^{\hat{A}}_{h_1}(\bar{z})\, Q_{h_2+\frac{1}{2}}(\bar{w}) = \sum_{h=-1}^{h_1+h_2-3} \lambda^h\, (-1)^{h-1} \times \left[f^{h_1,h_2+1,h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}})\, \frac{Q^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots,$$

$$W^{\hat{A}}_{h_1}(\bar{z})\, Q^{\hat{B}}_{h_2+\frac{1}{2}}(\bar{w}) = \sum_{h=-1}^{h_1+h_2-4} \lambda^h\, (-1)^{h-1}\, \hat{f}^{h_1,h_2+\frac{1}{2},h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{\frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}}\, Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \sum_{h=0}^{h_1+h_2-4} \lambda^h\, (-1)^{h-1}\, \check{f}^{h_1,h_2+\frac{1}{2},h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, Q^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h}(\bar{w}) + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, Q_{h_1+h_2-\frac{3}{2}-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \sum_{h=-1}^{h_1+h_2-4} \lambda^h\, (-1)^{h}\, \dot{f}^{h_1,h_2+\frac{1}{2},h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{\frac{i}{2}\, f^{\hat{A}\hat{B}\hat{C}}\, Q^{\hat{C}}_{h_1-1+h_2-\frac{3}{2}-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \sum_{h=0}^{h_1+h_2-4} \lambda^h\, (-1)^{h}\, \bar{f}^{h_1,h_2+\frac{1}{2},h}(\bar{\partial}_{\bar{z}}, \bar{\partial}_{\bar{w}}) \times \left[\frac{\frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, Q^{\hat{C}}_{h_1-1+h_2-\frac{3}{2}-h}(\bar{w}) + \frac{1}{K}\, \delta^{\hat{A}\hat{B}}\, Q_{h_1-1+h_2-\frac{3}{2}-h}(\bar{w})}{(\bar{z}-\bar{w})}\right] + \cdots. \tag{3.12}$$

The various differential operators coming from the structure constants act on the two complex coordinates $(\bar{z}, \bar{w})$. The currents in the right hand sides do depend on the coordinate $\bar{w}$.

From the structure constants where the quantity $N_h^{h_1,h_2}(m,n)$ in (2.4) is replaced by the quantity $M_h^{h_1,h_2}(m,n)$ in (3.11)

$$f_{\mathrm{F}}^{h_1,h_2,h}(m,r) \equiv \frac{(-1)^h}{4(h+2)!}\Big[(h_1-1)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,0) - (h_1-h-3)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,-\tfrac{1}{2})\Big]\, M_h^{h_1,h_2}(m,r),$$

$$f_{\mathrm{B}}^{h_1,h_2,h}(m,r) \equiv \frac{-1}{4(h+2)!}\Big[(h_1-h-2)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,0) - (h_1)\,\phi_{h+1}^{h_1,h_2+\frac{1}{2}}(0,-\tfrac{1}{2})\Big]\, M_h^{h_1,h_2}(m,r), \tag{3.13}$$

the previous structure constants together with (3.13) can be expressed as follows:

$$f^{h_1,h_2,h}(m,n) \equiv f_{\mathrm{B}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) + \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, f_{\mathrm{B}}^{h_1-1,h_2-\frac{1}{2},h-1} \times \left(m,n+\frac{1}{2}\right) + f_{\mathrm{F}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) - \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, f_{\mathrm{F}}^{h_1-1,h_2-\frac{1}{2},h-1} \times \left(m,n+\frac{1}{2}\right),$$

$$\tilde{f}^{h_1,h_2,h}(m,n) \equiv f_{\mathrm{B}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) + \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, f_{\mathrm{B}}^{h_1-1,h_2-\frac{1}{2},h-1} \times (m,n+\frac{1}{2}) - f_{\mathrm{F}}^{h_1,h_2-\frac{1}{2},h}\left(m,n+\frac{1}{2}\right) + \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, f_{\mathrm{F}}^{h_1-1,h_2-\frac{1}{2},h-1} \times \left(m,n+\frac{1}{2}\right),$$

$$\hat{f}^{h_1,h_2,h}(m,r) \equiv -f_{\mathrm{B}}^{h_1,h_2,h}(m,r) + f_{\mathrm{F}}^{h_1,h_2,h}(m,r),$$

$$\check{f}^{h_1,h_2,h}(m,r) \equiv f_{\mathrm{B}}^{h_1,h_2,h}(m,r) + f_{\mathrm{F}}^{h_1,h_2,h}(m,r),$$

$$\dot{f}^{h_1,h_2,h}(m,r) \equiv -\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1} \times f_{\mathrm{B}}^{h_1-1,h_2,h}(m,r) - \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, f_{\mathrm{F}}^{h_1-1,h_2,h}(m,r),$$

$$\bar{f}^{h_1,h_2,h}(m,r) \equiv \frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1} \times f_{\mathrm{B}}^{h_1-1,h_2,h}(m,r) - \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1} f_{\mathrm{F}}^{h_1-1,h_2,h}(m,r). \tag{3.14}$$

In Appendix B, we present the explicit OPEs between the currents W_4, $W_4^{\hat{A}}$, $Q_{\frac{7}{2}}$ and $Q_{\frac{7}{2}}^{\hat{A}}$. On the one hand, from the construction of free field realization, the OPEs can be obtained from either by hands or by Thielemans package [52] for fixed weights h_1 and h_2. On the other hand, we can write down (3.12) explicitly by substituting the structure constants which are differential operators into (3.12). The crucial point here is that from the transformation of (3.14) to the ones in (3.12) where $m \rightarrow \bar{\partial}_{\bar{z}}$ and $n, r \rightarrow \bar{\partial}_{\bar{w}}$, we should keep only the terms having a degree $(h+1)$ of the polynomial. In general, the structure constants in (3.14) do have the terms having a power sum of two variables less than the above $(h+1)$ of the polynomial, contrary to the case of (3.13). We will provide some details in Appendix B.

3.5 The possible realization in the $\mathcal{N}=1$ supersymmetric Einstein–Yang–Mills theory

It is known, in [5], that the leading OPE for two positive helicity gluons is given by the simple pole in the holomorphic sector with Euler beta function whose arguments depend on the two conformal weights appearing on the left hand side. They consider the scattering states in MHV (Maximally Helicity Violating) tree amplitudes. The celestial amplitudes can be written as the massless n particle amplitudes, which depend on the energies and the points on the celestial sphere, in the Mellin space. These amplitudes can be interpreted as correlation functions of n primary operators on the celestial sphere. The simplest nontrivial scattering process occurs in the $n=4$ gluons. Then the leading order behavior for the above massless four particle amplitudes can be obtained by taking the holomorphic collinear limit for the positive helicity (outgoing) gluons. Then it is possible to relate the full $n=4$ amplitude to $n=3$ amplitude and the corresponding celestial four point correlation function can be expressed as an integral over some integral parameter where the integrand contains the three point correlation function. Moreover the OPE of two positive helicity (outgoing) gluons can be described as a conformal block including all the antiholomorphic descendants. Then the leading OPE for two positive helicity gluons we mentioned before can be further simplified after performing a Taylor expansion. Finally, the OPE of the conformally soft gluon operators takes the simple pole in the holomorphic sector with finite sum of antiholomorphic derivatives acting on the second gluon appearing on the left hand side of the OPE. In this soft limit, the poles appearing in the OPE coefficients (Euler beta function) disappear and the rescaled soft gluon operators with specific weights occur in the OPE.

For the supersymmetric Einstein–Yang–Mills theory which is a generalization of previous paragraph, from the collinear limit of the respective Feynman matrix element, the OPEs can be obtained by performing the Mellin transforms [49]. For example, the OPE of the conformal primary graviton and the conformal primary gravitino of arbitrary weights takes the simple pole in the holomorphic sector and the OPE coefficient is given by Euler beta function which depends on the two previous weights on the left hand side of the OPE. By taking the soft limit, this OPE between the soft positive helicity graviton and the soft positive helicity gravitino can be written in terms of binomial coefficient which depends on the two weights of the soft operators on the left hand side and dummy variable for the finite sum for the antiholomorphic derivatives acting on the soft positive helicity gravitino [8]. The structure of this OPE looks like the one in previous case between two soft gluon operators in the sense that the numerical values appearing in the binomial coefficient and the power of the difference in the antiholomorphic coordinates are little different.

The MHV gluon amplitudes in previous paragraph are the simplest amplitudes in Yang–Mills theory. The next-to-simplest amplitudes, Next-to-MHV or NMHV sector is studied in [20] based on the non minimal couplings of gluons and gravitons by following the work of [32]. From the six-point NMHV analysis, the amplitude is no longer a finite polynomial in the complex coordinates (and its complex conjugated ones) of the soft particle. This leads to the fact that the lower limits of the holomorphic and the antiholomorphic mode expansions of the soft graviton, gravitino, gluon and gluino are given by $-\infty$ instead of finite values in MHV sector. Furthermore, the upper limits in the dummy variable appearing on the right hand sides of the OPEs between these soft operators are not finite values but ∞. This allows us to obtain the mode dependent function, which also depends on the three weights, when we express the OPEs in terms of the commutators between the soft operators.

According to [20,32], the OPE between the soft positive helicity graviton, where the weights are $h_1 = \frac{k-2}{2}$ and $h_2 = \frac{l-2}{2}$, is given by, after taking the soft limit,

$$H^k(z_1,\bar{z}_1)\, H^l(z_2,\bar{z}_2) = -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{n=0}^{\infty} \binom{2-2h-k-l-n}{1-h-l} \times \frac{\bar{z}_{12}^{n+h+1}}{n!}\, \bar{\partial}^n\, H^{k+l+h}(z_2,\bar{z}_2) + \cdots . \tag{3.15}$$

We can check the weights in the antiholomorphic sector both sides.[9] The left hand side has the weight $\frac{k-2}{2}+\frac{l-2}{2}$ while the right hand side has the weight $-(n+h+1)+n+\frac{k+l-2}{2}+h$ from the last three factors. We observe that for the n and h dependences in (3.15), there are no additional weight contributions. In other words, they are cancelled each other and the inclusion of h does not change the weight in the form of (3.15). Note that there exists the h-dependence on the right hand side of (3.15) and for $h=0$, we reproduce the result of [5] with proper upper limit for the summation variable n. It turns out that by performing the explicit contour integrals presented in [20], the following commutator relation from (3.15) is satisfied[10]

$$\begin{aligned}&\big[(\hat{w}_{h_1})_m,(\hat{w}_{h_2})_n\big]\\&\quad=(-1)^{h+1}\,N_h^{h_1,h_2}(m,n)\,(\hat{w}_{h_1+h_2-2-h})_{m+n}.\end{aligned}\tag{3.16}$$

Here $\hat{w}_h$ is the rescaled weight-h current and does not depend on the complex coordinate [8,20]. The mode dependent function $N_h^{h_1,h_2}(m,n)$ is given by the equation (2.4).

By using the two relations in (3.15) and (3.16) with associated other relations for the gravitino, gluon and gluino, we would like to construct the OPEs in the supersymmetric Einstein–Yang–Mills theory related to the previous seven commutator relations. Note how the weights (h_1,h_2) and the modes (m,n) of the left hand side appear on the right hand side of (3.16).

3.5.1 *The OPE between the soft positive helicity graviton and the soft positive helicity gravitino*

At vanishing deformation parameter, the OPE of the conformal primary graviton and the conformal primary gravitino of arbitrary weights is given by [49].[11] Then the question is how the contributions from the nonzero deformation parameter occur on the right hand side of this OPE.

(i) At least, we should have the OPE structure found by [32] which is the fact that the OPE contains the h dependence in the $\bar{z}_{12}$, the binomial coefficient and the weight of the operator appearing on the right hand side. These OPE coefficients are also obtained from the analysis in the collinear limits. See also the equation (3.2) of [32] in which the addition of fermions is also valid.

(ii) As mentioned before (in the non minimal couplings of gluons and gravitons), after taking the soft limit for the OPE in (i) in order to absorb the infinite number of poles appearing in the binomial coefficient, we also require that there is no restriction on the lower limits of the holomorphic and antiholomorphic mode expansions of the soft operators [20]. See also the equation $(A.17)$ of [20].

(iii) We also should sum over all the contributions from each fixed h on the right hand side of the OPE. Moreover, due to the construction of $\mathcal{N}=1$ supersymmetric theory in previous section, we should also consider the contributions from the antiholomorphic derivatives acting on the soft currents on the left hand sides of the OPEs. In some sense, we make the supersymmetric generalization of the work of [20].

Let us consider the case where one of the soft current on the left hand side in the OPE contains the fermionic current.

First of all, the OPE between the conformally soft positive helicity gravitons and the conformally soft positive helicity gravitinos, where the weights in the antiholomorphic sector are given by $h_1=\frac{k-2}{2}$ and $h_2=\frac{l-\frac{3}{2}}{2}$, can be generalized to the following expression

$$\begin{aligned}&H^k(z_1,\bar{z}_1)\,I^l(z_2,\bar{z}_2)=-\frac{\kappa}{2}\,\frac{1}{z_{12}}\sum_{h=0}^{h_1+h_2-3}(-1)^{h+1}\,\lambda^h\\&\times\Big[q_B^{h_1,h_2+\frac{1}{2},h}+q_F^{h_1,h_2+\frac{1}{2},h}\Big]\\&\times\sum_{n=0}^{\infty}\binom{\frac{3}{2}-2h-k-l-n}{\frac{1}{2}-h-l}\frac{\bar{z}_{12}^{n+h+1}}{n!}\,\bar{\partial}^n\,I^{k+l+h}(z_2,\bar{z}_2)\\&-\frac{\kappa}{2}\,\frac{1}{z_{12}}\sum_{h=0}^{h_1+h_2-3}(-1)^h\,\lambda^h\Big[\frac{2(h_1-2)}{2(h_1-2)+1}\,q_B^{h_1-1,h_2+\frac{1}{2},h-1}\\&-\frac{2(h_1-2)+2}{2(h_1-2)+1}\,q_F^{h_1-1,h_2+\frac{1}{2},h-1}\Big]\end{aligned}$$

[9] We denote the antiholomorphic weights as h_1,h_2 and h without barred notation by taking the notations in previous sections. For the notation of derivative, we are using the barred notation as in $\bar{\partial}_{\bar{z}}$. Taking the holomorphic and antiholomorphic expansions for the above soft graviton current we obtain: $H^k(z,\bar{z})=\sum_{n=-\infty}^{\frac{2-k}{2}}\frac{H_n^k(z)}{\bar{z}^{n+\frac{k-2}{2}}}=\sum_{m=-\infty}^{\frac{-2-k}{2}}\sum_{n=-\infty}^{\frac{2-k}{2}}\frac{H_{m,n}^k}{z^{m+\frac{k+2}{2}}\,\bar{z}^{n+\frac{k-2}{2}}}$. The operator $H_{m,n}^k$ is therefore independent of z and $\bar{z}$ and we focus on the case where the mode m is equal to $(1-h)$: $\hat{H}_n^k\equiv H_{m=-\frac{k}{2},n}^k$.

[10] In [20], there are two important identities around their equations (3.8) and (3.10). Their p corresponds to our $(h+1)$. In their equation (3.8), we see that the summation over dummy variable with the upper bound p reflects the mode dependent function $N_h^{h_1,h_2}(m,n)$ (2.4). They manage to express the infinite sum over their α variable in terms of a product of binomial coefficients. The final result is given by their equation (3.16) after changing the nontrivial transformation in the mode expansion [6]. Note that the general expression for the binomial coefficient is given by $\binom{-2h_1-2h_2-2(1+h)-n}{-2h_2-(1+h)}$ and their results will be used later.

[11] We have $\mathcal{O}_{\Delta_1,+2}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2)=-\frac{\frac{\kappa}{2}}{z_{12}}\sum_{n=0}^{\infty}B(\Delta_1-1+n,\Delta_2-\frac{1}{2})\,\frac{\bar{z}_{12}^{n+1}}{n!}\,\bar{\partial}^n\,\mathcal{O}_{\Delta_1+\Delta_2,+\frac{3}{2}}(z_2,\bar{z}_2)+\cdots$.

$$\times \sum_{n=0}^{\infty} \begin{pmatrix} \frac{3}{2}-2(h-1)-(k-2)-l-n \\ \frac{1}{2}-(h-1)-l \end{pmatrix} \frac{\bar{\partial}_{\bar{z}_1}\, \bar{z}_{12}^{n+(h-1)+1}}{n!}$$
$$\times \bar{\partial}^n\, I^{(k-2)+l+(h-1)}(z_2,\bar{z}_2)$$
$$+\cdots . \tag{3.17}$$

The numerical factor $\frac{3}{2}$ in the first line of the first binomial coefficient is given by $(2+\frac{3}{2})$ (which is the sum of numerical numbers with minus sign in the numerator of previous weights) minus 2. The numerical factor $\frac{1}{2}$ in the second line of the first binomial coefficient is given by $\frac{3}{2}$ (which is the numerical number for the gravitino with minus sign in the numerator of previous weights) minus 1.[12] The first binomial coefficient above can be obtained from (3.15) by taking the shift $l \rightarrow (l+\frac{1}{2})$.

In the above, we multiplied the factor $(-1)^{h+1}$ in order to absorb $(-1)^{h+1}$ factor on the right hand side of (3.16). Here the graviton current is replaced by the gravitino current on the left hand side of the OPE in (3.15).[13] Now we sum over the dummy variable h from 0 to (h_1+h_2-3) together with the structure constant, which depends on h_1, h_2 and h only without mode-dependent factor in (2.3), in order to obtain the first two lines of (3.17). In other words, from the first equation of (3.5) and (3.4), the full structure constant contains $q_B^{h_1,h_2+\frac{1}{2},h}(m,n)+q_F^{h_1,h_2+\frac{1}{2},h}(m,n)$, which is redefined as $(q_B^{h_1,h_2+\frac{1}{2},h}+q_F^{h_1,h_2+\frac{1}{2},h})\, N_h^{h_1,h_2+\frac{1}{2}}(m,n)$ together with (2.3). Then we can apply the property of (3.16) to the commutator, the first equation of (3.5), and arrive at the first two lines of above OPE. Here the mode-independent factor, $(q_B^{h_1,h_2+\frac{1}{2},h}+q_F^{h_1,h_2+\frac{1}{2},h})$, can be combined to other h-dependent factor, $(-1)^{h+1}\,\lambda^h$, inside the h summation. We observe that the weights in the antiholomorphic sector are preserved at both sides because the powers of the dummy variable n and the weight h are the same as before.

In the third and fourth lines of (3.17), we use the property between the mode of current and the current itself in (3.1) and (3.7). Note that there exists minus sign between them. Recall the remaining full structure constant contains $\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_B^{h_1-1,h_2+\frac{1}{2},h-1}(m,n) - \frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\, q_F^{h_1-1,h_2+\frac{1}{2},h-1}(m,n)$. As we did before, we factorize the mode-dependent term $(m+(h_1-2)+1)\, N_{h-1}^{h_1-1,h_2+\frac{1}{2}}(m,n)$ and the other mode-independent terms. The partial derivative $\bar{\partial}_{\bar{z}_1}$ of $H^{k-2}(z_1,\bar{z}_1)$ on the left hand side appears in the form of $\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}$ on the right hand side of the OPE. Further shifts in the weights h_1 and h in the structure constant affect the shifts in the corresponding the k and the weight h in the OPE. Note that the weight in the antiholomorphic sector in the last line of (3.17) behaves correctly because two additional weight 2 in the derivative term $\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}$ is cancelled by the corresponding additional -2 coming from the factors $(k-2)$ and $(h-1)$, compared to the second line of (3.17).

Then by performing the contour integrals as in [20] (See also [5,8]), the above OPE (3.17), from the analysis of two previous paragraphs, provides the first equation of (3.5) precisely.[14] Therefore, we have determined the OPE between the soft positive helicity graviton and the soft positive helicity gravitino which can be obtained from (or leads to) the corresponding commutator in two dimensions studied in previous section. We present the details for other OPEs including the soft positive helicity gravitinos or soft positive helicity gluinos in Appendix C explicitly where other structure constants (3.6) can be used appropriately.

3.5.2 The OPE between the soft positive helicity gravitons

Let us consider the second example, which is more nontrivial, for the appearance of the two dimensional symmetry algebra in the four dimensional Einstein–Yang–Mills theory. Based on the three features (i), (ii) and (iii), in previous subsection, we can write down the corresponding OPE for soft positive helicity graviton in the supersymmetric Einstein–Yang–Mills theory, by comparing (3.16) with the first equation of (3.2), as follows:

$$H^k(z_1,\bar{z}_1)\, H^l(z_2,\bar{z}_2) = -\frac{\kappa}{2}\,\frac{1}{z_{12}} \sum_{h=1,\mathrm{odd}}^{h_1+h_2-4} (-1)^h\,\lambda^h$$

12 Or we can use the formula in the Footnote 10.

13 That is, $H^k(z_1,\bar{z}_1)\, I^l(z_2,\bar{z}_2) = -\frac{\kappa}{2}\,\frac{1}{z_{12}} \sum_{n=0}^{\infty} \begin{pmatrix} \frac{3}{2}-2h-k-l-n \\ \frac{1}{2}-h-l \end{pmatrix} \frac{\bar{z}_{12}^{n+h+1}}{n!}\, \bar{\partial}^n\, I^{k+l+h}(z_2,\bar{z}_2)+\cdots$. For $h=0$ with proper upper limit of dummy variable n, we observe the result of [8].

14 In an expression of $\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}$, we can write down this as $(n+(h-1)+1)\,\bar{z}_{12}^{n+(h-1)}$ after a differentiation. Compared to the one without a derivative, there is an extra factor $(n+(h-1)+1)$ with different power of $\bar{z}_{12}$. Now we can combine this with the binomial coefficient $\begin{pmatrix} n+(h-1) \\ -m-h_1 \end{pmatrix}$ appearing in the $\bar{z}_1$ integral when we calculate the commutator from the OPE. Then it is easy to check that the following relation is satisfied $(n+(h-1)+1)\begin{pmatrix} n+(h-1) \\ -m-h_1 \end{pmatrix} = -(m+h_1-1)\left[\frac{(n+(h-1)+1)!}{(n+(h-1)+m+h_1)!(1-m-h_1)!}\right] = -(m+h_1-1)\left[\frac{(n+(h-1)+1)!}{(n+(h-1)+1+m+h_1)!(-m-h_1)!}\right]\Big|_{h_1\rightarrow h_1-1}$. This implies that the remaining calculation is the same as the one in [20] with an overall factor $-(m+h_1-1) = -(m+(h_1-2)+1)$ which appears in the second term of the first structure constant in (3.4) as we expected. Of course, the factor $(-\bar{z}_2)^{n+(h-1)+m+h_1}$ coming from the $\bar{z}_1$ integral can be written as $(-\bar{z}_2)^{n+(h-1)+1+m+(h_1-1)} = (-\bar{z}_2)^{n+(h-1)+1+m+h_1}\Big|_{h_1\rightarrow h_1-1}$. Moreover, the binomial coefficient is $\begin{pmatrix} \frac{3}{2}-2(h-1)-(k-2)-l-n \\ \frac{1}{2}-(h-1)-l \end{pmatrix} = \begin{pmatrix} \frac{3}{2}-2(h-1)-k-l-n \\ \frac{1}{2}-(h-1)-l \end{pmatrix}\Bigg|_{h_1\rightarrow h_1-1}$. Of course, $I^{(k-2)+l+(h-1)} = I^{k+l+(h-1)}\Big|_{h_1\rightarrow h_1-1}$.

$$
\begin{aligned}
&\times\Bigg[f_1^{h_1-1,h_2,h-1}\times\sum_{n=0}^{\infty}\begin{pmatrix}2-2(h-1)-(k-2)-l-n\\1-(h-1)-l\end{pmatrix}\\
&\times\frac{\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}}{n!}\,\bar{\partial}^n\,H^{(k-2)+l+(h-1)}(z_2,\bar{z}_2)\\
&+f_2^{h_1,h_2-1,h-1}\sum_{n=0}^{\infty}\begin{pmatrix}2-2(h-1)-k-(l-2)-n\\1-(h-1)-(l-2)\end{pmatrix}\\
&\times\frac{1}{n!}\,\bar{\partial}_{\bar{z}_2}\left[\bar{z}_{12}^{n+(h-1)+1}\,\bar{\partial}^n\,H^{k+(l-2)+(h-1)}(z_2,\bar{z}_2)\right]\Bigg]\\
&-\frac{\kappa}{2}\,\frac{1}{z_{12}}\sum_{h=0,\text{even}}^{h_1+h_2-4}(-1)^{h+1}\,\lambda^h\Bigg[f_3^{h_1,h_2,h}\\
&\times\sum_{n=0}^{\infty}\begin{pmatrix}2-2h-k-l-n\\1-h-l\end{pmatrix}\frac{\bar{z}_{12}^{n+h+1}}{n!}\,\bar{\partial}^n\,H^{k+l+h}(z_2,\bar{z}_2)\\
&+f_4^{h_1-1,h_2-1,h-2}\sum_{n=0}^{\infty}\begin{pmatrix}2-2(h-2)-(k-2)-(l-2)-n\\1-(h-2)-(l-2)\end{pmatrix}\\
&\times\frac{1}{n!}\,\bar{\partial}_{\bar{z}_1}\,\bar{\partial}_{\bar{z}_2}\left[\bar{z}_{12}^{n+(h-2)+1}\,\bar{\partial}^n\,H^{(k-2)+(l-2)+(h-2)}(z_2,\bar{z}_2)\right]\Bigg]+\cdots.
\end{aligned}
\tag{3.18}
$$

Here we introduce the following quantities

$$
\begin{aligned}
f_1^{h_1-1,h_2,h-1}&\equiv\Bigg[\frac{2(h_1-2)}{2(h_1-2)+1}\\
&\times\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h)+1}\,p_B^{h_1-1,h_2,h-1}\\
&-\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\,\frac{2(h_1-2)+2}{2(h_1-2)+1}\,p_F^{h_1-1,h_2,h-1}\Bigg],\\
f_2^{h_1,h_2-1,h-1}&\equiv\Bigg[\frac{2(h_2-2)}{2(h_2-2)+1}\,\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\\
&\times p_B^{h_1,h_2-1,h-1}-\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\\
&\frac{2(h_2-2)+2}{2(h_2-2)+1}\,p_F^{h_1,h_2-1,h-1}\Bigg],\\
f_3^{h_1,h_2,h}&\equiv\Bigg[\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\,p_B^{h_1,h_2,h}\\
&+\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\,p_F^{h_1,h_2,h}\Bigg],\\
f_4^{h_1-1,h_2-1,h-2}&\equiv\Bigg[\frac{2(h_1-2)}{2(h_1-2)+1}\,\frac{2(h_2-2)}{2(h_2-2)+1}\\
&\times\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\,p_B^{h_1-1,h_2-1,h-2}\\
&+\frac{2(h_1-2)+2}{2(h_1-2)+1}\,\frac{2(h_2-2)+2}{2(h_2-2)+1}\\
&\times\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\,p_F^{h_1-1,h_2-1,h-2}\Bigg].
\end{aligned}
\tag{3.19}
$$

Let us describe how we obtain this result. Now we return to the OPE (3.18) for the soft positive helicity gravitons. We multiply the factor $(-1)^{h+1}$ into the relation (3.15) and sum over the variable h from 0 to (h_1+h_2-4) together with the structure constant which depends on h_1, h_2 and h without mode-dependent factor in (2.3) as before. For the mode-dependent part, we want to use the previous relation (3.16). In the first three lines of (3.18), we again use the property between the mode of current and the current itself in (3.1) and (3.7). The partial derivative $\bar{\partial}_{\bar{z}_1}$ of $H^{k-2}(z_1,\bar{z}_1)$ appears in the form of $\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}$ on the right hand side of the OPE. The mode-independent coefficients come from the second and the fourth terms of the first equation in (C.2) with (3.19). Further shifts in the weights h_1 and h in the structure constant affect the shifts in the corresponding k and h. Note that the weight in the antiholomorphic sector here behaves correctly. See also the Footnote 14.

In the next line, the $\bar{\partial}_{\bar{z}_2}$ acts on both the coordinate $\bar{z}_{12}$ and $H^{k+(l-2)+(h-1)}(z_2,\bar{z}_2)$ (corresponding to $\bar{\partial}_{\bar{z}_2}$ of $H^{l-2}(z_2,\bar{z}_2)$ in the left hand side).[15] In this case, the mode-independent coefficients come from the first and the third terms of the first equation in (C.2).

For the remaining lines, we describe similarly by applying the first and the third terms in the second equation of (C.2) without any derivatives in the summation over even weight h. Finally, the second and the fourth terms in the second equation of (C.2) with two derivatives play the role in the second summation over even weight h.[16] The additional weight-2 from the two derivatives is cancelled by those from the power of $(k-2)$ and $(l-2)$ in the soft current.

[15] From the contribution of $\bar{\partial}_{\bar{z}_2}\,\bar{z}_{12}^{n'+(h-1)+1}$, there is an extra factor $-(n'+(h-1)+1)$ with $\bar{z}_{12}^{n'+(h-1)}$ where n' is a previous dummy variable n. By recalling the $\bar{z}_1$ contour integral, we have $-(n'+(h-1)+1)\begin{pmatrix}n'+(h-1)\\-m-h_1\end{pmatrix}(-1)^{n'+(h-1)+m+h_1}$ where $h_1=\frac{k-2}{2}$. Then this can be rewritten as $(n'+(h-1)+1+m+h_1)\begin{pmatrix}n'+(h-1)+1\\-m-h_1\end{pmatrix}(-1)^{n'+(h-1)+1+m+h_1}$. Note that the power of $\bar{z}_{12}$ is $n'+(h-1)$. This implies that there exists a factor $(n'+(h-1)+1+m+h_1)$ in the presence of $\bar{\partial}_{\bar{z}_2}$. Of course, the factor $(\bar{z}_2)^{n'+(h-1)+m+h_1+n+h_2-1}$ coming from the $\bar{z}_1$ integral and the exponent of $\bar{z}_2$ in the contour integral can be written as $(\bar{z}_2)^{n'+(h-1)+1+m+h_1+n+(h_2-1)-1}$ which can be obtained by taking $h_2\rightarrow(h_2-1)$ from the expression without a derivative. Furthermore, from the contribution of $\bar{\partial}_{\bar{z}_2}\,H^{k+(l-2)+(h-1)}(z_2,\bar{z}_2)$, we have the extra factor $(-m-n-h_1-(h_2-1)-(h-1)-1-n')$ (with $h_2=\frac{l-2}{2}$) which can be obtained by taking $h_2\rightarrow(h_2-1)$ from the expression without a derivative factor. The exponent of $\bar{z}_2$ is given by $(\bar{z}_2)^{n'+(h-1)+1+m+h_1+n+h_2-1}$ originally and from the above derivative $\bar{\partial}_{\bar{z}_2}$ we have an additional $\bar{z}_2^{-1}$, compared to the case without this derivative. Note that the power of $\bar{z}_{12}$ is $n'+(h-1)+1$. Therefore, the above power can be written as $n'+(h-1)+1+m+h_1+n+(h_2-1)-1$ which is obtained by taking $h_2\rightarrow h_2-1$. The binomial coefficient can be shifted similarly as in the footnote 14. Then finally we are left with the factor $[n'+(h-1)+1+m+h_1]+[-m-n-h_1-(h_2-1)-(h-1)-1-n']=-n-(h_2-1)$ by adding these two contributions.

[16] We can analyze the action of $\bar{\partial}_{\bar{z}_1}\,\bar{\partial}_{\bar{z}_2}$ appearing at the end of (3.18) by taking the procedures in the Footnote 14 and the Footnote 15.

Then by performing the contour integrals as in [20] similarly (See also [5,8]), the above OPEs (3.17) and (3.18) provide the first equations of (3.5) and (3.2) respectively. We can extract other five OPEs similarly by taking into account of the weights for the soft graviton, gravitino, gluon and gluino in the binomial coefficient above. We present the details in Appendix C explicitly. In particular, the useful structure constants $\tilde{q}^{h_1,h_2,h}(m,n)$ we are using, which appear in the third equation of (3.2), are given at the end of Appendix C. Therefore, we have found the precise correspondence between the OPEs between the conformally soft operators in the $\mathcal{N}=1$ supersymmetric Einstein–Yang–Mills theory and the two dimensional symmetry algebra.

4 Conclusions and outlook

The main result of this paper can be summarized by (3.2) and (3.5) together with (3.4) and (3.6). That is, we have determined the three commutator relations by combining the corresponding commutator relations from the bosonic currents made of the bosonic free fields with the corresponding commutator relations from the bosonic currents made of the fermionic free fields. Similarly, the remaining four commutator relations can be obtained from two different kinds of commutator relations. Their OPE version can be found in (3.12) with (3.13) and (3.14). We observe that there are also the nontrivial singular terms whose poles are greater than two and they play the role of the contributions from the deformation.

By using the celestial holography, we have obtained the results, summarized by (3.17), (3.18), Appendices (C.3), (C.4), (C.5), (C.6), and (C.7) in the supersymmetric Einstein–Yang–Mills theory at nonzero deformation parameter. The common behavior is as follows. There exist a simple pole in the holomorphic sector, the nontrivial structure constants which depend on the three weights, binomial coefficients containing the dummy variable also as well as the three weights and the descendant fields associated with the second soft currents on the left hand sides. Furthermore, after calculating the various commutator relations by using these seven OPEs between the soft currents, we have checked the above seven commutators in two dimensions discussed in previous paragraph.[17]

We have not discussed about the implications of the $\mathcal{N}=2$ supersymmetric $W_\infty^{K,K}$ algebra in the Sect. 2. It would be interesting to find out whether the possibility of the $\mathcal{N}=2$ supersymmetric $w_\infty^{K,K}$ algebra occurs or not by further examination. In the context of AdS_3/CFT_2 correspondence, the previous algebra in (2.1) is related to the case of vanishing 't Hooft-like coupling constant. Therefore, it is an open question how the another deformed case with nonvanishing 't Hooft-like coupling constant [56] will arise in the context of the present paper. We expect that the currents from the free field realization will depend on this nonzero coupling constant explicitly and they will become the currents we have described in this paper by taking this coupling constant to be zero.

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Appendix A: The remaining (anti)commutator relations in the $\mathcal{N}=2$ supersymmetric $W_\infty^{K,K}$ algebra with $U(K)\times U(K)$ symmetry

In this Appendix, we present some details which are related to the contents in Sect. 2.

[17] So far, we have focused on the soft currents as mentioned in the Footnote 1. According to the findings of [31,32], the leading tree level celestial OPEs from the cubic vertices of three spinning massless particles contains the OPE coefficients given by Euler beta function whose arguments are $(\Delta_1+s_2-s_3-1)$ and $(\Delta_2+s_1-s_3-1)$ where s_i is a helicity or spin. In particular, in [31], it is found that all the previous known seven nontrivial celestial OPEs are obtained from this general formula. We can ask whether the OPEs between the hard operators, which do not satisfy the conditions for the conformal dimensions in the footnote 1, provide any two dimensional symmetry algebra similar to the ones in this paper or not. It seems that the infinite number of poles appearing in the above Euler beta function can appear only for the corresponding conformal dimensions of the soft operators. In other words, for the conformal dimensions of the hard operators, we cannot cover the infinite number of poles and therefore we cannot remove all of them.

A.1 The commutators between the bosonic and the other fermionic currents

We can multiply the generators into the fourth equation of (2.1) and obtain the following commutator relations

$$\begin{aligned}
&\big[(W_{\mathrm{F},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times(\bar{Q}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
&\big[(W_{\mathrm{F},h_1})_m,(\bar{Q}^{\hat{A}}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m,(\bar{Q}^{\hat{B}}_{h_2+\frac{1}{2}})_r\big]=-\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times\frac{i}{2}\,f^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{F}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times\Big[\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,(\bar{Q}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big].
\end{aligned}\tag{A.1}$$

It is easy to check the following reduced commutator relations by taking (2.15)

$$\begin{aligned}
&\big[(W_{\mathrm{F},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=\frac{1}{4}\,(\bar{Q}_{h_1+h_2-\frac{1}{2}})_{m+r},\\
&\big[(W_{\mathrm{F},h_1})_m,(\bar{Q}^{\hat{A}}_{h_2+\frac{1}{2}})_r\big]=\frac{1}{4}\,(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=\frac{1}{4}\,(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{F},h_1})_m,(\bar{Q}^{\hat{B}}_{h_2+\frac{1}{2}})_r\big]=-\frac{i}{8}\,f^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r}\\
&+\frac{1}{4}\Big[\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r}+\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,(\bar{Q}_{h_1+h_2-\frac{1}{2}})_{m+r}\Big].
\end{aligned}\tag{A.2}$$

A.2 The commutators between the other bosonic and the other fermionic currents

The sixth equation of (2.1) with the addition of generators leads to the following commutator relations

$$\begin{aligned}
&\big[(W_{\mathrm{B},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\\
&\times q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\,(\bar{Q}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
&\big[(W_{\mathrm{B},h_1})_m,(\bar{Q}^{\hat{A}}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{3}{2}-h})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m,(\bar{Q}^{\hat{B}}_{h_2+\frac{1}{2}})_r\big]=\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times\frac{i}{2}\,f^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\sum_{h=-1}^{h_1+h_2-3}\lambda^h\,(-1)^h\,q_{\mathrm{B}}^{h_1,h_2+\frac{1}{2},h}(m,r)\\
&\times\Big[\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\\
&+\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,(\bar{Q}_{h_1+h_2-\frac{3}{2}-h})_{m+r}\Big].
\end{aligned}\tag{A.3}$$

By taking (2.15), the following reduced commutator relations hold

$$\begin{aligned}
&\big[(W_{\mathrm{B},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=-\frac{1}{4}\,(\bar{Q}_{h_1+h_2-\frac{1}{2}})_{m+r},\\
&\big[(W_{\mathrm{B},h_1})_m,(Q^{\hat{A}}_{h_2+\frac{1}{2}})_r\big]=-\frac{1}{4}\,(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m,(\bar{Q}_{h_2+\frac{1}{2}})_r\big]=-\frac{1}{4}\,(\bar{Q}^{\hat{A}}_{h_1+h_2-\frac{1}{2}})_{m+r},\\
&\big[(W^{\hat{A}}_{\mathrm{B},h_1})_m,(Q^{\hat{B}}_{h_2+\frac{1}{2}})_r\big]=-\frac{i}{8}\,f^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r}\\
&-\frac{1}{4}\Big[\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,(\bar{Q}^{\hat{C}}_{h_1+h_2-\frac{1}{2}})_{m+r}+\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,(\bar{Q}_{h_1+h_2-\frac{1}{2}})_{m+r}\Big].
\end{aligned}\tag{A.4}$$

A.3 The anticommutators between the fermionic currents

Finally, the seventh equation of (2.1) provides the following anticommutator relations

$$\begin{aligned}
&\{(Q_{h_1+\frac{1}{2}})_r,(\bar{Q}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1}\lambda^h\,o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s)\\
&\times(W_{\mathrm{F},h_1+h_2-h})_{r+s}+\sum_{h=0}^{h_1+h_2-2}\lambda^h\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s)\\
&\times(W_{\mathrm{B},h_1+h_2-h})_{r+s},\\
&+K\,c_{Q\bar{Q}_{h_1+\frac{1}{2}}}\,\delta^{h_1h_2}\,\lambda^{2(h_1+\frac{1}{2}-1)}\,\delta_{r+s},\\
&\{(Q_{h_1+\frac{1}{2}})_r,(\bar{Q}^{\hat{A}}_{h_2+\frac{1}{2}})_s\}=\sum_{h=0}^{h_1+h_2-1}\lambda^h\,o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s)\\
&\times(W^{\hat{A}}_{\mathrm{F},h_1+h_2-h})_{r+s}+\sum_{h=0}^{h_1+h_2-2}\lambda^h\,o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s)
\end{aligned}$$

$$\times (W^{\hat{A}}_{\mathrm{B},h_1+h_2-h})_{r+s},$$

$$\{(Q^{\hat{A}}_{h_1+\frac{1}{2}})_r, (\bar{Q}_{h_2+\frac{1}{2}})_s\} = \sum_{h=0}^{h_1+h_2-1} \lambda^h \, o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s) \times (W^{\hat{A}}_{\mathrm{F},h_1+h_2-h})_{r+s} + \sum_{h=0}^{h_1+h_2-2} \lambda^h o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s) \times (W^{\hat{A}}_{\mathrm{B},h_1+h_2-h})_{r+s},$$

$$\{(Q^{\hat{A}}_{h_1+\frac{1}{2}})_r, (\bar{Q}^{\hat{B}}_{h_2+\frac{1}{2}})_s\} = \sum_{h=0}^{h_1+h_2-1} \lambda^h \, o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s) \times \frac{i}{2} f^{\hat{A}\hat{B}\hat{C}} \, (W^{\hat{C}}_{F,h_1+h_2-h})_{r+s} + \sum_{h=0}^{h_1+h_2-1} \lambda^h \, o_{\mathrm{F}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s) \times \left[\frac{1}{2} d^{\hat{A}\hat{B}\hat{C}} \, (W^{\hat{C}}_{F,h_1+h_2-h})_{r+s} + \frac{1}{K} \delta^{\hat{A}\hat{B}} \, (W_{F,h_1+h_2-h})_{r+s}\right] - \sum_{h=0}^{h_1+h_2-2} \lambda^h \, o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s) \, \frac{i}{2} f^{\hat{A}\hat{B}\hat{C}} \, (W^{\hat{C}}_{B,h_1+h_2-h})_{r+s} + \sum_{h=0}^{h_1+h_2-2} \lambda^h \, o_{\mathrm{B}}^{h_1+\frac{1}{2},h_2+\frac{1}{2},h}(r,s) \left[\frac{1}{2} d^{\hat{A}\hat{B}\hat{C}} \, (W^{\hat{C}}_{B,h_1+h_2-h})_{r+s} + \frac{1}{K} \delta^{\hat{A}\hat{B}} \, (W_{B,h_1+h_2-h})_{r+s}\right] + c_{Q\bar{Q}_{h_1+\frac{1}{2}}} \, \delta^{\hat{A}\hat{B}} \, \delta^{h_1 h_2} \, \lambda^{2(h_1+\frac{1}{2}-1)} \delta_{r+s}. \tag{A.5}$$

Then the complete algebra consists of (2.10), (2.11), (2.12), (2.13), Appendix (A.1), Appendix (A.3) and Appendix (A.5).

Note that there are no nontrivial reduced anticommutator relations after taking (2.15). Then we have the complete results, (2.16), Appendix (A.2) and Appendix (A.4).

Appendix B: The operator product expansions in the $\mathcal{N}=1$ supersymmetric W^K_∞ algebra with $U(K)$ symmetry

In this Appendix, we present some details which are related to the contents in Sect. 3.

B.1 The seven OPEs for fixed h_1 and h_2

It is straightforward to calculate the following OPEs by using the Thielemans package [52] inside the mathematica [57]. We use the Eqs. (2.6), (3.7) and (3.8). For fixed h_1 and h_2, we perform each pole starting from the highest order pole. For each pole, we can consider the possible terms (descendant terms and new higher spin current). The higher spin currents are, in general, not quasiprimary. Although the current $W_h(\bar{w})$ for fixed h does not appear in the particular pole, its derivative term $\bar{\partial}\, W_h(\bar{w})$ appears at the next order pole. This derivative term plays the role of the quasiprimary current of the weight-$(h+1)$. Once we rearrange each OPE in terms of quasiprimary currents then the standard expressions in the right hand sides appear.[18] We have the following OPEs for fixed h_1 and h_2

$$W_4(\bar{z})\, W_4(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^6}\, \frac{9216\lambda^4}{5}\, W_2(\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^5}\, \frac{4608\lambda^4}{5}\, \bar{\partial}\, W_2(\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{1536\lambda^4}{5}\,\bar{\partial}^2\, W_2 + \frac{2304\lambda^3}{125}\,\bar{\partial}\, W_3 + \frac{8256\lambda^2}{25}\, W_4\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^3}\left[\frac{384\lambda^4}{5}\,\bar{\partial}^3\, W_2 + \frac{1152\lambda^3}{125}\,\bar{\partial}^2\, W_3 + \frac{4128\lambda^2}{25}\,\bar{\partial}\, W_4\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{384\lambda^4}{25}\,\bar{\partial}^4\, W_2 + \frac{2304\lambda^3}{875}\,\bar{\partial}^3\, W_3 + \frac{8256\lambda^2}{175}\,\bar{\partial}^2\, W_4 + \frac{8\lambda}{15}\,\bar{\partial}\, W_5 + 6\, W_6\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\left[\frac{64\lambda^4}{25}\,\bar{\partial}^5\, W_2 + \frac{96\lambda^3}{175}\,\bar{\partial}^4\, W_3 + \frac{344\lambda^2}{35}\,\bar{\partial}^3\, W_4 + \frac{4\lambda}{15}\,\bar{\partial}^2\, W_5 + 3\,\bar{\partial}\, W_6\right](\bar{w}) + \cdots,$$

$$W_4(\bar{z})\, W_4^{\hat{A}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^6}\, \frac{9216\lambda^4}{5}\, W_2^{\hat{A}}(\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^5}\, \frac{4608\lambda^4}{5}\, \bar{\partial}\, W_2^{\hat{A}}(\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{1536\lambda^4}{5}\,\bar{\partial}^2\, W_2^{\hat{A}} + \frac{2304\lambda^3}{125}\,\bar{\partial}\, W_3^{\hat{A}} + \frac{8256\lambda^2}{25}\, W_4^{\hat{A}}\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^3}\left[\frac{384\lambda^4}{5}\,\bar{\partial}^3\, W_2^{\hat{A}} + \frac{1152\lambda^3}{125}\,\bar{\partial}^2\, W_3^{\hat{A}} + \frac{4128\lambda^2}{25}\,\bar{\partial}\, W_4^{\hat{A}}\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^2}\left[\frac{384\lambda^4}{25}\,\bar{\partial}^4\, W_2^{\hat{A}} + \frac{2304\lambda^3}{875}\,\bar{\partial}^3\, W_3^{\hat{A}} + \frac{8256\lambda^2}{175}\,\bar{\partial}^2\, W_4^{\hat{A}} + \frac{8\lambda}{15}\,\bar{\partial}\, W_5^{\hat{A}} + 6\, W_6^{\hat{A}}\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\left[\frac{64\lambda^4}{25}\,\bar{\partial}^5\, W_2^{\hat{A}} + \frac{96\lambda^3}{175}\,\bar{\partial}^4\, W_3^{\hat{A}} + \frac{344\lambda^2}{35}\,\bar{\partial}^3\, W_4^{\hat{A}} + \frac{4\lambda}{15}\,\bar{\partial}^2\, W_5^{\hat{A}} + 3\,\bar{\partial}\, W_6^{\hat{A}}\right](\bar{w}) + \cdots,$$

$$W_4^{\hat{A}}(\bar{z})\, W_4^{\hat{B}}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})^6}\, \frac{9216\lambda^4}{5}\left[\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\, W_2 + \frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\, W_2^{\hat{C}}\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^5}\left[\frac{4608\lambda^4}{5}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\, W_2 + \frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\, W_2^{\hat{C}}\right) + \frac{1536\lambda^4}{25}\left(-\frac{i}{2}\right) f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\, W_2^{\hat{C}} + \frac{29952\lambda^3}{25}\left(-\frac{i}{2}\right) f^{\hat{A}\hat{B}\hat{C}}\, W_3^{\hat{C}}\right](\bar{w}) + \frac{1}{(\bar{z}-\bar{w})^4}\left[\frac{1536\lambda^4}{5}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\, W_2 + \frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\, W_2^{\hat{C}}\right) + \frac{2304\lambda^3}{125}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\, W_3 + \frac{1}{2}\, d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\, W_3^{\hat{C}}\right) + \frac{8256\lambda^2}{25}$$

[18] Note that W_4, $W_4^{\hat{A}}$, $Q_{\frac{7}{2}}$ and $Q^{\hat{A}}_{\frac{7}{2}}$ are not quasiprimary under the W_2 stress energy tensor which has zero central charge. We can make them to be quasiprimary by adding the derivatives of currents having lower weights. The weights for $Q_{\frac{7}{2}}$ and $Q^{\hat{A}}_{\frac{7}{2}}$ are 4 not $\frac{7}{2}$ after topological twisting.

$$\times\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,W_4+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,W_4^{\hat{C}}\right)$$
$$+\frac{768\lambda^4}{25}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,W_2^{\hat{C}}+\frac{14976\lambda^3}{25}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,W_3^{\hat{C}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^3}\Big[\frac{384\lambda^4}{5}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^3\,W_2+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,W_2^{\hat{C}}\right)$$
$$+\frac{1152\lambda^3}{125}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\,W_3+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,W_3^{\hat{C}}\right)$$
$$+\frac{4128\lambda^2}{25}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,W_4+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,W_4^{\hat{C}}\right)$$
$$+\frac{1152\lambda^4}{125}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,W_2^{\hat{C}}+\frac{22464\lambda^3}{125}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,W_3^{\hat{C}}$$
$$+\frac{96\lambda^2}{25}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,W_4^{\hat{C}}+\frac{1344\lambda}{25}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,W_5^{\hat{C}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^2}\Big[\frac{384\lambda^4}{25}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^4\,W_2+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,W_2^{\hat{C}}\right)$$
$$+\frac{2304\lambda^3}{875}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^3\,W_3+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,W_3^{\hat{C}}\right)$$
$$+\frac{8256\lambda^2}{175}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\,W_4+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,W_4^{\hat{C}}\right)$$
$$+\frac{8\lambda}{15}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,W_5+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,W_5^{\hat{C}}\right)$$
$$+6\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,W_6+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,W_6^{\hat{C}}\right)$$
$$+\frac{256\lambda^4}{125}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,W_2^{\hat{C}}+\frac{4992\lambda^3}{125}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,W_3^{\hat{C}}$$
$$+\frac{48\lambda^2}{25}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,W_4^{\hat{C}}+\frac{672\lambda}{25}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,W_5^{\hat{C}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})}\Big[\frac{64\lambda^4}{25}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^5\,W_2+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^5\,W_2^{\hat{C}}\right)$$
$$+\frac{96\lambda^3}{175}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^4\,W_3+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,W_3^{\hat{C}}\right)$$
$$+\frac{344\lambda^2}{35}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^3\,W_4+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,W_4^{\hat{C}}\right)$$
$$+\frac{4\lambda}{15}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\,W_5+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,W_5^{\hat{C}}\right)$$
$$+3\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,W_6+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,W_6^{\hat{C}}\right)$$
$$+\frac{64\lambda^4}{175}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^5\,W_2^{\hat{C}}+\frac{1248\lambda^3}{175}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,W_3^{\hat{C}}$$
$$+\frac{8\lambda^2}{15}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,W_4^{\hat{C}}+\frac{112\lambda}{15}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,W_5^{\hat{C}}$$
$$+\frac{3}{55}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,W_6^{\hat{C}}+\frac{1}{2\lambda}\left(-\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,W_7^{\hat{C}}\Big](\bar{w})+\cdots,$$

$$W_4(\bar{z})\,Q_{\frac{7}{2}}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\,\frac{9216\lambda^4}{5}\,Q_{\frac{3}{2}}(\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^5}\,\frac{4608\lambda^4}{5}\,\bar{\partial}\,Q_{\frac{3}{2}}(\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^4}\Big[\frac{1536\lambda^4}{5}\,\bar{\partial}^2\,Q_{\frac{3}{2}}+\frac{2304\lambda^3}{125}\,\bar{\partial}\,Q_{\frac{5}{2}}+\frac{8256\lambda^2}{25}\,Q_{\frac{7}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^3}\Big[\frac{384\lambda^4}{5}\,\bar{\partial}^3\,Q_{\frac{3}{2}}+\frac{1152\lambda^3}{125}\,\bar{\partial}^2\,Q_{\frac{5}{2}}+\frac{4128\lambda^2}{25}\,\bar{\partial}\,Q_{\frac{7}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^2}\Big[\frac{384\lambda^4}{25}\,\bar{\partial}^4\,Q_{\frac{3}{2}}+\frac{2304\lambda^3}{875}\,\bar{\partial}^3\,Q_{\frac{5}{2}}+\frac{8256\lambda^2}{175}\,\bar{\partial}^2\,Q_{\frac{7}{2}}$$
$$+\frac{8\lambda}{15}\,\bar{\partial}\,Q_{\frac{9}{2}}+6\,Q_{\frac{11}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})}\Big[\frac{64\lambda^4}{25}\,\bar{\partial}^5\,Q_{\frac{3}{2}}+\frac{96\lambda^3}{175}\,\bar{\partial}^4\,Q_{\frac{5}{2}}+\frac{344\lambda^2}{35}\,\bar{\partial}^3\,Q_{\frac{7}{2}}$$
$$+\frac{4\lambda}{15}\,\bar{\partial}^2\,Q_{\frac{9}{2}}+3\,\bar{\partial}\,Q_{\frac{11}{2}}\Big](\bar{w})$$
$$+\cdots,$$

$$W_4(\bar{z})\,Q^{\hat{A}}_{\frac{7}{2}}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\,\frac{9216\lambda^4}{5}\,Q^{\hat{A}}_{\frac{3}{2}}(\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^5}\,\frac{4608\lambda^4}{5}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{3}{2}}(\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^4}\Big[\frac{1536\lambda^4}{5}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{2304\lambda^3}{125}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{8256\lambda^2}{25}\,Q^{\hat{A}}_{\frac{7}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^3}\Big[\frac{384\lambda^4}{5}\,\bar{\partial}^3\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{1152\lambda^3}{125}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{4128\lambda^2}{25}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{7}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^2}\Big[\frac{384\lambda^4}{25}\,\bar{\partial}^4\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{2304\lambda^3}{875}\,\bar{\partial}^3\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{8256\lambda^2}{175}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{7}{2}}$$
$$+\frac{8\lambda}{15}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{9}{2}}+6\,Q^{\hat{A}}_{\frac{11}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})}\Big[\frac{64\lambda^4}{25}\,\bar{\partial}^5\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{96\lambda^3}{175}\,\bar{\partial}^4\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{344\lambda^2}{35}\,\bar{\partial}^3\,Q^{\hat{A}}_{\frac{7}{2}}$$
$$+\frac{4\lambda}{15}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{9}{2}}+3\,\bar{\partial}\,Q^{\hat{A}}_{\frac{11}{2}}\Big](\bar{w})$$
$$+\cdots,$$

$$W_4^{\hat{A}}(\bar{z})\,Q_{\frac{7}{2}}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\,\frac{9216\lambda^4}{5}\,Q^{\hat{A}}_{\frac{3}{2}}(\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^5}\,\frac{4608\lambda^4}{5}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{3}{2}}(\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^4}\Big[\frac{1536\lambda^4}{5}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{2304\lambda^3}{125}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{8256\lambda^2}{25}\,Q^{\hat{A}}_{\frac{7}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^3}\Big[\frac{384\lambda^4}{5}\,\bar{\partial}^3\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{1152\lambda^3}{125}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{4128\lambda^2}{25}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{7}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^2}\Big[\frac{384\lambda^4}{25}\,\bar{\partial}^4\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{2304\lambda^3}{875}\,\bar{\partial}^3\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{8256\lambda^2}{175}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{7}{2}}$$
$$+\frac{8\lambda}{15}\,\bar{\partial}\,Q^{\hat{A}}_{\frac{9}{2}}+6\,Q^{\hat{A}}_{\frac{11}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})}\Big[\frac{64\lambda^4}{25}\,\bar{\partial}^5\,Q^{\hat{A}}_{\frac{3}{2}}+\frac{96\lambda^3}{175}\,\bar{\partial}^4\,Q^{\hat{A}}_{\frac{5}{2}}+\frac{344\lambda^2}{35}\,\bar{\partial}^3\,Q^{\hat{A}}_{\frac{7}{2}}$$
$$+\frac{4\lambda}{15}\,\bar{\partial}^2\,Q^{\hat{A}}_{\frac{9}{2}}+3\,\bar{\partial}\,Q^{\hat{A}}_{\frac{11}{2}}\Big](\bar{w})$$
$$+\cdots,$$

$$W_4^{\hat{A}}(\bar{z})\,Q^{\hat{B}}_{\frac{7}{2}}(\bar{w})=\frac{1}{(\bar{z}-\bar{w})^6}\,\frac{9216\lambda^4}{5}\Big[\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,Q_{\frac{3}{2}}$$
$$+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,Q^{\hat{C}}_{\frac{3}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^5}\Big[\frac{4608\lambda^4}{5}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,Q_{\frac{3}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q^{\hat{C}}_{\frac{3}{2}}\right)$$
$$-\frac{1536\lambda^4}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q^{\hat{C}}_{\frac{3}{2}}-\frac{29952\lambda^3}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,W^{\hat{C}}_{\frac{5}{2}}\Big](\bar{w})$$
$$+\frac{1}{(\bar{z}-\bar{w})^4}\Big[\frac{1536\lambda^4}{5}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\,Q_{\frac{3}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q^{\hat{C}}_{\frac{3}{2}}\right)$$

$$
\begin{aligned}
&+\frac{2304\lambda^3}{125}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,Q_{\frac{5}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{5}{2}}^{\hat{C}}\right)+\frac{8256\lambda^2}{25}\\
&\times\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,Q_{\frac{7}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,Q_{\frac{7}{2}}^{\hat{C}}\right)\\
&-\frac{768\lambda^4}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q_{\frac{3}{2}}^{\hat{C}}-\frac{14976\lambda^3}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{5}{2}}^{\hat{C}}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^3}\Bigg[\frac{384\lambda^4}{5}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^3\,Q_{\frac{3}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,Q_{\frac{3}{2}}^{\hat{C}}\right)\\
&+\frac{1152\lambda^3}{125}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\,Q_{\frac{5}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q_{\frac{5}{2}}^{\hat{C}}\right)\\
&+\frac{4128\lambda^2}{25}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,Q_{\frac{7}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{7}{2}}^{\hat{C}}\right)\\
&-\frac{1152\lambda^4}{125}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,Q_{\frac{3}{2}}^{\hat{C}}-\frac{22464\lambda^3}{125}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q_{\frac{5}{2}}^{\hat{C}}\\
&-\frac{96\lambda^2}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{7}{2}}^{\hat{C}}-\frac{1344\lambda}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,Q_{\frac{9}{2}}^{\hat{C}}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})^2}\Bigg[\frac{384\lambda^4}{25}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^4\,Q_{\frac{3}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,Q_{\frac{3}{2}}^{\hat{C}}\right)\\
&+\frac{2304\lambda^3}{875}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^3\,Q_{\frac{5}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,Q_{\frac{5}{2}}^{\hat{C}}\right)+\frac{8256\lambda^2}{175}\\
&\times\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\,Q_{\frac{7}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q_{\frac{7}{2}}^{\hat{C}}\right)\\
&+\frac{8\lambda}{15}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,Q_{\frac{9}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{9}{2}}^{\hat{C}}\right)\\
&+6\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,Q_{\frac{11}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,Q_{\frac{11}{2}}^{\hat{C}}\right)\\
&-\frac{256\lambda^4}{125}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,Q_{\frac{3}{2}}^{\hat{C}}-\frac{4992\lambda^3}{125}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,Q_{\frac{5}{2}}^{\hat{C}}\\
&-\frac{48\lambda^2}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q_{\frac{7}{2}}^{\hat{C}}-\frac{672\lambda}{25}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{9}{2}}^{\hat{C}}\Bigg](\bar{w})\\
&+\frac{1}{(\bar{z}-\bar{w})}\Bigg[\frac{64\lambda^4}{25}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^5\,Q_{\frac{3}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^5\,Q_{\frac{3}{2}}^{\hat{C}}\right)\\
&+\frac{96\lambda^3}{175}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^4\,Q_{\frac{5}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,Q_{\frac{5}{2}}^{\hat{C}}\right)\\
&+\frac{344\lambda^2}{35}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^3\,Q_{\frac{7}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,Q_{\frac{7}{2}}^{\hat{C}}\right)\\
&+\frac{4\lambda}{15}\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}^2\,Q_{\frac{9}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q_{\frac{9}{2}}^{\hat{C}}\right)\\
&+3\left(\frac{1}{K}\,\delta^{\hat{A}\hat{B}}\,\bar{\partial}\,Q_{\frac{11}{2}}+\frac{1}{2}\,d^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{11}{2}}^{\hat{C}}\right)\\
&-\frac{64\lambda^4}{175}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^5\,Q_{\frac{3}{2}}^{\hat{C}}-\frac{1248\lambda^3}{175}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^4\,Q_{\frac{5}{2}}^{\hat{C}}\\
&-\frac{8\lambda^2}{15}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^3\,Q_{\frac{7}{2}}^{\hat{C}}-\frac{112\lambda}{15}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}^2\,Q_{\frac{9}{2}}^{\hat{C}}\\
&-\frac{3}{55}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,\bar{\partial}\,Q_{\frac{11}{2}}^{\hat{C}}-\frac{1}{2\lambda}\left(\frac{i}{2}\right)f^{\hat{A}\hat{B}\hat{C}}\,Q_{\frac{13}{2}}^{\hat{C}}\Bigg](\bar{w})+\cdots. \qquad \text{(B.1)}
\end{aligned}
$$

Note that the structure constants appearing in the third and last equations in Appendix (B.1) are common. Of course, the corresponding currents in the right hand sides are different from each other. We will see in next subsection that the sum of structure constants $\hat{q}^{h_1,h_2+\frac{1}{2},h}(m,n)$ and $\dot{q}^{h_1,h_2+\frac{1}{2},h-1}(m,n)$ is equal to $-\tilde{q}^{h_1,h_2+1,h}(m,n)$ up to the degree $(h+1)$ of the polynomial and the sum of structure constants $\check{q}^{h_1,h_2+\frac{1}{2},h}(m,n)$ and $\bar{q}^{h_1,h_2+\frac{1}{2},h-1}(m,n)$ is equal to the structure constant $q^{h_1,h_2+1,h}(m,n)$ up to the degree $(h+1)$ of the polynomial. This implies that in the OPE language, the independent structure constants are given by (3.4).

B.2 The structure constants for fixed h_1, h_2

Let us check whether the equations (3.12) are consistent with Appendix (B.1) obtained from the free field realizations for fixed h_1 and h_2. First of all, we need to obtain the following possible polynomials explicitly

$$
\begin{aligned}
q^{4,4,4}(m,n) &= \frac{4}{25}\Big(16m^5-16m^4n+32m^4+16m^3n^2\\
&-16m^3n-12m^3-16m^2n^3+12m^2n\\
&+4m^2+16mn^4+16mn^3-4mn-m-16n^5\\
&-32n^4-24n^3-8n^2-n\Big),\\
q^{4,4,3}(m,n) &= -\frac{6}{875}\Big(80m^4-64m^3n+288m^3\\
&-384m^2n-192m^2+64mn^3+384mn^2\\
&+336mn+80m-80n^4-288n^3-312n^2-136n-21\Big),\\
q^{4,4,2}(m,n) &= \frac{86}{175}\Big(20m^3-36m^2n+12m^2+36mn^2\\
&-9m-20n^3-12n^2+3n+2\Big),\\
q^{4,4,1}(m,n) &= \frac{1}{15}\Big(-4m^2-20m+4n^2+20n+9\Big),\\
q^{4,4,0}(m,n) &= 3(m-n),\\
\tilde{q}^{4,4,4}(m,n) &= -\frac{4}{875}\Big(80m^5-48m^4n+256m^4\\
&+16m^3n^2-208m^3n-108m^3+16m^2n^3\\
&+192m^2n^2+180m^2n+44m^2-48mn^4-208mn^3\\
&-240mn^2-108mn-17m\\
&+80n^5+256n^4+312n^3+184n^2+53n+6\Big),\\
\tilde{q}^{4,4,3}(m,n) &= \frac{78}{875}\Big(80m^4-128m^3n+96m^3+144m^2n^2\\
&-48m^2n-60m^2-128mn^3\\
&-48mn^2+48mn+20m+80n^4+96n^3+24n^2-8n-3\Big),\\
\tilde{q}^{4,4,2}(m,n) &= -\frac{2}{75}\Big(20m^3-12m^2n+84m^2-12mn^2\\
&-120mn-57m+20n^3+84n^2\\
&+69n+16\Big),\\
\tilde{q}^{4,4,1}(m,n) &= \frac{28}{75}\Big(20m^2-32mn+4m+20n^2+4n-3\Big),\\
\tilde{q}^{4,4,0}(m,n) &= -\frac{3}{55}(m+n+6), \qquad \tilde{q}^{4,4,-1}(m,n)=\frac{1}{2},\\
\hat{q}^{4,\frac{7}{2},4}(m,n) &= -\frac{32}{525}\Big(m^5-2m^4n+3m^3n^2\\
&-4m^2n^3+5mn^4-6n^5\Big),\\
\hat{q}^{4,\frac{7}{2},3}(m,n) &= -\frac{416}{875}
\end{aligned}
$$

$$\times\Big(5m^4-12m^3n+18m^2n^2-20mn^3+15n^4\Big),$$
$$\hat{q}^{4,\frac{7}{2},2}(m,n)=-\frac{4}{75}$$
$$\times\Big(5m^3-12m^2n+15mn^2-10n^3\Big),$$
$$\hat{q}^{4,\frac{7}{2},1}(m,n)=-\frac{112}{45}\Big(2m^2-4mn+3n^2\Big),$$
$$\hat{q}^{4,\frac{7}{2},0}(m,n)=\frac{1}{110}(6n-5m),$$
$$\hat{q}^{4,\frac{7}{2},-1}(m,n)=-\frac{1}{2},$$
$$\dot{q}^{4,\frac{7}{2},3}(m,n)=\frac{32}{375}(m+3)$$
$$\times\Big(5m^4-4m^3n+3m^2n^2-2mn^3+n^4\Big),$$
$$\dot{q}^{4,\frac{7}{2},2}(m,n)=-\frac{416}{875}(m+3)$$
$$\times\Big(10m^3-12m^2n+9mn^2-4n^3\Big),$$
$$\dot{q}^{4,\frac{7}{2},1}(m,n)=\frac{4}{25}(m+3)\Big(5m^2-6mn+3n^2\Big),$$
$$\dot{q}^{4,\frac{7}{2},0}(m,n)=-\frac{112}{225}(m+3)\Big(5m-4n\Big),$$
$$\dot{q}^{4,\frac{7}{2},-1}(m,n)=\frac{1}{10}(m+3),$$
$$\dot{q}^{4,\frac{7}{2},-2}(m,n)=0,$$
$$\check{q}^{4,\frac{7}{2},4}(m,n)=\frac{32}{75}$$
$$\times\Big(m^5-2m^4n+3m^3n^2-4m^2n^3+5mn^4-6n^5\Big),$$
$$\check{q}^{4,\frac{7}{2},3}(m,n)=\frac{32}{875}$$
$$\times\Big(5m^4-12m^3n+18m^2n^2-20mn^3+15n^4\Big),$$
$$\check{q}^{4,\frac{7}{2},2}(m,n)=\frac{172}{175}$$
$$\times\Big(5m^3-12m^2n+15mn^2-10n^3\Big),$$
$$\check{q}^{4,\frac{7}{2},1}(m,n)=\frac{4}{45}\Big(2m^2-4mn+3n^2\Big),$$
$$\check{q}^{4,\frac{7}{2},0}(m,n)=\frac{1}{2}(5m-6n),$$
$$\check{q}^{4,\frac{7}{2},-1}(m,n)=0,$$
$$\bar{q}^{4,\frac{7}{2},3}(m,n)=\frac{32}{75}(m+3)$$
$$\times\Big(5m^4-4m^3n+3m^2n^2-2mn^3+n^4\Big),$$
$$\bar{q}^{4,\frac{7}{2},2}(m,n)=-\frac{64}{875}(m+3)\Big(10m^3-12m^2n+9mn^2-4n^3\Big),$$
$$\bar{q}^{4,\frac{7}{2},1}(m,n)=\frac{172}{175}(m+3)\Big(5m^2-6mn+3n^2\Big),$$
$$\bar{q}^{4,\frac{7}{2},0}(m,n)=-\frac{4}{45}(m+3)(5m-4n),\quad \bar{q}^{4,\frac{7}{2},-1}(m,n)=\frac{1}{2}(m+3),$$
$$\bar{q}^{4,\frac{7}{2},-2}(m,n)=0. \tag{B.2}$$

Note that all the terms in some of these structure constants have the degree $(h+1)$ while some terms in other structure constants have the degree $(h+1)$.

Let us focus on the $W_2(\bar{w})$ in right hand side of the first OPE of Appendix (B.1). Then from the explicit form of the first OPE, we should calculate $\lambda^4\,(-1)^3\, f^{4,4,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}})\left[\frac{W_2(\bar{w})}{(\bar{z}-\bar{w})}\right]$. Here $f^{4,4,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}})$ can be obtained from $q^{4,4,4}(m,n)$ in Appendix (B.2) under the constraint by taking the terms having a degree $(h+1)=5$ and m and n are replaced by $\bar{\partial}_{\bar{z}}$ and $\bar{\partial}_{\bar{w}}$ respectively. Then we have

$$q^{4,4,4}(m,n)\rightarrow\frac{4}{25}\Big(16m^5-16m^4n+16m^3n^2-16m^2n^3+16mn^4-16n^5\Big). \tag{B.3}$$

This is due to the fact that $q^{h_1,h_2,h}(m,n)$ has the second and the fourth terms in (3.4) from the derivative terms in the current. The corresponding $f^{h_1,h_2,h}(m,n)$ has similar terms in (3.14).

The corresponding differential operator $f^{4,4,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}})$ from Appendix (B.3) is given by

$$f^{4,4,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}})\rightarrow\frac{4}{25}\Big(16\bar{\partial}_{\bar{z}}^5-16\bar{\partial}_{\bar{z}}^4\bar{\partial}_{\bar{w}}+16\bar{\partial}_{\bar{z}}^3\bar{\partial}_{\bar{w}}^2-16\bar{\partial}_{\bar{z}}^2\bar{\partial}_{\bar{w}}^3+16\bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}^4-16\bar{\partial}_{\bar{w}}^5\Big). \tag{B.4}$$

The next thing is to calculate $-\lambda^4\, f^{4,4,4}(\bar{\partial}_{\bar{z}},\bar{\partial}_{\bar{w}})\left[\frac{W_2(\bar{w})}{(\bar{z}-\bar{w})}\right]$ from Appendix (B.4). It turns out that

$$\frac{1}{(\bar{z}-\bar{w})^6}\,\frac{9216\lambda^4}{5}\,W_2(\bar{w})+\frac{1}{(\bar{z}-\bar{w})^5}\,\frac{4608\lambda^4}{5}\,\bar{\partial}\,W_2(\bar{w})+\frac{1}{(\bar{z}-\bar{w})^4}\,\frac{1536\lambda^4}{5}\,\bar{\partial}^2\,W_2(\bar{w})+\frac{1}{(\bar{z}-\bar{w})^3}\,\frac{384\lambda^4}{5}\,\bar{\partial}^3\,W_2(\bar{w})+\frac{1}{(\bar{z}-\bar{w})^2}\,\frac{384\lambda^4}{25}\,\bar{\partial}^4\,W_2(\bar{w})+\frac{1}{(\bar{z}-\bar{w})}\,\frac{64\lambda^4}{25}\,\bar{\partial}^5\,W_2(\bar{w}). \tag{B.5}$$

This Appendix (B.5) are exactly the terms with $h_1+h_2-2-h=2$ appearing in the first OPE of Appendix (B.1).

In this way, we can check that the equations (3.12) are right seven OPEs.

Appendix C: Other OPEs for soft currents in the supersymmetric Einstein–Yang–Mills theory

In this Appendix, we present some details which are related to the contents in Sect. 3.

According to (3.2) and (3.4), there is a shift in the weight h_2 coming from the quantity $(h_2 - \frac{1}{2})$ in the structure constant, therefore we cannot use the previous result in (3.16) directly. This is due to the fact that the corresponding structure constant is written in terms of the one in previous commutator in the first equation of (3.5). They are equivalent to each other [48]. We should find out the right basis where the structure constant can be obtained from the previous relation like as (3.16). Therefore, we return to the first equation of (3.2) and read off the structure constant in terms of p_B and p_F rather than q_B and q_F. By substituting the current in the first equation of (3.1) into the first equation of (3.2), then there are eight commutators in the left hand side and there are four current terms in the right hand side. We can substitute the commutators in the equations (2.10) and (2.11) into the above expression and collect each independent term. Then we obtain the structure constants in terms of p_B or p_F explicitly. That is,

$$
\begin{aligned}
q^{h_1,h_2,h}(m,n)\Big|_{h,\text{odd}} &= \left[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\right] \\
&\times p_B^{h_1-1,h_2,h-1}(m,n) \\
&+\left[\frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\right] \\
&\times p_B^{h_1,h_2-1,h-1}(m,n) \\
&-\left[\frac{2(h_1+h_2-2-h-1)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\right] \\
&\times q^{h_1,h_2,h-1}(m,n), \\
&= -\left[\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\right] p_F^{h_1-1,h_2,h-1}(m,n) \\
&-\left[\frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\right] p_F^{h_1,h_2-1,h-1}(m,n) \\
&+\left[\frac{(2(h_1+h_2-2-h-1)+2)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\right] \\
&\quad q^{h_1,h_2,h-1}(m,n), \\
\times q^{h_1,h_2,h}(m,n)\Big|_{h,\text{even}} &= p_B^{h_1,h_2,h}(m,n) \\
&+\left[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\,\frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\right] \\
&\times p_B^{h_1-1,h_2-1,h-2}(m,n) \\
&-\left[\frac{2(h_1+h_2-2-h-1)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\right] \\
&\times q^{h_1,h_2,h-1}(m,n) \\
&= p_F^{h_1,h_2,h}(m,n) + \left[\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\right. \\
&\times \left.\frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\right] p_F^{h_1-1,h_2-1,h-2}(m,n) \\
&+\left[\frac{(2(h_1+h_2-2-h-1)+2)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\right] \\
&\quad q^{h_1,h_2,h-1}(m,n). \qquad \text{(C.1)}
\end{aligned}
$$

Note that $q^{h_1,h_2,0}(m,n) = p_B^{h_1,h_2,0}(m,n) = p_F^{h_1,h_2,0}(m,n)$, which appears in (3.10), because the other terms in (C.1) vanish. Although there appear the unwanted terms $(m+n+h_1+h_2-2-h)\,q^{h_1,h_2,h-1}(m,n)$ in the right hand side of (C.1) because it is not obvious how we can deal with the mode dependent piece with a factor $(m+n+h_1+h_2-2-h+)$ and others, we can express this as the linear combination of p_B and p_F by realizing that the relative coefficients of these unwanted terms are different from each other and they have common behavior of above $(m+n+h_1+h_2-2-h)$-dependent factor. Then we can write down it in terms of other wanted terms by solving each two equations in (C.1). After substituting $(m+n+h_1+h_2-2-h)\,q^{h_1,h_2,h-1}(m,n)$ (for odd and even cases) written in terms of the structure constants p_B and p_F (Note that these structure constants terms contain only m or n dependence in their coefficients) into the above equations back then we determine the following relations[19]

$$
\begin{aligned}
&q^{h_1,h_2,h}(m,n)\Big|_{h,\text{odd}} \\
&= \left[\frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\right. \\
&\times \left.\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\right] p_B^{h_1,h_2-1,h-1}(m,n) \\
&\times\left[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\right. \\
&\times \left.\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\right] p_B^{h_1-1,h_2,h-1}(m,n) \\
&-\left[\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\right. \\
&\times \left.\frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\right] p_F^{h_1,h_2-1,h-1}(m,n) \\
&-\left[\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\right. \\
&\times \left.\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\right] p_F^{h_1-1,h_2,h-1}(m,n), \\
&q^{h_1,h_2,h}(m,n)\Big|_{h,\text{even}} \\
&= \left[\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\right] p_B^{h_1,h_2,h}(m,n) \\
&+\left[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\right. \\
&\times \frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}
\end{aligned}
$$

[19] Compared to the first equation of (2.10) and the first equation (2.11), as described before, the $h=0$ case leads to the result of $q^{h_1,h_2,0}(m,n) = p_B^{h_1,h_2,0}(m,n) = p_F^{h_1,h_2,0}(m,n)$. But for general h, they are different from each other. The range for h is also different. Note that the sum of coefficients of $p_B^{h_1,h_2,h}(m,n)$ and $p_F^{h_1,h_2,h}(m,n)$ in the second equation of (C.2) is equal to 1.

$$\times \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\Bigg] p_B^{h_1-1,h_2-1,h-2}(m,n)$$
$$+\Bigg[\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\Bigg] p_F^{h_1,h_2,h}(m,n)$$
$$+\Bigg[\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}$$
$$\times \frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}$$
$$\times \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\Bigg] p_F^{h_1-1,h_2-1,h-2}(m,n). \quad \text{(C.2)}$$

In (C.2), all the mode dependent terms are given by either $(m+(h_1-2)+1)$ or $(n+(h_2-2)+1)$. We remove the previous unwanted $(m+n+h_1+h_2-2-h)$ dependence completely. Instead of using the previous relations for the structure constants (3.4) which is appropriate for the first example or the OPEs including the gravitino and gluino, we use the above basis (C.2) for the structure constants which is more appropriate for the OPEs including the graviton and gluon. Therefore, the structure constant consists of both p_B and p_F coming from (2.11) and (2.10) respectively and the corresponding weights for the first current are either h_1 or (h_1-1) while the corresponding weights for the second current are either h_2 or (h_2-1). This is reasonable because from (3.7) we allow to have the first derivative terms. For the dummy variable h, the corresponding weight is given by h, $(h-1)$ or $(h-2)$. The last one occurs when we consider the commutators where the corresponding two currents contain each derivative term.

C.1 The OPE between the graviton and the gluino

The OPE between the conformally soft gravitons and the gluinos where the weights in the antiholomorphic sector are given by $h_1=\frac{k-2}{2}$ and $h_2=\frac{l-\frac{1}{2}}{2}$ can be expressed as

$$H^k(z_1,\bar{z}_1)\, L^{l,\hat{A}}(z_2,\bar{z}_2) = -\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{h=0}^{h_1+h_2-3}(-1)^{h+1}$$
$$\times\lambda^h\left[q_B^{h_1,h_2+\frac{1}{2},h}+q_F^{h_1,h_2+\frac{1}{2},h}\right]$$
$$\times\sum_{n=0}^{\infty}\begin{pmatrix}\frac{1}{2}-2h-k-l-n\\ -\frac{1}{2}-h-l\end{pmatrix}\frac{\bar{z}_{12}^{n+h+1}}{n!}\,\bar{\partial}^n\, L^{k+l+h,\hat{A}}(z_2,\bar{z}_2)$$
$$-\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{h=0}^{h_1+h_2-3}(-1)^h\,\lambda^h\left[\frac{2(h_1-2)}{2(h_1-2)+1}\, q_B^{h_1-1,h_2+\frac{1}{2},h-1}\right.$$
$$\left.-\frac{2(h_1-2)+2}{2(h_1-2)+1}\, q_F^{h_1-1,h_2+\frac{1}{2},h-1}\right]$$
$$\times\sum_{n=0}^{\infty}\begin{pmatrix}\frac{1}{2}-2(h-1)-(k-2)-l-n\\ -\frac{1}{2}-(h-1)-l\end{pmatrix}\frac{\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}}{n!}$$
$$\times\bar{\partial}^n\, L^{(k-2)+l+(h-1),\hat{A}}(z_2,\bar{z}_2)$$
$$+\cdots. \quad \text{(C.3)}$$

The numerical factor $\frac{1}{2}$ of the first binomial coefficient is given by $(2+\frac{1}{2})$ minus 2. The numerical factor $-\frac{1}{2}$ in the second line of the first binomial coefficient is given by $\frac{1}{2}$ minus 1.[20] The first binomial coefficient above can be obtained from (3.18) by taking $l\to(l+\frac{3}{2})$.

C.2 The OPE between the gluon and the gravitino

The OPE between the conformally soft gluons and the gravitinos where the weights in the antiholomorphic sector are given by $h_1=\frac{k-1}{2}$ and $h_2=\frac{l-\frac{3}{2}}{2}$ can be described as

$$R^{k,\hat{A}}(z_1,\bar{z}_1)\, I^l(z_2,\bar{z}_2) = -\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{h=0}^{h_1+h_2-3}(-1)^{h+1}$$
$$\times\lambda^h\left[q_B^{h_1,h_2+\frac{1}{2},h}+q_F^{h_1,h_2+\frac{1}{2},h}\right]$$
$$\times\sum_{n=0}^{\infty}\begin{pmatrix}\frac{1}{2}-2h-k-l-n\\ \frac{1}{2}-h-l\end{pmatrix}\frac{\bar{z}_{12}^{n+h+1}}{n!}\,\bar{\partial}^n\, L^{k+l+h,\hat{A}}(z_2,\bar{z}_2)$$
$$-\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{h=0}^{h_1+h_2-3}(-1)^h\,\lambda^h\left[\frac{2(h_1-2)}{2(h_1-2)+1}\, q_B^{h_1-1,h_2+\frac{1}{2},h-1}\right.$$
$$\left.-\frac{2(h_1-2)+2}{2(h_1-2)+1}\, q_F^{h_1-1,h_2+\frac{1}{2},h-1}\right]$$
$$\times\sum_{n=0}^{\infty}\begin{pmatrix}\frac{1}{2}-2(h-1)-(k-2)-l-n\\ \frac{1}{2}-(h-1)-l\end{pmatrix}\frac{\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}}{n!}$$
$$\times\bar{\partial}^n\, L^{(k-2)+l+(h-1),\hat{A}}(z_2,\bar{z}_2)$$
$$+\cdots. \quad \text{(C.4)}$$

The numerical factor $\frac{1}{2}$ in the first line of the first binomial coefficient is given by $(1+\frac{3}{2})$ minus 2. The numerical factor $\frac{1}{2}$ in the second line of the first binomial coefficient is given by $\frac{3}{2}$ minus 1. The first binomial coefficient above can be obtained from (3.18) by taking $k\to(k+1)$ and $l\to(l+\frac{1}{2})$.

C.3 The OPE between the gluon and the gluino

The OPE between the conformally soft gluons and the gluinos where the weights in the antiholomorphic sector are given by $h_1=\frac{k-1}{2}$ and $h_2=\frac{l-\frac{1}{2}}{2}$ can be written as

$$R^{k,\hat{A}}(z_1,\bar{z}_1)\, L^{l,\hat{B}}(z_2,\bar{z}_2) = -\frac{\kappa}{2}\left(\frac{i}{2}\right) f^{\hat{A}\hat{B}\hat{C}}\frac{1}{z_{12}}$$
$$\times\sum_{h=-1}^{h_1+h_2-3}(-1)^{h+1}\,\lambda^h\left[-q_B^{h_1,h_2+\frac{1}{2},h}+q_F^{h_1,h_2+\frac{1}{2},h}\right]$$
$$\times\sum_{n=0}^{\infty}\begin{pmatrix}-\frac{1}{2}-2h-k-l-n\\ -\frac{1}{2}-h-l\end{pmatrix}\frac{\bar{z}_{12}^{n+h+1}}{n!}\,\bar{\partial}^n\, L^{k+l+h,\hat{C}}(z_2,\bar{z}_2)$$

[20] As mentioned before, the general form for the element of binomial coefficient in the Footnote 10 can be used.

$$
\begin{aligned}
&-\frac{\kappa}{2}\left(\frac{1}{2}\right) d^{\hat{A}\hat{B}\hat{C}} \frac{1}{z_{12}} \sum_{h=-1}^{h_1+h_2-3} (-1)^{h+1} \\
&\times \lambda^h \left[q_B^{h_1,h_2+\frac{1}{2},h} + q_F^{h_1,h_2+\frac{1}{2},h} \right] \\
&\times \sum_{n=0}^{\infty} \binom{-\frac{1}{2}-2h-k-l-n}{-\frac{1}{2}-h-l} \frac{\bar{z}_{12}^{n+h+1}}{n!} \bar{\partial}^n L^{k+l+h,\hat{C}}(z_2,\bar{z}_2) \\
&-\frac{\kappa}{2}\left(\frac{1}{K}\right) \delta^{\hat{A}\hat{B}} \frac{1}{z_{12}} \sum_{h=-1}^{h_1+h_2-3} (-1)^{h+1} \\
&\times \lambda^h \left[q_B^{h_1,h_2+\frac{1}{2},h} + q_F^{h_1,h_2+\frac{1}{2},h} \right] \\
&\times \sum_{n=0}^{\infty} \binom{-\frac{1}{2}-2h-k-l-n}{-\frac{1}{2}-h-l} \frac{\bar{z}_{12}^{n+h+1}}{n!} \\
&\times \bar{\partial}^n I^{k+l+h}(z_2,\bar{z}_2) - \frac{\kappa}{2}\left(\frac{i}{2}\right) f^{\hat{A}\hat{B}\hat{C}} \frac{1}{z_{12}} \\
&\times \sum_{h=-1}^{h_1+h_2-3} (-1)^h \lambda^h \left[\frac{2(h_1-2)}{2(h_1-2)+1} q_B^{h_1-1,h_2+\frac{1}{2},h} \right. \\
&\left. + \frac{2(h_1-2)+2}{2(h_1-2)+1} q_F^{h_1-1,h_2+\frac{1}{2},h} \right] \\
&\times \sum_{n=0}^{\infty} \binom{-\frac{1}{2}-2h-(k-2)-l-n}{-\frac{1}{2}-h-l} \frac{\bar{\partial}_{\bar{z}_1} \bar{z}_{12}^{n+h+1}}{n!} \\
&\times \bar{\partial}^n L^{(k-2)+l+h,\hat{C}}(z_2,\bar{z}_2) - \frac{\kappa}{2}\left(\frac{1}{2}\right) d^{\hat{A}\hat{B}\hat{C}} \frac{1}{z_{12}} \\
&\times \sum_{h=-1}^{h_1+h_2-3} (-1)^h \lambda^h \left[-\frac{2(h_1-2)}{2(h_1-2)+1} q_B^{h_1-1,h_2+\frac{1}{2},h} \right. \\
&\left. + \frac{2(h_1-2)+2}{2(h_1-2)+1} q_F^{h_1-1,h_2+\frac{1}{2},h} \right] \\
&\times \sum_{n=0}^{\infty} \binom{-\frac{1}{2}-2h-(k-2)-l-n}{-\frac{1}{2}-h-l} \frac{\bar{\partial}_{\bar{z}_1} \bar{z}_{12}^{n+h+1}}{n!} \\
&\times \bar{\partial}^n L^{(k-2)+l+h,\hat{C}}(z_2,\bar{z}_2) - \frac{\kappa}{2}\left(\frac{1}{K}\right) \delta^{\hat{A}\hat{B}} \frac{1}{z_{12}} \\
&\times \sum_{h=-1}^{h_1+h_2-3} (-1)^h \lambda^h \left[-\frac{2(h_1-2)}{2(h_1-2)+1} q_B^{h_1-1,h_2+\frac{1}{2},h} \right. \\
&\left. + \frac{2(h_1-2)+2}{2(h_1-2)+1} q_F^{h_1-1,h_2+\frac{1}{2},h} \right] \\
&\times \sum_{n=0}^{\infty} \binom{-\frac{1}{2}-2h-(k-2)-l-n}{-\frac{1}{2}-h-l} \frac{\bar{\partial}_{\bar{z}_1} \bar{z}_{12}^{n+h+1}}{n!} \\
&\times \bar{\partial}^n I^{(k-2)+l+h}(z_2,\bar{z}_2).
\end{aligned} \tag{C.5}
$$

The various structure constants in (3.6) are used. The numerical factor $-\frac{1}{2}$ is given by $(1+\frac{1}{2})$ minus 2. The numerical factor $-\frac{1}{2}$ in the second line of the first binomial coefficient is given by $\frac{1}{2}$ minus 1. The first binomial coefficient above can be obtained from (3.18) by taking $k \to (k+1)$ and $l \to (l+\frac{3}{2})$.

C.4 The OPE between the graviton and the gluon

The OPE between the conformally soft gravitons and gluons where the weights in the antiholomorphic sector are given by $h_1 = \frac{k-2}{2}$ and $h_2 = \frac{l-1}{2}$ can be summarized by

$$
\begin{aligned}
&H^k(z_1,\bar{z}_1)\, R^{l,\hat{A}}(z_2,\bar{z}_2) = -\frac{\kappa}{2} \frac{1}{z_{12}} \sum_{h=1,\text{odd}}^{h_1+h_2-4} (-1)^h \lambda^h \\
&\times \left[\frac{2(h_1-2)}{2(h_1-2)+1} \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h)+1} p_B^{h_1-1,h_2,h-1} \right. \\
&\left. - \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} \frac{2(h_1-2)+2}{2(h_1-2)+1} p_F^{h_1-1,h_2,h-1} \right] \\
&\times \sum_{n=0}^{\infty} \binom{1-2(h-1)-(k-2)-l-n}{-(h-1)-l} \frac{\bar{\partial}_{\bar{z}_1} \bar{z}_{12}^{n+(h-1)+1}}{n!} \\
&\times \bar{\partial}^n R^{(k-2)+l+(h-1),\hat{A}}(z_2,\bar{z}_2) \\
&-\frac{\kappa}{2} \frac{1}{z_{12}} \sum_{h=1,\text{odd}}^{h_1+h_2-4} (-1)^h \lambda^h \left[\frac{2(h_2-2)}{2(h_2-2)+1} \right. \\
&\times \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1,h_2-1,h-1} \\
&\left. - \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} \frac{2(h_2-2)+2}{2(h_2-2)+1} p_F^{h_1,h_2-1,h-1} \right] \\
&\times \sum_{n=0}^{\infty} \binom{1-2(h-1)-k-(l-2)-n}{-(h-1)-(l-2)} \\
&\times \frac{1}{n!} \bar{\partial}_{\bar{z}_2} \left[\bar{z}_{12}^{n+(h-1)+1} \bar{\partial}^n R^{k+(l-2)+(h-1),\hat{A}}(z_2,\bar{z}_2) \right] \\
&-\frac{\kappa}{2} \frac{1}{z_{12}} \sum_{h=0,\text{even}}^{h_1+h_2-4} (-1)^{h+1} \lambda^h \left[\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} \right. \\
&\left. p_B^{h_1,h_2,h} + \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1,h_2,h} \right] \\
&\times \sum_{n=0}^{\infty} \binom{1-2h-k-l-n}{-h-l} \frac{\bar{z}_{12}^{n+h+1}}{n!} \bar{\partial}^n R^{k+l+h,\hat{A}}(z_2,\bar{z}_2) \\
&-\frac{\kappa}{2} \frac{1}{z_{12}} \sum_{h=0,\text{even}}^{h_1+h_2-4} (-1)^{h+1} \lambda^h \\
&\times \left[\frac{2(h_1-2)}{2(h_1-2)+1} \frac{2(h_2-2)}{2(h_2-2)+1} \right. \\
&\times \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1-1,h_2-1,h-2} \\
&+ \frac{2(h_1-2)+2}{2(h_1-2)+1} \frac{2(h_2-2)+2}{2(h_2-2)+1} \\
&\left. \times \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1-1,h_2-1,h-2} \right] \\
&\times \sum_{n=0}^{\infty} \binom{1-2(h-2)-(k-2)-(l-2)-n}{-(h-2)-(l-2)} \\
&\times \frac{1}{n!} \bar{\partial}_{\bar{z}_1} \bar{\partial}_{\bar{z}_2} \left[\bar{z}_{12}^{n+(h-2)+1} \bar{\partial}^n R^{(k-2)+(l-2)+(h-2),\hat{A}}(z_2,\bar{z}_2) \right] \\
&+ \cdots .
\end{aligned} \tag{C.6}
$$

According to the second equation of (3.2), the structure constants are the same as the first equation of (3.2). So we can repeat what we have done in (3.18). The difference appears in the first binomial coefficient in Appendix (C.6) in the sense that the numerical factor 1 is given by 2+1 minus 2. Similarly, the numerical factor 0 in the second line of the first binomial coefficient is given by 1 minus 1. The above binomial coefficients can be obtained from (3.18) by taking $l \rightarrow (l+1)$.

C.5 The OPE between the gluons

Finally, the OPE between the conformally soft gluons where the weights in the antiholomorphic sector are given by $h_1 = \frac{k-1}{2}$ and $h_2 = \frac{l-1}{2}$ can be expressed as

$$
\begin{aligned}
& R^{k,\hat{A}}(z_1,\bar{z}_1)\, R^{l,\hat{B}}(z_2,\bar{z}_2) = \left(-\frac{i}{2}\right) f^{\hat{A}\hat{B}\hat{C}} \\
& \times \Bigg\{ -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=0,\text{even}}^{h_1+h_2-4} (-1)^h \lambda^h \\
& \times \left[\frac{2(h_1-2)}{2(h_1-2)+1} \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1-1,h_2,h-1} \right. \\
& \left. -\frac{2(h_1-2)+2}{2(h_1-2)+1} \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1-1,h_2,h-1} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2(h-1)-(k-2)-l-n}{-(h-1)-l} \frac{\bar{\partial}_{\bar{z}_1} \bar{z}_{12}^{n+(h-1)+1}}{n!} \\
& \times \bar{\partial}^n R^{(k-2)+l+(h-1),\hat{C}}(z_2,\bar{z}_2) \\
& -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=0,\text{even}}^{h_1+h_2-4} (-1)^h \lambda^h \\
& \times \left[\frac{2(h_2-2)}{2(h_2-2)+1} \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1,h_2-1,h-1} \right. \\
& \left. -\frac{2(h_2-2)+2}{2(h_2-2)+1} \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1,h_2-1,h-1} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2(h-1)-k-(l-2)-n}{-(h-1)-(l-2)} \\
& \times \frac{1}{n!} \bar{\partial}_{\bar{z}_2} \left[\bar{z}_{12}^{n+(h-1)+1} \bar{\partial}^n R^{k+(l-2)+(h-1),\hat{C}}(z_2,\bar{z}_2) \right] \\
& -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=-1,\text{odd}}^{h_1+h_2-4} (-1)^{h+1} \lambda^h \\
& \times \left[\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1,h_2,h} \right. \\
& \left. +\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1,h_2,h} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2h-k-l-n}{-h-l} \frac{\bar{z}_{12}^{n+h+1}}{n!} \bar{\partial}^n R^{k+l+h,\hat{C}}(z_2,\bar{z}_2) \\
& -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=-1,\text{odd}}^{h_1+h_2-4} (-1)^{h+1} \lambda^h \\
& \times \left[\frac{2(h_1-2)}{2(h_1-2)+1} \frac{2(h_2-2)}{2(h_2-2)+1} \right. \\
& \times \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1-1,h_2-1,h-2} \\
& +\frac{2(h_1-2)+2}{2(h_1-2)+1} \frac{2(h_2-2)+2}{2(h_2-2)+1} \\
& \left. \times \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1-1,h_2-1,h-2} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2(h-2)-(k-2)-(l-2)-n}{-(h-2)-(l-2)} \\
& \times \frac{1}{n!} \bar{\partial}_{\bar{z}_1} \bar{\partial}_{\bar{z}_2} \left[\bar{z}_{12}^{n+(h-2)+1} \bar{\partial}^n R^{(k-2)+(l-2)+(h-2),\hat{C}}(z_2,\bar{z}_2) \right] \Bigg\} \\
& +\left(\frac{1}{2}\right) d^{\hat{A}\hat{B}\hat{C}} \Bigg\{ -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=1,\text{odd}}^{h_1+h_2-4} (-1)^h \lambda^h \\
& \times \left[\frac{2(h_1-2)}{2(h_1-2)+1} \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h)+1} p_B^{h_1-1,h_2,h-1} \right. \\
& \left. -\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} \frac{2(h_1-2)+2}{2(h_1-2)+1} p_F^{h_1-1,h_2,h-1} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2(h-1)-(k-2)-l-n}{-(h-1)-l} \\
& \times \frac{\bar{\partial}_{\bar{z}_1} \bar{z}_{12}^{n+(h-1)+1}}{n!} \bar{\partial}^n R^{(k-2)+l+(h-1),\hat{C}}(z_2,\bar{z}_2) \\
& -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=1,\text{odd}}^{h_1+h_2-4} (-1)^h \lambda^h \\
& \times \left[\frac{2(h_2-2)}{2(h_2-2)+1} \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1,h_2-1,h-1} \right. \\
& \left. -\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} \frac{2(h_2-2)+2}{2(h_2-2)+1} p_F^{h_1,h_2-1,h-1} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2(h-1)-k-(l-2)-n}{-(h-1)-(l-2)} \\
& \times \frac{1}{n!} \bar{\partial}_{\bar{z}_2} \left[\bar{z}_{12}^{n+(h-1)+1} \bar{\partial}^n R^{k+(l-2)+(h-1),\hat{C}}(z_2,\bar{z}_2) \right] \\
& -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=0,\text{even}}^{h_1+h_2-4} (-1)^{h+1} \lambda^h \\
& \times \left[\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} p_B^{h_1,h_2,h} \right. \\
& \left. +\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1,h_2,h} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2h-k-l-n}{-h-l} \frac{\bar{z}_{12}^{n+h+1}}{n!} \bar{\partial}^n R^{k+l+h,\hat{C}}(z_2,\bar{z}_2) \\
& -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=0,\text{even}}^{h_1+h_2-4} (-1)^{h+1} \lambda^h \\
& \times \left[\frac{2(h_1-2)}{2(h_1-2)+1} \frac{2(h_2-2)}{2(h_2-2)+1} \frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1} \right. \\
& \times p_B^{h_1-1,h_2-1,h-2} + \frac{2(h_1-2)+2}{2(h_1-2)+1} \frac{2(h_2-2)+2}{2(h_2-2)+1} \\
& \left. \times \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1} p_F^{h_1-1,h_2-1,h-2} \right] \\
& \times \sum_{n=0}^{\infty} \binom{-2(h-2)-(k-2)-(l-2)-n}{-(h-2)-(l-2)} \\
& \times \frac{1}{n!} \bar{\partial}_{\bar{z}_1} \bar{\partial}_{\bar{z}_2} \left[\bar{z}_{12}^{n+(h-2)+1} \bar{\partial}^n R^{(k-2)+(l-2)+(h-2),\hat{C}}(z_2,\bar{z}_2) \right] \Bigg\} \\
& +\left(\frac{1}{K}\right) \delta^{\hat{A}\hat{B}} \Bigg\{ -\frac{\kappa}{2}\frac{1}{z_{12}} \sum_{h=1,\text{odd}}^{h_1+h_2-4} (-1)^h \lambda^h
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg[\frac{2(h_1-2)}{2(h_1-2)+1}\,\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h)+1}\,p_B^{h_1-1,h_2,h-1}\\
&-\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\,\frac{2(h_1-2)+2}{2(h_1-2)+1}\,p_F^{h_1-1,h_2,h-1}\Bigg]\\
&\times\sum_{n=0}^{\infty}\binom{-2(h-1)-(k-2)-l-n}{-(h-1)-l}\,\frac{\bar{\partial}_{\bar{z}_1}\,\bar{z}_{12}^{n+(h-1)+1}}{n!}\,\bar{\partial}^n\\
&\times R^{(k-2)+l+(h-1)}(z_2,\bar{z}_2)-\frac{\kappa}{2}\,\frac{1}{z_{12}}\sum_{h=1,\mathrm{odd}}^{h_1+h_2-4}(-1)^h\\
&\times\lambda^h\Bigg[\frac{2(h_2-2)}{2(h_2-2)+1}\,\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\,p_B^{h_1,h_2-1,h-1}\\
&-\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\,\frac{2(h_2-2)+2}{2(h_2-2)+1}\,p_F^{h_1,h_2-1,h-1}\Bigg]\\
&\times\sum_{n=0}^{\infty}\binom{-2(h-1)-k-(l-2)-n}{-(h-1)-(l-2)}\\
&\times\frac{1}{n!}\,\bar{\partial}_{\bar{z}_2}\left[\bar{z}_{12}^{n+(h-1)+1}\,\bar{\partial}^n\,R^{k+(l-2)+(h-1)}(z_2,\bar{z}_2)\right]\\
&-\frac{\kappa}{2}\,\frac{1}{z_{12}}\sum_{h=0,\mathrm{even}}^{h_1+h_2-4}(-1)^{h+1}\,\lambda^h\Bigg[\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\,p_B^{h_1,h_2,h}\\
&+\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\,p_F^{h_1,h_2,h}\Bigg]\\
&\times\sum_{n=0}^{\infty}\binom{-2h-k-l-n}{-h-l}\,\frac{\bar{z}_{12}^{n+h+1}}{n!}\\
&\times\bar{\partial}^n\,R^{k+l+h}(z_2,\bar{z}_2)\\
&-\frac{\kappa}{2}\,\frac{1}{z_{12}}\sum_{h=0,\mathrm{even}}^{h_1+h_2-4}(-1)^{h+1}\,\lambda^h\\
&\times\Bigg[\frac{2(h_1-2)}{2(h_1-2)+1}\,\frac{2(h_2-2)}{2(h_2-2)+1}\,\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\\
&\times p_B^{h_1-1,h_2-1,h-2}\\
&+\frac{2(h_1-2)+2}{2(h_1-2)+1}\,\frac{2(h_2-2)+2}{2(h_2-2)+1}\,\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\\
&\times p_F^{h_1-1,h_2-1,h-2}\Bigg]\\
&\times\sum_{n=0}^{\infty}\binom{-2(h-2)-(k-2)-(l-2)-n}{-(h-2)-(l-2)}\\
&\times\frac{1}{n!}\,\bar{\partial}_{\bar{z}_1}\,\bar{\partial}_{\bar{z}_2}\left[\bar{z}_{12}^{n+(h-2)+1}\,\bar{\partial}^n\,R^{(k-2)+(l-2)+(h-2)}(z_2,\bar{z}_2)\right]\Bigg\}\\
&+\cdots. \qquad\qquad (\mathrm{C.7})
\end{aligned}
$$

The numerical factor 0 is given by $1+1$ minus 2. The numerical factor 0 in the second line of the first binomial coefficient is given by 1 minus 1. The first binomial coefficient above can be obtained from (3.18) by taking $k\rightarrow(k+1)$ and $l\rightarrow(l+1)$. We use the following relations between the structure constants

$$
\begin{aligned}
&\tilde{q}^{h_1,h_2,h}(m,n)\Big|_{h,\mathrm{odd}}=p_B^{h_1,h_2,h}(m,n)\\
&\quad+\Bigg[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\\
&\quad\times\frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\Bigg]\,p_B^{h_1-1,h_2-1,h-2}(m,n)\\
&\quad-\Bigg[\frac{2(h_1+h_2-2-h-1)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad\times\tilde{q}^{h_1,h_2,h-1}(m,n)\\
&=p_F^{h_1,h_2,h}(m,n)+\Bigg[\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\\
&\quad\times\frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\Bigg]\,p_F^{h_1-1,h_2-1,h-2}(m,n)\\
&\quad+\Bigg[\frac{(2(h_1+h_2-2-h-1)+2)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad\tilde{q}^{h_1,h_2,h-1}(m,n),\ \tilde{q}^{h_1,h_2,h}(m,n)\Big|_{h,\mathrm{even}}\\
&=\Bigg[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\Bigg]\,p_B^{h_1-1,h_2,h-1}(m,n)\\
&\quad+\Bigg[\frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\Bigg]\\
&\quad\times p_B^{h_1,h_2-1,h-1}(m,n)\\
&\quad-\Bigg[\frac{2(h_1+h_2-2-h-1)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad\tilde{q}^{h_1,h_2,h-1}(m,n)\\
&=-\Bigg[\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\Bigg]\,p_F^{h_1-1,h_2,h-1}(m,n)\\
&\quad-\Bigg[\frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\Bigg]\\
&\quad\times p_F^{h_1,h_2-1,h-1}(m,n)\\
&\quad+\Bigg[\frac{(2(h_1+h_2-2-h-1)+2)(m+n+h_1+h_2-2-h-1+1)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad\times\tilde{q}^{h_1,h_2,h-1}(m,n). \qquad\qquad (\mathrm{C.8})
\end{aligned}
$$

Note that $\tilde{q}^{h_1,h_2,-1}(m,n)=p_B^{h_1,h_2,h-1}(m,n)=\frac{1}{2}$. From Appendix (C.8), finally we obtain

$$
\begin{aligned}
&\tilde{q}^{h_1,h_2,h}(m,n)\Big|_{h,\mathrm{odd}}\\
&=\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\,p_B^{h_1,h_2,h}\\
&\quad+\Bigg[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\\
&\quad\times\frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\,\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad\times p_B^{h_1-1,h_2-1,h-2}\\
&\quad+\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\,p_F^{h_1,h_2,h}\\
&\quad+\Bigg[\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\\
&\quad\times\frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\,\frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad p_F^{h_1-1,h_2-1,h-2},\\
&\tilde{q}^{h_1,h_2,h}(m,n)\Big|_{h,\mathrm{even}}\\
&=\Bigg[\frac{2(h_2-2)(n+(h_2-2)+1)}{2(h_2-2)+1}\,\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad\times p_B^{h_1,h_2-1,h-1}\\
&\quad\times\Bigg[\frac{2(h_1-2)(m+(h_1-2)+1)}{2(h_1-2)+1}\,\frac{(h_1+h_2-2-h)}{2(h_1+h_2-2-h-1)+1}\Bigg]\\
&\quad\times p_B^{h_1-1,h_2,h-1}
\end{aligned}
$$

$$
-\left[\frac{(2(h_2-2)+2)(n+(h_2-2)+1)}{2(h_2-2)+1}\ \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\right]
\times p_F^{h_1,h_2-1,h-1}
-\left[\frac{(2(h_1-2)+2)(m+(h_1-2)+1)}{2(h_1-2)+1}\ \frac{(h_1+h_2-2-h-1)}{2(h_1+h_2-2-h-1)+1}\right]
\times p_F^{h_1-1,h_2,h-1}. \quad \text{(C.9)}
$$

By considering Appendix (C.9), we can write down the OPE in Appendix (C.7). The first one of Appendix (C.9) is the same as the functional form of the second one of (C.2) while the second one of Appendix (C.9) is the same as the functional form of the first one of (C.2).

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