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2D Integrable systems, 4D Chern–Simons theory and affine Higgs bundles

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Abstract We compare the construction of 2D integrable models through two gauge field theories. The first one is the 4D Chern–Simons (4D-CS) theory proposed by Costello and Yamazaki. The second one is the 2D generalization of the Hitchin integrable systems constructed by means of affine Higgs bundles (AHB). We illustrate the latter approach by considering 1 + 1 field versions of integrable systems including the Calogero–Moser field theory, the Landau–Lifshitz model and the field theory generalization of the elliptic Gaudin model.

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1 Introduction

In the 1990s, we attempted to construct two-dimensional (2D) classical integrable field theories starting with a 2D Wess–Zumino–Witten (WZW) action [1,2]. The corresponding equations of motion coincide with the Zakharov–Shabat equations. These equations are the hallmark of 2D integrable systems. But that approach had one essential drawback—the Lax operator did not depend on the spectral parameter. This parameter is a necessary ingredient for constructing the infinite number of commuting integrals of motion. A class of integrable theories derived from the WZW models was considered in papers by Fehér et al. (see the review [3]). Also, the interrelations between gauge theories and integrable systems were considered in the mid-1990s in [4,5]. Later, Nekrasov and Shatashvili derived quantum integrable systems from four-dimensional gauge theories [6,7].

The problem with the spectral parameter was overcome in the works of Costello and Yamazaki [8] by considering the so-called four-dimensional Chern–Simons theory (4D-CS).

Here we compare 4D-CS construction with the construction of 2D integrable systems based on the affine Higgs bundles (AHB) model proposed in [9]. The AHB model is the 2D analogue of the Hitchin systems [10]. To compare the AHB theory with the 4D-CS approach, we rewrite the AHB theory in the form of a special 4D-CS model. This allows us to establish a correspondence between the field content from both constructions.

The first formal difference between these two approaches is that AHB theory is free, and the nontrivial integrable models appear as a result of the symplectic reduction. The latter procedure is similar to what happens in the finite-dimensional case for the Hitchin systems. Symplectic reduction is defined by two types of constraints. The first one is given by the moment map constraints (the Gauss law analogue in the Yang–Mills theory). The second one is the gauge-fixing conditions. After imposing these constraints, we obtain the symplectic phase spaces of 2D integrable systems. Using the AHB, we constructed in [9] the 2D field generalization of the elliptic (spin) Calogero–Moser (CM) model. It was proved by A. Shabat (unpublished) and in [11] that this model is gauge-equivalent to the Landau–Lifshitz (LL) equation [12]. The gauge transformation is obtained from the so-called symplectic Hecke correspondence. Another example of 2D generalization of the Hitchin systems is the 2D elliptic Gaudin model. In particular, the principal chiral model is reproduced in this way.[1]

Another construction similar to the AHB approach is the algebra-geometric derivation of the Zakharov–Shabat equation proposed by Krichever [14,15]. Using the Kadomtsev–Petviashvili (KP) hierarchy, he constructed a 2D version of the Calogero–Moser model. This approach can also be extended to the field version of the Ruijsenaars–Schneider models [16].

In contrast to AHB construction, the 4D-CS theory is not free. The equations of motion have the form of the moment map constraint equations, which are similar to the moment map constraints in the AHB theory. It only remains to impose some gauge fixation to obtain 2D integrable systems. To compare these constructions, we rewrite the equations of motion and the moment map constraints in the AHB models in CS form.

In the standard approach to the 2D integrable in [8,9] the 3D space has the form $\mathbb{R}\times\mathbb{C}P^1$ or $S^1\times\mathbb{C}P^1$, or with an elliptic curve instead of $\mathbb{C}P^1$. More generally, these 3D spaces can be replaced by an arbitrary Seifert surface [17]. The Seifert surface is a $U(1)$ bundle over the Riemann curve Σ_g of genus g. Seifert surfaces have two topological characteristics (n, g), where n is the degree of the line bundle corresponding to the $U(1)$ bundle. Although the moduli space of the Higgs bundles over the Seifert surfaces depends on n, the invariant Hamiltonians do not depend on it. The reason is that there exists singular gauge transformation $\Xi(k)$ of the Lax operator $L(n)$ such that $\Xi(k) : L(n) \to L(n+k)$.

The AHB construction allows one to define 2D analogues of the additional structures in the Hitchin systems. The first structure is the affine analogue of the symplectic Hecke correspondence [9,11]. Another structure that appears in the AHB model is the affine version of the Nahm equations describing the surface defects. Both of these structures will be considered in a forthcoming publication [18].

This paper is organized as follows. In the next section we briefly explain 4D-CS construction of 2D integrable models based on the articles [8,19]. In Sect. 3 the AHB construction is given following notations from [9,20]. Some examples are given in Sect. 4. Finally, we establish the correspondence between the two construction in Sect. 5.

2 4D Chern–Simons model and integrable systems

Let us describe the field content of the 4D Chern–Simons model. Consider a Riemann curve C and the spacetime $M = \mathbb{R}^2\times C$ with the local coordinates (x,t), $(z.\bar z)$.[2] On $\mathbb{R}^2\sim\mathbb{C}$ introduce the complex coordinates $w = x+t$, $\bar w = x-t$. Let G be a complex simple Lie group. Consider a principal G bundle $\mathcal{P}$ over M and equip it with the connections

$$\begin{aligned} d+A &= (\partial_w + A_w)\otimes dw + (\partial_{\bar w}+A_{\bar w})\otimes d\bar w + (\bar\partial + \bar A)\otimes d\bar z \\ &= A_t dt + A_x dx + A_{\bar z} d\bar z. \end{aligned} \tag{2.1}$$

Let ω be a 1-form on C ($\omega = \varphi(z)dz$). It is a section of the canonical class $\mathcal{K}_C$ on C. The 4D-CS action is defined as

$$S_{4D} = \frac{1}{2\pi\hbar}\int_M \omega\wedge \mathrm{CS}(A), \tag{2.2}$$

where $\mathrm{CS}(A)$ is the standard CS action

$$\mathrm{CS}(A) := \mathrm{tr}\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\Big)$$

and A is the above-defined connection (2.1).

Beyond the points where the form ω vanishes, the equations of motion corresponding to (2.2) take the form:

$$\begin{aligned} &1.\ [D_{A_w}, D_{A_{\bar w}}] = 0,\\ &2.\ [D_{A_w}, D_{A_{\bar z}}] = 0,\\ &3.\ [D_{A_{\bar w}}, D_{A_{\bar z}}] = 0. \end{aligned} \tag{2.3}$$

These equations are invariant under the gauge transformations

$$A \to A^f = f(d+A)f^{-1}, \tag{2.4}$$
$$f\in\mathcal{G} = C^\infty(M\to G). \tag{2.5}$$

Let f be the gauge transformation fixing the gauge as $A^f_{\bar z} = A^0_{\bar z}$. We identify $A^f_w = L(w,\bar w, z)$ with the Lax operator, and $A^f_{\bar w} = M(w,\bar w,z)$ with the evolution operator M. Then the first equation in (2.3) turns into the Zakharov–Shabat type equation for some 2D integrable system:

$$\partial_{\bar w}L - \partial_w M + [M, L] = 0. \tag{2.6}$$

[1] In a recent paper [13], the authors proposed an approach to the affine Gaudin models based on the three-dimensional (3D) BF theory that is very close to the AHB construction.

[2] Here we follow notations from [8].

In most of the paper [8] it is assumed that there is a gauge choice

$$A_{\bar z} = 0, \tag{2.7}$$

or, put differently, that the moduli space of holomorphic bundles over C is empty. This is indeed true if C is a rational curve, but almost never true in the general case. For example, if C is an elliptic curve, this is possible for topologically nontrivial bundles.

If it is the case, then equations 2 and 3 from (2.3) mean that A_w and $A_{\bar w}$ are holomorphic on C and in this way they are constants. Therefore, we are left with the Zakharov–Shabat equation, where the operators L and M are independent of the spectral parameter z.

In order to come to meaningful cases with L and M depending on the spectral parameter, one should consider higher-genus curves. One more possibility is to consider additional degrees of freedom by introducing surface defects in the 4D-CS model. The surface defects come from the poles and zeros of the meromorphic 1-form ω in (2.2). The zeros of ω mean that the Lax operator has poles at these points, and the corresponding coefficients (residues) define additional degrees of freedom in the theory. These defects are called the disorder defects.

The poles of ω lead to restrictions of the gauge fields at these poles and also add degrees of freedom. These defects are called the order defects. Below we consider these defects in terms of AHB theory in greater detail.

3 Affine Higgs bundle

3.1 Three-dimensional space

Consider a principal $U(1)$-bundle W over Riemann curve Σ:

$$W \xrightarrow{\pi} \Sigma, \quad (W = U(1) \to \Sigma). \tag{3.1}$$

The total space of the bundle is called the *Seifert surface*. Let $(z, \bar z, \theta)$ be local coordinates on W and $\Omega^{(m,n,k)}(W)$ the space of corresponding (m, n, k)-forms. Redefine the 1-forms as

$$d\tilde{\bar z} = d\bar z, \quad d\tilde\theta = d\theta - n\bar\mu(z,\bar z)d\bar z. \tag{3.2}$$

Here, n is the degree of the U_1-bundle and $\bar\mu(z,\bar z) \in \Omega^{(0,-1,1)}$ is the Beltrami differential. Consider $\Omega^{(1,0)}(\Sigma)$-form dz on Σ and let $\pi^*(dz) \in \Omega^{(1,0,0)}(W)$.

Define two vector fields on W, which annihilate the form $\pi^* dz$:

$$1.\ \partial_\theta, \quad 2.\ \partial_{\bar z}^{\bar\mu}.$$

The first field ∂_θ acts along the S^1 fibers and thereby annihilates the form $\pi^* dz$. For the second field $\partial_{\bar z}^{\bar\mu}$, this condition means that

$$\partial_{\bar z}^{\bar\mu} = \partial_{\bar z} + n\bar\mu(z,\bar z)\partial_\theta. \tag{3.3}$$

Let

$$\tilde\theta = \theta - n\int^{\bar z} \bar\mu(z,\bar z) \tag{3.4}$$

be a local coordinate in the bundle W. Then for a smooth function f,

$$\partial_{\bar z}^{\bar\mu} f(\tilde\theta) = 0. \tag{3.5}$$

Consider a line bundle $\mathcal{L}$ over Σ_g, which is a complexification of the $U(1)$-bundle. Let $D_z \subset \Sigma_g$ be a small disc with the center $z = 0$ and $D'_z \subset D_z$ The degree n of the bundle is defined by a holomorphic nonvanishing transition function $f(z)$ on $D_z \backslash D'_z$. The degree can be changed by the multiplication $f(z) \to f(z)w(z)$ as follows.

$$\theta \to \theta - k \cdot \arg(w). \tag{3.6}$$

This procedure is called the *modification* of the $U(1)$-bundle.

If the bundle W is trivial, then one can take $n = 0$. In the examples below we assume $n = 0$.

Let G be a complex Lie group and $\mathcal{P}$ a principal G-bundle over W. We first define the *affine Higgs bundle* (AHB) over W as a pair of connections

$$(D_{\bar A,\bar\mu} = \partial_{\bar z}^{\bar\mu} + A_{\bar z},\ \partial_\theta + A_\theta). \tag{3.7}$$

The first component $\partial_{\bar z}^{\bar\mu} + A_{\bar z}$ defines the complex structure on the sections of $\mathcal{P}$ in the $(\bar z, \theta)$ direction. The precise definition of the AHB is given below (3.17). The second component is the *Higgs connection*. It is an affine analogue of the Higgs field introduced by Hitchin [21].

3.2 Affine holomorphic bundles

The affine Higgs bundles are the cotangent bundles to the affine holomorphic bundles, which we will define.

In the previous subsection we introduced the connection acting on the sections $\Gamma(\mathcal{P})$ (3.7):

$$D_{\bar A,\bar\mu} = (\partial_{\bar z} + \bar\mu(z,\bar z)\partial + \bar A(z,\bar z,x)) \otimes d\bar z.$$

Consider, in addition, a line bundle $\mathcal{L}$ over Σ with the connection $(\partial_{\bar z} + \bar k_{\bar z}) \otimes d\bar z$. The anti-holomorphic connection on $\mathcal{P} \oplus \mathcal{L}$ is the pair of operators

$$\nabla_{\bar A,\bar\mu,\bar k} = \begin{pmatrix} D_{\bar A,\bar\mu} \\ (\partial_{\bar z} + \bar k(z,\bar z)) \otimes d\bar z \end{pmatrix}. \tag{3.8}$$

Let $G(W)$ be a smooth map of W to G

$$G(W) = C^\infty(W \to G),$$

$$G(W) = \left\{ \sum_j f_j(z,\bar z)e^{j\theta},\ |\ f_j \in C^\infty(\Sigma \to G) \right\}.$$

It can be considered as a map of the spectral curve Σ to the loop group

$$G(W) = C^\infty(\Sigma \to L(G)). \tag{3.9}$$

The structure group of the bundle $\mathcal{P} \oplus \mathcal{L}$ (the gauge group) is defined by replacing $L(G)$ with its central and cocentral extensions (A.7):

$$\check{\mathcal{G}} := C^\infty(\Sigma \to \check{L}(G)).$$

More precisely,

$$\check{\mathcal{G}} = (G(W), \{\exp(\varepsilon_3(z,\bar{z})\}) \rtimes \{\exp(\varepsilon_2(z,\bar{z})\partial_\theta\}, \tag{3.10}$$

$$\varepsilon_2(z,\bar{z})\partial_\theta \in C^\infty(\Sigma \to \mathbb{C}),\ \varepsilon_3(z,\bar{z}) \in C^\infty(\Sigma \to \mathbb{C}).$$

Consider its infinitesimal action on $\nabla_{\bar{A},\bar{\mu},\bar{k}}$. As a vector space the Lie algebra Lie$(\hat{\mathcal{G}}^G)$ has three components:

$$\mathrm{Lie}(\hat{\mathcal{G}}) = M_1 \oplus M_2 \oplus M_3, \tag{3.11}$$

$$M_1 = C^\infty(M \to \mathfrak{g}) = \{\epsilon_1(z,\bar{z},\theta)\},$$
$$M_2 = C^\infty(\Sigma \to \mathbb{C}) = \{\varepsilon_2(z,\bar{z})\partial_\theta\},$$
$$M_3 = C^\infty(\Sigma \to \mathbb{C})) = \{\varepsilon_3(z,\bar{z})\}).$$

Their action on $\nabla_{\bar{A}}$ takes the form:

$$\begin{array}{lll} 1.\ \delta_{\epsilon_1}\bar{A} = -(\partial_{\bar{z}} + \bar{\mu}\partial)\epsilon_1 & & \\ \quad +[\epsilon_1, \bar{A}], & \delta_{\epsilon_1}\bar{\mu} = 0, & \delta_{\epsilon_1}\bar{k} = \langle \bar{A}\partial\epsilon_1\rangle, \\ 2.\ \delta_{\varepsilon_2}\bar{A} = \varepsilon_2\partial_\theta\bar{A}, & \delta_{\varepsilon_2}\bar{\mu} = -\partial_{\bar{z}}\varepsilon_2, & \delta_{\varepsilon_2}\bar{k} = 0, \\ 3.\ \delta_{\varepsilon_3}\bar{A} = 0, & \delta_{\varepsilon_3}\bar{\mu} = 0, & \delta_{\varepsilon_3}\bar{k} = -\partial_{\bar{z}}\varepsilon_3. \end{array} \tag{3.12}$$

The moduli of holomorphic structures on $\mathcal{P}(M) \oplus \mathcal{L}$ is the quotient space

$$\mathrm{Bun}_{G,M} = \nabla_{\bar{A},\bar{\mu},\bar{k}}/\check{\mathcal{G}} = \nabla_{\bar{L},\bar{\mu},\bar{k}}, \tag{3.13}$$

where we fix the gauge as $\bar{A} \to \bar{A}^f = \bar{L}$, i.e.,

$$\bar{L} = \bar{A}^f = f\partial_{\bar{z}}f^{-1} + f\bar{A}f^{-1}. \tag{3.14}$$

One can fix the action of the abelian subgroups $\{\exp(\varepsilon_3)\}$, $\{\exp(\varepsilon_3(z,\bar{z})\partial)\}$ on $\bar{\mu}$ and $\bar{k}$ (3.12) in a similar way. We preserve the notations for the gauge-transformed variables $\bar{\mu}$ and $\bar{k}$.

3.2.1 Affine Higgs bundles

Introduce the Higgs field $\Phi(z,\bar{z},\theta)$. Let $\mathcal{K}$ be a canonical class of Σ. Then the Higgs field is $\Phi(z,\bar{z},\theta) \in C^\infty(\Sigma \to (L(\mathfrak{g}) \otimes \mathrm{d}\theta) \otimes \mathcal{K}$.

Let $\nu(z,\bar{z}),\ r(z,\bar{z}) \in \Omega^{(1,0)}(\Sigma)$. Define

$$\nabla_{\Phi,\nu,r} = \begin{pmatrix} D_{\Phi,\nu} \\ r(z,\bar{z}) \end{pmatrix} \otimes \mathcal{K}, \tag{3.15}$$

$$D_{\Phi,\nu} = (\nu(z,\bar{z})\partial_\theta + \Phi(z,\bar{z},x))\mathrm{d}\theta. \tag{3.16}$$

Table 1 Dimensions of fields

	z	$\bar{z}$	θ
$\bar{A}$	0	1	0
A_θ	0	0	1
$\bar{\mu}$	0	1	−1
$\bar{k}$	0	1	0
Φ	1	0	1
ν	1	0	0
r	1	0	1

The affine Higgs bundle is the pair

$$\mathcal{H}^{\mathrm{aff}}(G) = (\nabla_{\bar{A},\bar{\mu},\bar{k}},\ \nabla_{\Phi,\nu,r}) \sim T^*\nabla_{\bar{A},\bar{\mu},\bar{k}} = \{\bar{A},\ \bar{\mu},\ \bar{k},\ \Phi,\ \nu,\ r\}. \tag{3.17}$$

The connection form A_θ in (3.7) is related to the Higgs field Φ as

$$A_\theta = \frac{\Phi}{\nu}. \tag{3.18}$$

The fields of the Higgs bundles have the following dimensions (Table 1):

The cotangent bundle structure of the AHB comes from the pairing (A.10) $\mathcal{H}^{\mathrm{aff}}(G) = T^*\nabla_{\bar{A},\bar{\mu},\bar{k}}$.
Define the symplectic form Ω on $\mathcal{H}^{\mathrm{aff}}(G)$

$$\Omega = \frac{1}{\pi}\int_\Sigma |\mathrm{d}^2 z|\Big(\langle\delta\Phi,\delta\bar{A}\rangle + \delta r\delta\bar{\mu} + \delta\nu\delta\bar{k}\Big), \tag{3.19}$$

where

$$\langle\delta\Phi,\delta\bar{A}\rangle = \frac{1}{2\pi}\int_{S^1}(\delta\Phi,\delta\bar{A}).$$

The form is invariant under the action of the gauge group $\hat{\mathcal{G}}$ (3.10). Along with (3.12), the corresponding Hamiltonian vector fields are as follows:

$$\begin{array}{ll} 1.\ \delta_{\epsilon_1}\Phi = \nu\partial_\theta\epsilon_1 + [\Phi,\epsilon_1], & \delta_{\epsilon_1}\nu = 0,\ \delta_{\epsilon_1}r = \langle\Phi,\partial\epsilon_1\rangle, \\ 2.\ \delta_{\varepsilon_2}\Phi = \varepsilon_2\partial_\theta\Phi, & \delta_{\varepsilon_2}\nu = 0,\ \delta_{\varepsilon_2}r = 0, \\ 3.\ \delta_{\varepsilon_3}\Phi = 0, & \delta_{\varepsilon_3}\nu = 0,\ \delta_{\varepsilon_3}r = 0. \end{array} \tag{3.20}$$

The action of $\hat{\mathcal{G}}$ is generated by the moment maps m_j : $\mathcal{H}(G) \to \mathrm{Lie}^*(\hat{\mathcal{G}})$, where

$$Lie^*(\hat{\mathcal{G}}^G) = M_1^* \oplus (M_2^* \sim M_3) \oplus (M_3^* \sim M_2). \tag{3.21}$$

More explicitly,

$$m_1 = (\partial_{\bar{z}} + \bar{\mu}\partial_\theta))\Phi - \nu\partial_\theta\bar{A} + [\bar{A},\Phi] \in M_1^*,$$
$$(m_1 = [D_{\bar{A},\bar{\mu},\bar{k}}, D_{\Phi,\nu,r}]),$$
$$m_2 = \int_{S^1}\langle\partial_\theta\Phi,\bar{A}\rangle - \partial_{\bar{z}}r \in M_2^*,$$
$$m_3 = \partial_{\bar{z}}\nu \in M_3^*.$$

Let $\mathcal{C}^{\text{aff}}(\bar{A}, \bar{\mu}, \bar{k}|\Phi, \nu, r)$ be the set of solutions of the moment equations $m_j = 0$, $(j = 1, 2, 3)$

$$\begin{cases} (\partial_{\bar{z}} + \bar{\mu}\partial_\theta)\Phi - \nu\partial_\theta \bar{A} + [\bar{A}, \Phi] = 0, \\ ([D_{\bar{A},\bar{\mu},\bar{k}}, D_{\Phi,\nu,r}] = 0), \\ m_2 = \int_{S^1}\langle \partial_\theta \Phi, \bar{A}\rangle - \partial_{\bar{z}} r = 0, \\ m_3 = \partial_{\bar{z}}\nu = 0. \end{cases} \tag{3.22}$$

The quotient of $\mathcal{C}^{\text{aff}}$ under the action of the gauge group $\check{\mathcal{G}}$ (3.10) is the moduli space of the affine Higgs bundles:

$$\mathfrak{M}^{\text{aff}}(G) = \mathcal{H}^{\text{aff}}(G)//\check{\mathcal{G}} \sim \mathcal{C}^{\text{aff}}/\check{\mathcal{G}}. \tag{3.23}$$

We can first fix the gauge and then solve the moment map equations. In this respect, $\mathfrak{M}^{\text{aff}}(G)$ is defined as the set of solutions of equations

$$(\partial_{\bar{z}} + \bar{\mu}\partial_\theta)L - \nu\partial_\theta \bar{L} + [\bar{L}, L] = 0, \tag{3.24}$$

$$\partial_{\bar{z}} r = \int_{S^1}\langle \partial_\theta L, \bar{L}\rangle, \quad \partial_{\bar{z}}\nu = 0.$$

3.2.2 *Parabolic structures. The order defects*

To introduce the parabolic structure, we attach the coadjoint orbits $\mathcal{O}_a = \mathcal{O}(p_a^{(0)}, c_a^{(0)})$ of the loop group $L(G)$ (A.14) to the marked points $z_a \in \Sigma$, $a = 1, \ldots, n$. This means that we add the order defects in the theory. The disorder defects correspond to the reduction of the gauge group $\check{\mathcal{G}}$ (3.10) to the subgroup $\check{\mathcal{G}}(\times_a Fl_a) \subset \check{\mathcal{G}}$, which preserves the affine flags Fl_a at the marked points. It was proved in [22] that these constructions are equivalent. Here we follow the order defects description.

The affine parabolic Higgs bundle has the following field content:

$$\mathcal{H}^{\text{aff, par}}(G) = (\bar{A},\ \bar{\mu},\ \bar{k},\ \Phi,\ \nu,\ r,\ \cup_{a=1}^n \mathcal{O}_a). \tag{3.25}$$

The coadjoint orbits (A.14) are equipped with the Kirillov–Kostant symplectic form (A.15). Thereby, the symplectic form on the reduced parabolic Higgs bundle $\mathcal{H}^{\text{aff par}}(G)$ is equal to

$$\Omega - \sum_{a=1}^n \omega_a(p_a^{(0)}, c_a^{(0)}), \tag{3.26}$$

where Ω is the form (3.19) and ω_a are the Kirillov–Kostant forms (A.15). Due to the presence of new terms in the form, the moment map constraints (3.22) are upgraded as

$$m_1 = \sum_{a=1}^n S(p_a^{(0)}, c_a^{(0)})\delta(z - z_a, \bar{z} - \bar{z}_a),$$

$$m_3 = \sum_{a=1}^n c_a^{(0)}\delta(z - z_a, \bar{z} - \bar{z}_a),$$

so that

$$\partial_{\bar{z}}\Phi - \nu\partial_\theta A_{\bar{z}} + [\bar{A}, \Phi] = \sum_{a=1}^n S(p_a^{(0)}, c_a^{(0)})\delta(z - z_a, \bar{z} - \bar{z}_a), \tag{3.27}$$

$$\partial_{\bar{z}}\nu = \sum_{a=1}^n c_a^{(0)}\delta(z - z_a, \bar{z} - \bar{z}_a). \tag{3.28}$$

This means that ν is not a constant in (3.27) but a meromorphic $(1, 0)$-form on Σ with the first-order poles at $z = z_a$:

$$\nu|_{z\to z_a} \sim \frac{c_a^0}{z - z_a}. \tag{3.29}$$

In other words, $\nu =$ const implies that we deal only with orbits without central extension, i.e.,

$$S_a = g p_a^{(0)} g^{-1}. \tag{3.30}$$

Since $\sum_{a=1}^n c_a^{(0)} = 0$, in the case of a single marked point (likewise for the LL equation), the orbit has the form (3.30), and $\nu = \nu^0$ is a constant.

Next, we pass to the symplectic quotient (the moduli space). Let us fix a gauge as in (3.14) and

$$L = \nu f^{-1}\partial_\theta f + f^{-1}\Phi f, \quad (f \in \hat{\mathcal{G}}), \tag{3.31}$$

$$L/\nu = f^{-1}\partial_\theta f + f^{-1}A_\theta f. \tag{3.32}$$

The moment map constraint equation (3.27) with $m_1 = 0$ is modified as

$$\partial_{\bar{z}} L - \nu\partial_\theta \bar{L} + [\bar{L}, L] = \sum_{a=1}^n \delta(z - z_a)S_a,$$

$$\left([D_{\bar{L},\bar{\mu}}, D_{L,\nu}] = \sum_{a=1}^n \delta(z - z_a)S_a\right). \tag{3.33}$$

Solutions of this equation along with (3.28) define the moduli space of the affine parabolic bundles as the symplectic quotient space

$$\mathcal{H}^{\text{aff, par}}(G)//\mathcal{G} \sim \mathfrak{M}^{\text{aff, par}}(G). \tag{3.34}$$

It is a phase space of 2D integrable systems. The symplectic form (3.26) on $\mathcal{M}^{\text{aff, par}}(G)$ turns into (see (3.26))

$$\Omega^{\text{par}} = \int_\Sigma \left(\langle \delta L|\delta \bar{L}\rangle + \delta\nu\delta\bar{k} + \delta r\delta\bar{\mu}\right) - \sum_{\alpha=1}^n \omega_\alpha. \tag{3.35}$$

3.3 Equations of motion

Let $W = S^1 \times \Sigma$ be a trivial bundle. The measure on W is $\varpi(z, \bar{z})d\theta$, where $\varpi(z, \bar{z}) \in \Omega^{(1,1)}(\Sigma)$ is a $(1, 1)$-form on Σ. The gauge-invariant integrals are generated by the traces of the monodromies of the Higgs field A_θ. We take the Hamiltonian in the form:

$$H(\Phi,\nu)=\int_\Sigma \varpi(z,\bar z)\left(\mathrm{tr}\exp\oint_{S^1}A_\theta(z,\bar z,\theta)\right)$$
$$=\int_\Sigma \varpi(z,\bar z)\left(\mathrm{tr}\exp\frac{1}{\nu(z,\bar z)}\oint_{S^1}\Phi(z,\bar z,\theta)\right). \tag{3.36}$$

Consider equations of motion on the "upstairs" space $\mathcal{H}^{\mathrm{aff}}(G)$ (3.25). They are derived by means of the symplectic form (3.26) and the Hamiltonians (3.36). In this way we obtain the following free system:

$$\dot\Phi=0, \tag{3.37}$$
$$\dot{\bar A}(z,\bar z)=\frac{\delta\mathcal{H}}{\delta\Phi(z,\bar z,\theta)}$$
$$=\frac{\varpi(z,\bar z)}{\nu(z,\bar z)}\exp\frac{1}{\nu(z,\bar z)}\oint_{S^1}\Phi(z,\bar z,\theta)\mathrm{d}\theta, \tag{3.38}$$
$$\dot\nu=0,\quad \dot{\bar\mu}=0,\quad \dot r=0, \tag{3.39}$$
$$\dot{\bar k}(z,\bar z)=\frac{\delta\mathcal{H}}{\delta\nu(z,\bar z)}$$
$$=-\frac{\varpi(z,\bar z)}{\nu^2(z,\bar z)}\oint_{S^1}\Phi(z,\bar z,\theta))\mathrm{d}\theta\exp\int_{S^1}\frac{\Phi(z,\bar z,\theta)}{\nu(z,\bar z)}\mathrm{d}\theta. \tag{3.40}$$

Recall that after the symplectic reduction we obtain the fields $\bar L$ (3.14) and L (3.31). For simplicity, we keep the same notation for the coadjoint orbit variables S_α, so they are transformed as in (A.17). This yields

$$H(L,\nu)=\int_\Sigma \omega(z,\bar z)\left(\mathrm{tr}\exp\frac{1}{\nu(z,\bar z)}\oint_{S^1}\mathrm{d}\theta L(z,\bar z,\theta)\right). \tag{3.41}$$

Let W be a nontrivial bundle ($n\neq 0$). It follows from (3.5) that $\bar L$ depends on $\tilde\theta$ (3.4). The moment equation (3.33) takes the form

$$(\partial_{\bar z}+n\bar\mu\partial_\theta)L-\nu\partial_\theta\bar L+[\bar L,L]=\sum_{a=1}^n\delta(z-z_a)S_a,$$
$$\left([D_{\bar L,\bar\mu},D_{L,\nu}]=\sum_{a=1}^n\delta(z-z_a)S_a\right).$$

Its solution L has the same form as for $n=0$, but the angle parameter θ is replaced with $\tilde\theta$.

The corresponding monodromy matrix is conjugated to the original monodromy matrix

$$\exp\frac{1}{\nu(z,\bar z)}\oint_{S^1}\mathrm{d}\theta L(z,\bar z,\tilde\theta)$$
$$=\Xi(n)\left(\exp\frac{1}{\nu(z,\bar z)}\oint_{S^1}\mathrm{d}\theta L(z,\bar z,\theta)\right)\Xi(n)^{-1},$$

where the gauge transformation assumes the form

$$\Xi(n)=\exp\int_0^\delta \mathrm{d}\theta L(z,\bar z,\theta),\quad \delta=n\int^{\bar z}\bar\mu.$$

In this way, as we claimed in the Introduction, the invariants of the monodromy matrix and, in particular, the Hamiltonian are independent of n.

It follows from the moment map equation (3.33) that for the parabolic bundles, the Lax operator L has first-order poles at the marked points z_a. Let $w_a=z-z_a$. The generating function of the Hamiltonians (3.41) has the expansion:

$$H(L,\nu)=\sum_{a\in I}\sum_{j=-1}^{+\infty}H_j^a w_a^j. \tag{3.42}$$

Consider the set of times $T_{a,j}=\{t_{a,j}\}$ corresponding to the Hamiltonians H_j^a. The one-dimensional spaces $T_{a,j}$ are isomorphic to $\mathbb{R}$. Let $\partial_{a,j}=\{H_j^a,\ \}$ be the Poisson vector field on the moduli space $\mathfrak{M}^{\mathrm{aff,\,par}}(G)$ (3.34). Assume that the gauge transformation f comes from the gauge fixation (3.14). Define the connection form $M_{a,j}=\partial_{a,j}ff^{-1}$. From (3.31) we have $\Phi=-\nu\partial ff^{-1}+fLf^{-1}$. Plugging it into (3.37) we obtain the Zakharov–Shabat equation

$$\partial_{a,j}L-\nu\partial_\theta M_{a,j}+[M_{a,j},L]=0,$$
$$\left([D_{M_{a,j}},D_L]=0\right), \tag{3.43}$$

where $D_{M_{a,j}}=\partial_{a,j}+M_{a,j}$. Notice that the variables on the moduli space L, $\bar L$, S_a do not depend on $\bar k$. In this way the dynamics of $\bar k$ (3.40) is inessential. The operators $M_{a,j}$ can be restored partly from Eq. (3.38):

$$\bar\partial M_{a,j}-\partial_{a,j}\bar L+[M_{a,j},\bar L]=\frac{\delta H_j^a}{\delta L},$$
$$\left([D_{\bar L},D_{M_{a,j}}]=\frac{\delta H_j^a}{\delta L}\right), \tag{3.44}$$

where

$$\frac{\delta H_j^a(L)}{\delta L}=f\frac{\delta H_j^a(\Phi)}{\delta\Phi}f^{-1}. \tag{3.45}$$

Equations (3.43) and (3.44) along with the moment constraint equation (3.33) yield the system:

$$\begin{aligned}&1.\ [D_{M_{a,j}},D_{L,\nu}]=0,\\ &2.\ [D_{\bar L},D_{M_{a,j}}]=\tfrac{\delta H_j^a}{\delta L},\\ &3.\ [D_{\bar L},D_{L,\nu}]=\textstyle\sum_{a=1}^n\delta(z-z_a)S_a.\end{aligned} \tag{3.46}$$

Let V be a module of the Lie algebra $\mathfrak{g}$.

Consider the associated bundle $E=\mathcal{P}\times_G V$, where $\mathcal{P}$ is the principal G-bundle over W. Equivalently, we can consider the associated vector $L(G)$-bundle over Σ. Let Ψ be a section of E. Consider the linear system

$$
\begin{aligned}
&1.\ (\nu\partial + L)\Psi = 0,\\
&2.\ (\partial_{\bar z} + \bar L)\Psi = 0,\\
&3.\ (\partial_{a,j} + M_{a,j})\Psi = 0.
\end{aligned}
\tag{3.47}
$$

Then Eq. (1.3.46) is the consistency condition for equations 1 and 3, and Eq. (3.3.46) is the consistency conditions for equations 1 and 2.

3.4 Conservation laws

The matrix equation (1.3.47) allows one to write down the conservation laws. The eigenvalues of the monodromy matrix of solutions Ψ are gauge-invariant. Represent solutions of (1.3.47) as the P-exponent

$$
\Psi(\theta, z) = \mathcal{R}(\theta, z) P \exp\left(\frac{\imath}{\nu(z)} \int_0^\theta L(\theta', z) \mathrm{d}\theta'\right) \quad (x = -t e^{\imath\theta}),
\tag{3.48}
$$

where $\mathcal{R}$ is periodic in θ. The monodromy of $\Psi(\theta, z)$ is

$$
\exp\left(\frac{1}{\nu(z)} \int_0^{2\pi} L(\theta, z) \mathrm{d}\theta\right).
$$

Consider the monodromy in a neighborhood of a pole $z_a \in \Sigma$ of L/ν with a local coordinate $w_a = z - z_a$. If

$$
\frac{1}{\nu(w_a)} L(\theta, w_a) = \left(\frac{L}{\nu}\right)^a_{-1} w_a^{-1} + \left(\frac{L}{\nu}\right)^a_0 + \left(\frac{L}{\nu}\right)^a_1 w_a + \cdots .
$$

The Hamiltonians

$$
H^a_j \sim \mathrm{tr}_V \exp\left(\imath \int_0^{2\pi} \left(\frac{L}{\nu}\right)^a_j \mathrm{d}\theta\right).
\tag{3.49}
$$

are all in involution. Thus, we have an infinite set of Poisson-commuting integrals of motion.

Let us "diagonalize" generic element $L \to h^{-1}\nu\partial h + h^{-1}Lh = \mathcal{S}$, where $\mathcal{S}$ is an element of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then the solutions of the equation (1.3.47) can be represented in the form

$$
\Psi(\theta, z) = \mathcal{R}(\theta, z) \exp\left(\frac{\imath}{\nu} \int_0^\theta \mathcal{S}(\theta', z) \mathrm{d}\theta'\right).
\tag{3.50}
$$

Let

$$
\frac{1}{\nu(w_a)} \mathcal{S}(\theta, w_a) = \left(\frac{\mathcal{S}(\theta)}{\nu}\right)^a_{-1} w_a^{-1} + \left(\frac{\mathcal{S}(\theta)}{\nu}\right)^a_0 + \left(\frac{\mathcal{S}(\theta)}{\nu}\right)^a_1 w_a + \cdots .
$$

Substitute (3.50) into (1.3.47). It follows from (3.41), (3.42) and (3.50) that the diagonal matrix elements of $\mathcal{S}^m_j$ are the densities of the conservation laws

$$
H^a_j \sim \mathrm{tr}_V \exp\left\{\imath \int_{S^1} \left(\frac{\mathcal{S}(\theta)}{\nu}\right)^a_j \mathrm{d}\theta\right\}.
\tag{3.51}
$$

There is a recurrence procedure to define the matrices $\mathcal{S}^a_j$. Details can be found in [9,23].

3.5 The action

Consider the $4D$ action on the space

$$
\mathcal{M}_{a,j} = T_{a,j} \times W
\tag{3.52}
$$

corresponding to the Hamiltonian system defined above[3]:

$$
\mathcal{S}^{AHB} = \frac{1}{2\pi\hbar} \sum_{a,j} \left(\int_{\mathcal{M}_{a,j}} (\Phi, D\bar A) - \sum_{a=1}^n \mathcal{S}^{WZW}(S_a)\delta(z_a, \bar z_a) - H^a_j(\Phi) D t_{a,j}\right).
$$

Here, H^a_j are the Hamiltonians (3.36) and $\mathcal{S}^{WZW}$ is the Wess–Zumino–Witten action

$$
\mathcal{S}^{WZW} = \frac{1}{2}\int_{S^1} \mathrm{d}\theta (S, Dgg^{-1}) + \frac{c^0}{2}\left(\int_{S^1} \mathrm{d}\theta (Dgg^{-1}, \partial gg^{-1}) + D^{-1}(\partial gg^{-1}, (Dgg^{-1})^2)\right).
$$

To come to the action on the moduli space of the affine Higgs bundles $\mathcal{H}^{\mathrm{aff,\,par}}(G)$ (3.25), we need to impose the moment map constraints (3.27) and fix the gauge. To do this, one should introduce in the action the terms containing the ghost and the anti-ghost fields. Instead, we first fix the gauge and rewrite the action in terms of the fields L and $\bar L$. The action takes the form

$$
\mathcal{S}^{AHB} = \frac{1}{2\pi\hbar} \sum_{j,a} \left(\int_{\mathcal{M}^a_j} (L, D\bar L) - \sum_{a=1}^n \mathcal{S}^{WZW}(S_a)\delta(z_a, \bar z_a) - \sum_{a,j} H^a_j D t_{a,j}\right),
$$

and then we impose the moment constraints (3.33).

4 Examples

In all examples, we consider the trivial S^1 bundles and put $\bar\mu = 0$.

4.1 Hamiltonians in the sl_2 case

Consider the one marked point case. Then, $c^{(0)} = 0$. Due to (3.28), $\partial_{\bar z}\nu = 0$ and, therefore, $\nu(z, \bar z) = \mathrm{const} = \nu_0$.

Let us perform the gauge transformation

$$
f^{-1} L f + \nu_0 f^{-1} \partial_\theta f = L',
\tag{4.1}
$$

[3] We omit the term $\nu D\bar k$ since, as we argued above, it is inessential.

with f defined as follows:

$$f=\begin{pmatrix}\sqrt{L_{12}} & 0\\ -\frac{L_{11}}{\sqrt{L_{12}}}-\nu_0\frac{\partial_\theta\sqrt{L_{12}}}{L_{12}} & \frac{1}{\sqrt{L_{12}}}.\end{pmatrix}. \tag{4.2}$$

Then the Lax matrix L is transformed into

$$L'=\begin{pmatrix}0 & 1\\ T & 0\end{pmatrix}, \tag{4.3}$$

where

$$T=L_{21}L_{12}+L_{11}^2+\nu_0\frac{L_{11}\partial_\theta L_{12}}{L_{12}}-\nu_0\partial_\theta L_{11} -\frac{1}{2}\nu_0^2\frac{\partial_\theta^2 L_{12}}{L_{12}}+\frac{3}{4}\nu_0^2\frac{(\partial_\theta L_{12})^2}{L_{12}^2}. \tag{4.4}$$

The linear problem

$$\begin{cases}(\nu_0\partial_\theta+L')\psi=0,\\ (\partial_j+M'_j)\psi=0,\end{cases} \tag{4.5}$$

where ψ is the Bloch wave function $\psi=\exp\{-i\oint\chi\}$, leads to the Riccati equation:

$$i\nu_0\partial_\theta\chi-\chi^2+T=0. \tag{4.6}$$

The decomposition of $\chi(z)$ provides densities of the conservation laws (see [24]):

$$\chi=\sum_{k=-1}^{\infty}z^k\chi_k, \tag{4.7}$$

$$H_k\sim\oint d\theta\,\chi_{k-1}. \tag{4.8}$$

The values of χ_k can be found from (4.6) using the expression (4.4) for $T(z)=\sum_{k=-2}^{\infty}z^kT_k$ in a neighborhood of zero. For $k=-2,\ -1$ and 0 we have:

$$\begin{cases}\chi_{-1}=\sqrt{T_{-2}}=\sqrt{h},\\ 2\sqrt{h}\chi_0=T_{-1}+i\nu_0\partial_\theta\chi_{-1}=T_{-1},\\ 2\sqrt{h}\chi_1=T_0+i\nu_0\partial_\theta\chi-\chi_0^2.\end{cases} \tag{4.9}$$

4.2 Landau–Lifshitz equation (LL)

In this case, $G=\mathrm{SL}(2,\mathbb{C})$. Let $\Sigma=\Sigma_\tau=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be the elliptic curve with one marked point $z=0$. Then the orbit has the form $\mathcal{O}=\{S=gp^{(0)}g^{-1}\}$, and $c^{(0)}=0$ (3.30), i.e., S is a traceless 2×2 matrix.

Impose the following quasiperiodic properties (boundary conditions) on the fields. Here we use the basis of the Pauli matrices σ_a $(a=0,\dots,3)$:

$$\sigma_0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-\imath\\ \imath&0\end{pmatrix},\ \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Table 2 Quasiperiodicities of LL fields

		$z\to z+1$	$z\to z+\tau$
1	$\bar{A}$	$\mathrm{Ad}_{\sigma_3}\bar{A}(z,\bar{z},\theta)$	$\mathrm{Ad}_{\sigma_1}\bar{A}(z,\bar{z},\theta)$
2	Φ	$\mathrm{Ad}_{\sigma_3}\Phi(z,\bar{z},\theta)$	$\mathrm{Ad}_{\sigma_1}\Phi(z,\bar{z},\theta)$
3	ϵ	$\mathrm{Ad}_{\sigma_3}\epsilon(z,\bar{z},\theta)$	$\mathrm{Ad}_{\sigma_1}\epsilon(z,\bar{z},\theta)$

By the gauge transformations $f(z,\bar{z},\theta)$, the field $\bar{A}$ can be made z-independent. Due to the boundary conditions in Table 2, $\bar{L}=0$, so that

$$f(z,\bar{z},\theta)(\partial_{\bar{z}}+\bar{A}(z,\bar{z},\theta))f^{-1}(z,\bar{z},\theta)=0.$$

Then the Lax operator of the LL equation is defined as

$$f(z,\bar{z},\theta)(\nu_0\partial_\theta+\Phi(z,\bar{z},\theta))f^{-1}(z,\bar{z},\theta) =\nu_0\partial_\theta+L^{LL}(z,\bar{z},\theta).$$

It satisfies the moment map equation

$$\partial_{\bar{z}}L^{LL}(z,\bar{z},\theta)=S(\theta)\delta(z,\bar{z})$$

and has the quasiperiodicities as the Higgs field Φ in Table 2.

To write it down we use the Kronecker elliptic function related to the curve Σ_τ:

$$\phi(u,z)=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}, \tag{4.10}$$

where $\vartheta(z)$ is the theta-function

$$\vartheta(z|\tau)=q^{\frac{1}{8}}\sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)},\quad q=\exp 2\pi\imath\tau. \tag{4.11}$$

The Kronecker function has the following quasiperiodicities:

$$\phi(u,z+1)=\phi(u,z),\quad \phi(u,z+\tau)=e^{-2\pi\imath u}\phi(u,z), \tag{4.12}$$

and has the first-order pole at $z=0$

$$\phi(u,z)=\frac{1}{z}+\frac{\vartheta'(u)}{\vartheta(u)}+O(z). \tag{4.13}$$

It is related to the Weierstrass function $\wp$ as follows:

$$\phi(u,z)\phi(-u,z)=\wp(z)-\wp(u). \tag{4.14}$$

Let

$$\varphi_1(z)=\phi\left(\frac{1}{2},z\right),\quad \varphi_2(z)=\exp(\pi\imath z)\phi\left(\frac{1+\tau}{2},z\right),\quad \varphi_3(z)=\exp(\pi\imath z)\phi\left(\frac{\tau}{2},z\right).$$

The Lax operator assumes the form

$$L^{LL}(z,\bar{z},\theta)=\sum_{\alpha=1}^{3}L_\alpha(z,\theta)\sigma_\alpha,\quad L_\alpha(z,\theta)=S_\alpha(\theta)\varphi_\alpha(z). \tag{4.15}$$

The symplectic form Ω (3.26) is reduced to the symplectic form on the orbit $\mathcal{O}(p^{(0)},0)$ (A.15):

$$\tilde{\Omega} = \omega(p^{(0)},0) = -\int_{S^1} D(S(p^{(0)},0)g^{-1}Dg). \tag{4.16}$$

The Hamiltonian H_2^{LL} (4.8) assumes the form

$$H_2 = \frac{1}{2}\int_{S^1} d\theta \sum_\alpha \left(S_\alpha(\theta)\wp_\alpha S_\alpha(\theta) + \left(\frac{\nu_0}{2p^{(0)}}\partial_\theta S_\alpha(\theta)\right)^2\right),$$

where $\wp_\alpha$ are the values of the Weierstrass functions at the half-periods. It is the Hamiltonian of the Euler–Arnold top on the group $L(G)$ defined by the inverse inertia tensor

$$J = \sum_\alpha \left(-\left(\frac{\nu_0}{2p^{(0)}}\partial_\theta\right)^2 + \wp_\alpha \right) : L^*(G) \to L(G).$$

The corresponding equations of motion (see (A.16)) are the LL equations:

$$\partial_t S = [S, J(S)] + \left[S, \left(\frac{\nu_0}{2p^{(0)}}\partial_\theta\right)^2 S\right]. \tag{4.17}$$

4.3 Calogero–Moser field theory (CM)

Again, consider the one-point case on the elliptic curve Σ_τ and the trivial $\widehat{\mathrm{SL}}(2,\mathbb{C})$ bundle over Σ_τ. It has a moduli space $Bun_{\mathrm{SL}(2,\mathbb{C})} \sim \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. Let $u = u(\theta)$ be a coordinate on the moduli space $Bun_{\mathrm{SL}(2,\mathbb{C})}$, and denote $\boldsymbol{e}(u) = \exp 2\pi\imath u\sigma_3$. Assume that the fields have the following quasiperiodicities (Table 3):

For stable bundles, the orbits of the gauge transformations (3.14) $\bar{A} \xrightarrow{f} \bar{L}$ are parameterized by the z-independent diagonal matrices $\bar{L}$. Let us take them in the form

$$\bar{L} = \frac{2\pi\imath}{\tau-\bar{\tau}}\,\mathrm{diag}(u,-u). \tag{4.18}$$

As above, we have $\nu = \nu_0$. The solution of the moment map equation (3.33)

$$\partial_{\bar{z}} L - \nu_0\partial_\theta \bar{L} + [\bar{L}, L] = \delta(z,\bar{z})S,$$

where $L = f^{-1}\nu_0\partial f + f^{-1}\Phi f$ is the Lax operator. We should the factorized solutions of this equation by to the action of the residual gauge group that preserves the gauge fixing (4.18). It is the group constant diagonal matrices

Table 3 Quasiperiodicities of CM fields

		$z \to z+1$	$z \to z+\tau$
1	$\bar{A}(z,\bar{z},\theta)$	$\bar{A}(z,\bar{z},\theta)$	$\mathrm{Ad}_{\boldsymbol{e}(-u)}\bar{A}(z,\bar{z},\theta)$
2	$\Phi(z,\bar{z},\theta)$	$\Phi(z,\bar{z},\theta)$	$\mathrm{Ad}_{\boldsymbol{e}(-u)}\Phi(z,\bar{z},\theta)$
3	$\epsilon(z,\bar{z},\theta)$	$\epsilon(z,\bar{z},\theta)$	$\mathrm{Ad}_{\boldsymbol{e}(-u)}\epsilon(z,\bar{z},\theta)$

$\mathcal{G}^{\mathrm{res}} = \mathcal{H}$– the Cartan subgroup of SL$(2,\mathbb{C})$. It acts on the symplectic form (3.26)

$$\frac{1}{\pi}\int_{S^1}\left(\int_{\Sigma_\tau}(DL, D\bar{L}) - D(S(p^{(0)},0)g^{-1}Dg)\right)$$

producing the moment map constraint

$$S_3 - 2\pi\imath u_\theta = 0, \quad u_\theta = \partial_\theta u.$$

In addition, the gauge fixing of the $\mathcal{G}^{\mathrm{res}}$ action allows one to choose $S^+ = S^- = l(\theta)$. Then

$$S = \begin{pmatrix} u_\theta & l \\ l & -u_\theta \end{pmatrix}.$$

Then the solution of the moment equation assumes the form

$$L^{CM} = \begin{pmatrix} -\frac{1}{4\pi\imath}v - u_\theta E_1(z) & l\phi(2u,z) \\ l\phi(-2u,z) & \frac{1}{4\pi\imath}v + u_\theta E_1(z)\end{pmatrix}, \tag{4.19}$$

where $E_1(z) = \partial_z\vartheta(z)/\vartheta(z)$ is the first Eisenstein function.

The Hamiltonian of the elliptic Calogero–Moser (ECM) field theory is the integrable 2D continuation of the standard two-particle ECM Hamiltonian (a motion of particle in the Lamé potential)

$$H = -\frac{v^2}{16\pi^2} - l^2\wp(2u), \quad (\{v,u\}=1), \tag{4.20}$$

where $\wp(2u)$ is the Weierstrass function. In the field case we have the canonical Poisson bracket $\{v(\theta), u(\theta')\} = \delta(\theta-\theta')$. From (4.8) and (4.9) one finds

$$H_0^{CM} = \int_{S^1} \mathrm{d}\theta 2\sqrt{h}\chi_1 = \int_{S^1}\mathrm{d}\theta(T_0 - \frac{1}{4h}T_{-1}^2)$$
$$= \int_{S^1}\mathrm{d}\theta\left(-\frac{v^2}{16\pi^2}(1-\frac{u_\theta^2}{h}) + (3u_\theta^2 - h)\wp(2u) - \frac{u_{\theta\theta}^2}{4l^2}\right), \tag{4.21}$$

where $h = u_\theta^2 + l^2$. For v and u it is the Hamiltonian (4.20). The equations of motion produced by H_0^{CM} are of the form:

$$\begin{cases} u_t = -\frac{v}{8\pi^2}(1-\frac{u_\theta^2}{h}), \\ v_t = \frac{1}{8\pi^2 h}\partial_\theta(v^2u_\theta) - 2(3u_\theta^2 - h)\wp'(2u) \\ \quad +6\partial_\theta(u_\theta\wp(2u)) + \frac{1}{2}\partial_\theta(\frac{u_{\theta\theta\theta}l - l_\theta u_{\theta\theta}}{l^3}). \end{cases} \tag{4.22}$$

There exists a transformation Ξ of the Lax operators:

$$\Xi \circ (\nu_0\partial_\theta + L^{CM}) = (\nu_0\partial_\theta + L^{LL}) \circ \Xi,$$

such that solutions of (4.22) become solutions of the LL equation $(u,v) \to (S_\alpha,\ \alpha = 1,2,3)$ [11]. It was called the symplectic Hecke correspondence for integrable systems [9] and can be described in terms of solutions of the extended Bogomolny equation [25,26]. In the 2D case, one should define the affine version of the extended Bogomolny equation. We will address this point in a separate publication.

4.4 Gaudin field theory and principal chiral model

The Gaudin models in classical mechanics are described by the Higgs fields (i.e., the Lax matrices) with a set of simple poles at punctures on a base curve with local coordinate z. For elliptic models the latter is the elliptic curve Σ_τ with punctures z_a. Then the Lax matrix is fixed by a chose of coadjoint orbits

$$S^a = \operatorname*{Res}_{z=z_a} L(z)$$

attached to punctures together with some boundary conditions (or quasiperiodic behavior). See [27] for a review of models related to SL-bundles and [28] for a generic complex Lie group G. Similarly, in the $1+1$ field case, the Gaudin type models are generalizations of the previously given examples for a multi-pole Higgs field.

Principal chiral model The rational 2D field Gaudin model corresponding to the Riemann sphere with two punctures is the widely known principal chiral model. Indeed, consider the Zakharov–Shabat equation[4]

$$\partial_t L(z) - \partial_\theta M(z) = [L(z), M(z)], \tag{4.23}$$

with

$$L(z) = \frac{S^1}{z-z_1} + \frac{S^2}{z-z_2}, \quad M(z) = \frac{S^1}{z-z_1} - \frac{S^2}{z-z_2}. \tag{4.24}$$

Then we have equations of motion

$$\begin{cases} \partial_t S^1 - \partial_\theta S^1 = -\frac{2}{z_1-z_2}[S^1, S^2], \\ \partial_t S^2 + \partial_\theta S^2 = \frac{2}{z_1-z_2}[S^1, S^2], \end{cases} \tag{4.25}$$

which are generated by the Poisson brackets

$$\{S^a_\alpha(x), S^b_\beta(y)\} = 2\sqrt{-1}\delta^{ab}\varepsilon_{\alpha\beta\gamma}S^a_\gamma(x)\delta(x-y) \tag{4.26}$$

and the Hamiltonian

$$H = \int_{\mathbb{S}^1} d\theta \left(P_1 - P_2 - \frac{\langle S^1 S^2\rangle}{z_1-z_2}\right). \tag{4.27}$$

Here, $\int_{S^1} d\theta \; P_a$ is the shift operator in the loop algebra $\hat{\mathrm{sl}}(N,\mathbb{C})$:

$$\left\{ \int_{S^1} d\theta' \; P_a(\theta'), S^b(\theta)\right\} = \delta_{ab}\partial_\theta S^b(\theta). \tag{4.28}$$

The substitution $S^1 = \frac{1}{2}(l_0+l_1)$ and $S^2 = \frac{1}{2}(l_0-l_1)$ transforms (4.25) into an equation of the principal chiral model:

$$\begin{cases} \partial_t l_1 - \partial_\theta l_0 + \frac{2}{z_1-z_2}[l_1, l_0] = 0, \\ \partial_t l_0 - \partial_\theta l_1 = 0. \end{cases} \tag{4.29}$$

[4] In this subsection we put $\nu_0 = 1$ for simplicity.

Also, by changing the coordinates (θ, t) to the "light-cone" coordinates $\xi = \frac{t+\theta}{2}$, $\eta = \frac{t-\theta}{2}$, one gets

$$\begin{cases} \partial_\eta S^1 = -\frac{2}{z_1-z_2}[S^1, S^2], \\ \partial_\xi S^2 = \frac{2}{z_1-z_2}[S^1, S^2]. \end{cases} \tag{4.30}$$

Elliptic 1+1 Gaudin model: first flows Let us proceed to the elliptic case. The multi-pole extensions of the (spin) Calogero–Moser field theory were studied in [9]. Here we briefly review the results of [20] on the multi-pole generalization of $\hat{\mathrm{sl}}(2,\mathbb{C})$-valued Lax matrix (4.15) with the quasiperiodic properties (4.10):

$$L(z) = \sum_{c=1}^{n}\sum_{\gamma=1}^{3} \sigma_\gamma S^c_\gamma \varphi_\gamma(z-z_c). \tag{4.31}$$

Using (4.6)–(4.8) one gets the following "first flow" Hamiltonians:

$$H_{a,1} = \oint_{\mathbb{S}^1} d\theta \; (P_a + H_a), \tag{4.32}$$

$$\begin{aligned} H_a &= -\frac{1}{2}\sum_{c\neq a}\langle S^a \hat{\varphi}_{ac}(S^c)\rangle \\ &= -\sum_{c\neq a} S^a_1 S^c_1 \varphi_1(z_a-z_c) \\ &\quad + S^a_2 S^c_2 \varphi_2(z_a-z_c) + S^a_3 S^c_3 \varphi_3(z_a-z_c). \end{aligned} \tag{4.33}$$

Here and below we use the following notations for the linear operators:

$$\hat{\wp}: \; S_\alpha \to S_\alpha \wp(\omega_\alpha), \quad \hat{\varphi}_{ab}: \; S_\alpha \to S_\alpha \varphi_\alpha(z_a-z_b), \quad \hat{F}_{ab}: \; S_\alpha \to S_\alpha F_\alpha(z_a-z_b), \tag{4.34}$$

where $F_\alpha(z) = \varphi_\alpha(z)(E_1(z) + E_1(\omega_\alpha) - E_1(z+\omega_\alpha))$.

The Hamiltonians (4.33) generate dynamics described by the following equations:

$$\begin{cases} \partial_{t_a} S^a - \partial_\theta S^a = -\sum_{c\neq a}[S^a, \hat{\varphi}_{ac}(S^c)], \\ \partial_{t_a} S^b = [S^b, \hat{\varphi}_{ba}(S^a)]. \end{cases} \tag{4.35}$$

These equations are equivalent to the Zakharov–Shabat equation (4.23) with $L(z)$ (4.31) and

$$M_a(z) = \sum_{\gamma=1}^{3} \sigma_\gamma S^a_\gamma \varphi_\gamma(z-z_a). \tag{4.36}$$

Elliptic version of the principal chiral model Consider the case of two punctures (i.e. $n=2$). Then $L(z) = M_1(z) + M_2(z)$. Let us choose $M(z) = M_1(z) - M_2(z)$. The above equations yield (with $\partial_t = \partial_{t_1} - \partial_{t_2}$)

$$\begin{cases} \partial_t S^1 - k\partial_\theta S^1 = -2[S^1, \hat{\varphi}_{12}(S^2)], \\ \partial_t S^2 + k\partial_\theta S^2 = 2[S^2, \hat{\varphi}_{21}(S^1)]. \end{cases} \tag{4.37}$$

or by analogy with (4.30):

$$\begin{cases} \partial_\eta S^1 = -2[S^1, \hat{\varphi}_{12}(S^2)], \\ \partial_\xi S^2 = 2[S^2, \hat{\varphi}_{21}(S^1)]. \end{cases} \tag{4.38}$$

Elliptic 1+1 Gaudin model: second flows (coupled LL equations) The second flows are described by the following set of Hamiltonians:

$$H_{a,2}=\int_{S^1}d\theta\,\left(\frac{1}{4}\langle S^a\hat{\wp}(S^a)\rangle+\frac{1}{2}\sum_{c\neq a}\langle S^a\hat{F}(S^c)\rangle\right.$$
$$-\frac{1}{4}\left\langle\left(\sum_{c\neq a}\hat{\varphi}_{ac}(S^c)\right)^2\right\rangle++\frac{1}{8\lambda_a^2}\left(\sum_{c\neq a}\langle S^a\hat{\varphi}_{ac}(S^c)\rangle\right)^2$$
$$\left.-\frac{1}{4\lambda_a^2}\sum_{c\neq a}\langle\hat{\varphi}_{ac}(S^c)\partial_\theta S^a S^a\rangle+\frac{1}{16\lambda_a^2}\langle(\partial_\theta S^a)^2\rangle\right),\quad(4.39)$$

where λ_a are the eigenvalues of S^a (i.e., spectrum of S^a is diag$(\lambda_a,-\lambda_a)$), and it is assumed that $\partial_\theta\lambda_a=0$. The equations of motion take the form

$$\begin{cases}\partial_{\bar{t}_a}S^a-\partial_\theta\eta^a=[S^a,\hat{\wp}(S^a)]+\sum_{c\neq a}[\eta^a,\hat{\varphi}_{ca}(S^c)]\\ \quad-\hat{\varphi}_{ca}([S^c,\hat{\varphi}_{ca}(S^a)]),\\ \partial_{\bar{t}_a}S^b=[\hat{\varphi}_{ab}(\eta^a),S^b]+\hat{\varphi}_{ba}([\hat{\varphi}_{ba}(S^b),S^a]),\end{cases}\qquad(4.40)$$

where

$$\eta^a=-\frac{1}{4\lambda_a^2}[S^a,\partial_\theta S^a]+\sum_{c\neq a}\hat{\varphi}_{ac}(S^c)+\frac{H_a}{\lambda_a^2}S^a.\qquad(4.41)$$

In the case of a single marked point ($n=1$) we obtain the LL equation in the form:

$$\partial_t S+\frac{1}{4\lambda^2}[S,S_{\theta\theta}]=[S,\hat{\wp}(S)],\qquad(4.42)$$

described by the Hamiltonian

$$H=\oint_{\mathbb{S}^1}d\theta\,\left(\frac{1}{4}\langle S\hat{\wp}(S)\rangle+\frac{1}{16\lambda^2}\langle(\partial_\theta S)^2\rangle\right).\qquad(4.43)$$

One can write its trigonometric and rational degenerations. For example, in the straightforward rational limit (related to XXX 6-vertex R-matrix), the above equations provide the model of coupled Heisenberg magnets. The rational 11-vertex deformation was described in [29]. Trigonometric 6-vertex and 7-vertex models are described in the same way.

5 Correspondence between 4D-CS and AHB

Consider expansion (3.42) of the Hamiltonian $\mathcal{H}(L)$ (3.41):

$$H(L)=\sum_a\sum_{j=-1}^{+\infty}H_j^a(L)w_a^j.$$

Let us pass to the following new field:

$$\bar{L}'_{a,j}=\bar{L}-\frac{\delta H_j^a(L)}{\delta L}t_{a,j}.\qquad(5.1)$$

Table 4 Correspondence between fields

4D CS	AHB
$M=R^2\times\Sigma$	$\mathcal{M}_{a,j}$ (3.52)
$(w,\bar{w})\times(z,\bar{z})$	$(t_{a,j},\theta)\times(z,\bar{z})$
$\bar{A}=0$	$\bar{L}'_{a,j}$ (5.1)
$A_w,\ A_{\bar{w}}$	$A_\theta,\ M_{a,j}$
ω	ν (3.16)
ϕ_a	$S_a\in\mathcal{O}_a$

Since $\bar{L}$ satisfies 2.(3.46), then $\bar{L}'_{a,j}$ satisfies the equation

$$\bar{\partial}M_{a,j}-\partial_{a,j}\bar{L}'_{a,j}+[M_{a,j},\bar{L}'_{a,j}]=0,$$
$$\left([D_{\bar{L}'},D_{M_{a,j}}]=0\right).\qquad(5.2)$$

To prove it we use the equation

$$\partial_{a,j}\frac{\delta H_j^a(L)}{\delta L}+\left[M_{a,j},\frac{\delta H_j^a(L)}{\delta L}\right]=0.$$

The latter follows from (3.37) and from (3.45).

Consider a family of 3D spaces with coordinates

$$\mathcal{W}_{a,j}=\{(\bar{z},T_{a,j},\theta\in S^1)\}\subset\mathcal{M}_{a,j}\ (3.52)\qquad(5.3)$$

and the $\mathcal{P}$-bundle over $\mathcal{W}_{a,j}$ with connections

$$D_{\mathcal{A}_{a,j}}=(D_{A_\theta}d\theta,\ D_{M_{a,j}}dt_{a,j},\ D_{\bar{L}'_{a,j}}d\bar{z}),$$
$$D_{A_\theta}=\partial_\theta+A_\theta,\quad D_{M_{a,j}}=\partial_{a,j}+M_{a,j},$$
$$D_{\bar{L}'_{a,j}}=\bar{\partial}+\partial_\theta+\bar{L}'_{a,j}.\qquad(5.4)$$

It follows from (3.32) that the system (3.46) assumes the form:

$$\begin{aligned}&1.\ \nu[D_{M_{a,j}},D_{A_\theta}]=0,\\&2.\ [D_{\bar{L}'_{a,j}},D_{M_{a,j}}]=0,\\&3.\ \nu[D_{\bar{L}'_{a,j}},D_{A_\theta}]=\textstyle\sum_{a=1}^n\delta(z-z_a)S_a.\end{aligned}\qquad(5.5)$$

The delta-functions in the right-hand side of (3.5.5) mean that the connection form (i.e. L) has the first-order poles. Equations (5.5) are the equations of motion for the 4D-CS action on the 4D spaces $\mathcal{M}_{a,j}$ (3.52)

$$S_{4D}=\frac{1}{2\pi\hbar}\int_{\mathcal{M}_{a,j}}\nu\cdot CS(\mathcal{A}_{a,j}),$$

where $\mathcal{A}_{a,j}=(D_{M_{a,j}},D_{\bar{L}'_{a,j}},D_{A_\theta})$ and $CS(\mathcal{A}_{a,j}):=\mathrm{tr}\left(\mathcal{A}_{a,j}\wedge d\mathcal{A}_{a,j}+\frac{2}{3}\mathcal{A}_{a,j}\wedge\mathcal{A}_{a,j}\wedge\mathcal{A}_{a,j}\right)$. Thereby, we rewrite the equations (3.46) of the AHB theory in the Chern–Simons form (2.2).

Comparing the system (5.5) with the system (2.3) in 4D-CS theory, we obtain the following relations between the fields in these two constructions (Table 4):

Thus, we established the equivalence of two constructions at the classical level in the case when the surface defects correspond to the first-order poles, and the W bundles (3.1) are trivial.

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6 Appendix

6.1 Affine Lie algebras [30]

Let $\mathfrak{g}$ be a simple complex Lie algebra and $L(\mathfrak{g}) = \mathfrak{g}\otimes\mathbb{C}(x)$, $x \in \mathbb{C}^*$ be the loop algebra of Laurent polynomials. Let $(\, ,\,)$ be an invariant form on $\mathfrak{g}$, and let res be the coefficient c_{-1} in the Laurent expansion of $X = \sum c_k x^k \in L(\mathfrak{g})$. Define the form on $L(\mathfrak{g})$

$$\langle X,\ Y\rangle = \int_{S^1} (X, Y)d\theta.$$

Consider its central extension $\hat{L}(\mathfrak{g}) = \{(X(x), k)\}, k \in \mathbb{C}$. The commutator in $\hat{L}(\mathfrak{g})$ assumes the form

$$[(X_1, k_1), (X_2, k_2)] = ([(X_1, X_2)]_0, \langle X_1, \partial X_2\rangle), \quad (\partial = \imath x\partial_x),$$

where $[(X_1, X_2)]_0$ is a commutator in $\mathfrak{g}$,

The cocentral extension $\check{L}(\mathfrak{g})$ of $\hat{L}(\mathfrak{g})$ is the algebra

$$\check{L}(\mathfrak{g}) = \{\mathcal{X} = (X, k, \mu) = (\mu\partial + X, k),\ X \in L(\mathfrak{g}),\ \ k \in \mathbb{C},\ \ \mu \in \mathbb{C}\}. \tag{A.1}$$

The commutator in $\check{L}$ assumes the form

$$[\mathcal{X}_1, \mathcal{X}_2] = [(X_1, k_1, \mu_1), (X_2, k_2, \mu_2)] = (\mu_1\partial X_2 - \mu_2\partial X_1 + [X_1, X_2]_0, \quad \langle X_1, \partial X_2\rangle, 0). \tag{A.2}$$

There is invariant non-degenerate form on $\check{L}$

$$(\mathcal{X}_1, \mathcal{X}_2) = \langle X_1, X_2\rangle + k_1\mu_2 + k_2\mu_1. \tag{A.3}$$

Let K be a generator of the central charge and $\mathfrak{h}^0$ the Cartan subalgebra of $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}$ of $\check{L}$ takes the form

$$\mathfrak{h} = \mathfrak{h}^0 \oplus \mathbb{C}\partial \oplus \mathbb{C}K. \tag{A.4}$$

Let $L(G)$ be the loop group corresponding to the loop Lie algebra $L(\mathfrak{g})$

$$L(G) = G \otimes \mathbb{C}(t)) = \left\{ \sum_k g_k x^k,\ \ g_k \in G \right\}, \tag{A.5}$$

The central extension $\hat{L}(G) = \{g(x), \zeta\}$ is defined by the 2-cocycle $\mathcal{C}(g, g')$ on $L(G)$ providing the associativity of the multiplication

$$(g, \zeta) \times (g', \zeta') = \left(gg', \zeta\zeta'\mathcal{C}(g, g')\right), \tag{A.6}$$

Consider the shift operators $T_\mu = \exp(\mu\partial)$, $\mu \in \mathbb{C}$ acting on $L(G)$. The semidirect product is the cocentral extension of $\hat{L}(G)$

$$\check{L}(G) = \hat{L}(G) \rtimes \{T_\mu\}. \tag{A.7}$$

The adjoint action of $f \in L(G)$ is defined as

$$\mathrm{Ad}_f\mathcal{X} = \mathrm{Ad}_f(X, k, \mu) = (fXf^{-1} - \mu\partial f f^{-1}, k + \langle f^{-1}\partial f, X\rangle - \frac{1}{2}\mu\langle (f^{-1}\partial f)^2\rangle, \mu). \tag{A.8}$$

The coalgebra

$$\check{L}^*(\mathfrak{g}) = \{\mathcal{Y} = (Y, r, \nu) \sim (\nu\partial + Y, r)\} \tag{A.9}$$

is defined by the pairing

$$(\mathcal{X}, \mathcal{Y}) = \langle X, Y\rangle + k\nu + \mu r. \tag{A.10}$$

Here, Y is a 1-form $Y d\theta$ on S^1.

The coadjoint action of $L(G)$ assumes the form

$$\mathrm{Ad}^*_f\mathcal{Y} = \mathrm{Ad}^*_f(Y, r, \nu) = (f^{-1}Yf + \nu f^{-1}\partial f, r - \langle \partial f f^{-1}, Y\rangle - \frac{1}{2}\nu\langle (f^{-1}\partial f)^2\rangle,\ \nu). \tag{A.11}$$

The corresponding Lie algebra $L(\mathfrak{g}) \otimes \mathbb{C}[x, x^{-1}]\{\epsilon\}$ acts as

$$\mathrm{ad}_\epsilon\mathcal{X} = ([\epsilon, X]_0 - \mu\partial\epsilon, k + \langle\partial\epsilon, X\rangle, 0). \tag{A.12}$$

$$\mathrm{ad}^*_\epsilon\mathcal{Y} = ([Y, \epsilon]_0 + \nu\partial\epsilon, r - \langle\partial\epsilon, Y\rangle, 0). \tag{A.13}$$

6.2 Coadjoint orbits

Coadjoint orbits are the result of coadjoint action (A.11) of $L(G)$ on a fixed element

$$\mathcal{Y}^{(0)} = (c^{(0)}\partial + p^{(0)}, 0) = (p^{(0)}, 0, c^{(0)})$$

of the Lie coalgebra $\hat{L}^*(\mathfrak{g})$ (A.9).

Consider the orbit of the loop group orbit passing through $\mathcal{Y}^{(0)}$

$$\mathrm{Ad}^*_g\mathcal{Y}^{(0)} = \Big(\mathcal{O}(p^{(0)}, c^{(0)}),\ -\langle\partial g g^{-1} p^{(0)}\rangle - \frac{1}{2}c^{(0)}\langle(g^{-1}\partial g)^2\rangle,\ c^{(0)}\Big),$$

where

$$\mathcal{O}(p^{(0)}, c^{(0)}) = \{S = g^{-1}p^{(0)}g + c^{(0)}g^{-1}\partial g,\ g \in L(G)\}, \tag{A.14}$$

The symplectic form on the orbit is the Kirillov–Kostant form

$$\omega^{KK} = -\int_{S^1}\left(p^{(0)}, Dgg^{-1}Dgg^{-1}\right) + \frac{c^{(0)}}{2}\int_{S^1}\left(Dgg^{-1}, \partial(Dgg^{-1})\right) = \int_{S^1}(S(p^{(0)}, c^{(0)}), g^{-1}Dgg^{-1}Dg). \tag{A.15}$$

The corresponding Poisson brackets are

$$\{S_\alpha(x), S_\beta(y)\} = \delta(x/y)c^\gamma_{\alpha\beta}S_\gamma(x) + c^{(0)}\kappa_{\alpha\beta}\partial\delta(x/y), \tag{A.16}$$

where $\kappa_{\alpha\beta}$ is invariant form on $\mathfrak{g}$. The form ω^{KK} is invariant under transformations

$$g \to gf,\ \ f \in L(G). \tag{A.17}$$

The corresponding moment is $S(p^{(0)}, c^{(0)})$. The action the $\{\exp(\varepsilon_2(z,\bar{z})\partial\}$) component takes the form (3.20)

$$\delta_{\varepsilon_2}g = \varepsilon_2\partial g.$$

The central element $\{\exp(\varepsilon_3)\}$ (3.10) does not act on $\mathcal{Y}^{(0)}$.

We assume that $p^{(0)}$ is a semi-simple element in the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}$. Its centralizer is the Cartan subgroup $H^{\mathbb{C}}$. The invariants defining the orbit $\mathcal{O}(p^{(0)}, c^{(0)})$ are the conjugacy classes of the monodromy operator corresponding to the connection $c^{(0)}\partial + S$ along a contour in $\mathbb{C}^*$. In fact, there is a one-to-one correspondence between the set of $L(G)$-orbits and the set of conjugacy classes in the group G. The orbit is the coset space $\mathcal{O}(p^{(0)}, c^{(0)}) \sim L(G)/H^{\mathbb{C}}$ for $c^{(0)} \neq 0$, and $\mathcal{O}(p^{(0)}, 0) \sim L(G)/L(H^{\mathbb{C}})$, where $H^{\mathbb{C}}$ is the Cartan subgroup of G.

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