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$\mathcal{N} = (2, 2)$ extended $\mathfrak{sl}(3|2)$ Chern–Simons AdS_3 supergravity with new boundaries

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Abstract We present the first example of $\mathcal{N} = (2, 2)$ formulation for the extended higher-spin AdS_3 supergravity with the most general boundary conditions as an extension of the $\mathcal{N} = (1, 1)$ work, discovered recently by us (Özer and Filiz in Eur Phys J C 80(11):1072, 2020). Using the method proposed by Grumiller and Riegler, we restrict a consistent class of the most general boundary conditions to extend it. An important consequence of our method is that, for the loosest set of boundary conditions it ensures that their asymptotic symmetry algebras consist of two copies of the $\mathfrak{sl}(3|2)_k$. Moreover, we impose some restrictions on the gauge fields for the most general boundary conditions, leading to the supersymmetric extensions of the Brown and Henneaux boundary conditions. Based on these results, we finally find out that the asymptotic symmetry algebras are two copies of the super $\mathcal{W}_3$ algebra for $\mathcal{N} = (2, 2)$ extended higher-spin supergravity theory in AdS_3.

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1 Introduction

In modern theoretical physics, unquestionably, one of the forefront achievements in the past few decades is the discovery of the AdS/CFT correspondence, which was first presented concretely by Maldacena [2,3] in late 1997. This remarkable duality has profound implications ranging from a better understanding of many aspects of theoretical (and even experimental) physics, especially general relativity, quantum gravity, quantum field theory, higher spin theory, and black holes. Moreover, the AdS/CFT correspondence is an important manifestation of the holographic principle that posits a relation between a certain classical gravitational theory and a lower-dimensional non-gravitational one. The AdS_3/CFT_2 correspondence which is also a useful testing arena in this respect, implies an equivalence between pure Einstein AdS_3 gravity with a negative cosmological constant in 3D and a 2-dimensional conformal field theory. As a matter of fact, the main advantage of this eminent correspondence in three dimensions is to allow Einstein's gravity can be reformulated as a Chern–Simons gauge theory in such a way that all the structure is considerably simplified [4,5]. What they discovered in their pioneer work is that, in three dimen-

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sions, the action and equations of motions are equivalent to a Chern–Simons theory for an appropriate gauge group.

Despite the simplicity owing to its topological nature, besides being a very rich and spectacular theory, it is well known that three dimensional gravity has outstanding holographic properties. In this context, the striking feature of 3-dimensional Einstein's Gravity is the absence of any local, propagating degrees of freedom, which means that any negatively curved Einstein space is locally AdS_3. There are only global degrees of freedom and hence no graviton in three dimensions. Notwithstanding that there are no local propagating degrees of freedom in the theory, its dynamic content is far from being insignificant due to the existence of boundary conditions. In other words, this means that the theory is wholly determined by global effects, since general relativity turns into a topological field theory, whose dynamics can be portrayed holographically by a 2-dimensional conformal field theory at the boundary. That is a Chern–Simons theory in an equivalent formulation. At this point, it would be fair to say that the dynamics of the theory is totally presided over by boundary conditions. Furthermore, one should call attention here that there is an infinite number of degrees of freedom living on the boundary under an appropriate choice of boundary conditions. These boundary conditions are requisite in order to provide that the action has a well-defined variational principle. Nevertheless, their choice is not unique. Essentially, the dynamic features of the theory take shape according to the choice of these boundary conditions. So the residual gauge symmetry on the boundary within this framework emerges as global symmetry(asymptotic symmetry).

One of the most crucial results this story aforementioned tells us is that the asymptotic boundary conditions play a vital role in AdS_3 gravity. In their seminal paper [6], Brown and Hanneux proposed that under a convenient choice of boundary conditions, asymptotic symmetry algebras of AdS_3 gravity yields two copies of the Virasoro algebras with a classical central extension. The reason why this significant result is pointed out as a pioneer work is the fact that it is actually the first realization of AdS/CFT correspondence, and also an important realization of holographic duality. Incidentally, another notable strength of $2+1$ dimensional gravity is that it also contains black hole solutions such as the famous BTZ black hole in which Einstein equations admit in the presence of the negative cosmological constant [7,8]. Parenthetically, since the BTZ geometry is algebraically simple, it has provided quite a useful playground for studying features of black holes, which has a tremendous importance in exploring both classical and quantum gravitational physics.

It is worth mentioning that the Chern–Simons formulation of higher spin gravity in three dimensions has attracted more attention by the discovery of the Chern–Simons theories based on gauge algebras such as $\mathfrak{sl}(N|\mathbb{R}))$ and $hs(\lambda)$ are versions of Vasiliev higher spin theories [9,10] and also these are purely bosonic theories [11,12] with higher spin fields of integer spin. Moreover, the Chern–Simons higher spin theories could pan out with a realization of the classical $\mathcal{W}_N$ asymptotic symmetry algebras as in the related two-dimensional CFT's [13–18]. The promising results obtained from this perspective are adapted to extend the theory to the supergravity [19,20] as well as higher spin theory [11,12]. Additionally, a supersymmetric generalization of these bosonic theories can be accomplished by keeping in view Chern–Simons theories based on superalgebras such as $\mathfrak{sl}(N|N-1)$, see, e.g. [19–24], or $\mathfrak{osp}(N|N-1)$ [25] which can be obtained by truncating out all the odd spin generators and one copy of the fermionic operators in $\mathfrak{sl}(N|N-1)$.

As already noted, one can casually state that three dimensional gravity is all about the choice of boundary conditions. More precisely, the specification of boundary conditions is pivotal in comprehending how a theory that (locally) admits only a single solution. This analysis was first carried out by Brown and Henneaux [6] in their famous paper. Their study has also been encouraging to propose new sets of boundary conditions by sparking a vigorous research area which has gained in breadth over the years modifying [26–33] and generalizing [34–41] these bc$'$s. In [26], Grumiller and Riegler have considered the most general AdS_3 boundary conditions, as a consequence, they have derived the asymptotic symmetry algebra consists of two $\mathfrak{sl}(2)_k$ current algebras. Furthermore, they have recovered all other previously found boundary conditions, imposing some certain restrictions to their most general boundary conditions. It is pertinent to address that there are several papers recently inspired by them, i.e., flat space [42] and chiral higher spin gravity [43], which is shown a new class of boundary conditions for higher spin theories in AdS_3. The simplest extension of Grumiller and Riegler's procedure for the most general $\mathcal{N}=(1,1)$, and $\mathcal{N}=(2,2)$ extended higher spin supergravity is introduced by Valcarcel [44] where the asymptotic symmetry algebra for the loosest set of boundary conditions for (extended) supergravity obtained. The most general $\mathcal{N}=(1,1)$ extended AdS_3 higher spin supergravity theory has been similarly presented, including further in [1].

This paper is concerned with the previously unresolved phenomenon; we construct a candidate solution for the most general $\mathcal{N}=(2,2)$ extended higher spin supergravity theory in AdS_3. We show that our theory falls under the same metric class as [42], in which it is seen that the metric formulation could include even both charge and chemical potentials which are present in the Chern–Simons formalism. This can be considered as an alternative solution to the non-chiral Drinfeld–Sokolov type boundary conditions. Firstly, we focus on the simplest case $\mathcal{N}=(2,2)$ Chern–Simons theory based on the $\mathfrak{sl}(2|1)_k$ superalgebra. The related asymptotic symmetry algebra is two copies of the $\mathfrak{sl}(2|1)_k$ affine algebra. Then, we are dealing with the extended $\mathcal{N}=(2,2)$ Chern–

Simons theory based on $\mathfrak{sl}(3|2)$ superalgebra, as a result, we obtain asymptotic symmetry algebra consists of two copies of the $\mathfrak{sl}(3|2)_k$ affine algebra. Additionally, we impose some certain restrictions to the gauge fields on the most general boundary conditions, which leading us to the supersymmetric extensions of the Brown–Henneaux boundary conditions. We also show that the asymptotic symmetry algebras are reduced to two copies of the super $\mathcal{W}_3$ algebra for the most general $\mathcal{N}=(2,2)$ extended higher spin AdS_3 supergravity theory. It is useful to indicate that it would be an interesting problem in its own right perform a different class of boundary conditions for (super)gravity that emerges in the literature (see, e.g. [28,30–32]), since their higher spin generalization is not as clear as Grumiller and Riegler's boundary conditions. In light of all these results, it is inevitable to say that this method yields an excellent laboratory to investigate the rich asymptotic structure of extended higher spin supergravity.

This paper is organized as follows. We first introduce a fundamental formulation of $\mathcal{N}=(2,2)$ supergravity as $\mathfrak{sl}(2|1)\oplus\mathfrak{sl}(2|1)$ Chern–Simons gauge theory for both affine and superconformal boundaries, respectively. Then, in Sect. 3, maintain our calculations to extend the theory $\mathfrak{sl}(3|2)\oplus\mathfrak{sl}(3|2)$ higher-spin Chern–Simons supergravity in the case of both affine and superconformal boundaries, in where we reveal explicitly principal embedding of $\mathfrak{sl}(2|1)\oplus\mathfrak{sl}(2|1)$ and also we come up with how asymptotic symmetry and higher spin Ward identities arise from these bulk equations of motion coupled to spin s, $(s=1,\frac{3}{2},\frac{3}{2},2,2,\frac{5}{2},\frac{5}{2},3)$ currents. We dedicate this section to perform the asymptotic symmetry algebras as classical two copies of the $\mathfrak{sl}(3|2)_k$ affine algebra on the affine boundary and the super $\mathcal{W}_3$ symmetry algebra on the superconformal boundary, respectively. Besides, we describe the chemical potentials related to source fields appearing through the temporal components of the connection. In the final section, we conclude with a discussion, open issues and future research directions.

2 Review of Chern–Simons supergravity in three dimensions

In this section, we give a brief discussion for AdS_3 higher spin supergravity based on Chern–Simons formalism. We especially employ this formulation to analyze AdS_3 supergravity in the presence of $\mathfrak{sl}(2|1)$ superalgebra basis, belonging to the same metric class as Grumiller and Riegler's recently proposed, the most general AdS_3 boundary conditions [26].

2.1 Connection to Chern–Simons theory

In three dimensions, Einstein–Hilbert action for $\mathcal{N}=(2,2)$ supergravity with a negative cosmological constant, can be defined in an equivalent Chern–Simons formulation over a spacetime manifold $\mathcal{M}$ as

$$S=S_{CS}[\Gamma]-S_{CS}[\bar{\Gamma}] \tag{2.1}$$

where

$$S_{CS}[\Gamma]=\frac{k}{4\pi}\int_{\mathcal{M}}\mathfrak{str}\left(\Gamma\wedge d\Gamma+\frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma\right) \tag{2.2}$$

which was first noted by Achucarro and Townsend [4] and further developed by Witten [5].

The Chern–Simons level k will eventually be related to ratio of AdS_3 radius l and Newton constant G, and also the related central charge c of the superconformal field theory as $k=\frac{\ell}{8G\,\mathfrak{str}(\mathrm{L}_0\mathrm{L}_0)}=\frac{c}{12\,\mathfrak{str}(\mathrm{L}_0\mathrm{L}_0)}$. Notice that while the 1-forms $(\Gamma,\bar{\Gamma})$ connections are defined as to take values in the gauge group of $\mathfrak{sl}(2|1)$ superalgebra, the supertrace $\mathfrak{str}$ which shows a metric on the $\mathfrak{sl}(2|1)$ Lie superalgebra, is taken over the superalgebra generators.

It is convenient to get started with standard basis for $\mathfrak{sl}(2|1)$ Lie superalgebra. We denote the bosonic generators by L_i $(i=\pm1,0)$, J and the fermionic ones by G_r^M $(r=\pm\frac{1}{2},M=\pm)$, whose commutations relations read

$$\begin{aligned}
\left[\mathrm{L}_i,\mathrm{L}_j\right]&=(i-j)\,\mathrm{L}_{i+j},\\
\left[\mathrm{L}_i,\mathrm{G}_r^{\pm}\right]&=\left(\frac{i}{2}-r\right)\mathrm{G}_{i+r}^{\pm},\\
\left[\mathrm{J},\mathrm{G}_r^{\pm}\right]&=\pm\mathrm{G}_r^{\pm}, &(2.3)\\
\left\{\mathrm{G}_r^{\pm},\mathrm{G}_s^{\mp}\right\}&=2\mathrm{L}_{r+s}\pm(r-s)\,\mathrm{J} &(2.4)
\end{aligned}$$

except for zero commutators.

The Chern–Simons equations of motions, also known as the flatness conditions correspond to vanishing field strengths; $F=\bar{F}=0$ where

$$F=d\Gamma+\Gamma\wedge\Gamma=0,\quad \bar{F}=d\bar{\Gamma}+\bar{\Gamma}\wedge\bar{\Gamma}=0 \tag{2.5}$$

which is equivalent to Einstein's equation. The relation to the Einstein's equation is made by expressing Lie algebra valued generalizations of the vielbein and spin connection in terms of the gauge connections. Then, one can obtain the metric $g_{\mu\nu}$ from the vielbein $e=\frac{\ell}{2}(\Gamma-\bar{\Gamma})$ in the usual fashion

$$g_{\mu\nu}=\frac{1}{2}\mathfrak{str}(e_\mu e_\nu). \tag{2.6}$$

By the choice of the radial gauge, asymptotically AdS_3 connections can be taken to have the form

$$\begin{aligned}
\Gamma&=b^{-1}a\,(t,\phi)\,b+b^{-1}db,\\
\bar{\Gamma}&=b\bar{a}\,(t,\phi)\,b^{-1}+bdb^{-1} &(2.7)
\end{aligned}$$

with state-independent group element (called Grumiller-Riegler gauge);

$$b=e^{\mathrm{L}_{-1}}e^{\rho\mathrm{L}_0} \tag{2.8}$$

which yields a more general metric and means that it includes all $\mathfrak{sl}(2|1)$ charges and chemical potentials can be chosen accordingly. At this point, it is important to note that as long as $\delta b = 0$, the choice of b is irrelevant for asymptotic symmetries. Unlike the standard choice of b, this freedom enables a more general metric. Therefore, it is crucial to choose the most general boundary conditions preserving this most general metric form for supergravity.

Further, in the radial gauge $a\,(t,\phi)$ and $\bar{a}\,(t,\phi)$ connections are the $\mathfrak{sl}(2|1)$ Lie superalgebra valued fields which are independent of a radial coordinate as

$$\begin{aligned} a\,(t,\varphi) &= a_t\,(t,\varphi)\,\mathrm{d}t + a_\varphi\,(t,\varphi)\,\mathrm{d}\varphi \\ \bar{a}\,(t,\varphi) &= \bar{a}_t\,(t,\varphi)\,\mathrm{d}t + \bar{a}_\varphi\,(t,\varphi)\,\mathrm{d}\varphi \end{aligned} \tag{2.9}$$

Hereafter only focused on the unbarred sector, since the analysis of the barred sector works in complete analogy yielding the same outcomes with the barred sector and it can be figured out by the same algorithm thanks to the procedure used.

2.2 $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern–Simons $\mathcal{N} = (2,2)$ supergravity for affine boundary

We begin by reviewing asymptotically AdS_3 boundary conditions for $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern–Simons theory in the affine case. We present how the procedure mentioned in [26] can be used to evaluate the asymptotic symmetry algebra. According to the results obtained, the most general solution of Einstein's equation that is asymptotically AdS_3 is defined by the following general metric form:

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{d}\rho^2 &+ 2\left[e^{\rho}N_i^{(0)} + N_i^{(1)} + e^{-\rho}N_i^{(2)} + \mathcal{O}\left(e^{-2\rho}\right)\right]\mathrm{d}\rho\,\mathrm{d}x^i \\ &+ \left[e^{2\rho}g_{ij}^{(0)} + e^{\rho}g_{ij}^{(1)} + g_{ij}^{(2)} + \mathcal{O}\left(e^{-\rho}\right)\right]\mathrm{d}x^i\mathrm{d}x^j. \end{aligned} \tag{2.10}$$

So, it is important to define the most general $\mathcal{N} = (2,2)$ supergravity boundary conditions which preserve this form of the metric.

We start by proposing $\mathfrak{sl}(2|1)$ Lie superalgebra valued a_φ component of the gauge connection in the form:

$$a_\varphi = \rho\mathcal{J}\,\mathrm{J} + \gamma_i\mathcal{L}^i\,\mathrm{L}_i + \sigma_M^p\mathcal{G}_M^p\,\mathrm{G}_p^M \tag{2.11}$$

where $\rho = \frac{1}{k}, \frac{\gamma_0}{2} = -\gamma_{\pm 1} = \frac{2}{k}, \sigma_\pm^{-\frac{1}{2}} = -\sigma_\pm^{\frac{1}{2}} = \frac{1}{k}$ are some scaling parameters to be identified later. We have eight state-dependent functions consisting of four bosonic $\left(\mathcal{J},\mathcal{L}^i\right)$ and four fermionic $\mathcal{G}_M^p$, usually called as *charges*. The time component a_t of the connection $a\,(t,\varphi)$ can be given as

$$a_t = \eta\,\mathrm{J} + \mu^i\,\mathrm{L}_i + \nu_M^p\,\mathrm{G}_p^M. \tag{2.12}$$

In this case, we have eight independent functions (η,μ^i,ν_M^p), as chemical potentials which are not allowed to vary, $\delta a_t = 0$.

Using the flatness conditions (2.5), the equations of motions for fixed chemical potentials impose the following additional conditions on the charges $(\mathcal{J},\mathcal{L}^i,\mathcal{G}_M^p)$:

$$\begin{aligned} 2\partial_t\mathcal{L}^0 = \frac{k}{2}\partial_\varphi\mu^0 &+ 2\mathcal{L}^{+1}\mu^{-1} \\ &+ 2\mathcal{L}^{-1}\mu^{+1} - \mathcal{G}_-^{+\frac{1}{2}}\nu_+^{-\frac{1}{2}} + \mathcal{G}_-^{-\frac{1}{2}}\nu_+^{+\frac{1}{2}} \\ &- \mathcal{G}_+^{+\frac{1}{2}}\nu_-^{-\frac{1}{2}} + \mathcal{G}_+^{-\frac{1}{2}}\nu_-^{+\frac{1}{2}}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \partial_t\mathcal{L}^{\pm 1} = -\frac{k}{2}\partial_\varphi\mu^{\pm 1} &\pm \mathcal{L}^0\mu^{\pm 1} \pm \mathcal{L}^{\pm 1}\mu^0 \\ &+ \mathcal{G}_+^{\pm\frac{1}{2}}\nu_-^{\pm\frac{1}{2}} + \mathcal{G}_-^{\pm\frac{1}{2}}\nu_+^{\pm\frac{1}{2}}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \partial_t\mathcal{G}_\pm^{\pm\frac{1}{2}} = \pm k\partial_\varphi\nu_\pm^{\pm\frac{1}{2}} &\pm 2\mathcal{L}^\pm\nu_\pm^{\mp\frac{1}{2}} + 2\mathcal{L}^0\nu_\pm^{\pm\frac{1}{2}} \\ &\pm \mathcal{G}_\pm^{\mp\frac{1}{2}}\mu^\pm \pm \frac{1}{2}\mu^0\mathcal{G}_\pm^{\pm\frac{1}{2}} \mp \mathcal{J}\nu_\pm^{\pm\frac{1}{2}} - \eta\mathcal{G}_\pm^{\pm\frac{1}{2}}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \partial_t\mathcal{J} = k\partial_\varphi\eta &+ \mathcal{G}_-^{+\frac{1}{2}}\nu_+^{-\frac{1}{2}} + \mathcal{G}_-^{-\frac{1}{2}}\nu_+^{+\frac{1}{2}} \\ &- \mathcal{G}_+^{+\frac{1}{2}}\nu_-^{-\frac{1}{2}} - \mathcal{G}_+^{-\frac{1}{2}}\nu_-^{+\frac{1}{2}}, \end{aligned} \tag{2.16}$$

that represents the temporal evolution of the eight state-dependent source fields.

We want to derive asymptotic symmetry algebra for the most general boundary conditions through a canonical analysis. That's why we embark on by considering all gauge transformations:

$$\delta_\lambda\Gamma = \mathrm{d}\lambda + [\Gamma,\lambda] \tag{2.17}$$

which preserve the most general boundary conditions. At this point, it would be appropriate to single out the gauge parameter in terms of the $\mathfrak{sl}(2|1)$ Lie superalgebra basis

$$\lambda = b^{-1}\left[\varrho\,\mathrm{J} + \epsilon^i\,\mathrm{L}_i + \zeta_M^p\,\mathrm{G}_p^M\right]b. \tag{2.18}$$

Note that the gauge parameter includes four bosonic ϱ,ϵ^i and four fermionic ζ_M^p, arbitrary functions of boundary coordinates. And also, we are concerned with the gauge parameters that satisfy (2.17). One can now determine the boundary preserving gauge transformations. Accordingly, the infinitesimal gauge transformations are given by;

$$\begin{aligned} 2\partial_t\mathcal{L}^0 = \frac{k}{2}\partial_\varphi\epsilon^0 &+ 2\mathcal{L}^{+1}\epsilon^{-1} + 2\mathcal{L}^{-1}\epsilon^{+1} - \mathcal{G}_-^{+\frac{1}{2}}\zeta_+^{-\frac{1}{2}} \\ &+ \mathcal{G}_-^{-\frac{1}{2}}\zeta_+^{+\frac{1}{2}} - \mathcal{G}_+^{+\frac{1}{2}}\zeta_-^{-\frac{1}{2}} + \mathcal{G}_+^{-\frac{1}{2}}\zeta_-^{+\frac{1}{2}}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \partial_t\mathcal{L}^{\pm 1} = -\frac{k}{2}\partial_\varphi\epsilon^{\pm 1} &\pm \mathcal{L}^0\epsilon^{\pm 1} \pm \mathcal{L}^{\pm 1}\epsilon^0 \\ &+ \mathcal{G}_+^{\pm\frac{1}{2}}\zeta_-^{\pm\frac{1}{2}} + \mathcal{G}_-^{\pm\frac{1}{2}}\zeta_+^{\pm\frac{1}{2}}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} \partial_t\mathcal{G}_\pm^{\pm\frac{1}{2}} = \pm k\partial_\varphi\zeta_\pm^{\pm\frac{1}{2}} &\pm 2\mathcal{L}^\pm\zeta_\pm^{\mp\frac{1}{2}} + 2\mathcal{L}^0\zeta_\pm^{\pm\frac{1}{2}} \\ &\pm \mathcal{G}_\pm^{\mp\frac{1}{2}}\epsilon^\pm \pm \frac{1}{2}\epsilon^0\mathcal{G}_\pm^{\pm\frac{1}{2}} \mp \mathcal{J}\zeta_\pm^{\pm\frac{1}{2}} - \varrho\mathcal{G}_\pm^{\pm\frac{1}{2}}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \partial_t\mathcal{J} = k\partial_\varphi\varrho &+ \mathcal{G}_-^{+\frac{1}{2}}\zeta_+^{-\frac{1}{2}} + \mathcal{G}_-^{-\frac{1}{2}}\zeta_+^{+\frac{1}{2}} \\ &- \mathcal{G}_+^{+\frac{1}{2}}\zeta_-^{-\frac{1}{2}} - \mathcal{G}_+^{-\frac{1}{2}}\zeta_-^{+\frac{1}{2}}. \end{aligned} \tag{2.22}$$

One can also derive the following constraints for the chemical potentials analogously.

$$2\partial_t\mu^0 = \frac{k}{2}\partial_\varphi\epsilon^0 + 2\mu^{+1}\epsilon^{-1} + 2\mu^{-1}\epsilon^{+1} - \nu_-^{+\frac{1}{2}}\zeta_+^{-\frac{1}{2}} + \nu_-^{-\frac{1}{2}}\zeta_+^{+\frac{1}{2}} - \nu_+^{+\frac{1}{2}}\zeta_-^{-\frac{1}{2}} + \nu_+^{-\frac{1}{2}}\zeta_-^{+\frac{1}{2}}, \tag{2.23}$$

$$\partial_t\mu^{\pm1} = -\frac{k}{2}\partial_\varphi\epsilon^{\pm1} \pm \mu^0\epsilon^{\pm1} \pm \mu^{\pm1}\epsilon^0 + \nu_+^{\pm\frac{1}{2}}\zeta_-^{\pm\frac{1}{2}} + \nu_-^{\pm\frac{1}{2}}\zeta_+^{\pm\frac{1}{2}}, \tag{2.24}$$

$$\partial_t\nu_\pm^{\pm\frac{1}{2}} = \pm k\partial_\varphi\zeta_\pm^{\pm\frac{1}{2}} \pm 2\mu^\pm\zeta_\pm^{\mp\frac{1}{2}} + 2\mu^0\zeta_\pm^{\pm\frac{1}{2}} \pm \nu_\pm^{\mp\frac{1}{2}}\epsilon^\pm \pm \frac{1}{2}\epsilon^0\nu_\pm^{\pm\frac{1}{2}} \mp \eta\zeta_\pm^{\pm\frac{1}{2}} - \varrho\nu_\pm^{\pm\frac{1}{2}}, \tag{2.25}$$

$$\partial_t\eta = k\partial_\varphi\varrho + \nu_-^{+\frac{1}{2}}\zeta_+^{-\frac{1}{2}} + \nu_-^{-\frac{1}{2}}\zeta_+^{+\frac{1}{2}} - \nu_+^{+\frac{1}{2}}\zeta_-^{-\frac{1}{2}} - \nu_+^{-\frac{1}{2}}\zeta_-^{+\frac{1}{2}}. \tag{2.26}$$

As a final step, the canonical boundary charge $\mathcal{Q}[\lambda]$ that generates the transformations (2.19)–(2.22) can be defined. For this purpose, the variation of the canonical boundary charge $\mathcal{Q}[\lambda]$ [46–49] leading the asymptotic symmetry algebra is given by

$$\delta_\lambda\mathcal{Q} = \frac{k}{2\pi}\int d\varphi\ \mathfrak{str}\left(\lambda\delta\Gamma_\varphi\right). \tag{2.27}$$

Hence, the variation of the canonical boundary charge $\mathcal{Q}[\lambda]$ can be functionally integrated to yield

$$\mathcal{Q}[\lambda] = \int d\varphi\ \left[\mathcal{J}\varrho + \mathcal{L}^i\epsilon^{-i} + \mathcal{G}_M^p\zeta_M^{-p}\right]. \tag{2.28}$$

After both having determined the infinitesimal transformations and the canonical boundary charge, now we are in a position to derive the asymptotic symmetry algebra using the standard method [45], which can be obtained through the following relation

$$\delta_\lambda F = \{F, \mathcal{Q}[\lambda]\} \tag{2.29}$$

for any phase space functional F. The Poisson brackets of all fields can be calculated as

$$\{\mathcal{L}^i(z_1), \mathcal{L}^j(z_2)\}_{PB} = (i-j)\mathcal{L}^{i+j}(z_2)\delta(z_1-z_2) - k\eta_2^{ij}\partial_\varphi\delta(z_1-z_2), \tag{2.30}$$

$$\{\mathcal{L}^i(z_1), \mathcal{G}_\pm^p(z_2)\}_{PB} = \left(\frac{i}{2}-p\right)\mathcal{G}_\pm^{i+p}(z_2)\delta(z_1-z_2), \tag{2.31}$$

$$\{\mathcal{J}(z_1), \mathcal{G}_\pm^p(z_2)\}_{PB} = \pm\mathcal{G}_\pm^p(z_2)\delta(z_1-z_2), \tag{2.32}$$

$$\{\mathcal{J}(z_1), \mathcal{J}(z_2)\}_{PB} = -k\eta\partial_\varphi\delta(z_1-z_2), \tag{2.33}$$

$$\{\mathcal{G}_\pm^p(z_1), \mathcal{G}_\pm^q(z_2)\}_{PB} = \left(2\mathcal{L}^{p+q}(z_2) \pm (p-q)\mathcal{J}\right)\delta(z_1-z_2) + k\eta_{\frac{3}{2}}^{pq}\partial_\varphi\delta(z_1-z_2). \tag{2.34}$$

where $\eta = \mathfrak{str}(\mathrm{JJ})$, $\eta_2^{ij} = \mathfrak{str}(\mathrm{L}_i\mathrm{L}_j)$ and $\eta_{\frac{3}{2}}^{pq} = \mathfrak{str}(\mathrm{G}_p^M\mathrm{G}_q^M)$ are the bilinear forms in the fundamental representation of $\mathfrak{sl}(2|1)$ Lie superalgebra. $\mathcal{J}(z)$ and $\mathcal{G}_M^p(z)$ charges are also identified as the generators of the asymptotic symmetry algebra. Finally, the operator product algebra can be written as

$$\mathcal{L}^i(z_1)\mathcal{L}^j(z_2) \sim \frac{\frac{k}{2}\eta_2^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}}\mathcal{L}^{i+j}(z_2), \tag{2.35}$$

$$\mathcal{L}^i(z_1)\mathcal{G}_\pm^p(z_2) \sim \frac{(\frac{i}{2}-p)}{z_{12}}\mathcal{G}_\pm^{i+p}(z_2), \tag{2.36}$$

$$\mathcal{J}(z_1)\mathcal{G}_\pm^p(z_2) \sim \pm\frac{\mathcal{G}_\pm^p(z_2)}{z_{12}}, \tag{2.37}$$

$$\mathcal{J}(z_1)\mathcal{J}(z_2) \sim \frac{\frac{k}{2}\eta}{z_{12}^2}, \tag{2.38}$$

$$\mathcal{G}_\pm^p(z_1)\mathcal{G}_\pm^q(z_2) \sim \frac{\frac{k}{2}\eta_{\frac{3}{2}}^{pq}}{z_{12}^2} + \frac{2}{z_{12}}\left(\mathcal{L}^{p+q}(z_2) \pm \frac{(p-q)}{2}\mathcal{J}(z_2)\right). \tag{2.39}$$

where $z_{12} = z_1 - z_2$, or in the more compact form,

$$\mathfrak{J}^A(z_1)\mathfrak{J}^B(z_2) \sim \frac{\frac{k}{2}\eta^{AB}}{z_{12}^2} + \frac{\mathfrak{f}^{AB}{}_C\mathfrak{J}^C(z_2)}{z_{12}}. \tag{2.40}$$

Note that η^{AB} is the supertrace matrix and $\mathfrak{f}^{AB}{}_C$'s are the structure constants of the related algebra with $(A, B = 0, \pm1, \pm\frac{1}{2})$, i.e, $\eta^{ip} = 0$ and $\mathfrak{f}^{ij}{}_{i+j} = (i-j)$. Lastly, by repeating the same analysis for the barred sector also, the asymptotic symmetry algebra of $\mathcal{N} = (2,2)$ supergravity for the loosest set of boundary conditions is given by two copies of the affine $\mathfrak{sl}(2|1)_k$ algebra.

2.3 $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern–Simons $\mathcal{N} = (2,2)$ supergravity for superconformal boundary

In this section, our aim is to look into the asymptotic symmetry algebra for the supersymmetric extension of the Brown–Henneaux boundary conditions. We start on by imposing the Drinfeld–Sokolov highest weight gauge condition on the $\mathfrak{sl}(2|1)$ Lie superalgebra valued connection (2.11), in order to further restrict the coefficients. So, the Drinfeld–Sokolov reduction sets the fields such that

$$\mathcal{L}^0 = \mathcal{G}_\pm^{+\frac{1}{2}} = 0, \qquad \mathcal{L}^{-1} = \mathcal{L}, \qquad \mathcal{G}_\pm^{-\frac{1}{2}} = \mathcal{G}_\pm, \qquad \gamma_{+1}\mathcal{L}^{+1} = 1. \tag{2.41}$$

Meanwhile, it is worth noting that the super-conformal boundary conditions are the supersymmetric extension of the well-known Brown–Henneaux boundary conditions proposed in [6] for AdS_3 supergravity. As a result, the supersymmetric gauge connection takes the form

$$a_\varphi = \mathrm{L}_{+1} + \gamma_{-1}\mathcal{L}\mathrm{L}_{-1} + \sigma_{\pm}^{-\frac{1}{2}}\mathcal{G}_{\pm}\mathrm{G}^{\pm}_{-\frac{1}{2}} + \rho\mathcal{J}\mathrm{J}, \tag{2.42}$$

where $\gamma_{-1} = -\frac{1}{k}, \rho = \frac{1}{2k}, \sigma_+^{-\frac{1}{2}} = \frac{1}{2k}$, and $\sigma_-^{-\frac{1}{2}} = -\frac{1}{2k}$'s are some scaling parameters and we have four functions: two bosonic $(\mathcal{J}, \mathcal{L})$ and two fermionic $\mathcal{G}_\pm$ as *charges*. After performing these steps, we are now close to acquire the superconformal asymptotic symmetry algebra. Following the results implied by the Drinfeld–Sokolov reduction, the gauge parameter λ has only four independent functions $(\varrho, \epsilon \equiv \epsilon^{+1}, \zeta_\pm \equiv \zeta_\pm^{+\frac{1}{2}})$ and given as

$$\begin{aligned}\lambda = b^{-1}\Bigg[&\epsilon\mathrm{L}_1 - \epsilon'\mathrm{L}_0 + \left(\frac{1}{2}\epsilon'' + \frac{1}{k}\mathcal{L}\epsilon\right.\\ &\left.+\frac{1}{2k}\mathcal{G}_+\zeta_+ + \frac{1}{2k}\mathcal{G}_-\zeta_-\right)\mathrm{L}_{-1} + \zeta_+\mathrm{G}^+_{\frac{1}{2}} + \zeta_-\mathrm{G}^-_{\frac{1}{2}}\\ &-\left(\frac{1}{2k}\mathcal{G}_-\epsilon + \frac{1}{2k}\mathcal{J}\zeta_+ + \zeta_+'\right)\mathrm{G}^-_{-\frac{1}{2}}\\ &-\left(\frac{1}{2k}\mathcal{G}_+\epsilon + \frac{1}{2k}\mathcal{J}\zeta_- + \zeta_-'\right)\mathrm{G}^+_{-\frac{1}{2}} + \varrho\mathrm{J}\Bigg]b. \end{aligned}\tag{2.43}$$

Substituting this gauge parameter in the transformation of the fields expression (2.17), we obtain the infinitesimal gauge transformations:

$$\delta_\lambda\mathcal{J} = \frac{1}{2k}\varrho' - \mathcal{G}_+\zeta_+ + \mathcal{G}_-\zeta_-, \tag{2.44}$$

$$\begin{aligned}\delta_\lambda\mathcal{L} = &\frac{k}{2}\epsilon''' + 2\mathcal{L}\epsilon' + \mathcal{L}'\epsilon + \frac{3}{2}\mathcal{G}_+\zeta_+'\\ &+\frac{3}{2}\mathcal{G}_-\zeta_-' + \frac{1}{2}\mathcal{G}_+'\zeta_+ + \frac{1}{2}\mathcal{G}_-'\zeta_-\\ &+\frac{1}{2k}\left(\mathcal{J}\mathcal{G}_+\zeta_+ + \mathcal{J}\mathcal{G}_-\zeta_-\right),\end{aligned}\tag{2.45}$$

$$\begin{aligned}\delta_\lambda\mathcal{G}_\pm = &\frac{1}{2k}\zeta_\mp'' + 2\left(\mathcal{L} + \frac{1}{4k}(\mathcal{J}\mathcal{J})\right)\zeta_\pm\\ &\pm\left(\varrho - \frac{1}{2k}\mathcal{J}\epsilon\right)\mathcal{G}_\pm \mp\zeta_\pm\mathcal{J}' + \frac{3}{2}\mathcal{G}_\pm\epsilon'\\ &+\epsilon\mathcal{G}_\pm' \mp\mathcal{J}\zeta_\mp'.\end{aligned}\tag{2.46}$$

In fact, these gauge transformations give a cue for the asymptotic symmetry algebra [20]. By taking forward to see the asymptotic symmetries, one can integrate the variation of the canonical boundary charges, i.e., $\delta_\lambda\mathcal{Q}$ expression (2.27) such that

$$\mathcal{Q}[\lambda] = \int \mathrm{d}\varphi\ \left[\mathcal{L}\epsilon + \mathcal{G}_M\zeta_M + \mathcal{J}\varrho\right]. \tag{2.47}$$

But, these canonical boundary charges do not give a convenient asymptotic operator product algebra for $\mathcal{N} = (2, 2)$ superconformal boundary,

$$\mathcal{J}(z_1)\mathcal{J}(z_2) \sim \frac{2k}{z_{12}^2}, \quad \mathcal{J}(z_1)\mathcal{G}_\pm(z_2) \sim \frac{\mp\mathcal{G}_\pm}{z_{12}^2}, \tag{2.48}$$

$$\begin{aligned}\mathcal{L}(z_1)\mathcal{L}(z_2) &\sim \frac{3k}{z_{12}^4} + \frac{2\,\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}' - \frac{\mathcal{G}_+\mathcal{G}_-}{k}}{z_{12}},\\ &\times\mathcal{L}(z_1)\mathcal{J}(z_2) \sim 0\end{aligned}\tag{2.49}$$

$$\mathcal{L}(z_1)\mathcal{G}_\pm(z_2) \sim \frac{\frac{3}{2}\mathcal{G}_\pm}{z_{12}^2} + \frac{\mathcal{G}_\pm' \pm \frac{\mathcal{J}\mathcal{G}_\pm}{2k}}{z_{12}}, \tag{2.50}$$

$$\mathcal{G}_\pm(z_1)\mathcal{G}_\pm(z_2) \sim \frac{\mp 4k}{z_{12}^3} - \frac{2\mathcal{J}}{z_{12}^2} + \frac{\mp 2}{z_{12}}\left(\mathcal{L} + \frac{\mathcal{J}\mathcal{J}}{4k} \pm \frac{\mathcal{J}'}{2}\right). \tag{2.51}$$

because $\mathcal{L}(z)$ and $\mathcal{J}(z)$ do not transform like a primary conformal field, besides there exist some nonlinear terms such as $(\mathcal{J}\mathcal{J})(z), (\mathcal{G}_+\mathcal{G}_-)(z)$ and also $(\mathcal{J}\mathcal{G}_\pm)(z)$. Therefore, it is necessary to perform a shift on the boundary charge $\mathcal{L}$ and also make a redefinition on the gauge parameter ϱ as follows:

$$\mathcal{L} \to \mathcal{L} + \frac{3}{2c}(\mathcal{J}\mathcal{J}), \quad \varrho \to \varrho + \frac{3}{c}\mathcal{J}\epsilon. \tag{2.52}$$

Before closing this section, one should also emphasize that these new variables do not affect the boundary charges. Thus, this leads to operator product expansions of the convenient asymptotic symmetry algebra for $\mathcal{N} = (2, 2)$ superconformal boundary with a set of conformal generators $\mathcal{G}_\pm \to \mathcal{G}^+ \pm \mathcal{G}^-$ in the complex coordinates by using (2.29)

$$\mathcal{J}(z_1)\mathcal{J}(z_2) \sim \frac{\frac{c}{3}}{z_{12}^2}, \quad \mathcal{J}(z_1)\mathcal{G}^\pm(z_2) \sim \frac{\pm\mathcal{G}^\pm}{z_{12}^2}, \tag{2.53}$$

$$\begin{aligned}\mathcal{L}(z_1)\mathcal{L}(z_2) &\sim \frac{\frac{c}{2}}{z_{12}^4} + \frac{2\,\mathcal{L}}{z_{12}^2} + \frac{\mathcal{L}'}{z_{12}},\\ \mathcal{L}(z_1)\mathcal{J}(z_2) &\sim \frac{\mathcal{J}}{z_{12}^2} + \frac{\mathcal{J}'}{z_{12}}\end{aligned}\tag{2.54}$$

$$\mathcal{L}(z_1)\mathcal{G}^\pm(z_2) \sim \frac{\frac{3}{2}\mathcal{G}^\pm}{z_{12}^2} + \frac{\mathcal{G}^{\pm\prime}}{z_{12}}, \tag{2.55}$$

$$\mathcal{G}^\pm(z_1)\mathcal{G}^\mp(z_2) \sim \frac{\frac{2c}{3}}{z_{12}^3} + \frac{2\mathcal{J}}{z_{12}^2} + \frac{1}{z_{12}}\left(2\mathcal{L} \pm \mathcal{J}'\right). \tag{2.56}$$

When the same analysis repeated for the barred sector, it is seen that the asymptotic symmetry algebra for the loosest set of boundary conditions of $\mathcal{N} = (2, 2)$ supergravity, consists of two copies of the super-Virasoro algebra with central charge $c = 6k$.

3 $\mathcal{N} = (2, 2)$ $\mathfrak{sl}(3|2) \oplus \mathfrak{sl}(3|2)$ higher-spin Chern–Simons supergravity

Finally, after having laid the groundwork by performing the canonical analysis of the simplest case, now we are in a position to present the extended $\mathcal{N} = (2, 2)$ higher-spin Chern–Simons supergravity theory based on $\mathfrak{sl}(3|2)_k$ superalgebra.

3.1 For affine boundary

The objective of this section is to construct $\mathcal{N} = (2,2)$ extended higher-spin AdS_3 supergravity as $\mathfrak{sl}(3|2) \oplus \mathfrak{sl}(3|2)$ Chern–Simons gauge theory on the affine boundary. We proceed our calculations to elucidate the asymptotic symmetry algebra for the loosest set of boundary conditions.

As already stated in the previous section, we consider the principal embedding of $\mathfrak{sl}(2|1)$ into $\mathfrak{sl}(3|2)$ as a subalgebra, giving rise to the asymptotic symmetry, where the even-graded sector of the superalgebra decomposes into spin-2, the $\mathfrak{sl}(2)$ generators $\mathtt{L}_i, (i = \pm 1, 0)$, a spin-1 element $\mathtt{J}$, a spin-2 multiplet $\mathtt{A}_i, (i = \pm 1, 0)$ and a spin-3 multiplet $\mathtt{W}_i, (i = \pm 2, \pm 1, 0)$. All these generators together span the bosonic sub-algebra $\mathfrak{sl}(3) \oplus \mathfrak{sl}(2) \oplus \mathfrak{u}(1)$. Furthermore, the odd-graded elements decompose in two spin $\frac{3}{2}$ multiplets $\mathtt{G}_r^M, (r = \pm\frac{1}{2}), (M = \pm)$ and two spin $\frac{5}{2}$ multiplets $\mathtt{S}_r^M, (r = \pm\frac{3}{2}, \pm\frac{1}{2}), (M = \pm)$. Then, the bosonic sector of this algebra is given as follows:

$$[\mathtt{L}_i, \mathtt{L}_j] = (i-j)\mathtt{L}_{i+j}\,, \qquad [\mathtt{L}_i, \mathtt{A}_j] = (i-j)\mathtt{A}_{i+j}\,, \quad (3.1)$$

$$[\mathtt{L}_i, \mathtt{W}_j] = (2i-j)\mathtt{W}_{i+j}\,, \qquad [\mathtt{A}_i, \mathtt{A}_j] = (i-j)\mathtt{L}_{i+j}\,,$$
$$[\mathtt{A}_i, \mathtt{W}_j] = (2i-j)\mathtt{W}_{i+j}\,, \quad (3.2)$$

$$[\mathtt{W}_i, \mathtt{W}_j] = -\frac{1}{6}(i-j)(2i^2+2j^2-ij-8)(\mathtt{L}_{i+j}+A_{i+j})\,. \quad (3.3)$$

Additionally, the explicit commutation relations between the bosonic and fermionic sectors are given by

$$\left[\mathtt{L}_i, \mathtt{G}^{\pm}_p\right] = \left(\frac{i}{2}-p\right)\mathtt{G}^{\pm}_{i+p}, \qquad \left[\mathtt{L}_i, \mathtt{S}^{\pm}_p\right] = \left(\frac{3i}{2}-p\right)\mathtt{S}^{\pm}_{i+p}, \quad (3.4)$$

$$[\mathtt{J}, \mathtt{G}^{\pm}_r] = \pm\mathtt{G}^{\pm}_r\,, \qquad [\mathtt{J}, \mathtt{S}^{\pm}_r] = \pm\mathtt{S}^{\pm}_r\,, \quad (3.5)$$

$$[\mathtt{A}_i, \mathtt{G}^{\pm}_r] = \frac{5}{3}\left(\frac{i}{2}-r\right)\mathtt{G}_{i+r} \pm \frac{4}{3}\mathtt{S}^{\pm}_{i+r}\,, \quad (3.6)$$

$$[\mathtt{A}_i, \mathtt{S}^{\pm}_r] = \frac{1}{3}\left(\frac{3i}{2}-r\right)\mathtt{S}^{\pm}_{i+r} \mp \frac{1}{3}\left(3i^2-2ir+r^2-\frac{9}{4}\right)\mathtt{G}^{\pm}_{i+r}\,, \quad (3.7)$$

$$[\mathtt{W}_i, \mathtt{G}^{\pm}_r] = -\frac{4}{3}\left(\frac{i}{2}-2r\right)\mathtt{S}^{\pm}_{i+r}\,, \quad (3.8)$$

$$[\mathtt{W}_i, \mathtt{S}^{\pm}_r] = \mp\frac{1}{3}\left(2r^2-2ri+i^2-\frac{5}{2}\right)\mathtt{S}^{\pm}_{i+r} - \frac{1}{6}\left(4r^3-3r^2i+2ri^2-i^3-9r+\frac{19}{4}i\right)\mathtt{G}^{\pm}_{i+r}. \quad (3.9)$$

And lastly, the fermionic sector satisfy the following anti - commutation relations

$$\left\{\mathtt{G}^{\pm}_r, \mathtt{G}^{\mp}_s\right\} = 2\,\mathtt{L}_{r+s} \pm (r-s)\,\mathtt{J} \quad (3.10)$$

$$\{\mathtt{G}^{\pm}_r, \mathtt{S}^{\pm}_s\} = -\frac{3}{2}\mathtt{W}^{\mp}_{r+s} + \frac{3}{4}(3r-s)\mathtt{A}_{r+s} - \frac{5}{4}(3r-s)\mathtt{L}_{r+s}\,, \quad (3.11)$$

$$\{\mathtt{S}^{\pm}_r, \mathtt{S}^{\mp}_s\} = -\frac{3}{4}(r-s)\mathtt{W}_{r+s} + \frac{1}{8}\left(3s^2-4rs+3r^2-\frac{9}{2}\right)(L_{r+s}-3\mathtt{A}_{r+s}) - \frac{1}{4}(r-s)\left(r^2+s^2-\frac{5}{2}\right)\mathtt{J} \quad (3.12)$$

except for zero commutators.

Having discussed the principle embedding of $\mathfrak{sl}(2|1)$ into $\mathfrak{sl}(3|2)$, we are now able to formulate the most general boundary conditions for asymptotically AdS_3 spacetimes. In accordance with this purpose it is useful to define the gauge connection as

$$a_\varphi = \rho\mathcal{J}\mathtt{J} + \gamma_i\mathcal{L}^i\mathtt{L}_i + \vartheta_i\mathcal{A}^i\mathtt{A}_i + \omega_i\mathcal{W}^i\mathtt{W}_i + \sigma^p_M\mathcal{G}^p_M\mathtt{G}^M_p + \tau^p_M\mathcal{S}^p_M\mathtt{S}^M_p \quad (3.13)$$

$$a_t = \eta\mathtt{J} + \mu^i\mathtt{L}_i + \xi^i\mathtt{A}_i + f^i\mathtt{W}_i + \nu^p_M\mathtt{G}^M_p + \psi^p_M\mathtt{S}^M_p \quad (3.14)$$

where

$$\rho = \sigma^{\frac{1}{2}}_M = -\sigma^{\frac{1}{2}}_M = \frac{1}{k}$$
$$2\gamma_1 = 2\gamma_{-1} - 2\gamma_0 = 2\vartheta_1 = 2\vartheta_{-1} = -\vartheta_0 = \frac{4}{k}$$
$$3\tau^{-\frac{3}{2}}_M = -3\sigma^{\frac{3}{2}}_M = \tau^{\frac{1}{2}}_M = -\sigma^{\frac{1}{2}}_M = \frac{8}{3k}$$
$$6\omega_{-2} = 6\omega_2 = -\frac{3}{2}\omega_{-1} = -\frac{3}{2}\omega_1 = \omega_0 = \frac{3}{k}$$

are some scaling parameters. As a result, we have twenty four functions; twelve bosonic $\left(\mathcal{J}, \mathcal{L}^i, \mathcal{A}^i, \mathcal{W}^i\right)$ and twelve fermionic $\left(\mathcal{G}^p_M, \mathcal{S}^p_M\right)$ as *charges*. Also, we have in total twenty four independent functions $\left(\eta, \mu^i, \xi^i, f^i, \nu^p_M, \psi^p_M\right)$ too, as *chemical potentials* for the time component.

In the presence of the loosest set of boundary conditions, thanks to the flatness conditions (2.5), the equations of motion for fixed chemical potentials impose the additional conditions as the temporal evolution of the twenty four independent source fields $\left(\mathcal{J}, \mathcal{L}^i, \mathcal{A}^i, \mathcal{W}^i, \mathcal{G}^p_M, \mathcal{S}^p_M\right)$ as calculated in Appendix A. Thus, the consequences of our calculations to derive the relevant superalgebra for the loosest set of boundary conditions can now be evaluated through a canonical analysis. We now consider the boundary preserving gauge transformations (encompassing all) (2.17) generated by the $\mathfrak{sl}(3|2)$ Lie superalgebra-valued gauge parameter λ, which we choose as

$$\lambda = b^{-1}\left[\varrho\mathtt{J} + \epsilon^i\mathtt{L}_i + \phi^i\mathtt{A}_i + \upsilon^i\mathtt{W}_i + \varsigma^p_M\mathtt{G}^M_p + \omega^p_M\mathtt{S}^M_p\right]b. \quad (3.15)$$

Note that there are in total twenty four arbitrary functions on the boundary, consist of twelve bosonic $(\varrho, \epsilon^i, \phi^i, \upsilon^i)$

and twelve fermionic $(\varsigma_M^p, \omega_M^p)$. Inserting this expression into (2.17) imposes the gauge transformations. For further details, see Appendix B. Analogously, the gauge transformations for the chemical potentials are calculated whose details can again be found in Appendix C. Following a similar approach as previous section, we act on to make out the canonical boundary charges $\mathcal{Q}[\lambda]$ that generates the transformations (B.1)–(B.11). As is well known, it is convenient to express the variation of the canonical boundary charge $\delta_\lambda \mathcal{Q}$ (2.27), to reach out the asymptotic symmetry algebra [46–49]. Hence, the canonical boundary charge $\mathcal{Q}[\lambda]$ can be obtained which reads

$$\mathcal{Q}[\lambda] = \int \mathrm{d}\varphi \,\Big[\mathcal{J}\varrho + \mathcal{L}^i\epsilon^{-i} + \mathcal{A}^i\phi^{-i} + \mathcal{W}^i\upsilon^{-i} + \mathcal{G}_M^p\varsigma^{-p} + \mathcal{S}_M^p\omega^{-p}\Big]. \quad (3.16)$$

The next step to derive the asymptotic symmetry algebra is to calculate Poisson bracket algebra using the standard method [45], which is acquired by the relation (2.29) given for any phase space functional F.

In light of the above statement, the operator product algebra for the bosonic sector is then achieved as

$$\mathcal{L}^i(z_1)\mathcal{L}^j(z_2) \sim \frac{\frac{k}{2}\eta_2^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}}\mathcal{L}^{i+j},$$
$$\mathcal{J}(z_1)\mathcal{J}(z_2) \sim \frac{\frac{k}{2}\eta}{z_{12}^2}, \quad (3.17)$$
$$\mathcal{L}^i(z_1)\mathcal{A}^j(z_2) \sim \frac{(i-j)}{z_{12}}\mathcal{A}^{i+j},$$
$$\mathcal{L}^i(z_1)\mathcal{W}^j(z_2) \sim \frac{(2i-j)}{z_{12}}\mathcal{W}^{i+j}, \quad (3.18)$$
$$\mathcal{A}^i(z_1)\mathcal{A}^j(z_2) \sim \frac{\frac{k}{2}\eta_2^{ij}}{z_{12}^2} + \frac{(i-j)}{z_{12}}\mathcal{A}^{i+j},$$
$$\mathcal{A}^i(z_1)\mathcal{W}^j(z_2) \sim \frac{(2i-j)}{z_{12}}\mathcal{A}^{i+j}, \quad (3.19)$$
$$\mathcal{W}^i(z_1)\mathcal{W}^j(z_2) \sim \frac{\frac{k}{2}\eta_3^{ij}}{z_{12}^2} + \frac{1}{z_{12}}\left(\frac{1}{3}(i-j)\left(2i^2-ij+2j^2-8\right) \times \left(\mathcal{A}^{i+j}+\mathcal{L}^{i+j}\right)\right). \quad (3.20)$$

Furthermore, the explicit operator product algebra between the bosonic and fermionic sectors is given by

$$\mathcal{J}(z_1)\mathcal{G}_\pm^p(z_2) \sim \pm\frac{\mathcal{G}_\pm^p}{z_{12}},$$
$$\mathcal{L}^i(z_1)\mathcal{G}_\pm^p(z_2) \sim \frac{(\frac{i}{2}-p)}{z_{12}}\mathcal{G}_\pm^{i+p}, \quad (3.21)$$
$$\mathcal{L}^i(z_1)\mathcal{S}_\pm^p(z_2) \sim \frac{(\frac{3i}{2}-p)}{z_{12}}\mathcal{S}_\pm^{i+p},$$
$$\mathcal{J}(z_1)\mathcal{S}_\pm^p(z_2) \sim \pm\frac{\frac{k}{2}}{z_{12}}\mathcal{S}_\pm^{p+i}, \quad (3.22)$$
$$\mathcal{G}_\pm^p(z_1)\mathcal{A}^i(z_2) \sim \mp\frac{\mathcal{S}_\mp^p}{z_{12}},$$
$$\mathcal{G}_\pm^p(z_1)\mathcal{W}^i(z_2) \sim -\frac{\frac{4}{3}(2i-\frac{p}{2})}{z_{12}}\mathcal{S}_\mp^{p+i}, \quad (3.23)$$
$$\mathcal{A}(z_1)\mathcal{S}_\pm^p(z_2) \sim \frac{1}{z_{12}}\left(\frac{1}{3}\left(\frac{3i}{2}-p\right)\mathcal{S}_\pm^{i+p} \mp\frac{1}{4}\left(3i^2-2ip+p^2-\frac{9}{4}\right)\mathcal{G}_\pm^{i+p}\right), \quad (3.24)$$
$$\mathcal{S}_\pm^p(z_1)\mathcal{W}^i(z_2) \sim \mp\frac{1}{z_{12}}\left(\frac{1}{3}\left(i^2-2ip+2p^2-\frac{5}{2}\right)\mathcal{S}_\mp^{p+i} + \frac{1}{8}\left(4p^3-i^3+2i^2p-3ip^2-9p+\frac{19}{4}i\right) \times\mathcal{G}_\mp^{p+i}\right). \quad (3.25)$$

Finally, the explicit operator product algebra for the fermionic sector yields

$$\mathcal{G}_\pm^p(z_1)\mathcal{G}_\pm^q(z_2) \sim \frac{\frac{k}{2}\eta_{\frac{3}{2}}^{pq}}{z_{12}^2} + \frac{2}{z_{12}} \times\left(\mathcal{L}^{p+q}+\frac{5}{3}\mathcal{A}^{p+q}\pm\frac{(p-q)}{2}\mathcal{J}\right), \quad (3.26)$$
$$\mathcal{G}_\pm^p(z_1)\mathcal{S}_\pm^q(z_2) \sim \frac{2}{z_{12}}\left(\frac{3}{4}\mathcal{W}^{p+q}-\left(\frac{3p}{2}-\frac{q}{2}\right)\mathcal{A}^{p+q}\right), \quad (3.27)$$
$$\mathcal{S}_\pm^p(z_1)\mathcal{S}_\pm^q(z_2) \sim \frac{\frac{k}{2}\eta_{\frac{5}{2}}^{pq}}{z_{12}^2} \pm \frac{1}{z_{12}}\left(\frac{1}{8}\left(3p^2-4pq+3q^2-\frac{9}{2}\right)\left(\mathcal{A}^{p+q}+3\mathcal{L}^{p+q}\right) \mp\frac{3}{4}(p-q)\mathcal{W}^{p+q} \pm\frac{3}{16}(p-q)\left(p^2+q^2-\frac{5}{2}\right)\mathcal{J}\right), \quad (3.28)$$

where $z_{12} = z_1 - z_2$, or in the more compact form,

$$\mathfrak{J}^A(z_1)\mathfrak{J}^B(z_2) \sim \frac{\frac{k}{2}\eta^{AB}}{z_{12}^2} + \frac{\mathfrak{f}^{AB}{}_C\mathfrak{J}^C(z_2)}{z_{12}}. \quad (3.29)$$

Note that η^{AB} is the supertrace matrix and $\mathfrak{f}^{AB}_C$'s are the structure constants of the related algebra with $(A, B = 0, \pm1, \pm\frac{1}{2}, 0, \pm1, \pm\frac{1}{2}, \pm\frac{3}{2})$, i.e, $\eta^{ip} = 0$ and $\mathfrak{f}^{ij}{}_{i+j} = (i-j)$.

Since the barred sector is completely analogous, the same results are obtained. Eventually, it follows that the asymptotic symmetry algebra for the loosest set of boundary conditions of $\mathcal{N} = (2, 2)$ supergravity is two copies of the affine $\mathfrak{sl}(3|2)_k$ algebra.

3.2 For superconformal boundary

As already discussed, it is convenient to point out that the super-conformal boundary conditions are the supersymmetric extension of the well-known Brown–Henneaux boundary conditions, presented in [6] for AdS_3 supergravity. In this section, our main goal is to construct the asymptotic symmetry algebra for the most general boundary conditions as the supersymmetric extension of the Brown–Henneaux boundary conditions. In accordance for this purpose, we launch into our section by imposing the Drinfeld–Sokolov heighest weight gauge condition on the $\mathfrak{sl}(3|2)$ Lie superalgebra valued connection (3.13), setting the fields as

$$\begin{aligned}&\mathcal{L}^0=\mathcal{A}^0=\mathcal{A}^{+1}=\mathcal{G}_M^{+\frac12}=\mathcal{S}_M^{+\frac12}=\mathcal{S}_M^{+\frac32}=0,\\&\mathcal{L}^{-1}=\mathcal{L},\ \mathcal{A}^{-1}=\mathcal{A},\ \mathcal{G}_M^{-\frac12}=\mathcal{G}_M,\\&\mathcal{S}_M^{-\frac32}=\mathcal{S}_M,\ \gamma_{+1}\mathcal{L}^{+1}=1.\end{aligned}\tag{3.30}$$

Correspondingly, the supersymmetric gauge connection is taken to be

$$\begin{aligned}a_\varphi&=\mathtt{L}_1+\gamma_{-1}\mathcal{L}\mathtt{L}_{-1}+\vartheta_{-1}\mathcal{A}\mathtt{A}_{-1}+\omega_{-2}\mathcal{W}\mathtt{W}_{-2}+\rho\mathcal{J}\mathtt{J}\\&\quad+\sigma_M^{-\frac12}\mathcal{G}_M\mathtt{G}^M_{-\frac12}+\tau_M^{-\frac32}\mathcal{S}_M\mathtt{S}^M_{-\frac32},\end{aligned}\tag{3.31}$$

$$\begin{aligned}a_t&=\eta\mathtt{J}+\mu\mathtt{L}_1+\xi\mathtt{A}_1+f\mathtt{W}_2+\nu_M\mathtt{G}^M_{+\frac12}+\psi_M\mathtt{S}^M_{+\frac32}\\&\quad+\sum_{i=-1}^{0}\mu^i\mathtt{L}_i+\sum_{i=-1}^{0}\xi^i\mathtt{A}_i\\&\quad+\nu_M^{-\frac12}\mathtt{G}^M_{-\frac12}+\sum_{p=-\frac32}^{\frac12}\psi_M^p\mathtt{S}^M_p,\end{aligned}\tag{3.32}$$

where η, $\mu\equiv\mu^{+1}$, $\xi\equiv\xi^{+1}$, $f\equiv f^{+2}$, $\nu_M\equiv\nu_M^{+\frac12}$, and $\psi_M\equiv\psi_M^{+\frac32}$ can be interpreted as the independent *chemical potentials*.

When we request to gather up our next steps yielding the asymptotic symmetry algebra in a one paragraph, it is an appropriate option to give the following brief summary. The all functions except the *chemical potentials* can be fixed by the flattness conditions (2.5) in the usual manner. The equations of motion for the fixed chemical potentials can also be obtained conventionally as the time evolution of the canonical boundary charges. But unfortunately, moving from the $\mathfrak{sl}(2|1)$-case to $\mathfrak{sl}(3|2)$-extension brings along the technical cumbersome although we have overcomen. Our preference is to ignore presenting these calculations here, because they take up too much space. We make choice to spare space for the calculations of the gauge parameter λ and further.

In line with all these results to be obtained, we are now able to derive the superconformal asymptotic symmetry algebra. Using the Drinfeld–Sokolov reduction we have only six independent parameters as ϱ, $\epsilon\equiv\epsilon^{+1}$, $\phi\equiv\phi^{+1}$, $\upsilon\equiv\upsilon^{+2}$, $\varsigma_M\equiv\varsigma_M^{+\frac12}$ and $\omega_M\equiv\omega_M^{+\frac32}$ It is now possible to compute the gauge transformations by considering all transformations (2.17) that preserve the boundary conditions with the $\mathfrak{sl}(3|2)$ Lie superalgebra valued gauge parameter λ. See Appendix D, for more details.

Since we are dealing with the asymptotic symmetries, it is natural to demand obtaining the canonical boundary charges $\mathcal{Q}[\lambda]$. So, the variation of the canonical boundary charge, i.e., $\delta_\lambda\mathcal{Q}$ (2.27) can be integrated to yield

$$\mathcal{Q}[\lambda]=\int d\varphi\left[\mathcal{J}\varrho+\mathcal{L}\epsilon+\mathcal{A}\phi+\mathcal{W}\upsilon+\mathcal{G}_M\varsigma_M+\mathcal{S}_M\omega_M\right].\tag{3.33}$$

But, these canonical boundary charges do not give a convenient asymptotic operator product algebra in the complex coordinates by using (2.29) for $\mathcal{N}=(2,2)$ superconformal boundary,

$$\mathcal{L}(z_1)\mathcal{L}(z_2)\sim\frac{3k}{z_{12}^4}+\frac{2\mathcal{L}}{z_{12}^2}+\frac{\mathcal{L}'-\frac{\mathcal{G}_+\mathcal{G}_-}{k}}{z_{12}}\tag{3.34}$$

$$\mathcal{L}(z_1)\mathcal{J}(z_2)\sim0\tag{3.35}$$

$$\mathcal{L}(z_1)\mathcal{G}_\pm(z_2)\sim\frac{3\mathcal{G}_+}{2z_{12}^2}+\frac{\mathcal{G}'_+\pm\frac{\mathcal{J}\mathcal{G}_+}{2k}}{z_{12}}\tag{3.36}$$

$$\mathcal{L}(z_1)\mathcal{A}(z_2)\sim\frac{\mathcal{A}'}{z_{12}}+\frac{2\mathcal{A}}{z_{12}^2}\tag{3.37}$$

$$\mathcal{L}(z_1)\mathcal{S}_\pm(z_2)\sim\frac{5\mathcal{S}_+}{2z_{12}^2}+\frac{\mathcal{S}'_+\pm\frac{\mathcal{J}\mathcal{S}_+}{2k}}{z_{12}}\tag{3.38}$$

$$\mathcal{L}(z_1)\mathcal{W}(z_2)\sim+\frac{3\mathcal{W}}{z_{12}^2}+\frac{1}{z_{12}}\left(\mathcal{W}'+\frac{\mathcal{G}_+\mathcal{S}_-}{k}-\frac{\mathcal{S}_+\mathcal{G}_-}{k}\right)\tag{3.39}$$

$$\mathcal{J}(z_1)\mathcal{J}(z_2)\sim\frac{2k}{z_{12}^2}\tag{3.40}$$

$$\mathcal{J}(z_1)\mathcal{G}_\pm(z_2)\sim\mp\frac{\mathcal{G}_\pm}{z_{12}}\tag{3.41}$$

$$\mathcal{J}(z_1)\mathcal{A}(z_2)\sim0\tag{3.42}$$

$$\mathcal{J}(z_1)\mathcal{S}_\pm(z_2)\sim\mp\frac{\mathcal{S}_\pm}{z_{12}}\tag{3.43}$$

$$\mathcal{J}(z_1)\mathcal{W}(z_2)\sim0\tag{3.44}$$

$$\begin{aligned}\mathcal{G}_\pm(z_1)\mathcal{G}_\pm(z_2)&\sim\frac{\mp4k}{z_{12}^3}-\frac{2\mathcal{J}}{z_{12}^2}\\&\quad+\frac{\mp2}{z_{12}}\left(\mathcal{L}+\frac{\mathcal{J}\mathcal{J}}{4k}\pm\frac{\mathcal{J}'}{2}\mp\frac{10\mathcal{A}}{3}\right)\end{aligned}\tag{3.45}$$

$$\mathcal{G}_\pm(z_1)\mathcal{G}_\mp(z_2)\sim0\tag{3.46}$$

$$\mathcal{G}_\pm(z_1)\mathcal{A}(z_2)\sim\mp\frac{15\mathcal{S}_\pm}{4z_{12}}\tag{3.47}$$

$$\mathcal{G}_\pm(z_1)\mathcal{S}_\pm(z_2)\sim\pm\frac{16\mathcal{A}}{15z_{12}^2}\pm\frac{1}{z_{12}}\left(\frac{4\mathcal{A}'}{15}\pm\frac{8\mathcal{A}\mathcal{J}}{15k}\mp4\mathcal{W}\right)\tag{3.48}$$

$$\mathcal{G}_\pm(z_1)\mathcal{S}_\mp(z_2)\sim0\tag{3.49}$$

$$\mathcal{G}_\pm(z_1)\mathcal{W}(z_2)\sim-\frac{5\mathcal{S}_\mp}{4z_{12}^2}-\frac{1}{z_{12}}\left(\frac{\mathcal{S}'_\mp}{4}-\frac{2\mathcal{A}\mathcal{G}_\mp}{15k}\mp\frac{\mathcal{J}\mathcal{S}_\mp}{2k}\right)\tag{3.50}$$

$$\mathcal{A}(z_1)\mathcal{A}(z_2)\sim\frac{3k}{z_{12}^4}+\frac{2\mathcal{L}}{z_{12}^2}+\frac{\mathcal{L}'-\frac{\mathcal{G}_+\mathcal{G}_-}{4k}}{z_{12}}\tag{3.51}$$

$$\mathcal{S}_\pm(z_1)\mathcal{S}_\mp(z_2)\sim-\frac{2\mathcal{G}_\mp\mathcal{G}'_\mp}{5kz_{12}}\tag{3.52}$$

$$\begin{aligned}\mathcal{A}(z_1)\mathcal{S}_\pm(z_2)&\sim\frac{4\mathcal{G}_+}{5z_{12}^3}+\frac{1}{z_{12}^2}\left(\frac{5\mathcal{S}_\pm}{6}+\frac{4\mathcal{G}'_\pm}{15}\pm\frac{2\mathcal{G}_\pm\mathcal{J}}{15k}\right)\\&\quad+\frac{1}{z_{12}}\left(\frac{\mathcal{G}_\pm\mathcal{J}^2}{60k^2}-\frac{3\mathcal{A}\mathcal{G}_\pm}{5k}\pm\frac{\mathcal{G}_\pm\mathcal{J}'}{30k}\right.\end{aligned}$$

$$\pm\frac{\mathcal{J}\mathcal{G}'_\pm}{15k}+\frac{3\mathcal{G}_\pm\mathcal{L}}{5k}\pm\frac{\mathcal{J}\mathcal{S}_\pm}{6k}+\frac{\mathcal{S}'_\pm}{3}+\frac{\mathcal{G}''_\pm}{15}\Bigg) \tag{3.53}$$

$$\begin{aligned}\mathcal{W}(z_1)\mathcal{W}(z_2)\sim\ &\frac{2k}{z_{12}^6}+\frac{2(\mathcal{A}+\mathcal{L})}{z_{12}^4}-\frac{1}{z_{12}^3}\\
&\times\left(\mathcal{A}'+\frac{13\mathcal{G}_-\mathcal{G}_+}{12k}+\mathcal{L}'\right)\\
&+\frac{1}{z_{12}^2}\Bigg(\frac{3\mathcal{A}''}{10}-\frac{13\mathcal{G}_+\mathcal{G}'_-}{24k}-\frac{13\mathcal{G}_-\mathcal{G}'_+}{24k}\\
&+\frac{16\mathcal{A}^2}{15kz_{12}^2}+\frac{32\mathcal{A}\mathcal{L}}{15kz_{12}^2}+\frac{16\mathcal{L}^2}{15k}+\frac{3\mathcal{L}''}{10}\Bigg)\\
&-\frac{1}{z_{12}}\Bigg(\frac{\mathcal{G}_-\mathcal{G}_+\mathcal{J}^2}{30k^3}+\frac{32\mathcal{A}\mathcal{G}_-\mathcal{G}_+}{45k^2}-\frac{\mathcal{A}^{(3)}}{15}\\
&-\frac{\mathcal{G}_+\mathcal{J}\mathcal{S}_-}{3k^2}+\frac{\mathcal{G}_-\mathcal{J}\mathcal{S}_+}{3k^2}-\frac{\mathcal{G}_+\mathcal{J}\mathcal{G}'_-}{15k^2}\\
&+\frac{\mathcal{G}_-\mathcal{J}\mathcal{G}'_+}{15k^2}+\frac{16\mathcal{G}_-\mathcal{G}_+\mathcal{L}}{15k^2}\\
&-\frac{16\mathcal{L}\mathcal{A}'}{15k}-\frac{16\mathcal{A}\mathcal{A}'}{15k}-\frac{16\mathcal{A}\mathcal{L}'}{15k}-\frac{5\mathcal{S}_+\mathcal{G}'_-}{12k}\\
&+\frac{\mathcal{G}_+\mathcal{S}'_-}{4k}-\frac{5\mathcal{S}_-\mathcal{G}'_+}{12k}+\frac{\mathcal{G}_-\mathcal{S}'_+}{4k}+\frac{7\mathcal{G}'_-\mathcal{G}'_+}{30k}\\
&+\frac{11\mathcal{G}_+\mathcal{G}''_-}{60k}+\frac{11\mathcal{G}_-\mathcal{G}''_+}{60k}+\frac{10\mathcal{S}_-\mathcal{S}_+}{k}\\
&-\frac{16\mathcal{L}\mathcal{L}'}{15k}-\frac{\mathcal{L}^{(3)}}{15}\Bigg)\end{aligned} \tag{3.54}$$

$$\begin{aligned}\mathcal{S}_\pm(z_1)\mathcal{W}(z_2)\sim\ &\mp\frac{3\mathcal{G}_\mp}{4z_{12}^4}\pm\frac{1}{z_{12}^3}\left(\frac{5\mathcal{S}_\mp}{4}\mp\frac{\mathcal{G}_\mp\mathcal{J}}{4k}-\frac{\mathcal{G}'_\mp}{4}\right)\\
&\mp\frac{1}{z_{12}^2}\Bigg(\frac{\mathcal{G}''_\mp}{16}+\frac{3\mathcal{G}_\mp\mathcal{J}^2}{64k^2}+\frac{91\mathcal{A}\mathcal{G}_\mp}{240k}\pm\frac{5\mathcal{G}_\mp\mathcal{J}'}{32k}\\
&\pm\frac{\mathcal{J}\mathcal{G}'_\mp}{16k}+\frac{11\mathcal{G}_\mp\mathcal{L}}{16k}\mp\frac{3\mathcal{J}\mathcal{S}_\mp}{8k}-\frac{\mathcal{S}'_-}{2}\Bigg)\\
&-\frac{1}{z_{12}}\Bigg(\frac{\mathcal{G}_\mp\mathcal{J}^3}{160k^3}\mp\frac{\mathcal{G}_\mp{}^{(3)}}{80}+\frac{13\mathcal{A}\mathcal{G}_\mp\mathcal{J}}{120k^2}\\
&\pm\frac{3\mathcal{J}^2\mathcal{G}'_-}{320k^2}\pm\frac{9\mathcal{G}_\mp\mathcal{J}\mathcal{J}'}{160k^2}\pm\frac{9\mathcal{G}_\mp\mathcal{J}\mathcal{L}}{40k^2}\mp\frac{\mathcal{J}^2\mathcal{S}_\mp}{16k^2}\end{aligned} \tag{3.55}$$

$$\begin{aligned}&\pm\frac{13\mathcal{G}_\mp\mathcal{A}'}{80k}\pm\frac{13\mathcal{A}\mathcal{G}'_\mp}{80k}\mp\frac{11\mathcal{A}\mathcal{S}_\mp}{4k}\\
&+\frac{9\mathcal{G}_\mp\mathcal{J}''}{160k}+\frac{7\mathcal{G}'_\mp\mathcal{J}'}{160k}\\
&+\frac{\mathcal{J}\mathcal{G}''_\mp}{80k}-\frac{5\mathcal{G}_\mp\mathcal{W}}{2k}\pm\frac{27\mathcal{G}_\mp\mathcal{L}'}{80k}\\
&\pm\frac{19\mathcal{L}\mathcal{G}'_\mp}{80k}-\frac{\mathcal{S}_\mp\mathcal{J}'}{4k}-\frac{\mathcal{J}\mathcal{S}'_\mp}{8k}\mp\frac{5\mathcal{S}_\mp\mathcal{L}}{4k}\mp\frac{\mathcal{S}''_\mp}{8}\Bigg)\end{aligned} \tag{3.56}$$

$$\begin{aligned}\mathcal{S}_\pm(z_1)\mathcal{S}_\pm(z_2)\sim\ &\frac{12k}{5z_{12}^5}\pm\frac{6\mathcal{J}}{5z_{12}^4}+\frac{1}{z_{12}^3}\left(\frac{2\mathcal{A}}{3}+\frac{3\mathcal{J}^2}{10k}+2\mathcal{L}\right)\\
&+\frac{1}{z_{12}^2}\Bigg(\frac{\mathcal{A}'}{3}\pm\frac{\mathcal{J}''}{5}\pm\frac{3\mathcal{J}'}{5}\pm\frac{\mathcal{J}^3}{20k^2}\pm\frac{\mathcal{A}\mathcal{J}}{3k}\\
&-\frac{13\mathcal{G}_-\mathcal{G}_+}{20k}+\frac{3\mathcal{J}\mathcal{J}'}{10k}\pm\frac{\mathcal{J}\mathcal{L}}{k}\mp4\mathcal{W}+\mathcal{L}'\Bigg)\\
&+\frac{1}{z_{12}}\Bigg(\frac{\mathcal{A}''}{10}\pm\frac{\mathcal{J}^{(3)}}{20}+\frac{\mathcal{J}^4}{160k^3}\\
&+\frac{\mathcal{A}\mathcal{J}^2}{12k^2}\mp\frac{\mathcal{G}_-\mathcal{G}_+\mathcal{J}}{5k^2}+\frac{\mathcal{J}^2\mathcal{L}}{4k^2}\\
&\pm\frac{3\mathcal{J}^2\mathcal{J}'}{40k^2}-\frac{3\mathcal{A}^2}{2k}\pm\frac{\mathcal{J}\mathcal{A}'}{6k}\\
&\pm\frac{\mathcal{A}\mathcal{J}'}{6k}+\frac{3\mathcal{A}\mathcal{L}}{5k}\mp\frac{2\mathcal{G}_+\mathcal{S}_-}{k}\\
&\pm\frac{2\mathcal{G}_-\mathcal{S}_+}{k}-\frac{9\mathcal{G}_\pm\mathcal{G}'_\mp}{20k}-\frac{\mathcal{G}_-\mathcal{G}'_+}{5k}\\
&+\frac{\mathcal{J}\mathcal{J}''}{10k}\pm\frac{\mathcal{L}\mathcal{J}'}{2k}+\frac{3\mathcal{J}'^2}{40k}\\
&-\frac{2\mathcal{J}\mathcal{W}}{k}\pm\frac{\mathcal{J}\mathcal{L}'}{2k}+\frac{9\mathcal{L}^2}{10k}\mp2\mathcal{W}'+\frac{3\mathcal{L}''}{10}\Bigg)\end{aligned} \tag{3.57}$$

because some boundary charges do not transform like a primary conformal field, and also there exist some nonlinear terms such as $(\mathcal{J}\mathcal{J})(z)$, $(\mathcal{G}_+\mathcal{G}_-)(z)$, and $(\mathcal{J}\mathcal{G}_\pm)(z)$, as already discussed in the previous section (2.3). Therefore, it is required to consider some redefinitions on the boundary charges and gauge parameters as

$$\begin{aligned}\gamma_{-1}\mathcal{L}&\to\frac{6}{c}\left(\mathcal{L}-\frac{3}{2c}(\mathcal{J}\mathcal{J})+\frac{\kappa}{2}\mathcal{A}\right),\\
\epsilon&\to\epsilon+\frac{\kappa}{2}\left(\phi+\frac{6}{c}\upsilon\mathcal{J}\right)\end{aligned} \tag{3.58}$$

$$\begin{aligned}\vartheta_{-1}\mathcal{A}&\to-\frac{9\kappa}{5c}\mathcal{A},\\
\phi&\to\phi-\frac{3\kappa}{10}\left(\phi+\frac{6}{c}\upsilon\mathcal{J}\right)\end{aligned} \tag{3.59}$$

$$\begin{aligned}\omega_{-2}\mathcal{W}&\to\frac{3\kappa}{5c}\left(\mathcal{W}-\frac{6}{c}\mathcal{J}\mathcal{A}\right),\\
\upsilon&\to\frac{3\kappa}{10}\upsilon\end{aligned} \tag{3.60}$$

$$\begin{aligned}\rho\mathcal{J}&\to\frac{3}{c}\mathcal{J},\\
\varrho&\to\varrho+\frac{3}{c}\left(\epsilon\mathcal{J}+2\upsilon\mathcal{A}\right)\end{aligned} \tag{3.61}$$

$$\sigma_\pm^{-\frac{1}{2}}\mathcal{G}_\pm\to\mp\frac{3}{c}\mathcal{G}_\pm,\qquad \varsigma_\pm\to\pm\varsigma_\pm \tag{3.62}$$

$$\tau_\pm^{-\frac{3}{2}}\mathcal{S}_\pm\to\pm\frac{4\kappa}{5c}\mathcal{S}_\pm,\qquad \omega_\pm\to\mp\frac{2\kappa}{5}\omega_\pm \tag{3.63}$$

where $\kappa=\pm\frac{5i}{2}$, which is defined to make a relation with the notation in [50] at the classical level.

It is important to emphasize that these new variables do not affect the boundary charges.

Finally, this leads to operator product expansions of convenient asymptotic symmetry algebra for $\mathcal{N}=(2,2)$ superconformal boundary with a set of conformal generators

$\mathcal{G}_\pm \rightarrow \mathcal{G}^+ \pm \mathcal{G}^-$ and $\mathcal{S}_\pm \rightarrow \mathcal{S}^+ \pm \mathcal{S}^-$ in the complex coordinates by using (2.29). After repeating the same procedure for the barred-sector, one can say that the asymptotic symmetry algebra for the loosest set of boundary conditions of $\mathcal{N} = (2, 2)$ supergravity is two copies of the super $\mathcal{W}_3$ algebra with central charge $c = 6k$. In this paper, we do not explicitly carry out the whole computation to obtain the classical $\mathcal{N} = (2, 2)$ super $\mathcal{W}_3$ algebra (see [50] for the entire quantum $\mathcal{N} = 2$ super $\mathcal{W}_3$ algebra and [51] for the classical case). Recently, a detailed derivation of the asymptotic symmetry algebra is also given in [52] for the $\mathfrak{sl}(3|2)$ case.

3.3 Remarks on charges and the nature of quantum gravity through holography

Before we move on to the last section, we make some significant remarks on the physical interpretation for the charges of the extended algebras. As previously mentioned, when one analyzes a theory with a boundary, it is essential to consider boundary conditions for the dynamical variables. In the covariant approach, the boundary conditions imposed on conformal gravity such that it entails to having a well-defined variational principle give the boundary conditions preserving transformations, and subsequently the asymptotic symmetry algebra. These conditions can be physically motivated, or determined to make into the theory self-consistent. As is well-known, if the original action is not differentiable in the presence of the boundary conditions in question, a boundary term is needed to be added it. In this context, the canonical boundary charges are defined as the boundary terms added to the smeared generators of the gauge transformations to make them differentiable. In order to be endow with the canonical boundary charges related to the mass and angular momentum of the BTZ black hole for pure gravity case, one would in principle work with the metric formulation or use the Chern–Simons description through the proposed holonomies which contain the gauge-invariant information [18,53]. It is also worth noting within this framework that the charges for extended algebras give rise to the symmetries of the dual two-dimensional CFT and so by evaluating the charges and their corresponding algebras, one can determine what chiral algebra the dual CFT has. In other words, the charges of these extended algebras have a physical meaning as conserved charges of the putative dual CFTs, and as such, one can define generalized partition functions with chemical potentials that couple to these conserved charges. Moreover, these could be corresponding to the black hole solutions that carry higher spin charges in AdS_3. Chiral and anti-chiral functions appearing in the canonical boundary charges characterize the physical state space in the black hole solutions. These are the vacuum expectation values of the corresponding operators in the holographic interpretation.

In addition, it would be appropriate to point out that these higher spin theories have been studied quite a bit over the last few years, and it is by now pretty clear which higher spin theory yields which two-dimensional chiral algebra. In brief: if the higher spin theory is described by a Chern-Simons theory based on Lie algebra g, then the dual CFT will be the Hamiltonian reduction of g. Which Hamiltonian reduction appears depends on the boundary conditions one imposes on the Chern–Simons theory, but also this has been studied in various examples by now.

Having studied the asymptotic symmetry algebra realization of the classical extended $\mathcal{N} = (2, 2)$ superalgebra in terms of higher-spin fields on AdS_3 with the most general boundary conditions, the corresponding dual field theory being two classical copies of the $\mathfrak{sl}(3|2)_k$ is highly suggestive because the physical Hilbert space, belongs to representations of two copies of the related superalgebra, as in the bosonic case. Therefore, it is of interest to investigate the holographic implications of our new encouraging boundary conditions, and also, this super extension will be novel in that it follows the boundary conditions discussed in [26].

It is also convenient to note that the choice of various boundary conditions enriches the holographic interpretation. Most prevalent works in the literature based on AdS_3/CFT_2 duality are those belonging to the class of Brown–Henneaux boundary conditions, obtained via Drinfeld–Sokolov reduction in the highest-weight gauge of the Chern–Simons theory. The holographic interpretation of each boundary condition is idiosyncratic for both Brown–Henneaux boundary conditions [6] and its (extended) supersymmetric extensions [8], and also other boundary conditions that have been proposed by altering and generalizing the Brown–Henneaux boundary conditions in the last years [28,31,32]. The most general boundary conditions and their supersymmetric extensions make it possible to look at the holographic picture from a much broader perspective in this sense. From this point of view, it is considerable to impose the most general boundary conditions, explore the asymptotic structure for (extended) supergravity and obtain new algebras. It is also important to check out whether these most general boundary conditions and their (extended) supersymmetric extensions encompassing or not the Brown–Henneaux boundary conditions in terms of consistency as a first test, or even other boundary conditions in the literature.

Furthermore, the analysis we performed is essentially classical. However, it is also possible to examine the corresponding dual conformal field theory at the quantum level. The usual way in this case is to compute Poisson brackets and then quantize them canonically by replacing with the commutators. An additional point worth noting here is that to make a relation with the holographic description, one needs to extract the semiclassical limit of the operator product expansion relations. This procedure involves taking a large-c and large-current limit, a procedure that is more subtle than a naive expansion in $1/c$.

3.4 Remarks on the correspondence of higher spin theories to the string theory and recent developments

It is pertinent to discuss how a supergravity theory constructed in this manner can be physically interesting. Infact, it is not exactly known how these higher spin theories (and their dual $CFTs$) sit inside string theory (and their dual $CFTs$) in general. Fortunately, there is one example in the literature that is understood well: this is the $\mathcal{N} = 4$ higher spin theory in the limit $\lambda \to 0$ which sits inside superstring theory on $AdS_3 \times S^3 \times T^4$ at the tensionless point, the dual CFT is then exactly the symmetric orbifold of T^4. This was originally explained in [54]; more recently, the relevant string theory has been managed to construct in detail, see in particular [55,56] and [57].

It would also be intriguing to see what the analogue of [54] should be for other theories, particularly with less supersymmetry. As a result, there can be made some suggestions in this direction for the questions such as: "Can one identify the higher spin theory corresponding to the theory in [58]?", or, alternatively, "Can one see what string theory could correspond to higher spin theories?." So, it could correspond to a limit of our higher spin theory at this very point.

4 Concluding remarks

In the present paper, it is clearly put forward that the Chern–Simons formulation of AdS_3 (super)gravity also allows a more convenient generalization of higher spin theories for fermionic states as well as for bosonic states. Furthermore, we also confirm explicitly that although the higher spin fields do not propagate any degrees of freedom, there exists a large class of intriguing nontrivial solutions. Specifically, we build up a candidate solution for $\mathcal{N} = (2, 2)$ extended higher spin AdS_3 supergravity and scrutinize its asymptotic symmetries.

To summarize, we give a brief discussion for AdS_3 higher spin supergravity based on Chern–Simons formulation. We first work out $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ Chern–Simons $\mathcal{N} = (2, 2)$ supergravity theory in detail. Then, we construct AdS_3 higher spin supergravity enlarging $\mathfrak{sl}(2|1) \oplus \mathfrak{sl}(2|1)$ to $\mathfrak{sl}(3|3) \oplus \mathfrak{sl}(3|2)$ in the presence of a tower of higher-spin fields up to spin-3. Thereafter we obtain two classical copies of the $\mathfrak{sl}(3|2)_k$ affine algebra on the affine boundary and two copies of super $\mathcal{W}_3$ symmetry algebra on the superconformal boundary as asymptotic symmetry algebras. We also go through the chemical potentials related to source fields appearing through the temporal components of the connection. On the other hand, we see that Chern–Simons action is compatible with our boundary conditions, resulting in a finite effect for higher spin fields and a well-defined variational principle. Consequently, this method can be considered as a good laboratory for researching the fertile asymptotic structure of extended supergravity. It also might be worthwhile to translate our outcomes into the metric formulation language because it will lift our boundary to higher dimensions, where is a Chern–Simons theory.

The results in our paper leave some further investigations which we put in order a few here:

It is a natural question to ask if another class of boundary conditions appearing in literature (see, e.g. [28,30–32]), whose higher-spin generalization is not as clear as the Grumiller and Riegler 's, are consistent with these most general ones, and it would be interesting to examine this. Besides, it has arousing curiosity to get two copies of $\mathcal{N} = (2, 2)$ warped superconformal algebras for the supersymmetric boundary conditions of [31] and also to check the supersymmetric extension of the Avery–Poojary–Suryanarayana boundary conditions [1,32]. In this point, it is also worth noting that the boundary conditions are more restrictive than the pure bosonic case [26]. While the first motivation is to extend them, it is worth mentioning that these generalizations about boundaries would also have a good potential for novel holographic applications.

Finally, we close scratching the limits in our debate of the most general boundary conditions of supergravity. Many open questions still exist for further investigations, e.g., what other boundary conditions from a similar starting point can be attained? Or how can be explained the puzzling result that the related geometries have an entropy? Overall, we think it is satisfying to see that even in specific instances of three dimensional gravity, the asymptotically AdS tale of the most general boundaries recently set forth by Grumiller and Riegler inspires new and unexpected innovations.

Last but not least, according to [26] and [1] there is $\mathcal{N} = (2, 2)$ extended supergravity with new boundaries. However, it is an open problem to decide whether we end up to its enough higher order $\mathcal{N}$ extension by taking the Grumiller-Riegler method as we perform in this paper. Therefore, our results presented in this paper can be extended in various ways. One possible extension is $\mathcal{N} = 3$ supergravity theory in AdS_3. In this context, the details of this possible extension will be examined in our forthcoming paper.

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A The temporal evolution of the twenty four independent source fields

In this appendix, we collect explicit calculations of additional conditions that the equations of motions impose for fixed chemical potentials as the temporal evolution of the twenty four source fields.

$$\begin{aligned}
\partial_t \mathcal{J} &= k\partial_\varphi \eta + \mathcal{G}_2^{\frac{1}{2}} \nu_1^{-\frac{1}{2}} + \mathcal{G}_2^{-\frac{1}{2}} \nu_1^{\frac{1}{2}} - \mathcal{G}_1^{\frac{1}{2}} \nu_2^{-\frac{1}{2}} \\
&-\mathcal{G}_1^{-\frac{1}{2}} \nu_2^{\frac{1}{2}} + \frac{4}{3}\mathcal{S}_2^{\frac{3}{2}} \psi_1^{-\frac{3}{2}} + \frac{4}{3}\mathcal{S}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} \\
&+\frac{4}{3}\mathcal{S}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} + \frac{4}{3}\mathcal{S}_2^{-\frac{3}{2}} \psi_1^{\frac{3}{2}} \\
&-\frac{4}{3}\mathcal{S}_1^{\frac{3}{2}} \psi_2^{-\frac{3}{2}} - \frac{4}{3}\mathcal{S}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} \\
&-\frac{4}{3}\mathcal{S}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} - \frac{4}{3}\mathcal{S}_1^{-\frac{3}{2}} \psi_2^{\frac{3}{2}}
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
\partial_t \mathcal{L}^0 &= \frac{k}{4}\partial_\varphi \mu^0 - \frac{1}{2}\mathcal{G}_2^{\frac{1}{2}} \nu_1^{-\frac{1}{2}} + \frac{1}{2}\mathcal{G}_2^{-\frac{1}{2}} \nu_1^{\frac{1}{2}} \\
&-\frac{1}{2}\mathcal{G}_1^{\frac{1}{2}} \nu_2^{-\frac{1}{2}} + \frac{1}{2}\mathcal{G}_1^{-\frac{1}{2}} \nu_2^{\frac{1}{2}} \\
&-\frac{5}{8}\mathcal{G}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} - \frac{5}{8}\mathcal{G}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} \\
&+\frac{5}{8}\mathcal{G}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} + \frac{5}{8}\mathcal{G}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} \\
&-\frac{5}{3}\nu_1^{-\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}} - \frac{5}{3}\nu_1^{\frac{1}{2}} \mathcal{S}_2^{-\frac{1}{2}} + \frac{5}{3}\nu_2^{-\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}} \\
&+\frac{5}{3}\nu_2^{\frac{1}{2}} \mathcal{S}_1^{-\frac{1}{2}} + \frac{1}{2}\mathcal{S}_2^{\frac{3}{2}} \psi_1^{-\frac{3}{2}} + \frac{1}{6}\mathcal{S}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} \\
&-\frac{1}{6}\mathcal{S}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} - \frac{1}{2}\mathcal{S}_2^{-\frac{3}{2}} \psi_1^{\frac{3}{2}} + \frac{1}{2}\mathcal{S}_1^{\frac{3}{2}} \psi_2^{-\frac{3}{2}} \\
&+\frac{1}{6}\mathcal{S}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} - \frac{1}{6}\mathcal{S}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} \\
&-\frac{1}{2}\mathcal{S}_1^{-\frac{3}{2}} \psi_2^{\frac{3}{2}} - \mathcal{A}^1\xi^{-1} + \mathcal{A}^{-1}\xi^1 \\
&+f^2\mathcal{W}^{-2} + \frac{1}{2}f^1\mathcal{W}^{-1} - \frac{1}{2}f^{-1}\mathcal{W}^1 \\
&-f^{-2}\mathcal{W}^2 - \mu^{-1}\mathcal{L}^1 + \mu^1\mathcal{L}^{-1}
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\partial_t \mathcal{L}^{\pm 1} &= -\frac{k}{2}\partial_\varphi \mu^{\pm 1} \pm \mathcal{G}_2^{\pm\frac{1}{2}} \nu_1^{\pm\frac{1}{2}} \pm \mathcal{G}_1^{\pm\frac{1}{2}} \nu_2^{\pm\frac{1}{2}} \\
&+\frac{15}{8}\mathcal{G}_2^{\mp\frac{1}{2}} \psi_1^{\pm\frac{3}{2}} + \frac{5}{8}\mathcal{G}_2^{\pm\frac{1}{2}} \psi_1^{\pm\frac{1}{2}} \\
&-\frac{15}{8}\mathcal{G}_{\mp}^{\frac{1}{2}} \psi_2^{\pm\frac{3}{2}} - \frac{5}{8}\mathcal{G}_1^{\pm\frac{1}{2}} \psi_2^{\pm\frac{1}{2}} - \frac{5}{3}\nu_1^{\pm\frac{1}{2}} \mathcal{S}_2^{\pm\frac{1}{2}} \\
&-\frac{5}{3}\nu_1^{\mp\frac{1}{2}} \mathcal{S}_2^{\pm\frac{3}{2}} + \frac{5}{3}\nu_2^{\pm\frac{1}{2}} \mathcal{S}_1^{\pm\frac{1}{2}} \\
&+\frac{5}{3}\nu_2^{\mp\frac{1}{2}} \mathcal{S}_1^{\pm\frac{3}{2}} \mp \mathcal{S}_2^{\mp\frac{1}{2}} \psi_1^{\pm\frac{3}{2}} \mp \frac{2}{3}\mathcal{S}_2^{\pm\frac{1}{2}} \psi_1^{\pm\frac{1}{2}} \\
&\mp\frac{1}{3}\mathcal{S}_2^{\pm\frac{3}{2}} \psi_1^{\mp\frac{1}{2}} \mp \mathcal{S}_1^{\mp\frac{1}{2}} \psi_2^{\pm\frac{3}{2}} + \frac{2}{3}\mathcal{S}_1^{\pm\frac{1}{2}} \psi_2^{\pm\frac{1}{2}} \\
&\mp\frac{1}{3}\mathcal{S}_1^{\pm\frac{3}{2}} \psi_2^{\mp\frac{1}{2}} \pm 2\mathcal{A}^0\xi^{\pm 1} \pm \mathcal{A}^{\pm 1}\xi^0 \\
&-\frac{1}{2}f^{\mp 1}\mathcal{W}^{\pm 2} \pm f^0\mathcal{W}^{\pm 1} \pm \frac{3}{2}f^{\pm 1}\mathcal{W}^0 \\
&\pm 2f^{\pm 2}\mathcal{W}^{\mp 1} \pm 2\mu^{\pm 1}\mathcal{L}^0 \pm \mu^0\mathcal{L}^{\pm 1}
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\partial_t \mathcal{A}^0 &= \frac{k}{4}\partial_\varphi \xi_0 + \frac{3}{8}\mathcal{G}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} + \frac{3}{8}\mathcal{G}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} \\
&-\frac{3}{8}\mathcal{G}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} - \frac{3}{8}\mathcal{G}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} \\
&+\nu_1^{-\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}} + \nu_1^{\frac{1}{2}} \mathcal{S}_2^{-\frac{1}{2}} - \nu_2^{-\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}} \\
&-\nu_2^{\frac{1}{2}} \mathcal{S}_1^{-\frac{1}{2}} - \frac{3}{2}\mathcal{S}_2^{\frac{3}{2}} \psi_1^{-\frac{3}{2}} - \frac{1}{2}\mathcal{S}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} \\
&+\frac{1}{2}\mathcal{S}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} + \frac{3}{2}\mathcal{S}_2^{-\frac{3}{2}} \psi_1^{\frac{3}{2}} - \frac{3}{2}\mathcal{S}_1^{\frac{3}{2}} \psi_2^{-\frac{3}{2}} \\
&-\frac{1}{2}\mathcal{S}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} + \frac{1}{2}\mathcal{S}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} + \frac{3}{2}\mathcal{S}_1^{-\frac{3}{2}} \psi_2^{\frac{3}{2}} - \mathcal{A}^1\mu^{-1} \\
&+\mathcal{A}^{-1}\mu^1 + f^2\mathcal{W}^{-2} + \frac{1}{2}f^1\mathcal{W}^{-1} \\
&-\frac{1}{2}f^{-1}\mathcal{W}^1 - f^{-2}\mathcal{W}^2 - \xi^{-1}\mathcal{L}^1 + \xi^1\mathcal{L}^{-1}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\partial_t \mathcal{A}^{\pm 1} &= -\frac{k}{2}\partial_\varphi \xi^{\pm 1} - \frac{9}{8}\mathcal{G}_2^{\mp\frac{1}{2}} \psi_1^{\pm\frac{3}{2}} \\
&\pm\frac{3}{8}\mathcal{G}_2^{\pm\frac{1}{2}} \psi_1^{\pm\frac{1}{2}} + \frac{9}{8}\mathcal{G}_1^{\mp\frac{1}{2}} \psi_2^{\pm\frac{3}{2}} \\
&\mp\frac{3}{8}\mathcal{G}_1^{\pm\frac{1}{2}} \psi_2^{\pm\frac{1}{2}} + \nu_1^{\pm\frac{1}{2}} \mathcal{S}_2^{\pm\frac{1}{2}} + \nu_1^{\mp\frac{1}{2}} \mathcal{S}_2^{\pm\frac{3}{2}} \\
&-\nu_2^{\pm\frac{1}{2}} \mathcal{S}_1^{\pm\frac{1}{2}} - \nu_2^{\mp\frac{1}{2}} \mathcal{S}_1^{\pm\frac{3}{2}} \pm 3\mathcal{S}_2^{\mp\frac{1}{2}} \psi_1^{\pm\frac{3}{2}} \\
&\pm 2\mathcal{S}_2^{\pm\frac{1}{2}} \psi_1^{\pm\frac{1}{2}} \pm \mathcal{S}_2^{\pm\frac{3}{2}} \psi_1^{\mp\frac{1}{2}} \\
&\pm 3\mathcal{S}_1^{\mp\frac{1}{2}} \psi_2^{\pm\frac{3}{2}} \pm 2\mathcal{S}_1^{\pm\frac{1}{2}} \psi_2^{\pm\frac{1}{2}} \\
&\pm\mathcal{S}_1^{\pm\frac{3}{2}} \psi_2^{\mp\frac{1}{2}} \pm 2\mathcal{A}^0\mu^{\pm 1} \pm \mathcal{A}^{\pm 1}\mu^0 \\
&\pm\frac{1}{2}f^{\mp 1}\mathcal{W}^{\pm 2} \pm f^0\mathcal{W}^{\pm 1} \\
&\pm\frac{3}{2}f^{\pm 1}\mathcal{W}^0 \pm 2f^{\pm 2}\mathcal{W}^{\mp 1} \pm 2\xi^{\pm 1}\mathcal{L}^0 \pm \xi^0\mathcal{L}^{\pm 1}
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\partial_t \mathcal{W}^0 &= \frac{k}{3}\partial_\varphi f^0 + \frac{1}{2}\mathcal{G}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} - \frac{1}{2}\mathcal{G}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} \\
&+\frac{1}{2}\mathcal{G}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} - \frac{1}{2}\mathcal{G}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} + \frac{4}{3}\nu_1^{-\frac{1}{2}} \mathcal{S}_2^{\frac{1}{2}} \\
&-\frac{4}{3}\nu_1^{\frac{1}{2}} \mathcal{S}_2^{-\frac{1}{2}} + \frac{4}{3}\nu_2^{-\frac{1}{2}} \mathcal{S}_1^{\frac{1}{2}} \\
&-\frac{4}{3}\nu_2^{\frac{1}{2}} \mathcal{S}_1^{-\frac{1}{2}} + \frac{2}{3}\mathcal{S}_2^{\frac{3}{2}} \psi_1^{-\frac{3}{2}} - \frac{2}{3}\mathcal{S}_2^{\frac{1}{2}} \psi_1^{-\frac{1}{2}} - \frac{2}{3}\mathcal{S}_2^{-\frac{1}{2}} \psi_1^{\frac{1}{2}} \\
&+\frac{2}{3}\mathcal{S}_2^{-\frac{3}{2}} \psi_1^{\frac{3}{2}} - \frac{2}{3}\mathcal{S}_1^{\frac{3}{2}} \psi_2^{-\frac{3}{2}} \\
&+\frac{2}{3}\mathcal{S}_1^{\frac{1}{2}} \psi_2^{-\frac{1}{2}} + \frac{2}{3}\mathcal{S}_1^{-\frac{1}{2}} \psi_2^{\frac{1}{2}} \\
&-\frac{2}{3}\mathcal{S}_1^{-\frac{3}{2}} \psi_2^{\frac{3}{2}} - 2\mathcal{A}^1 f^{-1} + 2\mathcal{A}^{-1} f^1
\end{aligned}$$

$$-2f^{-1}\mathcal{L}^{1}+2f^{1}\mathcal{L}^{-1}-2\mu^{-1}\mathcal{W}^{1}+2\mu^{1}\mathcal{W}^{-1}-2\xi^{-1}\mathcal{W}^{1}+2\xi^{1}\mathcal{W}^{-1} \tag{A.6}$$

$$\begin{aligned}\partial_t\mathcal{W}^{\pm1}=&-\frac{k}{2}\partial_\varphi f^{\pm1}\pm\frac{3}{4}\mathcal{G}_2^{\mp\frac12}\psi_1^{\pm\frac32}\\&\mp\frac{3}{4}\mathcal{G}_2^{\pm\frac12}\psi_1^{\pm\frac12}\pm\frac{3}{4}\mathcal{G}_1^{\mp\frac12}\psi_2^{\pm\frac32}\\&\mp\frac{3}{4}\mathcal{G}_1^{\pm\frac12}\psi_2^{\pm\frac12}\mp2\nu_1^{\pm\frac12}\mathcal{S}_2^{\pm\frac12}\pm\frac{2}{3}\nu_1^{\mp\frac12}\mathcal{S}_2^{\pm\frac32}\\&\mp2\nu_2^{\pm\frac12}\mathcal{S}_1^{\pm\frac12}\pm\frac{2}{3}\nu_2^{\mp\frac12}\mathcal{S}_1^{\pm\frac32}\mp2\mathcal{S}_2^{\mp\frac12}\psi_1^{\pm\frac32}\\&-\frac{2}{3}\mathcal{S}_2^{\pm\frac32}\psi_1^{\mp\frac12}\pm2\mathcal{S}_1^{\mp\frac12}\psi_2^{\pm\frac32}+\frac{2}{3}\mathcal{S}_1^{\pm\frac32}\psi_2^{\mp\frac12}\mp4\mathcal{A}^{1}f^{\pm2}\\&\pm2\mathcal{A}^{0}f^{\pm1}\pm2\mathcal{A}^{\pm1}f^{0}\mp4f^{\pm2}\mathcal{L}^{\mp1}\pm2f^{0}\mathcal{L}^{\pm1}\\&\pm2f^{\pm1}\mathcal{L}^{0}\pm3\mu^{\pm1}\mathcal{W}^{0}\pm\mu^{0}\mathcal{W}^{\pm1}+\mu^{\mp1}\mathcal{W}^{\pm2}\\&\pm3\xi^{\pm1}\mathcal{W}^{0}\pm\xi^{0}\mathcal{W}^{\pm1}\mp\xi^{\mp1}\mathcal{W}^{\pm2}\end{aligned} \tag{A.7}$$

$$\begin{aligned}\partial_t\mathcal{W}^{\pm2}=&\,2k\partial_\varphi f^{\pm2}\pm3\mathcal{G}_2^{\pm\frac12}\psi_1^{\pm\frac32}\pm3\mathcal{G}_1^{\pm\frac12}\psi_2^{\pm\frac32}\\&\mp\frac{8}{3}\nu_1^{\pm\frac12}\mathcal{S}_2^{\pm\frac32}\mp\frac{8}{3}\nu_2^{\pm\frac12}\mathcal{S}_1^{\pm\frac32}\\&+4\mathcal{S}_2^{\pm\frac12}\psi_1^{\pm\frac32}\mp\frac{4}{3}\mathcal{S}_2^{\pm\frac32}\psi_1^{\pm\frac12}\\&-4\mathcal{S}_1^{\pm\frac12}\psi_2^{\pm\frac32}-\frac{4}{3}\mathcal{S}_1^{\pm\frac32}\psi_2^{\pm\frac12}\mp16\mathcal{A}^{0}f^{\pm2}\\&\mp4\mathcal{A}^{\pm1}f^{\pm1}+16f^{\pm2}\mathcal{L}^{0}\mp4f^{\pm1}\mathcal{L}^{\pm1}\pm4\mu^{\pm1}\mathcal{W}^{\pm1}\\&\pm2\mu^{0}\mathcal{W}^{\pm2}\pm4\xi^{\pm1}\mathcal{W}^{\pm1}\pm2\xi^{0}\mathcal{W}^{\pm2}\end{aligned} \tag{A.8}$$

$$\begin{aligned}\partial_t\mathcal{G}_M^{\pm\frac12}=&\mp k\partial_\varphi\nu_M^{\pm\frac12}+\frac{10}{3}\mathcal{A}^{0}\nu_M^{\mp\frac12}\\&+\frac{10}{3}\mathcal{A}^{\pm1}\nu_M^{\mp\frac12}\mp4\mathcal{A}^{\mp1}\psi_M^{\pm\frac32}\\&\mp\frac{8}{3}\mathcal{A}^{0}\psi_M^{\pm\frac12}\mp\frac{4}{3}\mathcal{A}^{\pm1}\psi_M^{\mp\frac12}\mp\frac{8}{9}f^{\mp1}\mathcal{S}_M^{\pm\frac32}\\&\mp\frac{16}{9}f^{0}\mathcal{S}_M^{\pm\frac12}\mp\frac{8}{3}f^{\pm1}\mathcal{S}_M^{\mp\frac12}\mp\frac{32}{9}f^{\pm2}\mathcal{S}_M^{\mp\frac32}\\&-\eta\mathcal{G}_M^{\pm\frac12}\pm\frac{1}{2}\mu^{0}\mathcal{G}_M^{\pm\frac12}\pm\mu^{\pm1}\mathcal{G}_M^{\mp\frac12}\\&\pm\frac{5}{6}\xi^{0}\mathcal{G}_M^{\pm\frac12}\pm\frac{5}{3}\xi^{\pm1}\mathcal{G}_M^{\mp\frac12}\\&\mp\mathcal{J}\nu_M^{\pm\frac12}\mp\frac{16}{9}\xi^{\mp1}\mathcal{S}_M^{\pm\frac32}-\frac{16}{9}\xi^{0}\mathcal{S}_M^{\pm\frac12}\\&-\frac{16}{9}\xi^{\pm1}\mathcal{S}_M^{\mp\frac12}-2\mathcal{W}^{\mp1}\psi_M^{\pm\frac32}\\&-2\mathcal{W}^{0}\psi_M^{\pm\frac12}-2\mathcal{W}^{\pm1}\psi_M^{\mp\frac12}\\&-2\mathcal{W}^{\pm2}\psi_M^{\mp\frac32}+2\mathcal{L}^{0}\nu_M^{\pm\frac12}+2\mathcal{L}^{\pm1}\nu_M^{\mp\frac12}\end{aligned} \tag{A.9}$$

$$\begin{aligned}\partial_t\mathcal{S}_M^{\pm\frac12}=&\mp\frac{3}{8}k\psi_M^{\pm\frac12}\mp2\mathcal{A}^{0}\nu_M^{\pm\frac12}\pm\mathcal{A}^{\pm1}\nu_M^{\mp\frac12}\\&-\frac{3}{4}\mathcal{A}^{\mp1}\psi_M^{\pm\frac32}+\frac{1}{4}\mathcal{A}^{0}\psi_M^{\pm\frac12}\\&+\frac{1}{2}\mathcal{A}^{\pm1}\psi_M^{\mp\frac12}\mp\frac{1}{2}f^{0}\mathcal{G}_M^{\pm\frac12}\\&\mp\frac{3}{4}f^{\pm1}\mathcal{G}_M^{\mp\frac12}-\frac{2}{3}f^{\mp1}\mathcal{S}_M^{\pm\frac32}\\&-\frac{2}{3}f^{0}\mathcal{S}_M^{\pm\frac12}+\frac{4}{3}f^{\pm2}\mathcal{S}_M^{\mp\frac32}\\&-\frac{1}{2}\xi^{0}\mathcal{G}_M^{\pm\frac12}+\frac{1}{2}\xi^{\pm1}\mathcal{G}_M^{\mp\frac12}\mp\frac{3}{8}\mathcal{J}\psi_M^{\pm\frac12}\\&-\eta\mathcal{S}_M^{\pm\frac12}\pm\mu^{\mp1}\mathcal{S}_M^{\pm\frac32}\pm\frac{1}{2}\mu^{0}\mathcal{S}_M^{\pm\frac12}\\&-2\mu^{\pm1}\mathcal{S}_M^{\mp\frac12}\mp\frac{1}{3}\xi^{\mp1}\mathcal{S}_M^{\pm\frac32}\\&\pm\frac{1}{6}\xi^{0}\mathcal{S}_M^{\pm\frac12}\pm\frac{2}{3}\xi^{\pm1}\mathcal{S}_M^{\mp\frac12}\\&\pm\frac{3}{2}\mathcal{W}^{0}\nu_M^{\pm\frac12}-\frac{3}{2}\mathcal{W}^{\pm1}\nu_M^{\mp\frac12}\\&+\frac{3}{2}\mathcal{W}^{\mp1}\psi_M^{\pm\frac32}\mp\frac{3}{4}\mathcal{W}^{0}\psi_M^{\pm\frac12}\\&\pm\frac{3}{4}\mathcal{W}^{\pm2}\psi_M^{\mp\frac32}-\frac{9}{4}\mathcal{L}^{\mp1}\psi_M^{\pm\frac32}\\&+\frac{3}{4}\mathcal{L}^{0}\psi_M^{\pm\frac12}+\frac{3}{2}\mathcal{L}^{\pm1}\psi_M^{\mp\frac12}\end{aligned} \tag{A.10}$$

$$\begin{aligned}\partial_t\mathcal{S}_M^{\pm\frac32}=&\pm\frac{9}{8}k\psi_M^{\pm\frac32}\mp3\mathcal{A}^{-1}\nu_M^{\pm\frac12}\\&-\frac{9}{4}\mathcal{A}^{0}\psi_M^{\pm\frac32}-\frac{3}{4}\mathcal{A}^{\pm1}\psi_M^{\pm\frac12}\pm\frac{3}{4}f^{\pm1}\mathcal{G}_M^{\pm\frac12}\\&\pm3f^{\pm2}\mathcal{G}_M^{\mp\frac12}+\frac{2}{3}f^{0}\mathcal{S}_M^{\pm\frac32}\\&+2f^{\pm1}\mathcal{S}_M^{\pm\frac12}+4f^{\pm2}\mathcal{S}_M^{\mp\frac12}+\frac{3}{2}\xi^{\pm1}\mathcal{G}_M^{\pm\frac12}\pm\frac{9}{8}\mathcal{J}\psi_M^{\pm\frac32}\\&-\eta\mathcal{S}_M^{\pm\frac32}\pm\frac{3}{2}\mu^{0}\mathcal{S}_M^{\pm\frac32}\pm3\mu^{\pm1}\mathcal{S}_M^{\pm\frac12}\pm\frac{1}{2}\xi^{0}\mathcal{S}_M^{\pm\frac32}\\&\pm\xi^{\pm1}\mathcal{S}_M^{\pm\frac12}-\frac{3}{2}\mathcal{W}^{\pm1}\nu_M^{\pm\frac12}-\frac{3}{2}\mathcal{W}^{\pm2}\nu_M^{\mp\frac12}\\&+\frac{9}{4}\mathcal{W}^{0}\psi_M^{\pm\frac32}\mp\frac{3}{2}\mathcal{W}^{\pm1}\psi_M^{\pm\frac12}\\&\mp\frac{3}{4}\mathcal{W}^{\pm2}\psi_M^{\mp\frac12}-\frac{27}{4}\mathcal{L}^{0}\psi_M^{\pm\frac32}-\frac{9}{4}\mathcal{L}^{\pm1}\psi_M^{\pm\frac12}\end{aligned} \tag{A.11}$$

B The gauge transformations of the twenty four independent source fields

In this appendix, we present in more detail the gauge transformations of the twenty four independent source fields as:

$$\begin{aligned}\partial_\lambda\mathcal{J}=&\,k\partial_\varphi\varrho+\mathcal{G}_2^{\frac12}\nu_1^{-\frac12}+\mathcal{G}_2^{-\frac12}\nu_1^{\frac12}\\&-\mathcal{G}_1^{\frac12}\nu_2^{-\frac12}-\mathcal{G}_1^{-\frac12}\nu_2^{\frac12}\\&+\frac{4}{3}\mathcal{S}_2^{\frac32}\omega_1^{-\frac32}+\frac{4}{3}\mathcal{S}_2^{\frac12}\omega_1^{-\frac12}\\&+\frac{4}{3}\mathcal{S}_2^{-\frac12}\omega_1^{\frac12}+\frac{4}{3}\mathcal{S}_2^{-\frac32}\omega_1^{\frac32}\\&-\frac{4}{3}\mathcal{S}_1^{\frac32}\omega_2^{-\frac32}-\frac{4}{3}\mathcal{S}_1^{\frac12}\omega_2^{-\frac12}\\&-\frac{4}{3}\mathcal{S}_1^{-\frac12}\omega_2^{\frac12}-\frac{4}{3}\mathcal{S}_1^{-\frac32}\omega_2^{\frac32},\end{aligned} \tag{B.1}$$

$$\begin{aligned}\partial_\lambda\mathcal{L}^{0}=&\,\frac{k}{4}\partial_\varphi\epsilon^{0}-\frac{1}{2}\mathcal{G}_2^{\frac12}\nu_1^{-\frac12}\\&+\frac{1}{2}\mathcal{G}_2^{-\frac12}\nu_1^{\frac12}-\frac{1}{2}\mathcal{G}_1^{\frac12}\nu_2^{-\frac12}\\&+\frac{1}{2}\mathcal{G}_1^{-\frac12}\nu_2^{\frac12}-\frac{5}{8}\mathcal{G}_2^{\frac12}\omega_1^{-\frac12}-\frac{5}{8}\mathcal{G}_2^{-\frac12}\omega_1^{\frac12}\\&+\frac{5}{8}\mathcal{G}_1^{\frac12}\omega_2^{-\frac12}+\frac{5}{8}\mathcal{G}_1^{-\frac12}\omega_2^{\frac12}\end{aligned}$$

$$
\begin{aligned}
&-\frac{5}{3}\nu_1^{-\frac{1}{2}}\mathcal{S}_2^{\frac{1}{2}}-\frac{5}{3}\nu_1^{\frac{1}{2}}\mathcal{S}_2^{-\frac{1}{2}}\\
&+\frac{5}{3}\nu_2^{-\frac{1}{2}}\mathcal{S}_1^{\frac{1}{2}}+\frac{5}{3}\nu_2^{\frac{1}{2}}\mathcal{S}_1^{-\frac{1}{2}}\\
&+\frac{1}{2}\mathcal{S}_2^{\frac{3}{2}}\omega_1^{-\frac{3}{2}}+\frac{1}{6}\mathcal{S}_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}\\
&-\frac{1}{6}\mathcal{S}_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}-\frac{1}{2}\mathcal{S}_2^{-\frac{3}{2}}\omega_1^{\frac{3}{2}}\\
&+\frac{1}{2}\mathcal{S}_1^{\frac{3}{2}}\omega_2^{-\frac{3}{2}}+\frac{1}{6}\mathcal{S}_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}\\
&-\frac{1}{6}\mathcal{S}_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}-\frac{1}{2}\mathcal{S}_1^{-\frac{3}{2}}\omega_2^{\frac{3}{2}}\\
&-\mathcal{A}^1\phi^{-1}+\mathcal{A}^{-1}\phi^1\\
&+\upsilon^2\mathcal{W}^{-2}+\frac{1}{2}\upsilon^1\mathcal{W}^{-1}-\frac{1}{2}\upsilon^{-1}\mathcal{W}^1\\
&-\upsilon^{-2}\mathcal{W}^2-\epsilon^{-1}\mathcal{L}^1+\epsilon^1\mathcal{L}^{-1}
\end{aligned}\tag{B.2}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{L}^{\pm1}=&-\frac{k}{2}\partial_\varphi\epsilon^{\pm1}\pm\mathcal{G}_2^{\pm\frac{1}{2}}\nu_1^{\pm\frac{1}{2}}\pm\mathcal{G}_1^{\pm\frac{1}{2}}\nu_2^{\pm\frac{1}{2}}\\
&+\frac{15}{8}\mathcal{G}_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}+\frac{5}{8}\mathcal{G}_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}}\\
&-\frac{15}{8}\mathcal{G}_{\mp}^{\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}-\frac{5}{8}\mathcal{G}_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}}-\frac{5}{3}\nu_1^{\pm\frac{1}{2}}\mathcal{S}_2^{\pm\frac{1}{2}}\\
&-\frac{5}{3}\nu_1^{\mp\frac{1}{2}}\mathcal{S}_2^{\pm\frac{3}{2}}+\frac{5}{3}\nu_2^{\pm\frac{1}{2}}\mathcal{S}_1^{\pm\frac{1}{2}}\\
&+\frac{5}{3}\nu_2^{\mp\frac{1}{2}}\mathcal{S}_1^{\pm\frac{3}{2}}\mp\mathcal{S}_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}\\
&\mp\frac{2}{3}\mathcal{S}_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}}\mp\frac{1}{3}\mathcal{S}_2^{\pm\frac{3}{2}}\omega_1^{\mp\frac{1}{2}}\\
&\mp\mathcal{S}_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}+\frac{2}{3}\mathcal{S}_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}}\\
&\mp\frac{1}{3}\mathcal{S}_1^{\pm\frac{3}{2}}\omega_2^{\mp\frac{1}{2}}\pm2\mathcal{A}^0\phi^{\pm1}\pm\mathcal{A}^{\pm1}\phi^0\\
&-\frac{1}{2}\upsilon^{\mp1}\mathcal{W}^{\pm2}\pm\upsilon^0\mathcal{W}^{\pm1}\pm\frac{3}{2}\upsilon^{\pm1}\mathcal{W}^0\\
&\pm2\upsilon^{\pm2}\mathcal{W}^{\mp1}\pm2\epsilon^{\pm1}\mathcal{L}^0\\
&\pm\epsilon^0\mathcal{L}^{\pm1}
\end{aligned}\tag{B.3}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{A}^0=&\frac{k}{4}\partial_\varphi\phi_0+\frac{3}{8}\mathcal{G}_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}\\
&+\frac{3}{8}\mathcal{G}_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}-\frac{3}{8}\mathcal{G}_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}\\
&-\frac{3}{8}\mathcal{G}_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}+\nu_1^{-\frac{1}{2}}\mathcal{S}_2^{\frac{1}{2}}\\
&+\nu_1^{\frac{1}{2}}\mathcal{S}_2^{-\frac{1}{2}}-\nu_2^{-\frac{1}{2}}\mathcal{S}_1^{\frac{1}{2}}\\
&-\nu_2^{\frac{1}{2}}\mathcal{S}_1^{-\frac{1}{2}}-\frac{3}{2}\mathcal{S}_2^{\frac{3}{2}}\omega_1^{-\frac{3}{2}}\\
&-\frac{1}{2}\mathcal{S}_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}+\frac{1}{2}\mathcal{S}_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}+\frac{3}{2}\mathcal{S}_2^{-\frac{3}{2}}\omega_1^{\frac{3}{2}}-\frac{3}{2}\mathcal{S}_1^{\frac{3}{2}}\omega_2^{-\frac{3}{2}}\\
&-\frac{1}{2}\mathcal{S}_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}+\frac{1}{2}\mathcal{S}_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}\\
&+\frac{3}{2}\mathcal{S}_1^{-\frac{3}{2}}\omega_2^{\frac{3}{2}}-\mathcal{A}^1\epsilon^{-1}+\mathcal{A}^{-1}\epsilon^1+\upsilon^2\mathcal{W}^{-2}\\
&+\frac{1}{2}\upsilon^1\mathcal{W}^{-1}-\frac{1}{2}\upsilon^{-1}\mathcal{W}^1-\upsilon^{-2}\mathcal{W}^2\\
&-\phi^{-1}\mathcal{L}^1+\phi^1\mathcal{L}^{-1}
\end{aligned}\tag{B.4}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{A}^{\pm1}=&-\frac{k}{2}\partial_\varphi\phi^{\pm1}-\frac{9}{8}\mathcal{G}_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}\pm\frac{3}{8}\mathcal{G}_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}}\\
&+\frac{9}{8}\mathcal{G}_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}\mp\frac{3}{8}\mathcal{G}_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}}\\
&+\nu_1^{\pm\frac{1}{2}}\mathcal{S}_2^{\pm\frac{1}{2}}+\nu_1^{\mp\frac{1}{2}}\mathcal{S}_2^{\pm\frac{3}{2}}\\
&-\nu_2^{\pm\frac{1}{2}}\mathcal{S}_1^{\pm\frac{1}{2}}-\nu_2^{\mp\frac{1}{2}}\mathcal{S}_1^{\pm\frac{3}{2}}\\
&\pm3\mathcal{S}_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}\pm2\mathcal{S}_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}}\\
&\pm\mathcal{S}_2^{\pm\frac{3}{2}}\omega_1^{\mp\frac{1}{2}}\pm3\mathcal{S}_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}\pm2\mathcal{S}_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}}\\
&\pm\mathcal{S}_1^{\pm\frac{3}{2}}\omega_2^{\mp\frac{1}{2}}\pm2\mathcal{A}^0\epsilon^{\pm1}\pm\mathcal{A}^{\pm1}\epsilon^0\pm\frac{1}{2}\upsilon^{\mp1}\mathcal{W}^{\pm2}\\
&\pm\upsilon^0\mathcal{W}^{\pm1}\pm\frac{3}{2}\upsilon^{\pm1}\mathcal{W}^0\pm2\upsilon^{\pm2}\mathcal{W}^{\mp1}\pm2\phi^{\pm1}\mathcal{L}^0\\
&\pm\phi^0\mathcal{L}^{\pm1}
\end{aligned}\tag{B.5}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{W}^0=&\frac{k}{3}\partial_\varphi\upsilon^0+\frac{1}{2}\mathcal{G}_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}\\
&-\frac{1}{2}\mathcal{G}_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}+\frac{1}{2}\mathcal{G}_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}\\
&-\frac{1}{2}\mathcal{G}_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}+\frac{4}{3}\nu_1^{-\frac{1}{2}}\mathcal{S}_2^{\frac{1}{2}}\\
&-\frac{4}{3}\nu_1^{\frac{1}{2}}\mathcal{S}_2^{-\frac{1}{2}}+\frac{4}{3}\nu_2^{-\frac{1}{2}}\mathcal{S}_1^{\frac{1}{2}}\\
&-\frac{4}{3}\nu_2^{\frac{1}{2}}\mathcal{S}_1^{-\frac{1}{2}}+\frac{2}{3}\mathcal{S}_2^{\frac{3}{2}}\omega_1^{-\frac{3}{2}}\\
&-\frac{2}{3}\mathcal{S}_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}-\frac{2}{3}\mathcal{S}_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}+\frac{2}{3}\mathcal{S}_2^{-\frac{3}{2}}\omega_1^{\frac{3}{2}}\\
&-\frac{2}{3}\mathcal{S}_1^{\frac{3}{2}}\omega_2^{-\frac{3}{2}}+\frac{2}{3}\mathcal{S}_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}\\
&+\frac{2}{3}\mathcal{S}_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}\\
&-\frac{2}{3}\mathcal{S}_1^{-\frac{3}{2}}\omega_2^{\frac{3}{2}}-2\mathcal{A}^1\upsilon^{-1}+2\mathcal{A}^{-1}\upsilon^1-2\upsilon^{-1}\mathcal{L}^1\\
&+2\upsilon^1\mathcal{L}^{-1}-2\epsilon^{-1}\mathcal{W}^1+2\epsilon^1\mathcal{W}^{-1}-2\phi^{-1}\mathcal{W}^1\\
&+2\phi^1\mathcal{W}^{-1}
\end{aligned}\tag{B.6}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{W}^{\pm1}=&-\frac{k}{2}\partial_\varphi\upsilon^{\pm1}\pm\frac{3}{4}\mathcal{G}_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}\\
&\mp\frac{3}{4}\mathcal{G}_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}}\pm\frac{3}{4}\mathcal{G}_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}\mp\frac{3}{4}\mathcal{G}_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}}\\
&\mp2\nu_1^{\pm\frac{1}{2}}\mathcal{S}_2^{\pm\frac{1}{2}}\pm\frac{2}{3}\nu_1^{\mp\frac{1}{2}}\mathcal{S}_2^{\pm\frac{3}{2}}\\
&\mp2\nu_2^{\pm\frac{1}{2}}\mathcal{S}_1^{\pm\frac{1}{2}}\pm\frac{2}{3}\nu_2^{\mp\frac{1}{2}}\mathcal{S}_1^{\pm\frac{3}{2}}\mp2\mathcal{S}_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}\\
&-\frac{2}{3}\mathcal{S}_2^{\pm\frac{3}{2}}\omega_1^{\mp\frac{1}{2}}\pm2\mathcal{S}_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}\\
&+\frac{2}{3}\mathcal{S}_1^{\pm\frac{3}{2}}\omega_2^{\mp\frac{1}{2}}\mp4\mathcal{A}^1\upsilon^{\pm2}\\
&\pm2\mathcal{A}^0\upsilon^{\pm1}\pm2\mathcal{A}^{\pm1}\upsilon^0\mp4\upsilon^{\pm2}\mathcal{L}^{\mp1}\\
&\pm2\upsilon^0\mathcal{L}^{\pm1}\pm2\upsilon^{\pm1}\mathcal{L}^0\pm3\epsilon^{\pm1}\mathcal{W}^0\pm\epsilon^0\mathcal{W}^{\pm1}\\
&+\epsilon^{\mp1}\mathcal{W}^{\pm2}\\
&\pm3\phi^{\pm1}\mathcal{W}^0\pm\phi^0\mathcal{W}^{\pm1}\mp\phi^{\mp1}\mathcal{W}^{\pm2}
\end{aligned}\tag{B.7}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{W}^{\pm2}=&2k\partial_\varphi\upsilon^{\pm2}\pm3\mathcal{G}_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}\pm3\mathcal{G}_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}\\
&\mp\frac{8}{3}\nu_1^{\pm\frac{1}{2}}\mathcal{S}_2^{\pm\frac{3}{2}}\mp\frac{8}{3}\nu_2^{\pm\frac{1}{2}}\mathcal{S}_1^{\pm\frac{3}{2}}\\
&+4\mathcal{S}_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{3}{2}}\mp\frac{4}{3}\mathcal{S}_2^{\pm\frac{3}{2}}\omega_1^{\pm\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&-4\mathcal{S}_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{3}{2}}-\frac{4}{3}\mathcal{S}_1^{\pm\frac{3}{2}}\omega_2^{\pm\frac{1}{2}}\\
&\mp16\mathcal{A}^0\upsilon^{\pm2}\mp4\mathcal{A}^{\pm1}\upsilon^{\pm1}\\
&+16\upsilon^{\pm2}\mathcal{L}^0\mp4\upsilon^{\pm1}\mathcal{L}^{\pm1}\pm4\epsilon^{\pm1}\mathcal{W}^{\pm1}\\
&\pm2\epsilon^0\mathcal{W}^{\pm2}\pm4\phi^{\pm1}\mathcal{W}^{\pm1}\pm2\phi^0\mathcal{W}^{\pm2}
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{G}_M^{\pm\frac{1}{2}}=&\mp k\partial_\varphi\nu_M^{\pm\frac{1}{2}}+\frac{10}{3}\mathcal{A}^0\nu_M^{\mp\frac{1}{2}}\\
&+\frac{10}{3}\mathcal{A}^{\pm1}\nu_M^{\mp\frac{1}{2}}\mp4\mathcal{A}^{\mp1}\omega_M^{\pm\frac{3}{2}}\mp\frac{8}{3}\mathcal{A}^0\omega_M^{\pm\frac{1}{2}}\\
&\mp\frac{4}{3}\mathcal{A}^{\pm1}\omega_M^{\mp\frac{1}{2}}\mp\frac{8}{9}\upsilon^{\mp1}\mathcal{S}_M^{\pm\frac{3}{2}}\\
&\mp\frac{16}{9}\upsilon^0\mathcal{S}_M^{\pm\frac{1}{2}}\mp\frac{8}{3}\upsilon^{\pm1}\mathcal{S}_M^{\mp\frac{1}{2}}\mp\frac{32}{9}\upsilon^{\pm2}\mathcal{S}_M^{\mp\frac{3}{2}}\\
&-\varrho\mathcal{G}_M^{\pm\frac{1}{2}}\pm\frac{1}{2}\epsilon^0\mathcal{G}_M^{\pm\frac{1}{2}}\pm\epsilon^{\pm1}\mathcal{G}_M^{\mp\frac{1}{2}}\\
&\pm\frac{5}{6}\phi^0\mathcal{G}_M^{\pm\frac{1}{2}}\pm\frac{5}{3}\phi^{\pm1}\mathcal{G}_M^{\mp\frac{1}{2}}\\
&\mp\mathcal{J}\nu_M^{\pm\frac{1}{2}}\mp\frac{16}{9}\phi^{\mp1}\mathcal{S}_M^{\pm\frac{3}{2}}-\frac{16}{9}\phi^0\mathcal{S}_M^{\pm\frac{1}{2}}\\
&-\frac{16}{9}\phi^{\pm1}\mathcal{S}_M^{\mp\frac{1}{2}}-2\mathcal{W}^{\mp1}\omega_M^{\pm\frac{3}{2}}\\
&-2\mathcal{W}^0\omega_M^{\pm\frac{1}{2}}-2\mathcal{W}^{\pm1}\omega_M^{\mp\frac{1}{2}}\\
&-2\mathcal{W}^{\pm2}\omega_M^{\mp\frac{3}{2}}+2\mathcal{L}^0\nu_M^{\pm\frac{1}{2}}+2\mathcal{L}^{\pm1}\nu_M^{\mp\frac{1}{2}}
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{S}_M^{\pm\frac{1}{2}}=&\mp\frac{3}{8}k\omega_M^{\pm\frac{1}{2}}\mp2\mathcal{A}^0\nu_M^{\pm\frac{1}{2}}\pm\mathcal{A}^{\pm1}\nu_M^{\mp\frac{1}{2}}\\
&-\frac{3}{4}\mathcal{A}^{\mp1}\omega_M^{\pm\frac{3}{2}}\\
&+\frac{1}{4}\mathcal{A}^0\omega_M^{\pm\frac{1}{2}}+\frac{1}{2}\mathcal{A}^{\pm1}\omega_M^{\mp\frac{1}{2}}\mp\frac{1}{2}\upsilon^0\mathcal{G}_M^{\pm\frac{1}{2}}\\
&\mp\frac{3}{4}\upsilon^{\pm1}\mathcal{G}_M^{\mp\frac{1}{2}}-\frac{2}{3}\upsilon^{\mp1}\mathcal{S}_M^{\pm\frac{3}{2}}\\
&-\frac{2}{3}\upsilon^0\mathcal{S}_M^{\pm\frac{1}{2}}+\frac{4}{3}\upsilon^{\pm2}\mathcal{S}_M^{\mp\frac{3}{2}}\\
&-\frac{1}{2}\phi^0\mathcal{G}_M^{\pm\frac{1}{2}}+\frac{1}{2}\phi^{\pm1}\mathcal{G}_M^{\mp\frac{1}{2}}\\
&\mp\frac{3}{8}\mathcal{J}\omega_M^{\pm\frac{1}{2}}-\varrho\mathcal{S}_M^{\pm\frac{1}{2}}\\
&\pm\epsilon^{\mp1}\mathcal{S}_M^{\pm\frac{3}{2}}\pm\frac{1}{2}\epsilon^0\mathcal{S}_M^{\pm\frac{1}{2}}\\
&-2\epsilon^{\pm1}\mathcal{S}_M^{\mp\frac{1}{2}}\mp\frac{1}{3}\phi^{\mp1}\mathcal{S}_M^{\pm\frac{3}{2}}\pm\frac{1}{6}\phi^0\mathcal{S}_M^{\pm\frac{1}{2}}\\
&\pm\frac{2}{3}\phi^{\pm1}\mathcal{S}_M^{\mp\frac{1}{2}}\pm\frac{3}{2}\mathcal{W}^0\nu_M^{\pm\frac{1}{2}}\\
&-\frac{3}{2}\mathcal{W}^{\pm1}\nu_M^{\mp\frac{1}{2}}\\
&+\frac{3}{2}\mathcal{W}^{\mp1}\omega_M^{\pm\frac{3}{2}}\mp\frac{3}{4}\mathcal{W}^0\omega_M^{\pm\frac{1}{2}}\\
&\pm\frac{3}{4}\mathcal{W}^{\pm2}\omega_M^{\mp\frac{3}{2}}-\frac{9}{4}\mathcal{L}^{\mp1}\omega_M^{\pm\frac{3}{2}}\\
&+\frac{3}{4}\mathcal{L}^0\omega_M^{\pm\frac{1}{2}}+\frac{3}{2}\mathcal{L}^{\pm1}\omega_M^{\mp\frac{1}{2}}
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
\partial_\lambda\mathcal{S}_M^{\pm\frac{3}{2}}=&\pm\frac{9}{8}k\omega_M^{\pm\frac{3}{2}}\mp3\mathcal{A}^{-1}\nu_M^{\pm\frac{1}{2}}-\frac{9}{4}\mathcal{A}^0\omega_M^{\pm\frac{3}{2}}\\
&-\frac{3}{4}\mathcal{A}^{\pm1}\omega_M^{\pm\frac{1}{2}}\pm\frac{3}{4}\upsilon^{\pm1}\mathcal{G}_M^{\pm\frac{1}{2}}\\
&\pm3\upsilon^{\pm2}\mathcal{G}_M^{\mp\frac{1}{2}}+\frac{2}{3}\upsilon^0\mathcal{S}_M^{\pm\frac{3}{2}}\\
&+2\upsilon^{\pm1}\mathcal{S}_M^{\pm\frac{1}{2}}+4\upsilon^{\pm2}\mathcal{S}_M^{\mp\frac{1}{2}}+\frac{3}{2}\phi^{\pm1}\mathcal{G}_M^{\pm\frac{1}{2}}\\
&\pm\frac{9}{8}\mathcal{J}\omega_M^{\pm\frac{3}{2}}-\varrho\mathcal{S}_M^{\pm\frac{3}{2}}\pm\frac{3}{2}\epsilon^0\mathcal{S}_M^{\pm\frac{3}{2}}\\
&\pm3\epsilon^{\pm1}\mathcal{S}_M^{\pm\frac{1}{2}}\pm\frac{1}{2}\phi^0\mathcal{S}_M^{\pm\frac{3}{2}}\\
&\pm\phi^{\pm1}\mathcal{S}_M^{\pm\frac{1}{2}}-\frac{3}{2}\mathcal{W}^{\pm1}\nu_M^{\pm\frac{1}{2}}-\frac{3}{2}\mathcal{W}^{\pm2}\nu_M^{\mp\frac{1}{2}}\\
&+\frac{9}{4}\mathcal{W}^0\omega_M^{\pm\frac{3}{2}}\mp\frac{3}{2}\mathcal{W}^{\pm1}\omega_M^{\pm\frac{1}{2}}\mp\frac{3}{4}\mathcal{W}^{\pm2}\omega_M^{\mp\frac{1}{2}}\\
&-\frac{27}{4}\mathcal{L}^0\omega_M^{\pm\frac{3}{2}}-\frac{9}{4}\mathcal{L}^{\pm1}\omega_M^{\pm\frac{1}{2}}.
\end{aligned}
\tag{B.11}
$$

C The gauge transformations of the twenty four chemical potentials

This appendix consists of the gauge transformations for the twenty four chemical potentials.

$$
\begin{aligned}
\partial_\lambda\eta=&k\partial_\varphi\varrho+\nu_2^{\frac{1}{2}}\nu_1^{-\frac{1}{2}}+\nu_2^{-\frac{1}{2}}\nu_1^{\frac{1}{2}}\\
&-\nu_1^{\frac{1}{2}}\nu_2^{-\frac{1}{2}}-\nu_1^{-\frac{1}{2}}\nu_2^{\frac{1}{2}}\\
&+\frac{4}{3}\psi_2^{\frac{3}{2}}\omega_1^{-\frac{3}{2}}+\frac{4}{3}\psi_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}\\
&+\frac{4}{3}\psi_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}+\frac{4}{3}\psi_2^{-\frac{3}{2}}\omega_1^{\frac{3}{2}}\\
&-\frac{4}{3}\psi_1^{\frac{3}{2}}\omega_2^{-\frac{3}{2}}-\frac{4}{3}\psi_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}\\
&-\frac{4}{3}\psi_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}-\frac{4}{3}\psi_1^{-\frac{3}{2}}\omega_2^{\frac{3}{2}},
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
\partial_\lambda\mu^0=&\frac{k}{4}\partial_\varphi\epsilon^0-\frac{1}{2}\nu_2^{\frac{1}{2}}\nu_1^{-\frac{1}{2}}+\frac{1}{2}\nu_2^{-\frac{1}{2}}\nu_1^{\frac{1}{2}}\\
&-\frac{1}{2}\nu_1^{\frac{1}{2}}\nu_2^{-\frac{1}{2}}+\frac{1}{2}\nu_1^{-\frac{1}{2}}\nu_2^{\frac{1}{2}}\\
&-\frac{5}{8}\nu_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}-\frac{5}{8}\nu_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}\\
&+\frac{5}{8}\nu_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}+\frac{5}{8}\nu_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}\\
&-\frac{5}{3}\nu_1^{-\frac{1}{2}}\psi_2^{\frac{1}{2}}-\frac{5}{3}\nu_1^{\frac{1}{2}}\psi_2^{-\frac{1}{2}}\\
&+\frac{5}{3}\nu_2^{-\frac{1}{2}}\psi_1^{\frac{1}{2}}+\frac{5}{3}\nu_2^{\frac{1}{2}}\psi_1^{-\frac{1}{2}}\\
&+\frac{1}{2}\psi_2^{\frac{3}{2}}\omega_1^{-\frac{3}{2}}+\frac{1}{6}\psi_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}}\\
&-\frac{1}{6}\psi_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}}-\frac{1}{2}\psi_2^{-\frac{3}{2}}\omega_1^{\frac{3}{2}}\\
&+\frac{1}{2}\psi_1^{\frac{3}{2}}\omega_2^{-\frac{3}{2}}+\frac{1}{6}\psi_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}}\\
&-\frac{1}{6}\psi_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}}-\frac{1}{2}\psi_1^{-\frac{3}{2}}\omega_2^{\frac{3}{2}}-\xi^1\phi^{-1}+\xi^{-1}\phi^1\\
&+\upsilon^2f^{-2}+\frac{1}{2}\upsilon^1f^{-1}\\
&-\frac{1}{2}\upsilon^{-1}f^1-\upsilon^{-2}f^2-\epsilon^{-1}\mu^1+\epsilon^1\mu^{-1}
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\partial_\lambda \mu^{\pm 1} =& -\frac{k}{2}\partial_\varphi \epsilon^{\pm 1} \pm \nu_2^{\pm\frac{1}{2}}\nu_1^{\pm\frac{1}{2}} \pm \nu_1^{\pm\frac{1}{2}}\nu_2^{\pm\frac{1}{2}} \\
&+\frac{15}{8}\nu_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} + \frac{5}{8}\nu_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}} \\
&-\frac{15}{8}\nu_{\mp}^{\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} - \frac{5}{8}\nu_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}} \\
&-\frac{5}{3}\nu_1^{\pm\frac{1}{2}}\psi_2^{\pm\frac{1}{2}} - \frac{5}{3}\nu_1^{\mp\frac{1}{2}}\psi_2^{\pm\frac{3}{2}} \\
&+\frac{5}{3}\nu_2^{\pm\frac{1}{2}}\psi_1^{\pm\frac{1}{2}} + \frac{5}{3}\nu_2^{\mp\frac{1}{2}}\psi_1^{\pm\frac{3}{2}} \\
&\mp\psi_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} \mp \frac{2}{3}\psi_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}} \\
&\mp\frac{1}{3}\psi_2^{\pm\frac{3}{2}}\omega_1^{\mp\frac{1}{2}} \\
&\mp\psi_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} + \frac{2}{3}\psi_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}} \\
&\mp\frac{1}{3}\psi_1^{\pm\frac{3}{2}}\omega_2^{\mp\frac{1}{2}} \pm 2\xi^0\phi^{\pm 1} \pm \xi^{\pm 1}\phi^0 \\
&-\frac{1}{2}\upsilon^{\mp 1}f^{\pm 2} \pm \upsilon^0 f^{\pm 1} \pm \frac{3}{2}\upsilon^{\pm 1}f^0 \\
&\pm 2\upsilon^{\pm 2}f^{\mp 1} \\
&\pm 2\epsilon^{\pm 1}\mu^0 \pm \epsilon^0\mu^{\pm 1}
\end{aligned} \tag{C.3}
$$

$$
\begin{aligned}
\partial_\lambda \xi^0 =& \frac{k}{4}\partial_\varphi \phi_0 + \frac{3}{8}\nu_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}} + \frac{3}{8}\nu_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}} \\
&-\frac{3}{8}\nu_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}} - \frac{3}{8}\nu_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}} \\
&+\nu_1^{-\frac{1}{2}}\psi_2^{\frac{1}{2}} + \nu_1^{\frac{1}{2}}\psi_2^{-\frac{1}{2}} - \nu_2^{-\frac{1}{2}}\psi_1^{\frac{1}{2}} \\
&-\nu_2^{\frac{1}{2}}\psi_1^{-\frac{1}{2}} - \frac{3}{2}\psi_2^{\frac{3}{2}}\omega_1^{-\frac{3}{2}} \\
&-\frac{1}{2}\psi_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}} + \frac{1}{2}\psi_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}} \\
&+\frac{3}{2}\psi_2^{-\frac{3}{2}}\omega_1^{\frac{3}{2}} - \frac{3}{2}\psi_1^{\frac{3}{2}}\omega_2^{-\frac{3}{2}} \\
&-\frac{1}{2}\psi_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}} \\
&+\frac{1}{2}\psi_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}} + \frac{3}{2}\psi_1^{-\frac{3}{2}}\omega_2^{\frac{3}{2}} \\
&-\xi^1\epsilon^{-1} + \xi^{-1}\epsilon^1 + \upsilon^2 f^{-2} + \frac{1}{2}\upsilon^1 f^{-1} \\
&-\frac{1}{2}\upsilon^{-1}f^1 - \upsilon^{-2}f^2 \\
&-\phi^{-1}\mu^1 + \phi^1\mu^{-1}
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
\partial_\lambda \xi^{\pm 1} =& -\frac{k}{2}\partial_\varphi \phi^{\pm 1} - \frac{9}{8}\nu_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} \pm \frac{3}{8}\nu_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}} \\
&+\frac{9}{8}\nu_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} \mp \frac{3}{8}\nu_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}} \\
&+\nu_1^{\pm\frac{1}{2}}\psi_2^{\pm\frac{1}{2}} + \nu_1^{\mp\frac{1}{2}}\psi_2^{\pm\frac{3}{2}} \\
&-\nu_2^{\pm\frac{1}{2}}\psi_1^{\pm\frac{1}{2}} - \nu_2^{\mp\frac{1}{2}}\psi_1^{\pm\frac{3}{2}} \\
&\pm 3\psi_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} \pm 2\psi_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}} \\
&\pm\psi_2^{\pm\frac{3}{2}}\omega_1^{\mp\frac{1}{2}} \pm 3\psi_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} \\
&\pm 2\psi_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}} \\
&\pm\psi_1^{\pm\frac{3}{2}}\omega_2^{\mp\frac{1}{2}} \pm 2\xi^0\epsilon^{\pm 1} \\
&\pm\xi^{\pm 1}\epsilon^0 \pm \frac{1}{2}\upsilon^{\mp 1}f^{\pm 2} \pm \upsilon^0 f^{\pm 1} \pm \frac{3}{2}\upsilon^{\pm 1}f^0 \\
&\pm 2\upsilon^{\pm 2}f^{\mp 1} \pm 2\phi^{\pm 1}\mu^0 \\
&\pm\phi^0\mu^{\pm 1}
\end{aligned} \tag{C.5}
$$

$$
\begin{aligned}
\partial_\lambda f^0 =& \frac{k}{3}\partial_\varphi \upsilon^0 + \frac{1}{2}\nu_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}} - \frac{1}{2}\nu_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}} \\
&+\frac{1}{2}\nu_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}} - \frac{1}{2}\nu_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}} + \frac{4}{3}\nu_1^{-\frac{1}{2}}\psi_2^{\frac{1}{2}} \\
&-\frac{4}{3}\nu_1^{\frac{1}{2}}\psi_2^{-\frac{1}{2}} + \frac{4}{3}\nu_2^{-\frac{1}{2}}\psi_1^{\frac{1}{2}} \\
&-\frac{4}{3}\nu_2^{\frac{1}{2}}\psi_1^{-\frac{1}{2}} + \frac{2}{3}\psi_2^{\frac{3}{2}}\omega_1^{-\frac{3}{2}} \\
&-\frac{2}{3}\psi_2^{\frac{1}{2}}\omega_1^{-\frac{1}{2}} - \frac{2}{3}\psi_2^{-\frac{1}{2}}\omega_1^{\frac{1}{2}} \\
&+\frac{2}{3}\psi_2^{-\frac{3}{2}}\omega_1^{\frac{3}{2}} - \frac{2}{3}\psi_1^{\frac{3}{2}}\omega_2^{-\frac{3}{2}} \\
&+\frac{2}{3}\psi_1^{\frac{1}{2}}\omega_2^{-\frac{1}{2}} + \frac{2}{3}\psi_1^{-\frac{1}{2}}\omega_2^{\frac{1}{2}} \\
&-\frac{2}{3}\psi_1^{-\frac{3}{2}}\omega_2^{\frac{3}{2}} - 2\xi^1\upsilon^{-1} + 2\xi^{-1}\upsilon^1 - 2\upsilon^{-1}\mu^1 \\
&+2\upsilon^1\mu^{-1} - 2\epsilon^{-1}f^1 + 2\epsilon^1 f^{-1} - 2\phi^{-1}f^1 \\
&+2\phi^1 f^{-1}
\end{aligned} \tag{C.6}
$$

$$
\begin{aligned}
\partial_\lambda f^{\pm 1} =& -\frac{k}{2}\partial_\varphi \upsilon^{\pm 1} \pm \frac{3}{4}\nu_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} \\
&\mp\frac{3}{4}\nu_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{1}{2}} \pm \frac{3}{4}\nu_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} \mp \frac{3}{4}\nu_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{1}{2}} \\
&\mp 2\nu_1^{\pm\frac{1}{2}}\psi_2^{\pm\frac{1}{2}} \pm \frac{2}{3}\nu_1^{\mp\frac{1}{2}}\psi_2^{\pm\frac{3}{2}} \\
&\mp 2\nu_2^{\pm\frac{1}{2}}\psi_1^{\pm\frac{1}{2}} \pm \frac{2}{3}\nu_2^{\mp\frac{1}{2}}\psi_1^{\pm\frac{3}{2}} \mp 2\psi_2^{\mp\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} \\
&-\frac{2}{3}\psi_2^{\pm\frac{3}{2}}\omega_1^{\mp\frac{1}{2}} \pm 2\psi_1^{\mp\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} \\
&+\frac{2}{3}\psi_1^{\pm\frac{3}{2}}\omega_2^{\mp\frac{1}{2}} \mp 4\xi^1\upsilon^{\pm 2} \\
&\pm 2\xi^0\upsilon^{\pm 1} \pm 2\xi^{\pm 1}\upsilon^0 \mp 4\upsilon^{\pm 2}\mu^{\mp 1} \\
&\pm 2\upsilon^0\mu^{\pm 1} \pm 2\upsilon^{\pm 1}\mu^0 \pm 3\epsilon^{\pm 1}f^0 \\
&\pm\epsilon^0 f^{\pm 1} + \epsilon^{\mp 1}f^{\pm 2} \\
&\pm 3\phi^{\pm 1}f^0 \pm \phi^0 f^{\pm 1} \mp \phi^{\mp 1}f^{\pm 2}
\end{aligned} \tag{C.7}
$$

$$
\begin{aligned}
\partial_\lambda f^{\pm 2} =& 2k\partial_\varphi \upsilon^{\pm 2} \pm 3\nu_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} \pm 3\nu_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} \\
&\mp\frac{8}{3}\nu_1^{\pm\frac{1}{2}}\psi_2^{\pm\frac{3}{2}} \mp \frac{8}{3}\nu_2^{\pm\frac{1}{2}}\psi_1^{\pm\frac{3}{2}} \\
&+4\psi_2^{\pm\frac{1}{2}}\omega_1^{\pm\frac{3}{2}} \mp \frac{4}{3}\psi_2^{\pm\frac{3}{2}}\omega_1^{\pm\frac{1}{2}} \\
&-4\psi_1^{\pm\frac{1}{2}}\omega_2^{\pm\frac{3}{2}} \\
&-\frac{4}{3}\psi_1^{\pm\frac{3}{2}}\omega_2^{\pm\frac{1}{2}} \mp 16\xi^0\upsilon^{\pm 2} \mp 4\xi^{\pm 1}\upsilon^{\pm 1} \\
&+16\upsilon^{\pm 2}\mu^0 \mp 4\upsilon^{\pm 1}\mu^{\pm 1} \pm 4\epsilon^{\pm 1}f^{\pm 1} \\
&\pm 2\epsilon^0 f^{\pm 2} \pm 4\phi^{\pm 1}f^{\pm 1} \pm 2\phi^0 f^{\pm 2}
\end{aligned} \tag{C.8}
$$

$$
\begin{aligned}
\partial_\lambda \nu_M^{\pm\frac{1}{2}} =& \mp k\partial_\varphi \nu_M^{\pm\frac{1}{2}} + \frac{10}{3}\xi^0\nu_M^{\mp\frac{1}{2}} \\
&+\frac{10}{3}\xi^{\pm 1}\nu_M^{\mp\frac{1}{2}} \mp 4\xi^{\mp 1}\omega_M^{\pm\frac{3}{2}} \mp \frac{8}{3}\xi^0\omega_M^{\pm\frac{1}{2}} \\
&\mp\frac{4}{3}\xi^{\pm 1}\omega_M^{\mp\frac{1}{2}} \mp \frac{8}{9}\upsilon^{\mp 1}\psi_M^{\pm\frac{3}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\mp\frac{16}{9}\upsilon^{0}\psi_M^{\pm\frac{1}{2}}\mp\frac{8}{3}\upsilon^{\pm1}\psi_M^{\mp\frac{1}{2}}\\
&\mp\frac{32}{9}\upsilon^{\pm2}\psi_M^{\mp\frac{3}{2}}-\varrho\nu_M^{\pm\frac{1}{2}}\pm\frac{1}{2}\epsilon^{0}\nu_M^{\pm\frac{1}{2}}\\
&\pm\epsilon^{\pm1}\nu_M^{\mp\frac{1}{2}}\pm\frac{5}{6}\phi^{0}\nu_M^{\pm\frac{1}{2}}\pm\frac{5}{3}\phi^{\pm1}\nu_M^{\mp\frac{1}{2}}\\
&\mp\eta\nu_M^{\pm\frac{1}{2}}\mp\frac{16}{9}\phi^{\mp1}\psi_M^{\pm\frac{3}{2}}-\frac{16}{9}\phi^{0}\psi_M^{\pm\frac{1}{2}}\\
&-\frac{16}{9}\phi^{\pm1}\psi_M^{\mp\frac{1}{2}}-2f^{\mp1}\omega_M^{\pm\frac{3}{2}}\\
&-2f^{0}\omega_M^{\pm\frac{1}{2}}-2f^{\pm1}\omega_M^{\mp\frac{1}{2}}\\
&-2f^{\pm2}\omega_M^{\mp\frac{3}{2}}+2\mu^{0}\nu_M^{\pm\frac{1}{2}}+2\mu^{\pm1}\nu_M^{\mp\frac{1}{2}}
\end{aligned}
\tag{C.9}
$$

$$
\begin{aligned}
\partial_\lambda\psi_M^{\pm\frac{1}{2}}=&\mp\frac{3}{8}k\omega_M^{\pm\frac{1}{2}}\mp2\xi^{0}\nu_M^{\pm\frac{1}{2}}\pm\xi^{\pm1}\nu_M^{\mp\frac{1}{2}}\\
&-\frac{3}{4}\xi^{\mp1}\omega_M^{\pm\frac{3}{2}}+\frac{1}{4}\xi^{0}\omega_M^{\pm\frac{1}{2}}\\
&+\frac{1}{2}\xi^{\pm1}\omega_M^{\mp\frac{1}{2}}\mp\frac{1}{2}\upsilon^{0}\nu_M^{\pm\frac{1}{2}}\\
&\mp\frac{3}{4}\upsilon^{\pm1}\nu_M^{\mp\frac{1}{2}}-\frac{2}{3}\upsilon^{\mp1}\psi_M^{\pm\frac{3}{2}}-\frac{2}{3}\upsilon^{0}\psi_M^{\pm\frac{1}{2}}\\
&+\frac{4}{3}\upsilon^{\pm2}\psi_M^{\mp\frac{3}{2}}-\frac{1}{2}\phi^{0}\nu_M^{\pm\frac{1}{2}}\\
&+\frac{1}{2}\phi^{\pm1}\nu_M^{\mp\frac{1}{2}}\mp\frac{3}{8}\eta\omega_M^{\pm\frac{1}{2}}-\varrho\psi_M^{\pm\frac{1}{2}}\\
&\pm\epsilon^{\mp1}\psi_M^{\pm\frac{3}{2}}\pm\frac{1}{2}\epsilon^{0}\psi_M^{\pm\frac{1}{2}}-2\epsilon^{\pm1}\psi_M^{\mp\frac{1}{2}}\\
&\mp\frac{1}{3}\phi^{\mp1}\psi_M^{\pm\frac{3}{2}}\pm\frac{1}{6}\phi^{0}\psi_M^{\pm\frac{1}{2}}\pm\frac{2}{3}\phi^{\pm1}\psi_M^{\mp\frac{1}{2}}\\
&\pm\frac{3}{2}f^{0}\nu_M^{\pm\frac{1}{2}}-\frac{3}{2}f^{\pm1}\nu_M^{\mp\frac{1}{2}}\\
&+\frac{3}{2}f^{\mp1}\omega_M^{\pm\frac{3}{2}}\mp\frac{3}{4}f^{0}\omega_M^{\pm\frac{1}{2}}\\
&\pm\frac{3}{4}f^{\pm2}\omega_M^{\mp\frac{3}{2}}-\frac{9}{4}\mu^{\mp1}\omega_M^{\pm\frac{3}{2}}+\frac{3}{4}\mu^{0}\omega_M^{\pm\frac{1}{2}}\\
&+\frac{3}{2}\mu^{\pm1}\omega_M^{\mp\frac{1}{2}}
\end{aligned}
\tag{C.10}
$$

$$
\begin{aligned}
\partial_\lambda\psi_M^{\pm\frac{3}{2}}=&\pm\frac{9}{8}k\omega_M^{\pm\frac{3}{2}}\mp3\xi^{-1}\nu_M^{\pm\frac{1}{2}}-\frac{9}{4}\xi^{0}\omega_M^{\pm\frac{3}{2}}\\
&-\frac{3}{4}\xi^{\pm1}\omega_M^{\pm\frac{1}{2}}\pm\frac{3}{4}\upsilon^{\pm1}\nu_M^{\pm\frac{1}{2}}\\
&\pm3\upsilon^{\pm2}\nu_M^{\mp\frac{1}{2}}+\frac{2}{3}\upsilon^{0}\psi_M^{\pm\frac{3}{2}}\\
&+2\upsilon^{\pm1}\psi_M^{\pm\frac{1}{2}}+4\upsilon^{\pm2}\psi_M^{\mp\frac{1}{2}}+\frac{3}{2}\phi^{\pm1}\nu_M^{\pm\frac{1}{2}}\\
&\pm\frac{9}{8}\eta\omega_M^{\pm\frac{3}{2}}-\varrho\psi_M^{\pm\frac{3}{2}}\pm\frac{3}{2}\epsilon^{0}\psi_M^{\pm\frac{3}{2}}\\
&\pm3\epsilon^{\pm1}\psi_M^{\pm\frac{1}{2}}\pm\frac{1}{2}\phi^{0}\psi_M^{\pm\frac{3}{2}}\\
&\pm\phi^{\pm1}\psi_M^{\pm\frac{1}{2}}-\frac{3}{2}f^{\pm1}\nu_M^{\pm\frac{1}{2}}\\
&-\frac{3}{2}f^{\pm2}\nu_M^{\mp\frac{1}{2}}+\frac{9}{4}f^{0}\omega_M^{\pm\frac{3}{2}}\\
&\mp\frac{3}{2}f^{\pm1}\omega_M^{\pm\frac{1}{2}}\mp\frac{3}{4}f^{\pm2}\omega_M^{\mp\frac{1}{2}}\\
&-\frac{27}{4}\mu^{0}\omega_M^{\pm\frac{3}{2}}-\frac{9}{4}\mu^{\pm1}\omega_M^{\pm\frac{1}{2}}.
\end{aligned}
\tag{C.11}
$$

D The $\mathfrak{sl}(3|2)$ Lie superalgebra valued gauge parameter λ

This appendix contains an explicit calculated form of the $\mathfrak{sl}(3|2)$ Lie superalgebra valued gauge parameter λ as:

$$
\begin{aligned}
\lambda\;=\;b^{-1}\Bigg[&\varrho\mathtt{J}+\epsilon\mathtt{L}_1+\phi\mathtt{A}_1+\upsilon\mathtt{W}_2+\varsigma_-\mathtt{G}^-_{\frac{1}{2}}\\
&+\varsigma_+\mathtt{G}^+_{\frac{1}{2}}+\omega_-\mathtt{S}^-_{\frac{3}{2}}+\omega_+\mathtt{S}^+_{\frac{3}{2}}\\
&+\Bigg(-\frac{\epsilon\mathcal{G}_-}{2k}-\frac{5\phi\mathcal{G}_-}{6k}-\frac{20\upsilon\mathcal{S}_-}{3k}\\
&-\frac{2\mathcal{A}\omega_-}{k}+\frac{\mathcal{J}\varsigma_-}{2k}-\varsigma_-'\Bigg)\mathtt{G}^-_{-\frac{1}{2}}\\
&+\Bigg(\frac{\epsilon\mathcal{G}_+}{2k}+\frac{5\phi\mathcal{G}_+}{6k}-\frac{20\upsilon\mathcal{S}_+}{3k}+\frac{2\mathcal{A}\omega_+}{k}-\frac{\mathcal{J}\varsigma_+}{2k}-\varsigma_+'\Bigg)\mathtt{G}^+_{-\frac{1}{2}}\\
&+\frac{1}{4}\mathtt{L}_0\Bigg(-\frac{15\omega_-\mathcal{G}_+}{2k}-\frac{15\mathcal{G}_-\omega_+}{2k}-4\epsilon'\Bigg)-\mathtt{W}_1\upsilon'\\
&+\frac{1}{4}\mathtt{A}_0\Bigg(\frac{9\omega_-\mathcal{G}_+}{2k}+\frac{9\mathcal{G}_-\omega_+}{2k}-4\phi'\Bigg)\\
&+\mathtt{S}^-_{\frac{1}{2}}\Bigg(\frac{4\upsilon\mathcal{G}_-}{3k}+\frac{\mathcal{J}\omega_-}{2k}-\omega_-'\Bigg)\\
&+\mathtt{S}^+_{\frac{1}{2}}\Bigg(-\frac{4\upsilon\mathcal{G}_+}{3k}-\frac{\mathcal{J}\omega_+}{2k}-\omega_+'\Bigg)\\
&+\frac{1}{8}\mathtt{L}_{-1}\Bigg(\frac{8\mathcal{L}\epsilon}{k}-\frac{80\mathcal{W}\upsilon}{k}+\frac{8\mathcal{A}\phi}{k}-\frac{5\mathcal{J}\omega_-\mathcal{G}_+}{2k^2}\\
&-\frac{4\varsigma_-\mathcal{G}_+}{k}+\frac{15\omega_-\mathcal{S}_+}{k}+\frac{5\mathcal{J}\mathcal{G}_-\omega_+}{2k^2}\\
&+\frac{15\mathcal{S}_-\omega_+}{k}+\frac{4\mathcal{G}_-\varsigma_+}{k}+\frac{15\omega_+\mathcal{G}_-'}{2k}\\
&+\frac{25\mathcal{G}_+\omega_-'}{2k}+\frac{15\omega_-\mathcal{G}_+'}{2k}+\frac{25\mathcal{G}_-\omega_+'}{2k}+4\epsilon''\Bigg)\\
&+\frac{1}{4}\mathtt{W}_0\Bigg(\frac{8\mathcal{A}\upsilon}{k}+\frac{8\mathcal{L}\upsilon}{k}+\frac{3\omega_-\mathcal{G}_+}{2k}-\frac{3\mathcal{G}_-\omega_+}{2k}+2\upsilon''\Bigg)\\
&+\frac{1}{8}\mathtt{A}_{-1}\Bigg(\frac{8\mathcal{A}\epsilon}{k}-\frac{80\mathcal{W}\upsilon}{k}\frac{8\mathcal{L}\phi}{k}+\frac{3\mathcal{J}\omega_-\mathcal{G}_+}{2k^2}-\frac{45\omega_-\mathcal{S}_+}{k}\\
&--\frac{3\mathcal{J}\mathcal{G}_-\omega_+}{2k^2}-\frac{45\mathcal{S}_-\omega_+}{k}-\frac{9\omega_+\mathcal{G}_-'}{2k}-\frac{15\mathcal{G}_+\omega_-'}{2k}\\
&-\frac{9\omega_-\mathcal{G}_+'}{2k}-\frac{15\mathcal{G}_-\omega_+'}{2k}+4\phi''\Bigg)\\
&+\frac{1}{6}\mathtt{S}^-_{-\frac{1}{2}}\Bigg(\frac{3\omega_-\mathcal{J}^2}{4k^2}+\frac{2\upsilon\mathcal{G}_-\mathcal{J}}{k^2}-\frac{3\omega_-'\mathcal{J}}{k}+\frac{2\phi\mathcal{G}_-}{k}\\
&-\frac{20\upsilon\mathcal{S}_-}{k}+\frac{3\mathcal{A}\omega_-}{k}+\frac{9\mathcal{L}\omega_-}{k}-\frac{3\omega_-\mathcal{J}'}{2k}\\
&-\frac{7\mathcal{G}_-\upsilon'}{k}-\frac{4\upsilon\mathcal{G}_-'}{k}+3\omega_-''\Bigg)\\
&+\frac{1}{6}\mathtt{S}^+_{-\frac{1}{2}}\Bigg(\frac{3\omega_+\mathcal{J}^2}{4k^2}+\frac{2\upsilon\mathcal{G}_+\mathcal{J}}{k^2}+\frac{3\omega_+'\mathcal{J}}{k}\\
&+\frac{2\phi\mathcal{G}_+}{k}+\frac{20\upsilon\mathcal{S}_+}{k}+\frac{3\mathcal{A}\omega_+}{k}+\frac{9\mathcal{L}\omega_+}{k}\\
&+\frac{3\omega_+\mathcal{J}'}{2k}+\frac{7\mathcal{G}_+\upsilon'}{k}+\frac{4\upsilon\mathcal{G}_+'}{k}+3\omega_+''\Bigg)
\end{aligned}
$$

$$+\frac{1}{12}\mathtt{W}_{-1}\left(\frac{8\upsilon\mathcal{G}_-\mathcal{G}_+}{k^2}+\frac{3\mathcal{J}\omega_-\mathcal{G}_+}{2k^2}-\frac{9\omega'_-\mathcal{G}_+}{2k}+\frac{15\omega_-\mathcal{S}_+}{k}+\frac{3\mathcal{J}\mathcal{G}_-\omega_+}{2k^2}-\frac{15\mathcal{S}_-\omega_+}{k}-\frac{8\upsilon\mathcal{A}'}{k}-\frac{8\upsilon\mathcal{L}'}{k}-\frac{20\mathcal{A}\upsilon'}{k}-\frac{20\mathcal{L}\upsilon'}{k}+\frac{3\omega_+\mathcal{G}'_-}{2k}-\frac{3\omega_-\mathcal{G}'_+}{2k}+\frac{9\mathcal{G}_-\omega'_+}{2k}-2\upsilon^{(3)}\right)$$

$$+\frac{1}{54}\mathtt{S}^-_{-\frac{3}{2}}\left(\frac{9\omega_-\mathcal{J}^3}{8k^3}+\frac{3\upsilon\mathcal{G}_-\mathcal{J}^2}{k^3}-\frac{27\omega'_-\mathcal{J}^2}{4k^2}+\frac{3\phi\mathcal{G}_-\mathcal{J}}{k^2}-\frac{30\upsilon\mathcal{S}_-\mathcal{J}}{k^2}+\frac{21\mathcal{A}\omega_-\mathcal{J}}{2k^2}+\frac{63\mathcal{L}\omega_-\mathcal{J}}{2k^2}-\frac{27\omega_-\mathcal{J}'\mathcal{J}}{4k^2}-\frac{33\mathcal{G}_-\upsilon'\mathcal{J}}{2k^2}-\frac{12\upsilon\mathcal{G}'_-\mathcal{J}}{k^2}+\frac{27\omega''_-\mathcal{J}}{2k}+\frac{40\mathcal{A}\upsilon\mathcal{G}_-}{k^2}+\frac{72\mathcal{L}\upsilon\mathcal{G}_-}{k^2}-\frac{90\epsilon\mathcal{S}_-}{k}-\frac{30\phi\mathcal{S}_-}{k}-\frac{180\mathcal{W}\omega_-}{k}+\frac{24\mathcal{A}\varsigma_-}{k}+\frac{18\mathcal{G}_-\omega_-\mathcal{G}_+}{k^2}+\frac{9\mathcal{G}_-^2\omega_+}{k^2}-\frac{9\omega_-\mathcal{A}'}{k}-\frac{6\upsilon\mathcal{G}_-\mathcal{J}'}{k^2}-\frac{27\omega_-\mathcal{L}'}{k}+\frac{120\mathcal{S}_-\upsilon'}{k}-\frac{18\mathcal{G}_-\phi'}{k}-\frac{6\phi\mathcal{G}'_-}{k}+\frac{33\upsilon'\mathcal{G}'_-}{k}+\frac{60\upsilon\mathcal{S}'_-}{k}-\frac{21\mathcal{A}\omega'_-}{k}-\frac{63\mathcal{L}\omega'_-}{k}+\frac{27\mathcal{J}'\omega'_-}{2k}+\frac{9\omega_-\mathcal{J}''}{2k}+\frac{27\mathcal{G}_-\upsilon''}{k}+\frac{12\upsilon\mathcal{G}''_-}{k}-9\omega_-{}^{(3)}\right)$$

$$+\frac{1}{54}\mathtt{S}^+_{-\frac{3}{2}}\left(-\frac{9\omega_+\mathcal{J}^3}{8k^3}-\frac{3\upsilon\mathcal{G}_+\mathcal{J}^2}{k^3}-\frac{27\omega'_+\mathcal{J}^2}{4k^2}-\frac{3\phi\mathcal{G}_+\mathcal{J}}{k^2}-\frac{30\upsilon\mathcal{S}_+\mathcal{J}}{k^2}-\frac{21\mathcal{A}\omega_+\mathcal{J}}{2k^2}-\frac{63\mathcal{L}\omega_+\mathcal{J}}{2k^2}-\frac{27\omega_+\mathcal{J}'\mathcal{J}}{4k^2}-\frac{33\mathcal{G}_+\upsilon'\mathcal{J}}{2k^2}-\frac{12\upsilon\mathcal{G}'_+\mathcal{J}}{k^2}-\frac{27\omega''_+\mathcal{J}}{2k}+\frac{9\omega_-\mathcal{G}_+^2}{k^2}-\frac{40\mathcal{A}\upsilon\mathcal{G}_+}{k^2}-\frac{72\mathcal{L}\upsilon\mathcal{G}_+}{k^2}-\frac{90\epsilon\mathcal{S}_+}{k}-\frac{30\phi\mathcal{S}_+}{k}+\frac{180\mathcal{W}\omega_+}{k}+\frac{18\mathcal{G}_-\mathcal{G}_+\omega_+}{k^2}-\frac{24\mathcal{A}\varsigma_+}{k}-\frac{9\omega_+\mathcal{A}'}{k}-\frac{6\upsilon\mathcal{G}_+\mathcal{J}'}{k^2}-\frac{27\omega_+\mathcal{L}'}{k}-\frac{120\mathcal{S}_+\upsilon'}{k}-\frac{18\mathcal{G}_+\phi'}{k}-\frac{6\phi\mathcal{G}'_+}{k}-\frac{33\upsilon'\mathcal{G}'_+}{k}-\frac{60\upsilon\mathcal{S}'_+}{k}-\frac{21\mathcal{A}\omega'_+}{k}-\frac{63\mathcal{L}\omega'_+}{k}-\frac{27\mathcal{J}'\omega'_+}{2k}-\frac{9\omega_+\mathcal{J}''}{2k}-\frac{27\mathcal{G}_+\upsilon''}{k}-\frac{12\upsilon\mathcal{G}''_+}{k}-9\omega_+{}^{(3)}\right)$$

$$+\frac{1}{48}\mathtt{W}_{-2}\left(\frac{48\upsilon\mathcal{A}^2}{k^2}+\frac{96\mathcal{L}\upsilon\mathcal{A}}{k^2}+\frac{27\omega_-\mathcal{G}_+\mathcal{A}}{2k^2}-\frac{27\mathcal{G}_-\omega_+\mathcal{A}}{2k^2}+\frac{32\upsilon''\mathcal{A}}{k}+\frac{120\mathcal{W}\epsilon}{k}+\frac{48\mathcal{L}^2\upsilon}{k^2}+\frac{120\mathcal{W}\phi}{k}+\frac{10\upsilon\mathcal{S}_-\mathcal{G}_+}{k^2}+\frac{9\mathcal{J}^2\omega_-\mathcal{G}_+}{8k^3}+\frac{45\mathcal{L}\omega_-\mathcal{G}_+}{2k^2}+\frac{10\upsilon\mathcal{G}_-\mathcal{S}_+}{k^2}+\frac{15\mathcal{J}\omega_-\mathcal{S}_+}{k^2}-\frac{30\varsigma_-\mathcal{S}_+}{k}\right.$$

$$-\frac{9\mathcal{J}^2\mathcal{G}_-\omega_+}{8k^3}-\frac{45\mathcal{L}\mathcal{G}_-\omega_+}{2k^2}+\frac{15\mathcal{J}\mathcal{S}_-\omega_+}{k^2}-\frac{30\mathcal{S}_-\varsigma_+}{k}-\frac{15\omega_-\mathcal{G}_+\mathcal{J}'}{4k^2}-\frac{15\mathcal{G}_-\omega_+\mathcal{J}'}{4k^2}-\frac{29\mathcal{G}_-\mathcal{G}_+\upsilon'}{k^2}+\frac{28\mathcal{A}'\upsilon'}{k}+\frac{28\mathcal{L}'\upsilon'}{k}-\frac{14\upsilon\mathcal{G}_+\mathcal{G}'_-}{k^2}-\frac{3\mathcal{J}\omega_+\mathcal{G}'_-}{2k^2}+\frac{15\omega_+\mathcal{S}'_-}{k}-\frac{6\mathcal{J}\mathcal{G}_+\omega'_-}{k^2}-\frac{45\mathcal{S}_+\omega'_-}{k}-\frac{14\upsilon\mathcal{G}_-\mathcal{G}'_+}{k^2}-\frac{3\mathcal{J}\omega_-\mathcal{G}'_+}{2k^2}+\frac{6\omega'_-\mathcal{G}'_+}{k}-\frac{15\omega_-\mathcal{S}'_+}{k}-\frac{6\mathcal{J}\mathcal{G}_-\omega'_+}{k^2}+\frac{45\mathcal{S}_-\omega'_+}{k}-\frac{6\mathcal{G}'_-\omega'_+}{k}+\frac{8\upsilon\mathcal{A}''}{k}+\frac{8\upsilon\mathcal{L}''}{k}+\frac{32\mathcal{L}\upsilon''}{k}-\frac{3\omega_+\mathcal{G}''_-}{2k}+\frac{9\mathcal{G}_+\omega''_-}{k}+\frac{3\omega_-\mathcal{G}''_+}{2k}-\frac{9\mathcal{G}_-\omega''_+}{k}+2\upsilon^{(4)}\left.\right)\Big]b. \quad \text{(D.1)}$$

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