



On light surfaces in black hole thermodynamics

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Received: 2 March 2022 / Accepted: 23 April 2022 / Published online: 18 May 2022

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Abstract We discuss the fundamentals of classical black hole (**BH**) thermodynamics in a new framework determined by light surfaces and their frequencies. This new approach allows us to study **BH** transitions inside the Kerr geometry. In the case of **BH**s, we introduce a new parametrization of the metric in terms of the maximum extractable rotational energy or, correspondingly, the irreducible mass, which is an alternative to the spin parametrization. It turns out that **BH** spacetimes with spins $a/M = \sqrt{8/9}$ and $a/M = 1/\sqrt{2}$ show anomalies in the rotational energy extraction and surface gravity whereas the case $a/M = \sqrt{3}/2$ is of particular relevance to study the variations of the horizon area. We find the general conditions under which **BH** transitions can occur and express them in terms of the masses of the initial and final states. This shows that **BH** transitions in the Kerr geometry are not arbitrary but depend on the relationship between the mass and spin of the initial and final states. From an observational point of view, we argue that near the **BH** poles it is possible to detect photon orbits with frequencies that characterize the light surfaces analyzed in this work.

1 Introduction

A black hole (**BH**) singularity is mainly an “active” gravitational object in the sense that it can interact with its environment constituted usually by matter and fields. A **BH** transition from a state to another, accompanied by a change of the characteristic parameters, is regulated by the laws of **BH** thermodynamics. These changes are also crucial in the analysis of **BH** energy extraction detectable, for example, by observing jet emissions. The progenitor collapse into a **BH** appears as an irreversible process of information degeneration, where the information becomes inaccessible to the dis-

tant observers and the **BH** has no observable hair. A (stationary) **BH** macro-state is defined and determined only by the mass parameter M mass, spin J , and electric charge Q . However, the **BH** micro-states number can be enormously large, providing eventually a very high **BH** entropy. The number of **BH** micro-states increases with the **BH** size and the **BH** size, function of outer horizon area, becomes a measure of its entropy. The horizon area cannot decrease in any process with suitable conditions on energy and no repulsive gravity. Hawking’s area theorem, states that the total horizon area of classical **BH**s cannot decrease over time [1–3].

In the processes of **BH** energy and rotational energy extraction, underlying for instance jet emission processes, this result is maintained. The Penrose process is, for example, a classical method for extracting energy from a Kerr **BH** using its frame-dragging effects in the ergoregion. A small body may have in this region negative energy as measured by an observer at infinity. If the body is fragmented into two parts, and the negative-energy part is swallowed by the **BH**, decreasing its (total) mass, the other part can be ejected. In this process J decreases, consequently, its area increases (as determined by the outer horizon) [4]. The balance of the mass increase (which increases the **BH** area) and spin variations in this process is maintained within the constraint that the event horizon can increase or remains constant. For example, the recently observational confirmation of Hawking’s black-hole area theorem has been presented based on data from GW150914 [5].

Black holes are classical solutions of the Einstein field equations, representing spacetime regions bounded by the event horizon and causally disconnected from observers at infinity. Bekenstein first suggested the idea that the event horizon¹ defines the **BH** entropy [8, 9]. However, **BH** thermodynamic properties seem to be purely intrinsically geometric

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¹ It should be noted that the inner horizon has been shown to be unstable and it has been conjectured it could end into a singularity [6, 7].

characteristics, defined on the basis of the outer **BH** (event) horizon alone. Probably, an unknown quantum mechanism would explain and complete **BH** classical thermodynamics, leaving however open complex problems (for instance, the well-known information paradox as a consequence of Hawking thermal radiation), and touching the limit of applicability of quantum gravity.

However, if **BHs** have an entropy, we can expect them to have also a temperature. This implies that a **BH** must also radiate. This radiation was explained by effects of quantum fluctuation of matter fields in the vacuum close to the singularity as Hawking radiation. Fluctuations of the (matter) field vacuum produce couples of particles and antiparticles in the close vicinity of a black hole. A particle is captured by the singularity with negative energy, the other particle can escape to infinity with positive energy, constituting observable radiation with thermal profile.

In this semi-classical scenario, the **BH** would evaporate emitting only thermal Hawking radiation with non-vanishing (non constant) entropy; however, such an evolution appears not to be fully described by quantum mechanics (raising the so called information paradox). The derivation of Hawking radiation is indeed semi-classical and yet does not establish the **BH** situation at the Planck size, where quantum gravity is expected to play a major role. Specifically, it does not establish if, for example, the **BH** evaporates completely or viceversa some kind of remnants is left. A further question is how the emitted radiation, reaching infinity, may be entangled with such remnant, and what is its entropy. The **BH** may also continue the evaporation process completely, leaving no remnants. One could expect that (Hawking) radiation from the collapsing matter into a **BH** would carry out the information of the matter, the **BH** would evaporate away as a result of thermal emission and the process preserves the unitarity foreseen in quantum mechanics. This expectation, however, fails with the difficulties existing in obtaining a unitary description of **BH** evaporation (and **BH** formation). A concrete possibility is that the final radiation of the complete evaporation does not “bring” information on the initial matter state (contradicting quantum mechanics unitarity realized through the action of an unitary evolution operator). In this case, the quantum fluctuation would produce a mixed state of the outgoing emitted particles with the ingoing particles; consequently, the outgoing radiation would be entangled with the final hole state (the final state is a mixed state if the fate of the **BH** is to evaporate to nothing) – see, for example, [10] for a review on the **BH** information problem. All these are aspects of still open debated issues, at the heart of the quantization of the geometric theories of gravity, of the treatment of unified theories for matter field-quantized gravity, and very often addressed in quantum theories such as loop quantum gravity or in the context of string theories.

From the classical point of view the effects of a change of state in **BHs** are constrained by the laws of thermodynamics. One might ask if there are privileged or not allowed state transitions or if transitions are singled out by some anomalous characteristics. Constrains on these processes are generally determined by the initial state of the black hole, its spin J or by the specific mass ratios of the initial and final states. There are strong indications that the creation of a naked singularity (**NS**) through a conversion process from an extreme **BH** is forbidden and that no accretion process can bring a **BH** into a **NS** state, while the reverse process, assuming the **NS** existence, is still debated in different contexts. However, it is necessary to consider possible processes of rotational energy extraction. For example, geometrically thin accretion disks have been studied for possible mechanisms to convert asymptotically **NSs** into extreme **BH** states [11].

In this work, we study aspects of classic black hole thermodynamics reformulated in terms of light surfaces. We focus on Kerr singularities. Using some characteristics of specific null surfaces of the Kerr geometries, including the case of the horizons, we study the relation between the initial and final states of a **BH** transformation regulated by the laws of **BH** thermodynamics. **BHs** and **NSs** geometries are therefore represented in a plane called the extended plane. The surfaces are defined by a characteristic null frequency ω , which connect all the geometries described by the Kerr solution and all the points of these geometries where a null orbit with frequency ω is defined. These geometries always include **BHs** and in some cases naked singularities. Characteristic frequencies are frequencies of the Kerr horizons. A special metric line parametrization, describing only **BH** spacetimes (no naked singularities), is given in terms of the maximum extractable rotational energy ξ or, correspondingly, the inertial mass parameter M_{irr} , replacing the metric spin parameter. We focus on some particular transformations between one state and another. We identify some particular transition states, representing the initial spin of the black hole or a specific ratio of the masses. These emerge as properties of the classical thermodynamic variables, when viewed in the extended plane. Using the representation provided by the light surfaces, we obtain an overview of the possible states. Thus, we first introduce the new framework, writing the principal quantities in this new way, and then we study the laws of **BH** transformations in the extended plane in terms of the light surfaces.

The plan of the article is as follows. In Sect. 2, we introduce the Kerr geometries while, in Sect. 2.1, we discuss **BH** Killing horizons and the characteristic frequencies, building up the framework for the formulation of the laws of **BH** thermodynamics that are introduced in Sect. 3. In Sect. 3.1, masses and thermodynamic variables are explored in terms of the light surfaces. The analysis continues in Sect. 4, where we focus on the laws of **BH** thermodynamics. Rotational energy

and **BH** “rest” mass are the focus of Sect. 4.1 while the **BH** irreducible mass is addressed in Sect. 4.2. **BH** transformations in the new frame are investigated in Sect. 5. An in–deep discussion on the inner and outer horizons in **BH** transitions is presented in Sect. 5.1. The case of constant irreducible mass is described in Sect. 5.2. In Sect. 5.3, we discuss the the inner horizon relations in **BH** thermodynamics. In Sect. 6, we present the metric tensor re-parameterizations in the new framework in terms of inertial mass, extractable rotational energy, and surface gravity. Discussion and final remarks follow in Sect. 7. In Appendix A, there are further details of black holes in the extended plane. Some remarkable areas in the extended plane are described in Appendix B.

2 The Kerr geometry

The metric tensor of the Kerr geometry can be written in Boyer-Lindquist (BL) coordinates $\{t, r, \theta, \phi\}$ as follows

$$\begin{aligned}
 ds^2 = & -dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 \\
 & + \frac{2M}{\rho^2} r (dt - a \sin^2 \theta d\phi)^2 \\
 \rho^2 \equiv & r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2Mr + a^2, \\
 r_{\pm} \equiv & M \pm \sqrt{M^2 - a^2}; \quad r_{\epsilon}^{\pm} \equiv M \pm \sqrt{M^2 - a^2 \cos^2 \theta},
 \end{aligned}
 \tag{1}$$

where $r \in [0, +\infty[$, $t \in [0, +\infty[$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The radii r_+ and r_- are the outer and inner Killing horizons, respectively; r_{ϵ}^+ and r_{ϵ}^- represent the outer and inner ergosurfaces, respectively. Here M is the (ADM and Komar) mass parameter and the specific angular momentum is given as $a = J/M$, where J is the total angular momentum of the gravitational source. For simplicity, here and in the following we consider dimensionless parameters defined as $r \rightarrow r/M$ and $a \rightarrow a/M$. Moreover, we will use the notation $\sigma \equiv \sin^2 \theta$.

It is $r_+ < r_{\epsilon}^+$ on the planes $\theta \neq 0$ and $r_{\epsilon}^+ = 2M$ on the equatorial plane $\theta = \pi/2$, where $r_- = 0$. Moreover, $0 < r_{\epsilon}^- < r_- < r_+ < r_{\epsilon}^+$ for $\sigma \neq 0$ and $a \in]0, M[$. For $\sigma = 0$, the functions r_{ϵ}^{\pm} satisfy the properties $0 < r_{\epsilon}^- = r_- < r_+ = r_{\epsilon}^+$. For $a = \pm M$ (extreme Kerr **BH**), we have that $r_{\pm} = M$. The case $a = 0$ corresponds to the static spherically symmetric Schwarzschild solution, where $r_+ = 2M$ and there is no ergosurface. Naked singularities are defined for $a^2 > M^2$. The surfaces r_{ϵ}^{\pm} in **NSs** are well-defined for $\cos \theta \in [-M/a, M/a]$ or, equivalently, $a/M \in]1, 1/\sec^2 \theta]$. In the region $r \in]r_+, r_{\epsilon}^+[$ (outer *ergoregion* or simply *ergoregion*), it is $g_{tt} > 0$ and the t -Boyer–Lindquist coordinate becomes spacelike. This fact implies that a static observer cannot exist inside the ergoregion. In

the following analysis, we will refer to some properties of circular motion in these regions.

Since the metric is independent of ϕ and t , the covariant components p_{ϕ} and p_t of a particle four-momentum are conserved along its geodesic. Consequently, $E \equiv -g_{ab}\xi_t^a p^b$ and $L \equiv g_{ab}\xi_{\phi}^a p^b$ are constants of motion for test particle orbits, where $\xi_t = \partial_t$ is the Killing field representing the stationarity of the Kerr geometry and $\xi_{\phi} = \partial_{\phi}$ is the rotational Killing field (the vector ξ_t becomes spacelike in the ergoregion). The constant L may be interpreted as the axial component of the angular momentum of a test particle following timelike geodesics and E as representing the total energy of the test particle coming from radial infinity, as measured by a static observer at infinity. The motion on the fixed plane $\sigma = 1$ is restricted to that plane ($u^{\theta} = 0$) because the Kerr metric is symmetric under reflections with respect to the equatorial hyperplane $\theta = \pi/2$.

2.1 Killing horizons and characteristic frequencies

In this section, we build the framework for the **BH** thermodynamics laws by using the properties of special light surfaces and the **BH** horizons.

2.1.1 BH horizons

Let us introduce the Killing vector $\mathcal{L} = \partial_t + \omega \partial_{\phi}$. We will consider mainly $\omega = \text{constant}$. The quantity $\mathcal{L}_{\mathcal{N}} \equiv \mathcal{L} \cdot \mathcal{L}$ becomes null for photon-like particles with orbital frequencies ω_{\pm} .

The Killing vector $\mathcal{L}_{\pm} \equiv \xi_t + \omega_{\pm} \xi_{\phi}$, where ω_{\pm} satisfy the null condition on the norm of \mathcal{L} , can be interpreted as generator of null curves ($g_{\alpha\beta} \mathcal{L}_{\pm}^{\alpha} \mathcal{L}_{\pm}^{\beta} = 0$) as the Killing vectors \mathcal{L}_{\pm} are also generators of Killing event horizons.

The Kerr horizons are null (lightlike) hypersurfaces generated by the flow of a Killing vector, whose null generators coincide with the orbits of a one-parameter group of isometries, i.e., in general, there exists a Killing field \mathcal{L} , which is normal to the null surface. More precisely, the **BH** horizon r_+ is a non-degenerate (bifurcate) Killing horizon generated by the vector field \mathcal{L} . In the case $a = 0$ (where $\omega = 0$), ξ_t (now generator of r_+) is hypersurface-orthogonal.

Thus, the null frequencies ω_{\pm} evaluated on the **BH** horizons, are the **BH** inner and outer horizons frequencies ω_H^{\mp} respectively.

2.1.2 Light surfaces and characteristic frequencies

The results we discuss in this work follow from the investigation of the properties of the null vector \mathcal{L} . The framework of **BH** thermodynamics is constructed upon the solutions $\mathcal{MB} : \mathcal{L}_{\mathcal{N}} = 0$, with $\omega = \text{constant}$, where \mathcal{MB} is a spin parameter of the set $\mathcal{MB} \in \{a, a\sqrt{\sigma}\}$. Solutions \mathcal{MB} are

functions of (ω, r) , or (ω, r, σ) , and can be represented as a curve called metric bundle \mathcal{MB} on a plane $a - r$ or $\mathcal{A} - r$ (extended plane) where $\mathcal{A} \equiv a\sqrt{\sigma}$ [12–18].

The quantity $\omega : \mathcal{L}_N = 0$, for the null vector \mathcal{L} will be called characteristic \mathcal{MB} frequency.

A particular null vector \mathcal{L} is defined by $\omega = \omega(r_{\pm}) \equiv \omega_H^{\pm}$, defining the Killing horizons of the metric. In fact, the event horizons of a spinning **BH** are Killing horizons with respect to the Killing field $\mathcal{L}_H = \partial_t + \omega_H^{\pm} \partial_{\phi}$, where ω_H^{\pm} is the angular velocity (frequency) of the horizons representing the **BH** rigid rotation. The event horizon of a stationary asymptotically flat solution with matter, satisfying suitable hyperbolic equations, is a Killing horizon. The strong rigidity theorem connects the event horizon with a Killing horizon. In the limiting case of spherically symmetric, static spacetimes, the event horizons are Killing horizons with respect to the Killing vector ∂_t and the event, apparent, and Killing horizons with respect to the Killing field ξ_t coincide (we can say that $\omega_H^+ = 0$).

Therefore, in the extended plane, the curve $a_{\pm} \equiv \sqrt{r(2M - r)}$, is called horizon curve and it is tangent at any point to a \mathcal{MB} curve. The tangent point (a_g, r_g) , distinguishes the **BH** rotation a_g and its outer (inner) horizon $r_g \in [M, 2M]$ ($r_g \in [0, M]$). The tangent point defines the **BH** outer (inner) horizon frequency which, therefore, is the \mathcal{MB} characteristic frequency. In the extended plane, \mathcal{MB} are made by points (a, r) or $(a\sqrt{\sigma}, r)$ distinguishing a geometry a (which can be also a naked singularity) and an orbit (and plane σ) where there is light-like frequency ω corresponding to the \mathcal{MB} characteristic frequency.

Two important properties follow: all metric bundles are tangent to the horizon curves and the horizon curves are the envelope surfaces of the metric bundles. Thus, all the \mathcal{MB} characteristic frequencies, all the light surfaces frequencies (also in the naked singularity solutions) are frequencies of a **BH** (inner or outer) horizons, determined by the corresponding \mathcal{MB} tangent point. The point a_0 (\mathcal{A}_0) at the origin $r = 0$ (central singularity) is called the bundle origin.

Note that in this case we are assuming $\omega > 0$ as $a > 0$; however, the study of counter-rotating photon circular orbits with frequency $\omega < 0$ is possible with \mathcal{MB} s in the extended plane $a \in \mathcal{R}$ with $\omega > 0$, although in this analysis this extension is not necessary – see [13]. The **NS** light surfaces properties are defined by the **BH** light surfaces (through the condition of tangency of the bundles with the horizon curve). Vice versa it has been shown that most bundles \mathcal{MB} with **NS** origin are tangent to the inner horizons curve. The horizon curves point $a = 0, r = 0$ is the central singularity, ($a = M, r = M$) is the horizon of the extreme Kerr **BH**, while $a = 0$ with point $r = 2M$ is the horizon of the Schwarzschild solution.

Each line $a_{\pm} = \text{constant}$ in this extended plane is a **BH** geometry, where the inner horizons are for $r \in [0, M]$ on

the line a_{\pm} , while $r \in [M, 2M]$ contains the outer horizons. The maximum of the curve a_{\pm} is the point $a = M$ and $r = M$ which is the horizon in the extreme Kerr **BH**. The Schwarzschild **BH** is for $a = 0$; therefore, it corresponds to the zeros of the metric bundles curves, where $r = 2M$ is its horizon-and in the extended plane it coincides also with the outer ergosurface on the equatorial plane of all the Kerr **BH**s and **NSs**².

The characteristic frequency ω , the bundles origin a_0 , the tangent spin a_g and radius r_g to the horizon curve in the extended plane are related by the following conditions

$$\begin{aligned} \omega &= \frac{1}{\mathcal{A}_0} = \frac{1}{a_0\sqrt{\sigma}} = \frac{\sqrt{(2 - r_g)r_g}}{2r_g} = \omega_H^{\pm}(r), \\ r_g(\omega) &= \frac{2}{4\omega^2 + 1} = \frac{2\mathcal{A}_0^2}{\mathcal{A}_0^2 + 4}, \\ a_g(a_0) &= \frac{4\mathcal{A}_0}{\mathcal{A}_0^2 + 4} = 4\sqrt{\frac{\omega^2}{(4\omega^2 + 1)^2}}, \\ a_0 &= \frac{2}{\sqrt{\sigma}}\sqrt{\frac{r_g}{2 - r_g}}, \end{aligned} \tag{2}$$

where $r_g \in [0, 2M]$, $a_0 \in [0, +\infty[$, $a_g \in [0, M]$ (with the extension to negative values for the counter-rotating case), $\omega \in [0, +\infty]$ and $\sigma \in [0, 1]$ (see Fig. 1).

The bundle is tangent to the horizon curve at one point only; therefore, this defines uniquely the **BH** with the characteristic frequency ω and, consequently, bundles can contain either only **BH**s or **BH**s and **NS**s. Bundles are not defined in the region $]r_-, r_+[$ of **BH** spacetimes, but they are defined in the region $r \in [0, 2M]$ of **NS**s).

\mathcal{MB} curves in the extended plane connect points of different (**BH** or **BH** and **NS**) geometries having all the same characteristic null frequency ω . They are related to the light surface $r_s(\omega) : \mathcal{L}_N = 0$ defined in a fixed spacetime as the light surfaces r_s are the collections of all points, crossing of the \mathcal{MB} with a line $a = \text{constant}$ on the extended plane.

2.1.3 Constrains on matter distribution

\mathcal{MB} s, defined by light-surfaces, determine also the timelike (matter circular) motion providing the limiting orbital frequencies of stationary observers.

As the solutions $\omega : \mathcal{L}_N = 0$, for each point (a, r, θ) , are generally two frequencies ω_{\pm} , this implies that there are two bundles crossing at any point of the extended plane. The frequencies ω_{\pm} , at the crossing point r , bound the possible stationary observers frequencies. Specifically, the causal structure defined by timelike stationary observers is charac-

² \mathcal{MB} s are a conformal invariants of the metric as many other quantities considered in this analysis that inherit some of the properties of the Killing vector \mathcal{L}

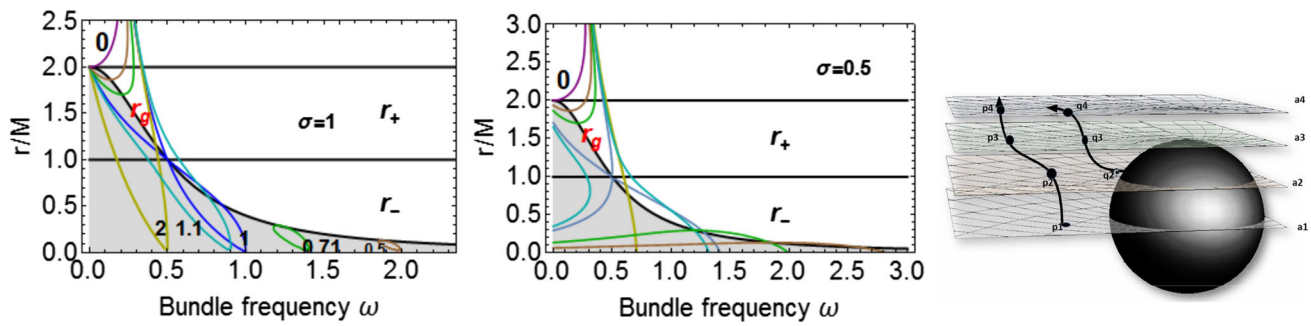


Fig. 1 Left and center panels: light surfaces r_s^\pm as functions of the bundles frequency ω for different values (denoted on each curve) of the dimensionless spin a/M , representing **NS** and **BH** geometries. The quantities r_s^\pm are solutions of $\mathcal{L} \cdot \mathcal{L} = 0$, where $\sigma = \sin^2 \theta$. Left-panel: equatorial plane $\sigma = 1$. Right-panel: Plane $\sigma = 0.5$. r_\pm are the inner and outer horizons in the extended plane, where $r_+ \in [M, 2M]$, and $r_- \in [0, M]$, while r_g represents the points of the metric bundles that are tangent to the horizon curve in the extended plane, i.e., it is the horizon curve as function of the bundles characteristic (and horizons) frequency. Gray region is $r < r_g$. There a_0 is the bundle origin spin as function of the frequency $a_0 = 1/\omega\sqrt{\sigma}$. Note that in slowly rotating

NSs, an increase of the spin a_0 leads to the appearance of the Killing bottleneck (a restriction of the light surfaces). Right panel: The central sphere represents **BHs** in the extended plane; each horizontal plane corresponds to a $a = \text{constant}$ surface, i.e., a particular Kerr geometry, at the points on the black surface represent the inner and outer **BH** horizons. Planes not crossing the sphere are **NSs**. The sphere considers also the two rotation directions of the **BH** spin and the central equatorial plane is the Schwarzschild spacetime. The points p_i and q_i correspond to replicas, where p_i denote geometries related by the same frequency ω and q_i include a point of the horizon

terized by a frequency bounded in the range $\omega \in]\omega_-, \omega_+[$. On the other hand, static observers are defined by the limiting condition $\omega = 0$ and cannot exist in the ergoregion.

The limiting frequencies ω_\pm , which are photon orbital frequencies, solutions of the condition $\mathcal{L}_N = 0$, determine the frequencies ω_H^\pm of the Killing horizons as well as the bundles characteristic frequencies.

The vector \mathcal{L} appears in the description of certain **BH** evolution processes because it enters the definitions of thermodynamic variables and stationary observers. The vector \mathcal{L} , the condition $\mathcal{L}_N = 0$ and \mathcal{MBs} are closely related to the definition of stationary observers, i. e., observers with a tangent vector which is a Killing vector, that is, whose four-velocity u^α is a linear combination of the two Killing vectors ξ_ϕ and ξ_t ; therefore, $u^\alpha = \gamma \mathcal{L}^\alpha = \gamma(\xi_t^\alpha + \omega \xi_\phi^\alpha)$, where γ is a normalization factor and $d\phi/dt = u^\phi/u^t \equiv \omega$. The dimensionless quantity ω is the orbital frequency of the stationary observer.

It is clear that \mathcal{MBs} relate different geometries through their light surfaces defined by the characteristic frequencies which are **BHs** frequency. Thus, they are particularly adapted to describe **BH** state transitions, constrained by the properties of the null vector \mathcal{L} , which builds up many of the thermodynamic properties of **BHs**. In the next section, we shall detail the **BH** thermodynamic properties in terms of the null vector \mathcal{L} and, therefore, of the metric bundles.

\mathcal{MBs} are structures that are related to the points of the light surfaces of different geometries and, therefore, provide relations between geometries. This different way of rewriting the properties of the null vector \mathcal{L} , which defines also the **BH** horizons, emphasizes some geometric characteris-

tics and properties, repeating in the same spacetime or different geometries of a \mathcal{MB} . One of these properties, from the light surfaces and \mathcal{MB} definitions is the frequency ω , all the points of a \mathcal{MB} curve are the replica of the **BH** horizon frequency ω_H^+ or ω_H^- of the **BH** individuated by the tangent point (a_g, r_g) . In some cases, these horizons (frequencies) replicas are in the same spacetime. In [17, 18], the **BH** poles $\sigma \approx 0$ have been studied for the observation of the photon orbits with \mathcal{MBs} characteristic frequency. These are light-like (circular) orbits having the same frequency as the black hole ω_H^\pm .

In the context of replicas, we also introduce the concept of horizon confinement. Indeed, we say that there is a replica when in the spacetime it is possible to find at least a couple of points having the same value for the property \mathcal{Q} . Then, we say that there is a *confinement*, when that value is not replicated. In the Kerr spacetime, part of the inner horizon frequencies are “confined”. The confinement analysis, which is the study of the topology of the curves $\omega = \text{constant}$ in the extended plane, provides information about the local properties of the spacetime replicated in regions more accessible to observers; for example, in the case of properties defined in the proximity of the **BH** poles or of the inner horizons. An observer can register the presence of a replica at the point p of the **BH** spacetime with spin a_p , belonging to a Killing bundle. The observer will find the replica of the **BH** horizon frequency $\omega_H^+(a_p)$ at the point p ; therefore, her/his orbital stationary frequency is $\omega_p \in]\omega_\bullet, \omega_\star[$ where one of $(\omega_\bullet, \omega_\star)$ is the horizon’s frequency ω_H^+ , replicated on a pair of orbits (r_+, r_p) . The second light-like frequency ω_\bullet is the frequency of a horizon in a **BH** spacetime. The relation between the two

frequencies $(\omega_*, \omega_\bullet)$ is determined by a characteristic ratio, which we also study [17, 18]. In general, we use in this work replicas to connect **BH** spacetimes related in a transition and governed by the thermodynamic laws in the extended plane. We reformulate **BH** thermodynamics on the replicas in terms of the light surfaces, exploring the thermodynamic properties of the geometries defined by the metric bundles.

3 Black hole thermodynamics

The norm $\mathcal{L}_N \equiv \mathcal{L} \cdot \mathcal{L}$ is constant on the horizon. We start here our analysis of **BH** thermodynamics by introducing the **BH** surface gravity as the constant (acceleration)³ $\ell : \nabla^\alpha \mathcal{L}_N = -2\ell \mathcal{L}^\alpha$, evaluated on the outer horizon r_+ , or equivalently, $\mathcal{L}^\beta \nabla_\alpha \mathcal{L}_\beta = -\ell \mathcal{L}_\alpha$ and $L_{\mathcal{L}} \ell = 0$, where $L_{\mathcal{L}}$ is the Lie derivative, a non affine geodesic equation, i.e., $\ell = \text{constant}$ on the orbits of \mathcal{L} .

The **BH** surface gravity ℓ , which is also a conformal invariant of the metric [21], may be defined as the rate at which the norm of the Killing vector \mathcal{L} vanishes from outside (i.e. $r > r_+$). For the Kerr spacetime it becomes $\ell_{Kerr} = (r_+^2 - a^2)/(r_+^2 + a^2)^2$.

The surface gravity re-scales with the conformal Killing vector, i.e., it is not the same on all generators but, because of the symmetries, it is constant along one specific generator.

The **BH** event horizon of stationary solutions has constant surface gravity, i. e., the surface gravity is constant on the horizon of stationary black holes, which is postulated as the zeroth **BH** law-area theorem (see for example [19, 20]).

More generally, the **BH** horizon area is non-decreasing, a property which is considered as the second law of **BH** thermodynamics, establishing the impossibility to achieve with any physical process a **BH** state with zero surface gravity.

Clearly, in the extreme Kerr spacetime ($a = M$), where $r_\pm = M$, the surface gravity is zero. This implies that the temperature is also null ($T_H = 0$), with a non-vanishing entropy [19, 20, 22]. A non-extremal **BH** cannot reach the extremal limit in a finite number of steps, which is implied by the third law. (This fact has consequences also regarding the stability against Hawking radiation.)

On the other hand, the condition (constance of) $\nabla^a \mathcal{L} = 0$ when $\ell = 0$ substantially constitutes the definition of the degenerate Killing horizon – degenerate **BH** –, in the case of Kerr geometries only the extreme **BH** case is degenerate; therefore, in the extended plane it corresponds to the point $a = M, r = M$. A fundamental theorem of Boyer shows that degenerate horizons are closed. This fact also establishes

³ We note that in this work we have used the notation ℓ for the surface gravity instead of the more usual κ , to emphasize the role of acceleration relating to the null vector \mathcal{L} .

a topological difference between black holes and extreme black holes.

Now, the first law of **BH** thermodynamics, $\delta M = (1/8\pi)\ell_H^+ \delta A_{area}^+ + \omega_H^+ \delta J$, relates the variation of the mass δM , the (outer) horizon area δA_{area}^+ , and angular momentum δJ with the surface gravity ℓ_H^+ and angular velocity ω_H^+ on the outer horizon. Here all the quantities, including the surface gravity ℓ , are evaluated on the outer Killing horizon so that the notation (+) can be omitted. The (Hawking) temperature term is related to the surface gravity by $T_H = \hbar c \ell / 2\pi k_B$ (k_B is the Boltzmann constant) and the horizon area A_{area}^+ to the entropy, $S = k_B A_{area}^+ / l_P^2$ (l_P is the Planck length, \hbar the reduced Planck constant, and c is the speed of light).⁴

We focus our analysis on two initial and final states for **BH** transition. We can interpret the terms $(\omega_H^+ \delta J)$ in the **BH** transition from one state (0) to a new state (1) considering the frequency as characteristic bundle frequency, therefore, with $\omega_H^+(0) = \omega(0)$ which is the \mathcal{MB} frequency tangent to the outer horizon of the **BH** at the initial state (0), the entropy term $(\ell(0) \delta A_{area}^+)$ is also considered on the bundle, writing the relation in the extended plane.

All the quantities are expressed in terms of bundles at $\omega(0) = \text{constant}$ and $(\delta A_{area}^+, \delta J, \delta M)$, describing the transition from the initial to final state. Furthermore, as in the extended plane, bundles tangent to the inner horizons are also relevant. We write a corresponding relation between the quantities $(\delta A_{area}^-, \delta J, \delta M)$ and $(\omega_H^-(0), \ell^-(0))$ evaluated on the inner horizon r_- . Alternately, we express $(\delta A_{area}^\pm, \delta J, \delta M)$ and $(\omega_H^\pm(0), \ell^\pm(0))$ and their relation in a unique form as function of a generic point r of the extended plane.

In the next sections we will consider quantities (ℓ, ω) and (M, A_{area}, J) and their variations in the extended plane, on the horizon curve. We express the relation between these in terms of the properties of the light surfaces.

3.1 Masses and **BH** thermodynamics

Here we explore **BH** thermodynamics in the extended plane in terms of the light surfaces using \mathcal{MB} curves and discuss variations of the total mass M , area A_{area} , and momentum J of the **BH** defined by the tangency property with the horizons curve in the extended plane, considering the bundle frequency ω and the acceleration ℓ . We emphasize some (“conformal”) properties, as defined below, in these relations.

⁴ If the **BH** temperature is $T = \ell/(2\pi)$, its entropy is $S = A_{area}/(4\hbar G)$, the pressure-term is $p = -\omega_H$, where the internal energy is $U = GM$ ($M = c^2 m/G = \text{mass}$, where m is a mass term). The term $\omega_H \delta J$ is interpreted as the “work”. The **BH** (horizon) area A_{area} is related to the outer horizon definition, $A_{area} = 8\pi m r_+$, while the volume term is $V = GJ/c^2$ (where $J = amc^3/G$). The **BH** horizon area will be considered here alternatively in the extended plane.

We will connect the **BHs** states before and after a transition by expressing their characteristics and parameters through the \mathcal{MBs} (curves defined by the properties of the light surfaces for different geometries) and their representation in the extended plane. Particularly the condition of tangency with the curve of the horizons provides the constraint for **BHs** transformations. Although the extended plane represents also **NSs** solutions as the origin of the bundles and the \mathcal{MBs} contain **BHs** and **BHs** and **NSs** (related to the tangency to the curve portion of the internal horizons) or depending on the σ angle (near the poles there are interesting properties of the ω frequencies explored in [17, 18]), these transformations seem to confirm cut off of any **BH-NS** transition.

The problem of writing the relation between main **BH** quantities before and after a transition (on the horizon curve) can be rephrased as the problem of finding replicas in the extended plane. More precisely, let us consider two points, r and r_p , along the horizon curve $a_{\pm} > 0$, which can also be in the entire range $r \in [0, 2M]$. We relate the parameters $\mathcal{P}_{\pm} = (\ell^{\pm}, \omega_{\pm}^{\pm})$, regulating the **BH** transition, as evaluated on the point r and r_p .

We search for a quantity κ such that $\mathcal{P}_{\pm}(r) = \kappa \mathcal{P}_{\pm}^*(r_p)$ (conformal property). Note that it can also be ℓ^{\pm} or $\pm \ell^{\pm}$, connecting, therefore, properties defined on an outer horizon with properties defined on an inner horizon; for this reason the notation $(*)$ represents a change in sign in one or more components of the couple. While the frequency ω is always positive (according to the sign of J), the surface gravity ℓ can be negative, when evaluated on a point of the inner horizon; therefore, it can be $\ell_{\pm}^{\pm}(r_p) = \ell_{\pm}^{\mp} < 0$.

We also investigate the case $\delta A_{area}^+ = -\delta A_{area}^-$ occurring for $\delta M = 0$ (as $\delta M^+ = \delta M^-$). In here we use notation (\pm) to indicate quantities evaluated on the horizons r_{\pm} . Moreover, $\delta Q \equiv Q(1) - Q(0)$ denotes the change of the quantity Q from the initial (0) to the final state (1) of the transition. The point $a = M$ and $r = r_p = M$ corresponds to the extreme Kerr **BH**, where $\omega(r_p) \neq \omega(r)$ for points (r, r_p) of the horizon curve. For the surface gravity the situation is different. There can be replicas on the curve of the horizons, which depend on a and r and connect inner and outer horizons; this is clearly illustrated in Fig. 2.

For convenience, we report here some relations for the frequencies:

$$\omega_{\pm}^{\pm} = \frac{\sqrt{2-r}}{2\sqrt{r}}, \quad \omega_{\pm}^{\mp} = \pm \frac{\sqrt{(r-1)^2 \pm 1}}{2\sqrt{(2-r)r}}. \tag{3}$$

In the first equation of (3), the frequency is defined as a function of a general point $r \in a_{\pm}$ of the inner horizon for $r \in [0, M]$ or the outer horizon for $r \in [M, 2M]$. In the second expression, we explicit the inner and outer horizon frequencies.

When calculated on the inner horizon curve in the extended plane, the surface gravity reads

$$\ell_{\pm}^{\pm} = \frac{a^2 \pm \sqrt{1-a^2} \mp 1}{2a^2} = \frac{1}{2 \left[1 \pm \frac{1}{\sqrt{(r-1)^2}} \right]}, \quad \text{and}$$

$$\ell_{\pm}^{\pm}(r) = \frac{r-1}{2r}. \tag{4}$$

In first expression of Eq. (4), we denote with ℓ_{\pm}^{\pm} the acceleration for the inner horizon ℓ_{\pm}^- and outer horizon ℓ_{\pm}^+ as functions of the spin a or as functions of the radius r on the horizon curve. The second function, instead, is the acceleration defined as function of a general point $r \in a_{\pm}$. We introduce the quantities $\ell_{\pm}^{\mp}(r) \equiv \ell_{\pm}^{\mp}(r)/\ell_{\pm}^{\pm}(r)$ and $\omega_{\pm}^{\mp}(r) \equiv \omega_{\pm}^{\mp}(r)/\omega_{\pm}^{\pm}(r)$. Using the relation $\ell_{\pm}^{\mp}(r) = -\omega_{\pm}^{\mp}(r)$, we obtain

$$\{\ell_{\pm}^{\mp}, \omega_{\pm}^{\mp}\} = \omega_{\pm}^{\mp}(r) \{-\ell_{\pm}^{\pm}, \omega_{\pm}^{\pm}\},$$

$$\{\ell_{\pm}^{\pm}, \omega_{\pm}^{\pm}\} = \omega_{\pm}^{\pm}(r) \{-\ell_{\pm}^{\mp}, \omega_{\pm}^{\mp}\},$$

where $\omega_{\pm}^{\mp}(r) = \frac{1}{\omega_{\pm}^{\pm}(r)}$. \tag{5}

see Fig. 3.

There is $\ell_{\pm}^{\mp} = \pm \omega_{\pm}^{\mp}$ for $a = a_{\gamma} \equiv 1/\sqrt{2}$.

An analogue relation holds for the quantities evaluated as functions of the spin a . Considering the dependence from the spin a we find,

$$\ell_{\pm}^{\mp}(a) = \mp \frac{\sqrt{1-a^2}}{2r_{\mp}}, \tag{6}$$

alternatively to Eq. (4). Defining $\ell_{\pm}^{\mp}(a) \equiv \ell_{\pm}^{\mp}(a)/\ell_{\pm}^{\pm}(a)$ and $\omega_{\pm}^{\mp}(a) \equiv \omega_{\pm}^{\mp}(a)/\omega_{\pm}^{\pm}(a)$, we obtain $\omega_{\pm}^{\mp}(a) = -\ell_{\pm}^{\mp}(a)$ and

$$\{\ell_{\pm}^{\mp}(a), \omega_{\pm}^{\mp}(a)\} = \omega_{\pm}^{\mp}(a) \{-\ell_{\pm}^{\pm}(a), \omega_{\pm}^{\pm}(a)\}, \tag{7}$$

$$\{\ell_{\pm}^{\pm}(a), \omega_{\pm}^{\pm}(a)\} = \omega_{\pm}^{\pm}(a) \{-\ell_{\pm}^{\mp}(a), \omega_{\pm}^{\mp}(a)\}, \tag{8}$$

see also Eq. (5) and Fig. 2.

Here, we search for the relation $\delta M(a_p)^* = s \delta M(a)^*$, where $a \in [0, M]$ and $a_p \in [0, M]$ are two **BH** geometries. For the particular case $a = a_p$, the couple (\star, \ast) denotes that \star or \ast are \pm , relating to the inner-inner horizons of the two geometries or outer-outer or inner-outer horizons of the two geometries (a, a_p) . The special case $a = a_p$ contains $(\star = \ast = \pm, s = 1)$. The case considered for the quantities as functions of r , where $\star = -\ast = \pm$ and s , satisfies conditions (5). The apparent contradiction in this case is that for a fixed spacetime $a \in \mathbf{BH}$, we have that $\delta M^+ = \delta M^-$, but the relation regulating this variation with the other characteristic quantities depends on the point as $\delta M^+ = s \delta M^-$ (where

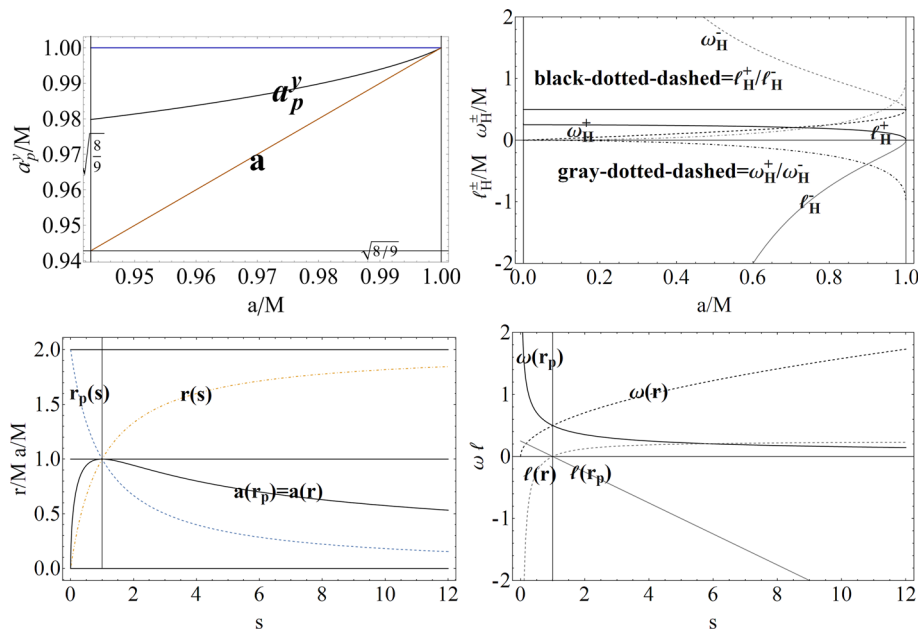


Fig. 2 Upper left panel: The BH spin $a_p = a_p^y$ defined in Eq. (10) as a function of the spin a , solution of $\ell(r_+(a_p)) = -\ell(r_-(a))$, where the acceleration $\ell(r_+)$ is the surface gravity. The outer and inner horizon are r_+ and r_- , respectively. Upper right panel: The frequencies ω_H^\pm of the outer and inner BH horizons. The quantities

ω_H^\pm , ℓ_H^\pm , and the ratios are plotted as functions of the BH dimensionless spin. Bottom right panel: The quantities $r(s)$, $r_p(s)$ and the spin $a(r) = a(r_p)$ from Eqs. (12) and (15) as functions of the parameter s . There $(\omega(r_p), \ell(r_p)) = s(\omega(r), -\ell(r))$. Bottom right panel: The frequency ω and the acceleration ℓ evaluated on r and r_p are plotted as functions of s – see Eq. (14)

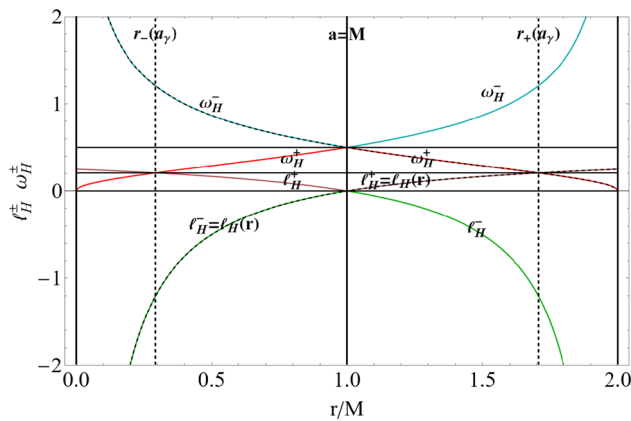


Fig. 3 The accelerations ℓ_H^\pm and the frequencies ω_H^\pm on the horizon curve in the extended plane, as functions of a point $r \in [0, 2M]$ on the horizon curve, defined in Eqs. (3) and (4). Dashed lines are the functions $\ell_H(r)$ and $\omega_H(r)$ defined along the horizon curve $a_\pm(r)$. The radius $r \in [0, M]$ corresponds to the inner horizons and the radius $r \in [M, 2M]$ to the outer horizon. Here $r_\pm = M$ is the horizon in the extreme Kerr BH. Note the symmetries of the functions ℓ_H^\pm in the entire range $r \in [0, 2M]$. For the spin $a_y \equiv \sqrt{1/2}$, we get $\omega_H^+ = \ell_H^+$ and $\omega_H^- = -\ell_H^-$

we have used the notation \pm to stress quantities in relations evaluated on the horizons r_\pm , respectively).

Then, we obtain

$$\begin{aligned} \delta M^+ &= \ell_H^+ \delta A_{area}^+ + \omega_H^+ \delta J^+ \\ &= \delta M = \ell_H^+ \delta A_{area}^+ + \omega_H^+ \delta J = \\ r \delta M^- &\equiv \ell_H^- \delta A_{area}^- + \omega_H^- \delta J^-. \end{aligned} \tag{9}$$

We note also the coincidence $\omega_H^\pm(r) = \ell_H^\pm(r) = Q_{\ell\omega}^+ \equiv 1/\sqrt{2} - 1/2$ in the form of Eqs. (3) and (4) for $r = r_+ = (\sqrt{2} + 2)/2$. This is the outer horizon for the BH geometry with spin $a/M = a_y = 1/\sqrt{2}$, specifically, $\ell_H^\pm = \pm\omega_H^\pm$. Therefore, for this spacetime the mass variation has a special form when written in the extended plane $\delta M^\pm = Q_{\ell\omega}^\pm \delta \tilde{M}^\pm$ (where $\delta \tilde{M}^\pm \equiv \pm \delta A_{area}^\pm + \delta J^\pm$), where $Q_{\ell\omega}^- = \omega_H^- = 1/2 + 1/\sqrt{2} = -\ell_H^-$.

We analyze the more general problem addressing firstly the special case $s = 1$ and secondly the general case with $s \neq 1$.

The case $s = 1$ This problem can be reduced to finding replicas for $\ell(r_+(a_p)) = -\ell(r_-(a))$, solved for

$$\begin{aligned} \frac{2\sqrt{2}}{3} < a \leq 1, \\ a_p = a_p^y \equiv \frac{\sqrt{4(1-a^2)^{3/2} + 12a^4 - 15a^2 + 4}}{4a^2 - 3} \end{aligned} \tag{10}$$

Figure 2. The cases $\ell(r_{\pm}(a_p)) = -\ell(r_{\pm}(a))$ are solved for $a = a_p = M$, while $\ell_H^{\pm}(a) = \ell_H^{\pm}(a_p)$ for $a = a_p$. The case $a/M = \sqrt{8/9}$ implies $(\ell_H^- = -1/4, \ell_H^+ = 1/8)$ and $(\omega_H^- = 1/\sqrt{2}, \omega_H^+ = 1/2\sqrt{2})$. Therefore $(\ell_H^-, \omega_H^-) = 2(-\ell_H^+, \omega_H^+)$.

The case $s \neq 1$. As seen for the **BH** spin $a/M = \sqrt{8/9}$, now we extend the problem of finding replicas by searching for orbits that relate the surface gravity and frequency as follows

$$\omega_H(r_p) = s\omega_H(r) \quad \text{and} \quad \ell_H(r_p) = -s\ell_H(r) \quad (11)$$

for $s \geq 0$ (the situation for $s < 0$, implying a change of the **BH** spin rotation, provides a result analogue to the case $s > 0$) that is a conformal transformation from an outer horizon r_+ to an inner horizon $r_- = r$. We obtain:

$$r = \frac{2s}{s+1} \quad \text{and} \quad r_p \equiv \frac{2}{s+1}, \quad (12)$$

therefore, $r/r_p = s$, and the conformal relation can be also expressed as

$$\omega_H(r_p) = (r/r_p)\omega_H(r) \quad \text{and} \quad -\ell_H(r_p) = (r/r_p)\ell_H(r). \quad (13)$$

The frequencies and accelerations are

$$\begin{aligned} \omega(r) &= \frac{1}{2\sqrt{s}}, & \omega(r_p) &= \frac{\sqrt{s}}{2}, \\ \ell(r) &= \frac{s-1}{4s}, & \ell(r_p) &= \frac{1-s}{4} \end{aligned} \quad (14)$$

then

$$\begin{aligned} (\omega(r_p), \ell(r_p)) &= s(\omega(r), -\ell(r)), \quad \text{where} \quad \frac{r}{r_p} = s \\ \text{with} \quad a(r) = a(r_p) &= 2\sqrt{\frac{s}{(s+1)^2}} = \sqrt{rr_p} = \sqrt{s}r_p \end{aligned} \quad (15)$$

The radii r and r_p correspond to the horizons of the **BH** spacetime with $a(r) = a(r_p)$. It is clear that (r, r_p) represent a horizon parametrization and $a(r_p)$ can be seen as analogue to the tangent curve of the horizon $a_g(\omega)$, function of the frequency ω . It is clear from Eqs. (12) and (14) that there are two ranges, $s < 1$ and $s > 1$, describing the same spacetime (horizontal line in the plot of $a(r_p) = a(r)$ as function of s). For $s < 1$, we have that $r = r_-$ and $r_p = r_+$ while for $s > 1$, we obtain $r = r_+$ and $r_p = r_-$. The limiting case $s = 1$ corresponds to the extreme Kerr **BH**.

Therefore, Eq. (11) hold for two points (r, r_p) only, the horizons of a **BH** spacetime with $a = a(r_p)$. There are

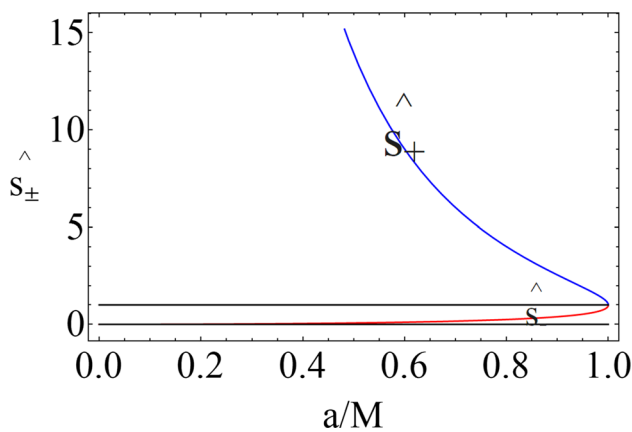


Fig. 4 The quantities \hat{s}_{\mp} as functions of the **BH** dimensionless spin a/M , solutions of Eq. (11): $\omega_H(r_p) = s\omega_H(r)$ and $\ell_H(r_p) = -s\ell_H(r)$, where ℓ_H and ω_H are the acceleration ℓ and the frequency ω , respectively, on the **BH** horizons r_{\pm} . The surface gravity is ℓ evaluated on r_+

two values of $s = \bar{s} > 1$ and $s = 1/\bar{s} < 1$ for the same spacetime. The remarkable aspect of this relation is that the conformal factor is different according with the spin $\hat{s}_{\mp} \equiv [(2 - a^2) \mp 2\sqrt{(1 - a^2)}]/a^2$, see Fig. 4.

In Sect. 5.1, we will study these relations in detail.

4 Exploring the laws of BH thermodynamics

Israel’s theorem establishes that the event horizon must be spherically symmetric, implying that the sphericity of an isolated and static **BH** cannot be broken. Information on the **BH** past, in the sense of **BH** “hair”, is to be considered as eliminated or inaccessible to observers. (Counterexamples might be considered in some multi-dimensional spacetimes with non-abelian fields).

In this section, we reformulate several aspects of **BH** thermodynamics in the extended plane and on the bundles.

To explore the laws of **BH** thermodynamics in the extended plane, we consider the Smarr’s formula connecting (M, J, A_{area}) , where A_{area} is the horizon area.

In this section, we expand on the analysis of Sect. 3.1, where we considered masses and **BH** thermodynamics in the extended plane. In Sect. 4.1, we introduce the **BH** rotational energy and “rest” mass in the extended plane, which is studied using the light surfaces in Sect. 4.2, where we investigate also horizon areas in the extended plane and **BH** thermodynamics. Some of the concepts discussed here are the subject of Sect. 5.2. Moreover, we analyze the constant irreducible mass and the quantities evaluated on the inner horizons in Sect. 5.3.

4.1 Rotational energy and BH “rest” mass

The horizon surface area (on the event horizon r_+) is $A_{area} = 16\pi M_{irr}^2$ and determines the BH irreducible (or rest) mass $M_{irr} = \sqrt{a^2 + r_+^2}/2$.

The total BH mass M of the Kerr BH can be decomposed into the mass M_{irr} and the rotational energy.

From the first law of BH thermodynamics, we obtain $M^2 = M_{irr}^2 + J^2/4M_{irr}^2$, where J is the BH angular momentum in units of mass M , as measured in the asymptotical flat region.

The maximum rotational energy which can be extracted from the black hole is $\xi \propto (M - M_{irr})$. A result of Christodolou and Ruffini sets an upper limit on the amount of energy that can be extracted from a Kerr BH as the total rotational energy, assuming a Schwarzschild BH as the final state, and the the bottom limit⁵ of M at the end of stationary process of the energy extraction, has to be M_{irr} .

Within these assumptions, the maximum rotational energy which can be extracted is the limit of $\xi = \xi_\ell \equiv (2 - \sqrt{2})/2$, hence $\xi \in [0, \xi_\ell]$, where at the state (0) (prior to the extraction) there is an extreme Kerr spacetime (with spin $a = M$). Considering (0) as the state prior the extraction, from the law $M_{irr}^2 = r_+(J, M)/2$, we obtain

$$M(0)^2 - M_{irr}(0)^2 = \left(\frac{J(0)M(0)}{M(0)(2M_{irr}(0))} \right)^2, \tag{16}$$

from the variation we find

$$\frac{\delta M_{irr}}{M_{irr}} = \frac{\delta M - \delta J \omega_H(0)}{\sqrt{M(0)^2 - \frac{J(0)^2}{M(0)^2}}},$$

where $\delta M_{irr} \geq 0$, thus $(\delta M - \delta J \omega_H(0)) \geq 0$ and $\omega_H \equiv \omega_H^+$ is the frequency of the outer Killing horizon for the initial BH, imposing limits on the BH spin-shift⁶. We obtain $M^2 = J(0)^2/(4M_{irr}^2) + M_{irr}^2$, and the (extracted rotational) energy is essentially

⁵ The upper limit for the energy extraction from a stationary process bringing the BH to the state (1) is $M(0) - M_{irr}(M(0), J(0))$, where $M(0)$ and $J(0)$ are the mass and angular momentum of the initial state and the BH angular momentum after the process, the state (1), is zero [23].

⁶ Measuring ξ , therefore, will provide and indication of the BH spin. This method, independent of the specific model of energy extraction, was introduced first in [24], and applied in other analysis in [23–26, 28], relating the dimensionless BH spin a/M to the dimensionless ratio ξ , where ξ represents the total released rotational energy versus BH mass (measured by an observer at infinity), assuming a process ending with the total extraction of the rotational energy of the central Kerr BH.

$$\xi = M(0) - \sqrt{\frac{r_+(0)}{2}}, \quad \text{or}$$

$$\frac{\xi}{M(0)} = 1 - \frac{\sqrt{1 + \sqrt{1 - \frac{J(0)^2}{M(0)^4}}}}{\sqrt{2}}.$$

Note that here ξ has units of mass, the dimensionless spin $a_g/M \equiv J(0)/M(0)^2$ refers to an initial state before the transition, defined by the mass $M(0)$ and spin $J(0)$, which is a function of the extracted rotational energy $\xi/M(0)$, that is, $1 - M_{irr}(0)/M(0)$, and, therefore, of the ratio $M_{irr}(0)/M(0)$.

On the other hand, the radius r/M on the extended plane can be expressed in terms of the total mass $M(0)$. We can write the dimensionless spin as $a_g = a_\xi$ as

$$a_\xi(\xi) \equiv 2\sqrt{(2 - \xi)(\xi - 1)^2\xi}. \tag{17}$$

The function $a_\xi(\xi)$ links the former state spin a_0 to the rotational extraction in the subsequent phase, where the BH is settled in a Schwarzschild spacetime. The function a_ξ is, therefore, an expression for the tangent spin in the extended plane as function of the rotational energy parameter ξ . Here and in the following, we shall use dimensionless quantities.

Equivalently, we can express the rotational energy parameter as

$$\xi_\pm^\mp = 1 \pm \sqrt{\frac{r_\pm}{2}}. \tag{18}$$

Considering $a_\xi(\xi) \equiv a_s^{(\pm)}$, solving for ξ there is

$$\xi_s^{(\pm)} = 1 \mathfrak{b} \frac{\sqrt{1 \mathfrak{q} \sqrt{1 - (a_s^{(\pm)})^2}}}{\sqrt{2}}, \quad \text{where } \mathfrak{b} = \pm; \quad \mathfrak{q} = \pm, \tag{19}$$

relating directly the energy parameter ξ to the horizons. As $\xi \propto M - M_{irr}$ (here ξ has units of mass M), only the solution ξ_\pm^+ has to be considered. The general solution $a_\xi(\xi) = a_s^{(\pm)}$, for a couple of spins $a_s^{(\pm)}$, provides the eight functions $\xi_s^{(\pm)}$. For the point $\xi = 1$, it is $a_\xi(\xi) = 0$ with $\xi = 0$. There is a maximum that depends on the spin, $\xi_\ell \equiv (2 - \sqrt{2})/2$ (and $\xi_m \equiv (2 + \sqrt{2})/2$), where $a_\xi(\xi) = M$ and $r_+ = M$ (the extreme Kerr BH). In general, it is $a_\xi(\xi) \in [0, M]$ (dimensionless) and the definition of the rotational energy as $1 - M_{irr}/M$, with restricted range $\xi \in [0, \xi_\ell]$, where $\xi_\ell \equiv (2 - \sqrt{2})/2$, limiting, therefore, the energy extracted to a maximum of $\approx 29\%$ of the mass M . We can express the extracted energy in terms of characteristic frequency of the bundle and through the tangency condition of the bundle in the extended plane. We introduce the curves $\xi_{\tau^+}^\mp$ and $\xi_{\tau^-}^\mp$ in the extension of the plane for extended values of ξ , considering the BH horizon frequencies ω_H^\pm , the horizon curves in the plane $\xi - r/M$, and the functions ξ_μ^\mp, ξ_ν^\mp from the other two solutions of $a = a_\xi(\xi)$:

$$\begin{aligned} \xi_\mu^\mp &= 1 \mp \sqrt{1 - \frac{r}{2}}, \quad \xi_\nu^\mp \equiv 1 \mp \sqrt{\frac{r}{2}} \quad \text{and} \\ \xi_\tau^\mp &\equiv 1 \mp \frac{1}{\sqrt{4\omega^2 + 1}} = 1 \mp \sqrt{\frac{r_g}{2}} = 1 \mp \sqrt{\frac{a_g}{4\omega}}, \\ \xi_{\tau\tau}^\mp &\equiv 1 \mp \frac{2\omega}{\sqrt{4\omega^2 + 1}} = 1 \mp 2\omega\sqrt{\frac{r_g}{2}} = 1 \mp \sqrt{2\omega a_g}, \end{aligned} \tag{20}$$

where r is a point of the horizon curve in the extended plane. The maximum rotational energy extractable is therefore expressed as function of the light-surface frequencies (or equivalently the \mathcal{MB} origin \mathcal{A}_0). For the extreme case $a = M$, where the frequency is $\omega = 1/2$, we obtain $\xi = 1 \pm 1/\sqrt{2}$.

The horizon frequencies, which are also the characteristic frequencies of the bundles, in terms of the dimensionless parameter ξ are

$$\omega_g^\pm \equiv \frac{1 \pm \sqrt{4(\xi - 2)\xi(\xi - 1)^2 + 1}}{4\sqrt{-(\xi - 2)(\xi - 1)^2\xi}}; \tag{21}$$

ω_g^- has a saddle point for energy $\xi = (1 \pm \sqrt{2/3})$, corresponding to the spin $a/M = 2\sqrt{2}/3$ and frequency $\omega = (1/2\sqrt{2}, 1/\sqrt{2})$. Frequency $\omega_H^+ = \pm 1/2\sqrt{2}$ for the outer horizon $r_g/M = 4/3$ is also the saddle point of ξ_τ^\pm as function of ω , the maximum extractable rotational energy decreases/increases faster with the horizon (\mathcal{MB}) frequencies constrained by the limiting values $\omega_H^+ = \pm 1/2\sqrt{2}$.

4.2 BH irreducible mass

We express the irreducible mass of a **BH** in terms of the horizon curve in the extended plane. Thus, using also the results given in Sect. 3.1, we rewrite the mass function in terms of the inner and outer horizon radii in the extended plane. We then study the areas and the relation (9) in this representation, wiring first law of **BH** thermodynamics in this new frame.

We can express the **BH** irreducible mass M_{irr} , a quantity defined on the outer **BH** horizon, in terms of its area $M_{irr} = \sqrt{a^2 + r_+^2}/2 = \sqrt{A_{area}/16\pi}$ as

⁷ Notable spins are for $a = 1/\sqrt{2}$: $r_\gamma^- = r_\epsilon^+$, and $a/M = 2(\sqrt{2} - 1)$: $r_{mbo}^- = r_\epsilon^+$, $a/M = \sqrt{8/9}$ $r_{mso}^- = r_\epsilon^+$, and $a/M = 2\sqrt{2}$ for naked singularities, where $r_{mso}^- = r_\epsilon^+$, and $(r_\gamma^-, r_{mbo}^-, r_{mso}^-)$ are the corotating last circular orbit (photon orbit), marginally bounded orbit, and marginally stable orbit, respectively. For $a/M \geq \sqrt{8/9}$ co-rotating accreting disks close to the outer ergosurface, centred on the **BH** equatorial plane must have their inner part (distance-torus center-inner edge) in the ergoregion [29]. This implies the peculiarities highlighted here in the extraction process should be evidenced for these discs whose inner part is subjected to the frame dragging of the ergoregion.

$$\begin{aligned} M_{irr}(all) &= \sqrt{\frac{\mathcal{A}_0^2}{\mathcal{A}_0^2 + 4}} = \sqrt{\frac{r_g}{2}} = \sqrt{\frac{1}{4\omega^2 + 1}}, \\ M_{irr}^\mp &= \sqrt{\frac{1}{2} \mp \frac{1}{2} \sqrt{\frac{(\mathcal{A}_0^2 - 4)^2}{(\mathcal{A}_0^2 + 4)^2}}} \\ &= M_{irr}(r_\mp) = \sqrt{\frac{r_\mp(a_g)}{2}}. \end{aligned} \tag{22}$$

in terms of the radius r_g , the frequency ω , the bundle origin $\mathcal{A}_0 = a_0\sqrt{\sigma}$ or the horizon curve a_g . (Note the dependence on θ through the bundle origin, does not contradict the rigidity of the black hole horizon, being a representation of the light-surfaces frequencies on different planes σ s). Therefore, the mass function is defined also in the inner horizon. The function $M_{irr}(all)$ denotes the irreducible mass evaluated on the horizon curve in the extended plane, as function of the bundle origin \mathcal{A}_0 , the tangent radius r_g or the frequencies ω . The notation \pm refers to the radii r_\pm .

The irreducible mass is, therefore, determined by the tangent radius in the extended plane. Note that

$$\begin{aligned} \frac{M_{irr}^+}{M_{irr}^-} &= \sqrt{\frac{r_+}{r_-}} = \sqrt{\frac{\omega_H^-}{\omega_H^+}} \\ &= \sqrt{\frac{-\ell_H^-}{\ell_H^+}} = \frac{\sqrt{r_+}}{\sqrt{2 - r_+}} = \sqrt{\frac{1}{4(\omega_H^+)^2}} \end{aligned} \tag{23}$$

relating the irreducible masses, the frequencies and the surfaces gravities. Considering the expression for the irreducible mass in terms of the radius r , we obtain the expressions

$$\tilde{\mathcal{M}}_{irr}^\pm = \frac{\sqrt{r_\pm}}{\sqrt{2}}, \quad \tilde{\mathcal{M}}_{irr}^\pm = \sqrt{\frac{1}{2} \mp \frac{1}{2} \sqrt{(r - 1)^2}}, \tag{24}$$

which are evaluated for $a = a_\pm$, respectively, see Fig. 5.

We use the concept of maximum extractable energy ξ introduced in Eq. (26), where $M_{irr} = M - \xi$, (ξ has units of mass M) and $(M_{irr}/M)_{\min} = 1 - \xi_{\max} = 1 - \xi_\ell = 1/\sqrt{2}$ for the extreme spacetime. We obtain

$$\begin{aligned} \frac{M_{irr}}{M} &= 1 - \xi_\pm^\mp = \mp \sqrt{\frac{r_\mp}{2}}; \quad \frac{M_{irr(v)}}{M} \equiv 1 - \xi_\nu^\mp = \pm \sqrt{\frac{r}{2}} \quad \text{and} \\ \frac{M_{irr}}{M} &= 1 - \xi_\tau^\mp = \pm \frac{1}{\sqrt{4\omega^2 + 1}} = \pm \sqrt{\frac{r_g}{2}} = \pm \sqrt{\frac{a_g}{4\omega}}. \end{aligned} \tag{25}$$

On the other hand, from the functions ξ_μ^\mp and $\xi_{\tau\tau}^\mp$ of Eq. (26), which are extensions of the rotational energy definition in the extended plane, we obtain

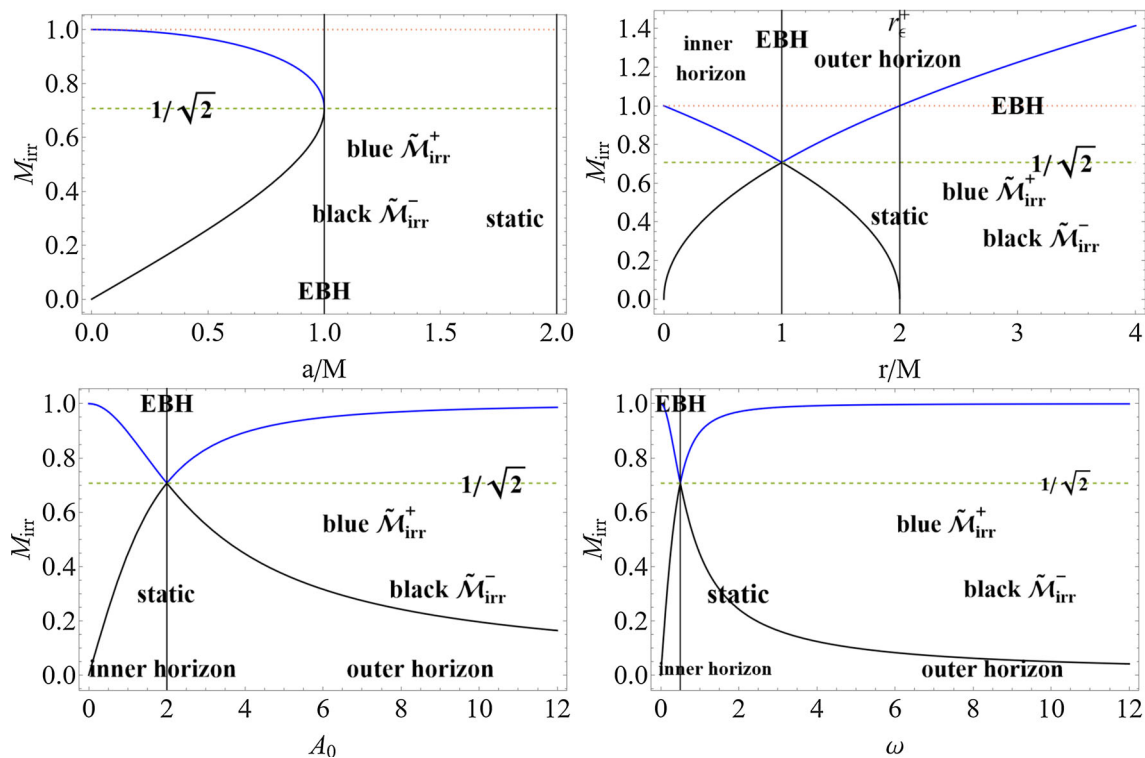


Fig. 5 Irreducible mass M_{irr} as function of the spin a/M , the radius on the extended plane r/M , the origin spin of the bundle $\mathcal{A}_0 = a_0\sqrt{\sigma}$, where $\sigma \equiv \sin^2\theta$, and the bundle characteristic frequency ω . Here,

EBH denotes the extreme **BH** spacetime. The regions of the inner and outer horizons in the extended plane are denoted, respectively, on the left and right of the **EBH** limit

$$\begin{aligned} \frac{M_{irr}(\mu)}{M} &= 1 - \xi_{\mu}^{\mp} \equiv \pm\sqrt{1 - \frac{r}{2}}, \quad \text{and} \\ \frac{M_{irr}}{M} &= 1 - \xi_{\tau\tau}^{\mp} = \pm\frac{2\omega}{\sqrt{4\omega^2 + 1}} = \pm 2\omega\sqrt{\frac{r_g}{2}} = \pm\sqrt{2\omega a_g}. \end{aligned} \tag{26}$$

Note that for $r = 0$, according to Eq. (26), we find $M_{irr}/M = \{1, 0\}$, with $M_{irr}/M = 1/\sqrt{2}$ for $r = M$, and $M_{irr}/M = 0$ for $r = 2M$. In Sect. 5, we analyze further aspects of the irreducible mass.

To enlighten the properties of the mass M_{irr} in the extended plane, we can consider some limiting values according to the \mathcal{MB} s structures. From Eq. (25), we obtain $M_{irr}/M = 1/\sqrt{2}$ for the extreme Kerr spacetime. On the ergosurface and on the Schwarzschild horizon, we find $M_{irr}/M = 1$. For $\omega = +\infty$ (equivalently $r = 0$), it is $M_{irr}/M = 0$. From Eq. (22), we find

$$\begin{aligned} \lim_{\mathcal{A}_0 \rightarrow \mathbf{X}} M_{irr}(all) &= \mathbf{Y} \quad \text{and} \quad \mathbf{Y} = \left\{ 1, 0, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{5}} \right\}, \\ \lim_{\mathcal{A}_0 \rightarrow \mathbf{X}} M_{irr}(r_+) &= \mathbf{Y} \quad \text{and} \quad \mathbf{Y} = \left\{ 1, 1, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{5}} \right\}, \\ \text{where respectively } \mathbf{X} &= \{+\infty, 0, 1, 2, 4\}. \end{aligned} \tag{27}$$

Limits $M_{irr} = 2/\sqrt{5}$ or $M_{irr} = 1/\sqrt{5}$ refer to the definitions for inner and outer horizons for the tangent spin $a_g/M = 4/5$.

To clarify the relation between bundle **NS** origin and the **BH** irreducible mass through the tangency properties of the bundles, we invert some of the previous relations as functions of M_{irr} , obtaining the bundle origin and the characteristic frequency as the expressions

$$\mathcal{A}_0(all) = \frac{2M_{irr}}{\sqrt{1 - M_{irr}^2}}, \tag{28}$$

$$\text{therefore } \omega = \frac{1}{\mathcal{A}_0(all)} = 4\mathcal{A}_0(r_+), \tag{29}$$

which are illustrated in Fig. 6. These expressions relate the origin spin \mathcal{A}_0 to the irreducible mass M_{irr} .

Below we discuss this limiting value in relation of the **BH** transitions, revisiting the considerations of Sect. 4.1.

From the relation $M_{irr}^2 = r_+/2$, it follows that

$$\frac{M_{irr}^+}{M} = +\frac{\sqrt{1 + \sqrt{1 - \frac{J(0)^2}{M(0)^4}}}}{\sqrt{2}}, \tag{30}$$

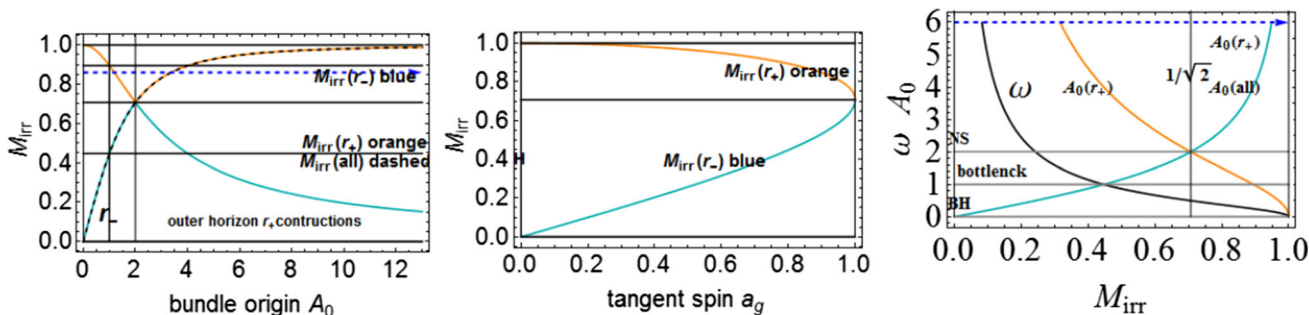


Fig. 6 The irreducible mass $M_{irr}(a, r)$ in terms of $\{r_+, r_-, r_g, a_g\}$, according to the analysis of Eq. (27). r_{\pm} are the **BH** horizons as functions of the **BH** spin, (r_g, a_g) are the tangent radius and tangent spin of the metric bundles, i.e., Killing horizons as functions of the origin

$\mathcal{A}_0 = a_0\sqrt{\sigma}$ (left panel) or tangent spin a_g (center panel). Right panel: Analysis of Eq. (28). The bundle origin spins \mathcal{A}_0 and frequency ω as functions of the irreducible mass. The **BH**, **NS** and bottleneck regions as well as the special value $1/\sqrt{2}$ are highlighted

and using Eq. (19)

$$\frac{M_{irr}^b}{M} = b \frac{\sqrt{1 \mp \sqrt{1 - (a_s^{(\pm)})^2}}}{\sqrt{2}}, \quad \text{where } b = \mp; \mp = \pm, \tag{31}$$

where we use dimensionless units and $(\delta M - \delta J(0)\omega_H^+) \geq 0$. In particular, we find $M_{irr}/M = \{0, 1\}$ for $a = 0$ and $M_{irr}/M = \pm 1/\sqrt{2}$ for $a = M$ –see discussion in Sect. 5.

4.2.1 BH horizon areas in the extended plane

The **BH** inertial mass can be found from the **BH** area:

$$A_{area} = \int_0^\pi \int_0^{2\pi} \sqrt{g_{\phi\phi}g_{\theta\theta}} d\theta d\phi = 4\pi (a^2 + r^2). \tag{32}$$

Here we use the extended definition $A_{area}(a_{\pm}) = 8\pi r$. Note that this expression depends explicitly on $r = r_g$. Considering also the relations analyzed in Sect. 3.1, we obtain

$$\delta M^\mp = \frac{\ell^\mp}{8\pi} \delta A_{area}^\mp + \omega_H^\mp \delta J^\mp, \quad da_{\pm} = \mp \frac{\ell(a_{\pm})}{\omega_{\pm}(a_{\pm})} dr, \tag{33}$$

(or $a_{\pm} \delta a_{\pm} = (M - r) \delta r$).

The last relation can be written also as $\delta a = [\delta M r + \delta r (M - r)] / \sqrt{r(2M - r)}$. In Eq. (33), we considered the variation of the horizon area (and the irreducible masses) for the ADM mass M , the area A_{area} and the momentum J . The horizons frequencies ω_H^{\pm} are the characteristic bundle frequencies. We consider other regions of the extended plane. In Sect. 5, there are also further considerations on the transitions with $\delta M_{irr} = 0$ and $\delta M_{irr} > 0$.

From the expressions for $A_{area}(\mathcal{A}_0)$ and $A_{area}(\omega)$, we obtain the saddle points

$$\text{for } A_{area}(\mathcal{A}_0) = 8\pi r_g(\mathcal{A}_0), \quad \text{it is} \\ \mathcal{A}_0 = \frac{2}{\sqrt{3}} \quad \text{where } A_{area}(\mathcal{A}_0) = 4\pi, \tag{34}$$

$$\text{for } A_{area}(\omega) = 8\pi r_g(\omega) \quad \text{it is} \\ \omega = \frac{1}{2\sqrt{3}} \quad \text{where } A_{area} = 12\pi \tag{35}$$

Frequency $\omega = \omega_H^+ = 1/2\sqrt{3}$ is the outer horizon frequency for the **BH** with spin $a/M = \sqrt{3}/2$, where $\omega_H^- = \sqrt{3}/2 = 1/\mathcal{A}_0 = 2/\sqrt{3}$. For $a/M = \sqrt{3}/2$ there is $r_+/M = 3/2$, $r_-/M = 1/2$, in this geometry there is $\partial_r \ell = 0$ on r_+ and $\partial_a \ell = 0$ on r_- where $\ell = (r^2 - a^2)/(a^2 + r^2)^2$ is the acceleration, defined for $r > 0$ equal the the **BHs** surfaces gravities when evaluated on the horizons.

The saddle point $\mathcal{A}_0 = 2/\sqrt{3}$ (**NS**) for the area is also the saddle point for the angular momentum, corresponding to the frequency $\omega = 1/(L_f)$ of the inner horizon $r_- = 1/2$ for the spacetime $a = \omega$, whose frequency of the outer horizon is a saddle point for the frequency function). It is easy to see that

$$A_{area}^{\pm}(r_{\pm}) = \pm 8\pi r_{\pm} \quad \text{and} \quad \delta A_{area}^- = -\delta A_{area}^+ \\ \text{i.e. } \partial_a A_{area}^+ \delta a = -\partial_a A_{area}^- \delta a \tag{36}$$

which follows from relation $\delta r_+ = -\delta r_-$, where $r_+ r_- = a^2$ and $r_+ + r_- = 2M$ and, therefore, it holds if $\delta M = 0$.

Analogously, we obtain

$$A_{area}(a_{\pm}) = 8\pi r, \quad \text{and} \quad \ell(r_b, a_{\pm}) = \frac{1}{2 + b \frac{2}{\sqrt{(r-1)^2}}}, \\ (b = \pm), \quad \omega_H^\mp(a_{\pm}) = \frac{\sqrt{r(2-r)}}{2 \mp 2\sqrt{(r-1)^2}}. \tag{37}$$

The total frequencies in terms of the radius r/M is

$$\omega = \frac{\sqrt{2-r}}{2\sqrt{r}}, \quad \ell(r, a_{\pm}) = \frac{r-1}{2r}, \quad \text{and there is} \quad (38)$$

$$\ell = \omega_{\text{EBH}}^2 - \omega^2,$$

or, equivalently, $\omega = \frac{1}{2}\sqrt{1-4\ell}$, that is, ω_{EBH} is the frequency of the extreme **BH** – see Eq. (B.3).

4.2.2 Notes on **BH** thermodynamics

We can now analyze **BH** thermodynamics in the extended plane, considering the mass variations and the interpretation of Eq. (33) for the inner horizon curve (–). However, in the extended plane, quantities relevant to **BH** transitions may be different, when referring to a point r on the horizon curve for a **BH** state. (For simplicity, in some of the expressions we do not consider the factor 8π . Moreover, we will consider in some expressions spin, radius, characteristic frequency and origin spin as dimensionless parameters). In these relations, we express the area (or the irreducible mass) in terms of the radius in the extended plane, considering dimensionless parameters.

Then,

$$\delta M = \frac{\delta A_{\text{area}}(r-1)}{2r} + \frac{\delta J\sqrt{2-r}}{2\sqrt{r}}, \quad \text{and}$$

$$\delta M = \delta A_{\text{area}} \left(\frac{1}{4} - \omega^2 \right) + \delta J\omega,$$

$$\delta M = \left(\frac{1}{4} - \frac{1}{\mathcal{A}_0^2} \right) \delta A_{\text{area}} + \frac{\delta J}{\mathcal{A}_0}. \quad (39)$$

This the the mass variation is constrained by the $\mathcal{M}B$ tangent point ($r = r_g$) or characteristic null frequency ω (origin \mathcal{A}_0) regulating therefore the **BHs** transitions. Where $r = r_g \in [M, 2M]$, when evaluated on the outer curve. In the extended plane the relations are satisfied also for the inner horizons terms and we use again this property in Sect. 6 where we will study the doubled metric for the inner and outer horizon by writing two different metrics.

In Sect. 3.1, we discussed the relation between quantities defined on inner and outer horizon points in the extended plane. Here, we consider this relation in terms of surface gravity and frequencies. From Eq. (23), it follows that $(\omega_+^-)^{-1} = \omega_-^+ = \omega_H^+/\omega_H^- = -\ell_H^+/\ell_H^- = s(a)$. Also, $A_{\text{area}}^- = -A_{\text{area}}^+ + 4M^2$; thus, we can write $\delta A_{\text{area}}^- = -\delta A_{\text{area}}^+ + 8M\delta M$, and $\delta A_{\text{area}}^+ = -\delta A_{\text{area}}^-$ for $\delta M = 0$, where i.f $\delta J = 0$ then $\delta M = \delta A_{\text{area}}^+ \ell_H^+ / 2M$

A more precise analysis is provided in Sect. 5.3. Here we consider the condition $\delta A_{\text{area}}^+ = -\delta A_{\text{area}}^-$ (i.e. $\delta M = 0$, invariant masses). Then,

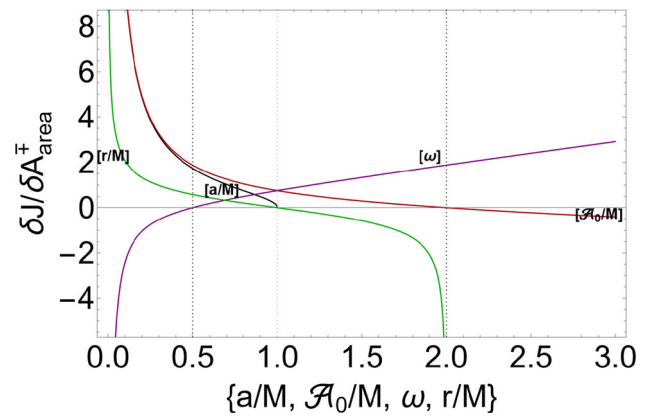


Fig. 7 Variation $\delta J/\delta A_{\text{area}}^{\mp}$ of Eq. (41), as functions of the tangent spin $a = a_g \in [0, M]$, the origin spin $\mathcal{A}_0 \in [0, +\infty]$, the characteristic **BH** horizons frequencies ω , the tangent radius $r = r_g \in [0, 2M]$. The **BH** area is $A_{\text{area}} (A_{\text{area}}^{\pm}$ for $r = r_{\pm}$ **BH** horizons), J is the **BH** spin

$$0 = \delta M = \frac{1}{2}\delta A_{\text{area}}^+(\ell^+ - \ell^-) + \frac{1}{2}\delta J(\omega_H^- + \omega_H^+),$$

$$\text{and } \frac{\delta J}{\delta A_{\text{area}}^+} = \frac{\ell^- + \ell^+}{\omega_H^- - \omega_H^+} \quad (40)$$

From the definition of horizon in the extended plane, we find $\delta M = (\omega\delta J + \ell\delta r)/(3M - r)$, with the distinguished value $r = 3M$, which is related to the Schwarzschild case [13]. Considering $\delta M = 0$, we obtain

$$\frac{\delta J}{\delta A_{\text{area}}^{\mp}} = (\pm) \frac{\sqrt{1-a^2}}{a} = \frac{1}{\mathcal{A}_0} - \frac{\mathcal{A}_0}{4} = \omega - \frac{1}{4\omega}$$

$$= \frac{1-r}{\sqrt{r(2-r)}} \quad (41)$$

see Fig. 7.

In this way, we can consider the variation of masses, areas and angular momentum of the **BHs** in terms of the quantities defined on the horizon curves, for instance, the first law of **BH** thermodynamics in terms of the tangent point r on the horizons, the origin \mathcal{A}_0 and the frequency ω , describing the outer horizon $r \in [M, 2M]$, $\mathcal{A}_0 \in [2M, +\infty]$, $\omega \in [0, 1/2]$ or the inner horizon for $r \in [0, M]$, $\mathcal{A} \in [0, 2M]$, $\omega \in [1/2, \infty]$, respectively. Such relations express the variations in terms of the angular momentum of the horizons, corresponding in some cases to naked singularities. We can write Eq. (39) in terms of quantities relative to the extreme (*ext*) Kerr **BH**, considered as reference with $r_{\text{ext}} = M$, $\omega = 1/2$ and $\mathcal{A}_0 = 2M$, as

$$\delta M = \frac{1}{2r}\delta\tilde{M} \quad \text{where } \delta\tilde{M} \equiv \delta A_{\text{area}}(r - r_{\text{ext}}) + \delta J a_{\pm}(r),$$

$$\delta M = \frac{1}{(L_f)_{\text{extr}}^2}\delta\tilde{M} = \left[\frac{1}{(L_f)_{\text{extr}}^2} - \frac{1}{(L_f)^2} \right] \delta A_{\text{area}} + \frac{\delta J}{(L_f)}, \quad (42)$$

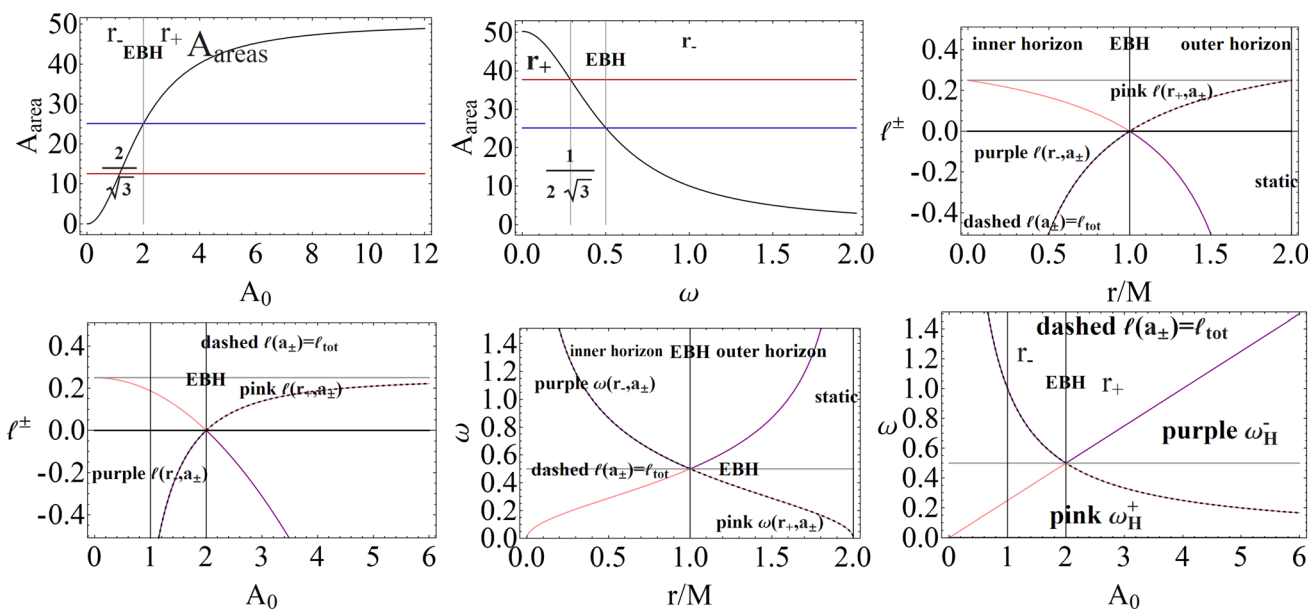


Fig. 8 The **BH** area A_{area} (A_{area}^+ for $r = r_+$) as function of the bundle origin spin $\mathcal{A}_0 \equiv a_0\sqrt{\sigma}$, where $\sigma \equiv \sin^2 \theta$, and ω is the bundle characteristic frequency and the horizon frequency. **EBH** denotes the extreme Kerr **BH**. The inner r_- and outer horizons r_+ in the extended plane are also shown. The points $\mathcal{A}_0 = 2/\sqrt{3}$ and $\omega = 1/(2\sqrt{3})$ are saddle points of the curves corresponding to inner and outer horizons, respectively. The accelerations ℓ^\pm are evaluated on the horizon curves r_\pm , where ℓ^+ is the **BH** surface gravity. They are represented as functions of r/M

on the extended plane and as functions of the bundle origin. The functions $\ell(a_\pm)$ represent the accelerations on the horizon curve a_\pm in the extended plane. Center and right panel of bottom line: The characteristic bundle frequency and horizon frequency ω in the extended plane as function of $r/M \in [0, 2]$. Here, $r_e^+ \equiv 2M$ represents the static limit for $\sigma = 1$ and the static Schwarzschild spacetime. ω_H^\pm are the horizon frequencies

where $(L_f) = 1/\omega$, **BH** angular momentum, coincident with the bundle origin. The case of extreme Kerr **BHs** in this sense is a limit where $\delta M = \delta J/2$.

The Figs. 5 and 8 outline the four zones identified through the inertial mass as functions of the spin, \mathcal{A}_0 , of the characteristic frequency of the bundles.

5 On BH transitions

5.1 Relating inner and outer horizons in BH transitions

Considering the discussion of Sect. 3.1, we use the expression $\ell_H^\pm(\omega)$ in the extended plane and solve the equation $\ell_H^\pm(\omega) = s\omega$ for a given frequency ω . Analyzing this special **BH**, transition we look for a solution ℓ in a fixed spacetime and obtain $s = (1 - 4\omega^2)/4\omega$, which is represented in Fig. 9 for any ω . The condition $s < 0$ implies that ℓ is evaluated on the inner horizon (we assume $\omega > 0$) and, therefore, $\omega^2 > 1/4$, bounded by the limiting value of the extreme Kerr spacetime. Then, $\ell = 1/4 - \omega^2$. However, we are interested in the cases $s = \text{constant}$: $(\ell, \omega) = \omega [(1 - 4\omega^2)/4\omega, 1]$. For $s = \pm 1$ it is $\omega = \pm(1 \pm \sqrt{2})/2$ corresponding to $a_\gamma/M = 1/\sqrt{2}$.

Here we solve the more general problem where the frequencies are not equal, that is, where quantities (ℓ, ω) and ω_l are related by $\ell(\omega) = s\omega_l$, with $\omega \neq \omega_l$, and obtain $\omega_l \neq 0$ and $\omega = \pm 1/2$ (extreme Kerr **BH**) or $\omega_l \neq 0$ and $s = (1 - 4\omega^2)/4\omega_l$; for $s = \pm 1$ we obtain $\omega_l = \pm(1 - 4\omega^2)/4$, which is represented in Fig. 9. In this way we connect **BHs**, as initial and final state of these particular transformations.

We now consider the relation between frequencies and surface gravity as a function of a radius r of the horizon curve in the extended plane (that is, in this way we express the conditions for the inner and outer horizons relations), where $\omega(r) = s\ell(r)$ at the same point r ($r = r_g$, tangent point on the horizon curve). We obtain $r = 1 \mp \sqrt{1/(s^2 + 1)}$ for $s < 0$ and $s \geq 0$, respectively. This relates a **BH** inner and outer horizon (at fixed a). For $s = \pm 1$ we obtain $r = r_p = r_\pm = 1 \pm 1/\sqrt{2}$, outer and inner horizons, respectively, for the **BH** with spin $a_\gamma/M = 1/\sqrt{2}$.

Consider the more general case where $r \neq r_p$.

There is: $\omega(r) = s\ell(r_p)$ for

$$\begin{aligned} &(r = 2M, s = 0, r_p > 0), \quad \text{and} \\ &\left(r = \frac{2r_p^2}{r_p^2 + (r_p - M)^2 s^2}, [r \in [0, M], s < 0, (s > 0, r_p \geq M)] \right) \end{aligned} \tag{43}$$

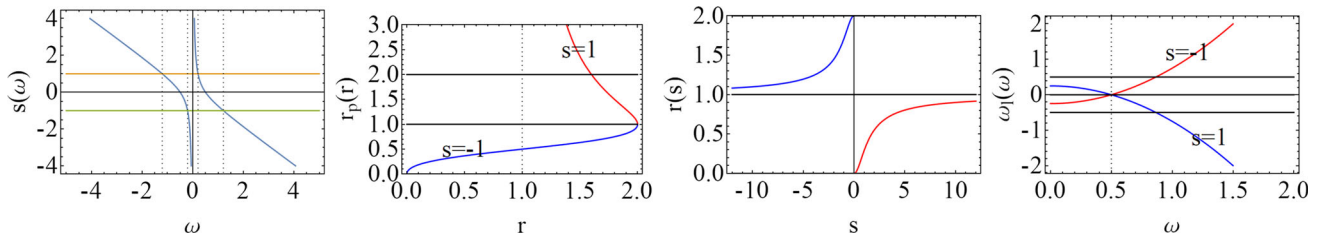


Fig. 9 Left panel: the function $s(\omega)$ introduced in Sect. 5.1, solution of $\ell_H^\pm(\omega) = s\omega$, is shown. Horizontal lines are for $s = \pm 1$. The function ℓ is the acceleration on the **BH** horizons, where ℓ_H^+ is the **BH** surface gravity, ω is the **BH** angular frequency and bundle characteristic frequency. Second panel: the functions $r_p(r)$ for $s = \pm 1$: $\omega(r) = s\ell(r)$

are defined in Eqs. (46) and (47). Third panel: the function $r = r_p(s)$ defined Sect. 5.1, solution of $\omega(r) = s\ell(r)$, is shown. The value $r = M$ corresponds to the extreme Kerr **BH**. Right panel: the function $\omega_l(\omega)$ defined in Sect. 5.1, solution of $\ell(\omega) = s\omega_l$ for $s = \pm 1$, is shown. $\omega = \pm 1/2$ corresponds to an extreme Kerr **BH**

alternatively, defining $\bar{s}_\beta \equiv \sqrt{\frac{2M-r}{r}}(r_p/(r_p - M))$ we can write the results as follows

$$s > 0 : (r = 2M, r_p = M, s > 0),$$

$$\text{and } (r_p > M, r \in]0, 2M[, s = \bar{s}_\beta) \tag{44}$$

$$s \leq 0 : (r = 2M, r_p = M, s \leq 0),$$

$$[(r_p \in]0, M[, r \in]0, 2M]), (r_p > M, r = 2M), s = \bar{s}_\beta]. \tag{45}$$

In particular, for the static case there is ($r = 2M, s = 0, r_p > 0$). For the special cases $s = \pm 1$ we obtain

$$\text{for } s = 1 \text{ it is } r \in]M, 2M] \text{ and } r_p = \frac{M}{M - \frac{\sqrt{2M-r}}{\sqrt{r}}}, \tag{46}$$

$$\text{for } s = -1 \text{ it is } r \in]0, 2M] \text{ and } r_p = \frac{M}{M + \frac{\sqrt{2M-r}}{\sqrt{r}}}. \tag{47}$$

represented in Fig. 9, where the relation between the **BH** horizons (where the range $r = r_g$ has been extended) connect couple of geometries as final and initial states of these special transitions.

5.2 Constant irreducible mass

A more complicated transition occurs when the final state is not a Schwarzschild static **BH**, implying $J(1) = 0$, where (1) is for final state of transition and (0) for the initial state. We start by noting that the condition $M_{irr} = \text{constant}$ and $M(1) = M(0) = M$ implies $J(1) = J(0) = J$, which can also be interpreted as $J(1)^2 = J(0)^2 = J^2$ and consider the case of change in rotation orientation, implying a state to be considered a static **BH**, that is, with a state $J = 0$ with a consequent increase of mass, (for $\delta M_{irr} \geq 0$).

Introducing the ratios (k_m, k_j), with $J_1 = k_j J_0$ and $M_1 = k_m M_0$ (we use notation $Q(0) = Q_0$ or $Q(1) = Q_1$), the condition $\delta M_{irr} = 0$ is verified for

$$k_j^\pm = (\pm) \sqrt{\frac{2(k_m^2 - 1)M_0^4}{J_0^2} \pm 2 \sqrt{\frac{(k_m^2 - 1)^2 M_0^4 (M_0^4 - J_0^2)}{J_0^4}} + 1},$$

(we consider with the sign (\pm) the possibility of $J_1 J_0 < 0$). Eventually, it can also be written in compact form as

$$k = \pm \frac{\sqrt{2(k_m^2 - 1)M_0 r_+(0) + a(0)^2}}{a(0)}, \tag{48}$$

where $r_+(0)$ is the outer horizon radius at state (0). Therefore, $k_j = \pm 1$ for $k_m = 1$ (unaltered masses). Assuming now $k_j \equiv k_m k_{jm}$ (with $k_m > 0$), the condition $\delta M_{irr} = 0$ implies

$$k_{jm} = \pm \frac{\sqrt{\frac{2(k_m^2 - 1)M_0 r_+(0) + a(0)^2}{k_m^2}}}{a(0)} \text{ where}$$

$$k_{jm} \in k_m \left[-\sqrt{\frac{M(0)^2}{a(0)^2}}, \sqrt{\frac{M(0)^2}{a(0)^2}} \right]. \tag{49}$$

Then, there are

$$\left(k_m = \frac{1}{\sqrt{2}}, k_{jm} = 0 \right),$$

$$\left(k_m \in \left] \frac{1}{\sqrt{2}}, 1 \right[, k_{jm} \pm \sqrt{2 - \frac{1}{k_m^2}} \right), \tag{50}$$

where it is described a total or partial energy extraction respectively.

Let us introduce the quantities

$$\tilde{k}_\pm \equiv \sqrt{-\frac{2M_0^2}{a(0)^2 k_m^2} \pm 2 \sqrt{\frac{(k_m^2 - 1)^2 M_0^2 (M_0^2 - a(0)^2)}{a(0)^4 k_m^4} + \frac{2M_0^2}{a(0)^2} + \frac{1}{k_m^2}}},$$

$$\text{and } c_a \equiv \sqrt{\frac{r_+(0)}{M_0}} \tag{51}$$

Note that $c_a/\sqrt{2}$ is related to the definition of irreducible mass – Sect. 4.2. For the Schwarzschild case we obtain ($a(0) = 0, k_m = 1$). For $a(0) \in]0, M_0[$ we find

$$(k_m = c_a/\sqrt{2}, k_{jm} = 0), \text{ or } (k_m \in c_a]1/\sqrt{2}, 1], k_{jm} = \pm \tilde{k}_{\pm}),$$

$$\text{for } a(0) = M_0, \text{ it is } (k_m = c_a, k_{jm} = 0),$$

$$\text{and } (k_m \in]c_a, 1], k_{jm} = -\tilde{k}_{\pm}).$$

In particular, the condition $k_m = 1$ implies $a(0) = 0$ or $a(0) \in]0, M]$ and $k_{jm} = \pm 1$. For $k_{jm} = 0$, that is, the final state of a Schwarzschild **BH**, or for $a(0) \in [0, M_0]$ it is $k_m = c_a/\sqrt{2}$, which for an extreme **BH**, $a(0) = M_0$, is $k_m = 1/\sqrt{2}$. For $k_{jm} = 1$, it is $k_m = 1$, implying $k_j = 1$; in other words, the condition $k_j = k_m$ implies that the **BH** state is immutable. Analogously, $k_m = 1$ for $k_{jm} = -1$.

Let us now introduce the spins

$$J_a \equiv M_1 \sqrt{2M_0^2 - M_1^2}, \quad J_i \equiv 2M_1 \sqrt{M_0^2 - M_1^2},$$

$$\tilde{J}_{\pm} \equiv \sqrt{J_0^2 \pm 2 \left[\sqrt{(M_0^4 - J_0^2)(M_0^2 - M_1^2)^2 + M_0^2(M_1^2 - M_0^2)} \right]}.$$

(52)

We obtain the following conditions for a transition $\delta M_{irr} = 0$:

$$\text{for } M_0 = \frac{M_1}{\sqrt{2}} : (J_0 = 0, J_1 = \pm M_1^2);$$

$$M_0 \in \left] \frac{M_1}{\sqrt{2}}, M_1 \right[: (J_0 \in]0, J_a[, J_1 = \pm \tilde{J}_+);$$

$$(J_0 = J_a, J_1 = \pm M_1^2)$$

$$\text{for } M_0 = M_1 : (J_0 = 0, J_1 = 0);$$

$$(J_0 \in]0, J_a[, J_1 = \pm \tilde{J}_-);$$

$$(J_0 = J_a, J_1 = \pm M_1^2);$$

$$\text{for } \frac{M_0}{M_1} \in]1, \sqrt{2}[: (J_0 = J_i, J_1 = 0);$$

$$(J_0 \in]J_i, M_0^2], J_1 = \pm \tilde{J}_-); \tag{53}$$

$$\text{for } M_0 = \sqrt{2}M_1 : (J_0 = M_0^2, J_1 = 0). \tag{54}$$

Finally, we see also the special case:

$$\text{for } \frac{M_0}{M_1} = \sqrt{\frac{3}{2}} : (J_0 = J_a, J_1 = 0);$$

$$(J_0 \in]J_a, M_0^2], J_1 = \pm \tilde{J}_-), \tag{55}$$

(note the symmetries in the state $(0) \leftrightarrow (1)$). In this case $J_a = J_i = \sqrt{8/9}M_0^2$ while $\tilde{J}_{\pm} = \sqrt{J_0^2 \pm \frac{2}{3} \left(\sqrt{M_0^8 - J_0^2 M_0^4 - M_0^4} \right)}$, and in this transition, $a = \sqrt{8/9}M_0$ is the *initial BH* state.

If the transition ends with a static **BH**, $J_1 = 0$, then $J_0 = 0$ and $M_1 = M_0$, that is, there is no transition with $M_1 = M_0/\sqrt{2}$ or with $J_0 = M_0^2$, i.e., an extreme Kerr **BH** as initial state. Viceversa, in the case $J_0 \in]0, M_0^2[$ we have $M_1 = \sqrt{r_+(0)}/\sqrt{2}$. If $J_0 = 0$, that is, the starting state is a Schwarzschild **BH**, then $J_1 = 0$ with $M_0 = M_1$. Therefore, there is no transition for $M_1 \in]M_0, \sqrt{2}M_0]$ with $J_1 = 2M_0\sqrt{M_1^2 - M_0^2}$ (note the symmetry with solution J_i in Eq. (52)). In the case, $M_1 = \sqrt{2}M_0$, we get $J_1 = 2M_0$ implying an extreme Kerr spacetime ($J_1 = M_1^2$).

In the extreme case, where $M_1^2 = J_1$, that is, a transition ending in a Kerr extreme **BH**, for $J_0 \in [0, M_0]$, we obtain $M_1 = \sqrt{r_+(0)}$. Then, for $J_0 = M_0^2$, that is, starting from a Kerr extreme **BH** and ending in Kerr extreme **BH**, we get $M_0 = M_1$. Note that the condition $J_1/M_1^2 = J_0/M_0^2$, that is, the **BH** dimensionless spin is invariant, implies that $J_0 = J_1$ and $M_1 = M_0$, that is, there is no transition from (J_0, M_0) with $M_{irr} = \text{constant}$ and $a/M = \text{constant}$. This is particularly relevant for the case of Kerr extreme **BH**. If the initial state is an extreme Kerr **BH**, i.e. $J_0 = M_0^2$, the transformation can lead to a Schwarzschild **BH**, i.e. $J_1 = 0$, if $M_1 = M_0/\sqrt{2}$; or can lead to a Kerr **BH** with

$$\frac{M_1}{M_0} \in \left] \frac{1}{\sqrt{2}}, 1 \right] \text{ and } J_1 = \pm M_0^2 \sqrt{\frac{2M_1^2}{M_0^2} - 1}, \tag{56}$$

(note the symmetry with respect to the case $J_0 = \{0, 1\}$ or $J_1 = \{0, 1\}$ considered above). In the particular case $M_1/M_0 = \sqrt{2/3}$, we obtain

$$\frac{J_0}{M_0^2} = \frac{2\sqrt{2}}{3}, \text{ and } J_1 = 0, \text{ or } \frac{J_0}{M_0^2} \in \left] \frac{2\sqrt{2}}{3}, 1 \right],$$

$$\text{and } J_1 = \pm \sqrt{J_0^2 - \frac{2M_0^2 r_+(0)}{3}}. \tag{57}$$

5.3 On the inner horizon relations in **BH** thermodynamics

Here and in the following analysis, we use the notation $M = M_{\pm}$ and $J = J_{\pm}$ to stress the quantities defined on a point of the outer or inner horizon, respectively. From the definition $M_{irr}^{\mp} \equiv A_{area}^{\mp}/2$ per inner and outer horizon, we obtain the two relations

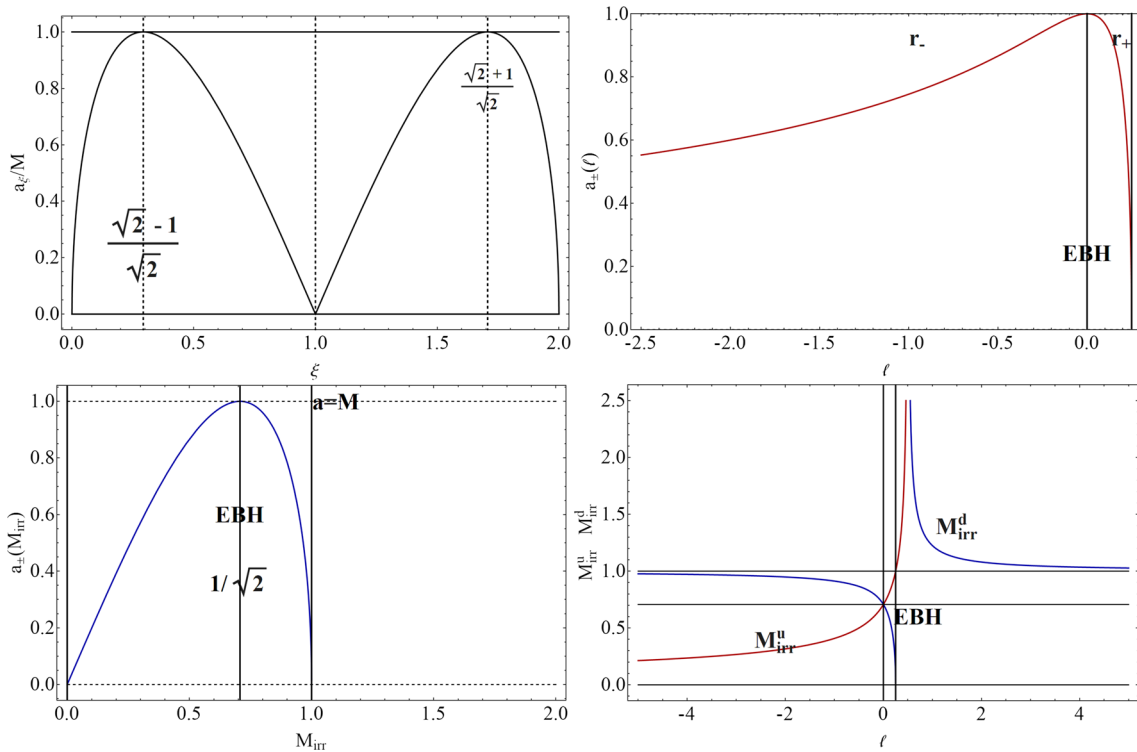


Fig. 10 Upper-left panel: the **BH** spin a_ξ of Eq. (17) versus extractable rotational energy ξ . The Schwarzschild geometry ($a = 0$) for $\xi = \{0, 1, 2\}$ and the extreme Kerr spacetime a/M for $\xi = \xi_\ell$ and ξ_m are shown. In Sect. 6, we used the extended range $\xi \in [0, 2]$ for the reparametrization of the metric in the extended plane. Upper-right panel: The **BH** spin function $a_\pm(\ell)$ as function of the surface gravity $\ell = \ell^\pm$

for the outer and inner horizons. Bottom-left panel: The spin $a_\pm(M_{irr})$ as function of irreducible mass M_{irr} . Bottom-right panel: The irreducible masses, (M_{irr}^u, M_{irr}^d) as functions of the surface gravity ℓ , see Eq. (69). The static spacetime and the limit of extreme Kerr **BH**, **EBH**, are represented

$$\xi_\pm \equiv M_\pm - \frac{1}{2} \sqrt{\left(\frac{J_\pm}{M_\pm}\right)^2 + r_\pm^2}, \quad \text{equivalently}$$

$$\xi_\mp(a) = 1 - \frac{\sqrt{1 \mp \sqrt{1 - a_\mp^2}}}{\sqrt{2}} \quad (58)$$

constituting the two branches of $a(\xi)$ functions and M_{irr} in Figs. 5 and 10. The second relation is in terms of the dimensionless spin of the **BH** and dimensionless energy ξ . These are solutions of $a(\xi) = 2\sqrt{-(\xi - 2)(\xi - 1)^2\xi}$ for $\xi_- \in [\frac{1}{2}(2 - \sqrt{2}), 1]$ and for $\xi_+ \in [0, \frac{1}{2}(2 + \sqrt{2})]$, respectively. Solving $a(\xi) = a$, we obtain the solutions:

$$\xi_\mp^+ \equiv M_\pm + \frac{1}{2} \sqrt{\left(\frac{J_\pm}{M_\pm}\right)^2 + r_\pm^2},$$

$$\text{equivalently } \xi_\pm^+(a) = 1 + \frac{\sqrt{1 \mp \sqrt{1 - a_\mp^2}}}{\sqrt{2}} \quad (59)$$

defining the new functions $\bar{M}_{irr}^\pm = -\sqrt{r_\pm/2}$ and constituting the other two branches of Figs. 5 and 10 (at $\xi > 1$).

Then, $(M_{irr}^+)^2 = M^2 - (M_{irr}^-)^2$ where $\delta A_{area}^+ = 2M\delta M - \delta A_{area}^-$. Condition $(<)_M : \delta M_{irr}^+ M_{irr}^+ < M\delta M$ occurs in the following cases: By using the ratio $\delta J/\delta M_+$, we find that condition $(<)_M$ holds for $J_+ \in]0, M^2]$ and $\delta M < 0$ (mass decreasing) for $\delta J/\delta M_+ < 2M_+^3 r_- / J_+$, and $\delta M > 0$ (mass increasing) for $\delta J/\delta M_+ > 2M_+^2 r_- / J_+ = 4\omega_H^+ = L_f^-$, or explicitly

$$J_+ \in]0, M_+^2[, \quad [\delta M_+ \leq 0, \delta J_+ > \delta J_+^+];$$

$$\text{and } [\delta M_+ > 0, \delta J_+ > \delta J_+^-],$$

$$\text{where } \delta J_+^\pm \equiv \frac{2M_+ \left(\delta M_+ M_+^2 \pm J_+ \sqrt{\delta M_+^2 \left(\frac{M_+^4}{J_+^2} - 1 \right)} \right)}{J_+}. \quad (60)$$

In these relations we assumed δM_{irr} , for the variations $(\delta M, \delta J)$, to be null. However, during the **BH** evolution the irreducible mass can also increase; in this case, we get the relation $\delta M_{irr}^+ \geq 0$ for $\delta M^+ - \omega_0^+ \delta J^+ \geq 0$ and then $\delta J^+/\delta M^+ \leq 1/\omega_0^+ = \mathcal{A}_0(0) = (L_f)$, that is, the origin of its bundle (where $\mathcal{A}(0) = a_0\sqrt{\sigma}$) constrains the transition.

This relation also holds for the angular momentum of the horizon.

However,

$$(M_{irr}^\pm)^2 = M^2 - (M_{irr}^\mp)^2, \text{ then}$$

$$\xi = M - \sqrt{M^2 - (M_{irr}^-)^2} = M - M_{irr}^+. \tag{61}$$

The condition $\delta M_{irr}^- > 0$ with $\delta M_{irr}^+ > 0$ holds for

$$M > M_{irr}^+, \quad \delta M_{irr}^+ \in \left[0, \frac{\delta M M}{M_{irr}^+} \right] \tag{62}$$

viceversa, the condition $\delta M_{irr}^- < 0$ with $\delta M_{irr}^+ > 0$ holds for

$$M > M_{irr}^+, \quad \left[(\delta M \leq 0, \delta M_{irr}^+ \geq 0), \left(\delta M > 0, \delta M_{irr}^+ \geq \frac{\delta M M}{M_{irr}^+} \right) \right].$$

While the condition $\delta M_{irr}^+ \geq 0$, from Eq. (61), holds for

$$(M = M_{irr}^+, \quad \delta M \geq 0);$$

$$M > M_{irr}^+, \quad \text{and} \quad \left(\delta M > 0; \quad \delta M_{irr}^- \leq -\sqrt{\frac{\delta M^2 M^2}{M^2 - (M_{irr}^+)^2}} \right);$$

$$\left(\delta M \leq 0; \quad \delta M_{irr}^- \leq \sqrt{\frac{\delta M^2 M^2}{M^2 - (M_{irr}^+)^2}} \right). \tag{63}$$

Interestingly, $\delta M_{irr}^+ > 0$ for $J_+ \in]0, M_+^2[$, $\delta M_+ = 0$, $\delta J_+ < 0$. In other words,

$$\delta M_{irr}^+ > 0, \quad \text{for}$$

$$J_+ = 0, \quad \text{and} \quad \delta M_+ > 0;$$

$$J_+ \in]0, M_+^2[, \quad \text{and} \quad \left(\delta M_+ < 0, \quad \frac{\delta J_+}{\delta M_+} > \frac{1}{\omega_H^+} \right);$$

$$\left(\delta M_+ > 0, \quad \frac{\delta J_+}{\delta M_+} < \frac{1}{\omega_H^+} \right)$$

and

$$\delta M_{irr}^- > 0, \quad \text{for}$$

$$J_- \in]0, M_-^2[, \quad \text{and} \quad \left(\delta M_- < 0, \quad \frac{\delta J_-}{\delta M_-} < \frac{1}{\omega_H^-} \right);$$

$$\left(\delta M_- > 0, \quad \frac{\delta J_-}{\delta M_-} > \frac{1}{\omega_H^-} \right)$$

$$J_- \in]0, M_-^2[, \quad \text{and} \quad \delta M_- = 0, \quad \delta J_- > 0. \tag{64}$$

where $\omega_H^\pm = J_\pm / 2M_\pm^2 r_\pm$ and we consider the cases $\delta M_{irr}^\pm \geq 0$. In these relation we note how the bundle origin \mathcal{A}_0 constrains the transition, distinguishing **NS** origin with $\mathcal{A}_0 > 2$, for the relations in terms of the **BH** inner horizon (in the extended plane inner and outer horizons relations have to be considered).

Below we summarize some properties of the **BH** spacetimes highlighted in this analysis relation to the \mathcal{MB} s characteristic. In the **BH** geometry with spin $a_\gamma \equiv \sqrt{1/2}$, it is $\ell_H^\pm = \pm \omega_H^\pm$. In this spacetime the transitions are essentially regulated by the characteristic bundle frequency. In this case, $r_\gamma^- = r_\epsilon^+$, that is, the outer ergosurface on the equatorial plane of the **BH** is a geodesic orbit of the (corotating) photon. In this geometry, $\omega(r) = s\ell(r)$ at the same point r \mathcal{MB} tangent point with the horizon curve. We obtain $r = 1 \mp \sqrt{1/(s^2 + 1)}$ for $s < 0$ and $s \geq 0$, respectively. For $s = \pm 1$, we obtain $r = r_\pm = 1 \pm 1/\sqrt{2}$, outer and inner horizons, respectively, for **BHs** with spin $a/M = 1/\sqrt{2}$ - Eq. (43). In other words, if we consider $\ell_H^\pm(\omega) = s\omega$, then for $s = \pm 1$, we obtain $\omega = \pm (1 \pm \sqrt{2})/2$, corresponding to $a/M = 1/\sqrt{2}$.

Furthermore, $\partial_a^{(2)} \ln s = 0$ for $a = a_\gamma \equiv M/\sqrt{2}$, where $s \equiv \omega_\pm^+ \equiv \omega_H^+/\omega_H^-$. The analysis of the extremes of the accelerations ℓ , solutions $\partial_r \ell = 0$, enlightens the role of the **BH** spin $a/M = \sqrt{3}/2$. The radius solution of $\partial_r \ell = 0$, is on the horizon curve in the extended plane, where $r = 3/2M$, which is the outer horizon of the **BH** spacetime with spin $a/M = \sqrt{3}/2$. Restricting to the horizon curve, we find the solution for $r = M/2$, which is the inner horizon of the **BH** spacetime with $a/M = \sqrt{3}/2$. Therefore, the extremes of the acceleration ℓ for a and r constitute, respectively, the inner and outer horizon for the **BH** with spin $a/M = \sqrt{3}/2$. (That is $\partial_a \ell = 0$ for $r_a = a/\sqrt{3}$, while $\partial_r \ell = 0$ for $r_r = \sqrt{3}a$; however, $r_a = r_-$ and $r_r = r_+$ for $a/M = \sqrt{3}/2$). The saddle point of $(L_f)(\omega^\pm)$ (**BH** angular momentum or bundle origin) as function of r is $r_-/M = 1/2$, which is the inner horizon of the **BH** spacetime, where the momentum and frequency are $(L_f)(\omega^\pm) = 2/\sqrt{3} = 1/\omega$. (Note that in this case $a = \omega_-$). The saddle point of the function $\omega(r)$ is for the same **BH** spacetime on the outer horizon $r_+/M = 3/2$ where $(L_f) = 2\sqrt{3} = 1/\omega$. The horizon (metric bundles and light surfaces) frequency ω_g^- of Eq. (21) has a saddle point for $\xi = (1 \pm \sqrt{2/3})$ corresponding to the spin $a = 2\sqrt{2}/3$

and frequency $\omega = (1/2\sqrt{2}, 1/\sqrt{2})$. For $a/M = 2\sqrt{2}/3$, we obtain also $r_{mso}^- = r_\epsilon^+$, that is, the marginally stable orbit of the corotating particle is the outer ergosurface on the equatorial plane. Spin $a/M = 2\sqrt{2}/3$ solves the problem of replicas for $\ell(r_+(a_p)) = -\ell(r_-(a))$ – see Eq. (10). On the other hand, Eq. (57) shows the relevance of this spin $J_0/M_0^2 = 2\sqrt{2}/3$ in relation to **BH** transition with masses ratio $M_1/M_0 = \sqrt{2/3}$. The origin spin $\mathcal{A}_0 = 2/\sqrt{3}$ is a saddle point for the area, Eq. (34), and it is also the saddle

point for the angular momentum corresponding to the frequency $\omega = 1/(L_H)$ of the inner horizon $r_-/M = 1/2$ for the spacetime $a = \omega$, whose frequency of the outer horizon is a saddle point for the frequency.

Rotational energy ξ

Considering the spin-function $a = a_\xi(\xi) = 2\sqrt{-\xi(\xi - 2)(\xi - 1)^2} \in [0, 1]$ of Eq. (17), we obtain up to four copies of the geometry for each ξ value. In this way we use the symmetries of the function to create copies. The metric components are

$$\begin{aligned}
 g_{tt} &= -\frac{4(\xi - 2)\xi(\xi - 1)^2(\sigma - 1) + (r - 2)r}{4(\xi - 2)(\xi - 1)^2\xi(\sigma - 1) + r^2}, & g_{t\phi} &= -\frac{4\sqrt{-(\xi - 2)(\xi - 1)^2\xi}r\sigma}{4(\xi - 2)(\xi - 1)^2\xi(\sigma - 1) + r^2} \\
 g_{rr} &= \frac{4(\xi - 2)(\xi - 1)^2\xi(\sigma - 1) + r^2}{(r - 2)r - 4(\xi - 2)(\xi - 1)^2\xi}, & g_{\theta\theta} &= 4(\xi - 2)(\xi - 1)^2\xi(\sigma - 1) + r^2 \\
 g_{\phi\phi} &= \frac{\sigma \left[(r^2 - 4(\xi - 2)(\xi - 1)^2\xi)^2 + 4(\xi - 2)\xi(\xi - 1)^2\sigma \left[(r - 2)r - 4(\xi - 2)(\xi - 1)^2\xi \right] \right]}{4(\xi - 2)(\xi - 1)^2\xi(\sigma - 1) + r^2}.
 \end{aligned}
 \tag{65}$$

6 Inertial mass, extractable rotational energy, and surface gravity

We can connect two **BH** states, before and after a transition, representing them in the same extended plane. We analyze the tensor $\mathbf{g}(\cdot, \cdot)$ in the extended plane by evaluating the metric along the horizon curve in terms of thermodynamic quantities. In the first case, we use the function $a = a_\xi(\xi)$ of Eq. (17) for the **BH** spin in terms of the maximum extractable rotational energy ξ . The tensor describes a set of **BH** geometries. Considering Fig. 10, where the function $a_\xi(\xi)$ is represented, there are multiple copies of the geometry, up to four, for the parameter ξ in the range $\xi \in]0, 2[$ and $a_\xi \in]0, 1[$. The maximum extractable energy is for an initial extreme Kerr **BH** with parameter ξ_ℓ , we consider however the range $\xi \in]0, 2[$. In the second case, we use the concept of irreducible mass with the relations Eqs. (22), (25), and (30). In the third case, we use the acceleration ℓ and the surface gravity. This re-parametrization provide a way to connect dif-

ferent states offer with multiple solutions, as in the extended plane, for different values of the parameters, creating a sort of **BH** cluster metric. Notably there are no **NS** solutions.

The metric describes a **BH** geometry for $\xi \in [0, \xi_\ell]$, where ξ_ℓ is the extreme **BH** spacetime. In the extended range $\xi \in]0, 2M]$, the metric tensor (65) describes multiple copies of the **BHs** with $a \in [0, M]$. The extreme Kerr **BH** corresponds to ξ_ℓ and $\xi_m \equiv (2 + \sqrt{2})/2 > \xi_\ell$. Then, the Kerr extreme **BH** and the Schwarzschild static metric are the limits

$$\begin{aligned}
 \lim_{\xi \rightarrow \xi_\ell} g &= \lim_{\xi \rightarrow \xi_m} g = \mathbf{g}_{\text{Kerr}}; & \lim_{\xi \rightarrow 0} g \\
 &= \lim_{\xi \rightarrow 1} g = \lim_{\xi \rightarrow 2} g = \mathbf{g}_{\text{Schw}}
 \end{aligned}$$

Irreducible mass M_{irr}

We use the function $a = a_{irr} \equiv 2\sqrt{M_{irr}^2(1 - M_{irr}^2)}$ for the spin; then, the metric tensor describes **BH** geometries⁸. Note that the condition $a_{irr} = 0$ is a limiting value in the extended plane, corresponding (as for M_{irr}) to the static limit. In fact, the function a_{irr} has been evaluated from the horizon curve in the extended plane, writing the irreducible mass for any point of the horizon curve. Therefore, it describes **BH** spacetimes ($a_{irr} \in [0, 1]$). From Fig. 10 it is also clear that there are two copies of one **BH** geometry a for $M_{irr} = M_{irr}^\pm(a)$. Explicitly,

⁸ The irreducible mass is defined in Eq. (22), where $M_{irr} = \sqrt{r^2 + a^2}/2$. Therefore, $M_{irr}^\pm = \sqrt{r_\pm}/2$. All the quantities are dimensionless.

$$\begin{aligned}
 g_{tt} &= \frac{2r}{4(M_{irr}^2 - 1)M_{irr}^2(\sigma - 1) + r^2} - 1, & g_{t\phi} &= -\frac{4\sqrt{M_{irr}^2(1 - M_{irr}^2)}r\sigma}{4(M_{irr}^2 - 1)M_{irr}^2(\sigma - 1) + r^2} \\
 g_{rr} &= \frac{4(M_{irr}^2 - 1)M_{irr}^2(\sigma - 1) + r^2}{4M_{irr}^2(1 - M_{irr}^2) + (r - 2)r}, & g_{\theta\theta} &= 4(M_{irr}^2 - 1)M_{irr}^2(\sigma - 1) + r^2 \\
 g_{\phi\phi} &= \frac{\sigma \left[[4M_{irr}^2(1 - M_{irr}^2) + r^2]^2 - 4M_{irr}^2(M_{irr}^2 - 1)\sigma(2M_{irr}^2 - r)(2M_{irr}^2 + r - 2) \right]}{4(M_{irr}^2 - 1)M_{irr}^2(\sigma - 1) + r^2}.
 \end{aligned} \tag{66}$$

The limits for the Schwarzschild spacetime and for the the extreme Kerr spacetime are

$$\lim_{M_{irr} \rightarrow 0} g = \lim_{M_{irr} \rightarrow 1} g = \mathbf{gSchw}, \quad \lim_{M_{irr} \rightarrow 1/\sqrt{2}} g = \mathbf{gKerr} \tag{67}$$

The singular points of the line elements are

$$\begin{aligned}
 g_{tt} = 0: & \quad [r = 2 : ((M_{irr} = 0, \sigma \in [0, 1]); (\sigma = 1, M_{irr} \in]0, 1[))]; \\
 [M_{irr} = \frac{1}{\sqrt{2}} : [r = 1, \sigma = 0); (\sigma \in]0, 1[, (r = 1 \pm \sqrt{\sigma}))]]; \\
 \sigma \geq 0: & \quad \left[r = r_{M_{irr}}^+ : \left[\sigma < 1, (M_{irr} \in]0, 1[, M_{irr} \neq \frac{1}{\sqrt{2}}) \right] \right],
 \end{aligned}$$

(note the possibility of $M_{irr} > 1$) where

$$\begin{aligned}
 r_{M_{irr}}^s &\equiv 2M_{irr}\sqrt{(M_{irr}^2 - 1)(1 - \sigma)}, \\
 r_{M_{irr}}^\mp &\equiv 1 \mp \sqrt{1 + (r_{M_{irr}}^s)^2}.
 \end{aligned} \tag{68}$$

Surface gravity ℓ

We use $a = a_{\pm}(\ell) = \sqrt{1 - 4\ell}/\sqrt{(2\ell - 1)^2}$, from the definition $\ell_H^\pm(a) = \ell$. Then, $a = M$ for $\ell = 0$ and $a = 0$ for $\ell = 1/4$. Note that the metric tensor describes **BH** geometries and $a_{\pm}(\ell)$ is not well defined for the asymptotic value $\ell = 1/2$. In terms of the irreducible mass using Eq. (66) with the two solutions $M_{irr}^u \equiv 1/\sqrt{2(1 - 2\ell)}$ and $M_{irr}^d \equiv 1/\sqrt{(1 - 4\ell)^{-1} + 1}$ (Fig. 10), the tensor is

$$\mathbf{g} = \begin{pmatrix} \frac{2(1-2\ell)^2r}{\tilde{p}} - 1 & 0 & 0 & -\frac{2\sqrt{1-4\ell}\sqrt{(1-2\ell)^2}r\sigma}{\tilde{p}} \\ 0 & \frac{\tilde{T}}{\tilde{\chi} - 2r} & 0 & 0 \\ 0 & 0 & \tilde{T} & 0 \\ -\frac{2\sqrt{1-4\ell}\sqrt{(1-2\ell)^2}r\sigma}{\tilde{p}} & 0 & 0 & \frac{\sigma \left[\tilde{\chi}^2 + \frac{(4\ell-1)\sigma(2\ell(r-2)-r+1)((2\ell-1)r+1)}{(1-2\ell)^4} \right]}{\tilde{T}} \end{pmatrix},$$

$$\tilde{p} \equiv 4\ell \left[(\ell - 1)r^2 + \sigma - 1 \right] + r^2 - \sigma + 1, \quad \tilde{T} \equiv \frac{(4\ell - 1)(\sigma - 1)}{(1 - 2\ell)^2} + r^2, \quad \tilde{\chi} \equiv \frac{1 - 4\ell}{(1 - 2\ell)^2} + r^2. \tag{69}$$

$(M_{irr} \geq 1, \sigma \leq 1)$,

$$\begin{aligned}
 [r = r_{M_{irr}}^- : \sigma < 1, (M_{irr} \in]0, 1[, M_{irr} \neq \frac{1}{\sqrt{2}})]; \\
 g_{rr} = 0: & \quad [r = 0, \sigma = 1, (M_{irr} \in]0, 1[, M_{irr} > 1)]; \\
 [r = r_{M_{irr}}^s, \sigma \in [0, 1[, M_{irr} > 1], \\
 g_{tt}^{-1} = 0: & \quad \sigma \in [0, 1[, M_{irr} > 1, r = r_{M_{irr}}^s, \\
 g_{rr}^{-1} = 0: & \quad \sigma \in [0, 1[, (M_{irr} = 0, r = 2); \\
 (M_{irr} = \frac{1}{\sqrt{2}}, r = 1); & \quad (r = 2M_{irr}^2, (M_{irr} \neq \frac{1}{\sqrt{2}})); \\
 (r = 2(1 - M_{irr}^2), & \quad (M_{irr} \in]0, 1[, M_{irr} \neq \frac{1}{\sqrt{2}}))] ,
 \end{aligned}$$

The extreme Kerr and the Schwarzschild **BH** limit are

$$\lim_{\ell \rightarrow 0} \mathbf{g} = \mathbf{gKerr}, \quad \lim_{\ell \rightarrow 1/4} \mathbf{g} = \lim_{\ell \rightarrow \pm\infty} \mathbf{g} = \mathbf{gSchw}. \tag{70}$$

There are the following singular points for the metric

$$\begin{aligned}
 g_{tt} = 0: \\
 \sigma = 0; & \quad \left((\ell = 0, r = 1); \left(r = r_{\alpha}^+, \ell \neq \left\{ \frac{1}{2}, 0 \right\} \right); \right. \\
 & \quad \left. \left(r = r_{\alpha}^-, \ell < \frac{1}{4}; \ell \neq 0 \right) \right)
 \end{aligned} \tag{71}$$

$$\begin{aligned}
 &\ell \neq \frac{1}{2} : (\sigma = 1, r = 2), \\
 &\left(\sigma \in]0, 1[, \left[r = r_\beta^+, \left(r = r_\beta^-, \ell < \frac{1}{4} \right) \right] \right), \\
 g_{rr} = 0 : &\left(\sigma = 1, \left[\left(\ell = \frac{1}{2}, r \geq 0 \right), \right. \right. \\
 &\left. \left. \left(r = 0, \ell \neq \left\{ \frac{1}{2}, \frac{1}{4} \right\} \right) \right] \right); \\
 &\left(\sigma \in [0, 1[, r = r_z, \ell > \frac{1}{4}, \ell \neq \frac{1}{2} \right), \\
 g_{tt}^{-1} = 0 : &\left(\sigma \in [0, 1[, \ell > \frac{1}{4}, \ell \neq \frac{1}{2}, r = r_z \right). \\
 g_{rr}^{-1} = 0 : &\left(\sigma \in [0, 1], (\ell = 0, r = 1), [r = r_\pm(\ell), \right. \\
 &\left. \left(\ell < \frac{1}{2}, \ell \neq 0 \right) \right], \\
 &\left[r = 2 - r_\pm(\ell), \left(\ell < \frac{1}{4}, \ell \neq 0, \ell > \frac{1}{2} \right) \right] \right) \quad (72)
 \end{aligned}$$

with

$$\begin{aligned}
 r_\alpha^\pm &\equiv 1 \pm 2\sqrt{\frac{\ell^2}{(1-2\ell)^2}}, & r_\beta^\pm &\equiv 1 \pm \sqrt{\frac{4\ell(\ell-\sigma) + \sigma}{(1-2\ell)^2}}, \\
 r_z &\equiv \sqrt{\frac{(4\ell-1)(1-\sigma)}{(2\ell-1)^2}}, & r_\pm(\ell) &\equiv \frac{1}{1-2\ell}, \quad (73)
 \end{aligned}$$

where $r_\pm(\ell)$ are the horizon curves as functions of the acceleration ℓ_H^\pm – see Fig. 11.

7 Discussion and final remarks

The laws of **BH** classical thermodynamics have a geometric nature determined by the **BH** horizon, which has a prominent role in **BH** physics. In this work, we studied **BH** thermodynamics by using certain light surfaces, grouped in structures defined in metric bundles (\mathcal{MB} s), tightly connected to the definitions of Killing horizons. The introduction of the extended plane allows an overview of all the possible **BH** (and possibly **NS**) states and their properties. It can be used to describe **BH** transitions from the \mathcal{MB} s view-point, for instance, a transition from the tangent spin $a_g(0)$ for the **BH**(0) to $a_g(1)$ for the **BH**(1), corresponding to a transition between the respective tangent bundles. Consequently, we can also describe transitions in terms of the origin frequency and the tangent radius of the bundle.

In this new frame we reformulated, **BH** thermodynamics on the replicas in terms of the light surfaces, exploring the thermodynamic properties of the **BH**s geometries. Fundamentals of classical **BH** thermodynamics are discussed interpreting **BH** transformations according to the character-

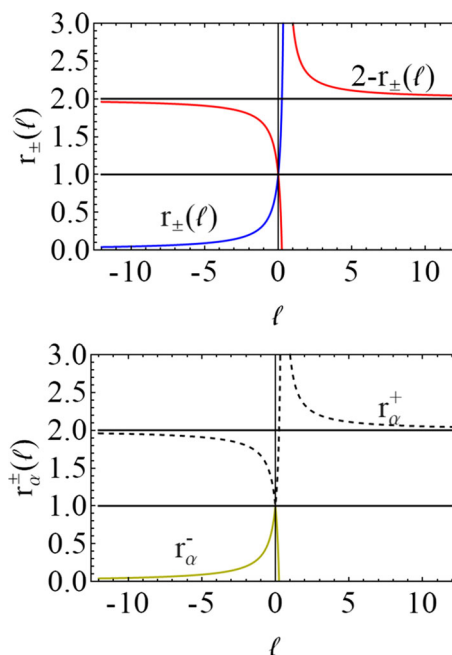


Fig. 11 The radii r_α^\pm and **BH** horizons $r_\pm(\ell)$ of Eq. (73) as functions of the acceleration ℓ . These are the singular points of the metric line (69). Note the horizons replicas

istic frequency which is also the **BH** horizon frequency. In particular, we considered the transformations with constant irreducible mass.

In the analysis of the thermodynamical properties, we apply the concepts of rotational energy parameter ξ , surface gravity, frequency ω , irreducible mass M_{irr} , which we use to re-express the first law as well as the metric line element. A special metric line parametrization, describing only **BH** spacetimes, is given in terms of the maximum extractable rotational energy ξ or, correspondingly, the inertial mass parameter M_{irr} , replacing the metric spin parameter. In this work, we change the perspective in the analysis of some aspects of the Kerr solutions as seen in the extended plane, that is, we study properties unfolding along all the solutions; therefore, we re-consider the line element adapting it to this new frame. In Sect. 6, we rewrite the metric tensor on parts of the extended plane, providing alternative representations of the **BH** geometries, creating clusters of **BH** copies with respect to a new metric parameter, recovered from the bundles representations as curves in the extended plane. We used the irreducible mass M_{irr} of the **BH** – Fig. 5, the **BH** rotational energy ξ (Fig. 10), the **BH** surface gravity (Fig. 8). The extended plane and \mathcal{MB} s concepts include **NS**s; however, no transformation allows a **BH-NS** transition. **NS**s, as \mathcal{MB} s components, can be used to represent some properties of the **BH** geometry. With these re-parameterizations with functions defined in the extended plane, we obtain structures constituted by replicas of the metric tensor describing the

same geometries or portions of the same geometry in the extended plane. We plan in a future work to study the cluster metric with extended range of parameters.

Replicas of light-surfaces with equal photon frequencies, are useful to connect different points of the spacetime, and different geometries following the **BH** transition with energy release. Here, we use the function $\xi(r)$, maximum extractable rotational energy – see [23–28]. In this context, we discuss how ω , the bundle characteristic frequency, is tied to the extracted rotational energy ξ . The frequency, in fact, also appears as a component of the “work term” in the laws of **BH** thermodynamics. We use extensively this correspondence to formulate **BH** thermodynamics in terms of metric bundles. More precisely, the energy extraction process can take place regulated by the light surfaces which are, in general, two surfaces, one surface located outside the horizon r_+ and one within the ergoregion $]r_+, r_e^+[$. (In the case of the inner horizon, that is, of points characterized by the frequency $\omega_H^\pm \equiv \omega_\pm(r_-)$, where r_- is the inner Killing horizon, the situation is more complex and it is not always possible to find a replica of the horizon [12]. Interestingly, the dependence of the orbit from the poloidal angle θ , allows a more accurate study of the regions close to the **BH** rotational axis, revealing interesting observational implications [12].

In Sect. 4, we reformulated aspects of **BH** thermodynamics on the bundles, exploring the properties of the rotational energy and “rest” mass in the extended plane. The transition from a state (0) to a new state (1) is essentially regulated by the characteristics of the initial state, such as the **BH** surface gravity $\ell_H^+(0)$ evaluated on the outer horizon, entering the entropy/temperature terms, and by the work term, which is expressed through the bundle frequency ω_H^+ . We have expressed the relation between these quantities in terms of the tangency conditions of the bundles with the horizon curve and in terms of the bundle curves in the extended plane, leading to the alternative view-point in terms of replicas, i.e., orbits of the bundles located at $r > r_+$ (and some being in the $r < r_-$ region). In this sense, we describe the state transitions in terms of metric bundles transitions. In this regard, as bundles connect different points and geometries in the extended plane, we evaluated the line element along the bundles and in other portions of the extended plane, thus, connecting metrics particularly from the state with $(\ell(0), \omega(0))$ to the state with parameters $(\ell(1), \omega(1))$.

We highlighted classes of **BHs** related to the fact that a transition between initial and final state is subject to particular laws. The results point out **BH** spacetimes with spins $a/M = \sqrt{8/9}$ and $a/M = 1/\sqrt{2}$ showing anomalies in the rotational energy extraction, and surface gravity variation after **BH** state transitions and **BH** spin $a/M = \sqrt{3}/2$ relevant in the **BH** area variations. We also enlightened **BHs** masses ratios $M_1 = M_0/\sqrt{2}$, and $M_1/M_0 = \sqrt{2/3}$ (distinguishing spin $a/M = \sqrt{8/9}$), regulating the **BH** transi-

tions studied here, and representing limiting values in the **BH** transformations—see for example Eqs. (55) and (56).

Acknowledgements This work was partially supported by UNAM-DGAPA-PAPIIT, Grant No. 114520, Conacyt-Mexico, Grant No. A1-S-31269.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

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Funded by SCOAP³.

Appendix A: Black holes in the extended plane

In this section, we discuss some aspects of the extended plane [12, 13]. Here, we present the set of functions

$$\begin{aligned}
 0_1 : \quad \mathcal{A}_0(r) &\equiv \mathcal{A}_0 \left(1 - \frac{r}{2M}\right), \\
 0_2 : \quad \mathcal{A}_r^*(r) &\equiv \frac{2M}{\mathcal{A}_0} r, \\
 0_3 : \quad \mathcal{A}_{tan}^0 &\equiv \frac{\mathcal{A}_0^2 - 2}{2a_g(\mathcal{A}_0)} + \left(\frac{4 - \mathcal{A}_0^2}{\mathcal{A}_0}\right) r, \quad \text{and} \\
 0_4 : \quad \mathcal{A}_{tangent}(r) &\equiv \frac{(\mathcal{A}_0^4 - 16) a_g(\mathcal{A}_0) + 32\mathcal{A}_0}{8(\mathcal{A}_0^2 + 4)} \\
 &\quad - \frac{a_g(\mathcal{A}_0)(\mathcal{A}_0^4 - 16)}{16\mathcal{A}_0^2} r, \quad (\text{A.1})
 \end{aligned}$$

see Fig. 12.

The four regions of Fig. 5, outlined in this analysis also through the inertial mass, the surface gravity ℓ and the **BH** area in the extended plane, can be obtained as functions of the origin spin \mathcal{A}_0 , the characteristic frequency of ω of the bundles, and the tangent radius (the first law of **BH** thermodynamics in the extended plane is defined along the boundary of these regions). Considering the curves of Eq. (A.1) and using $r_g = r_g(\mathcal{A}_0)$ and $a_g = a_g(\mathcal{A}_0)$, with the origin spin $\mathcal{A}_0 = 2r_\pm/a_g$ and $\mathcal{A}_0 = 2\sqrt{r_g}/\sqrt{2 - r_g}$, we can express the curves as

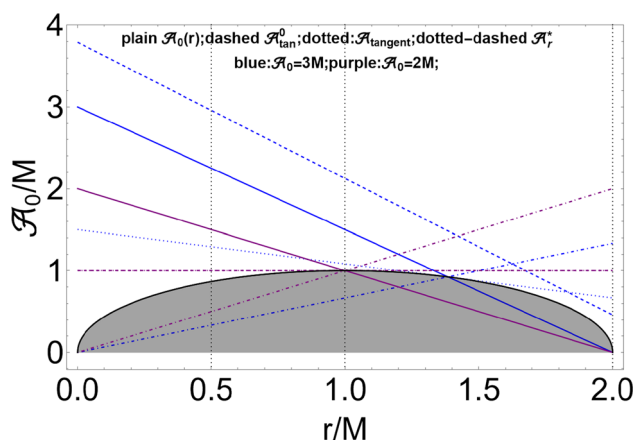


Fig. 12 Functions $(\mathcal{A}_0, \mathcal{A}_r^*, \mathcal{A}_{tan}^0, \mathcal{A}_{tangent})$ of Eq. (A.1) in the extended plane, for different bundle origin \mathcal{A}_0 . Horizons curve is also plotted (black curve) a_{\pm} , the region $a < a_{\pm}$ is shaded

$$\{O_{\sharp}\} \equiv \{O_1 : \mathcal{A}_0(r), O_2 : \mathcal{A}_r^*(r), O_3 : \mathcal{A}_{tan}^0, O_4 : \mathcal{A}_{tangent}(r)\}$$

and

$$\{O_{\sharp}\} = \left\{ \begin{array}{l} \frac{(2-r)\sqrt{r_g}}{\sqrt{2-r_g}}, \frac{r\sqrt{2-r_g}}{\sqrt{r_g}}, \\ \frac{r_g(3r_g-2)-4r(r_g-1)}{\sqrt{(2-r_g)r_g}}, \frac{r_g(1-r)+r}{\sqrt{(2-r_g)r_g}} \end{array} \right\},$$

in terms of the tangent radius $r_g \in [0, 2M]$ on the horizons. By inverting Eq. (22), we obtain $\mathcal{A}_0 = 2M_{irr}(all)/\sqrt{1 - M_{irr}(all)^2}$ and

$$\{O_{\sharp}\} = \left\{ \begin{array}{l} -\frac{M_{irr}(all)(r-2)}{\sqrt{1 - M_{irr}(all)^2}}, \frac{\sqrt{1 - M_{irr}(all)^2}r}{M_{irr}(all)}, \frac{2[3M_{irr}(all)^4 - M_{irr}(all)^2(2r+1) + r]}{M_{irr}(all)\sqrt{1 - M_{irr}(all)^2}}, \\ \frac{4M_{irr}(all)^4 - 2M_{irr}(all)^2(r+1) - 4\sqrt{M_{irr}(all)^2}(M_{irr}(all)^2 - 1)M_{irr}(all) + r}{2\sqrt{M_{irr}(all)^2}(1 - M_{irr}(all)^2)} \end{array} \right\}.$$

Using the irreducible mass M_{irr}^{\pm} to express the bundle origin \mathcal{A}_0 , we find $\mathcal{A}_0 = 2\sqrt{1 - (M_{irr}^{\pm})^2}/M_{irr}^{\pm}$ and

$$\{O_{\sharp}\} = \left\{ \begin{array}{l} -\frac{\sqrt{1 - (M_{irr}^{\pm})^2}(r-2)}{M_{irr}^{\pm}}, \frac{M_{irr}^{\pm}r}{\sqrt{1 - (M_{irr}^{\pm})^2}}, \\ \frac{2[(M_{irr}^{\pm})^2(2r-5) + 3(M_{irr}^{\pm})^4 - r + 2]}{M_{irr}^{\pm}\sqrt{1 - (M_{irr}^{\pm})^2}}, \\ \frac{2M_{irr}^{\pm} \left\{ 2(M_{irr}^{\pm})^3 - 2\sqrt{(M_{irr}^{\pm})^2}[(M_{irr}^{\pm})^2 - 1] + M_{irr}^{\pm}(r-3) \right\} - r + 2}{2\sqrt{(M_{irr}^{\pm})^2}[1 - (M_{irr}^{\pm})^2]} \end{array} \right\}.$$

It is evident the role of radii $r = 2M$, the Schwarzschild horizon, and $r = 3M$, the ergosurface on the equatorial plane. The limit $M_{irr} = 1$ and $r = 2/5$, which is the tangent point to an inner horizon of the space $a/M = 4/5$, is related to the horizon confinement [13].

Appendix B: Notable areas in the extended plane

Table 1 contains the evaluation of the areas in the extended plane,⁹ according to the regions of Fig. 5, which constitute the integral form of the relations in Sect. 4. Therefore, considering also the results of [13], we obtain

$$\int_0^{2M} a_0(r)dr = 2\pi, \quad \int a_{\epsilon}^{\pm} d\sigma dr = \pi, \quad (B.2)$$

$$\int_0^M \omega_H^{\pm} \delta a = \frac{(1 - r_{\pm} + \ln[r_{\pm}])}{2}, \quad \text{where}$$

$$\int_0^M \omega_H^+ da = -\int_{\omega_H^+} a_g(\omega)d\omega + \frac{1}{2} = \frac{1 - \log(2)}{2}, \quad (B.3)$$

where a_0 is the bundle spin origin, a_{ϵ}^{\pm} are the ergosurfaces curves in the extended plane, relating characteristic quantities of the bundles in the extended plane.

⁹ It is $\int \ell \delta \omega = \arcsin(M_{irr(v)}) + M_{irr(\mu)}M_{irr(v)}$, where, $2M_{irr(\mu)}M_{irr(v)} = a_{\pm}$, $(\int \ell \delta r = (r - \log r)/2)$.

Table 1 A_{areas} represent the **BH** areas in the extended plane, a_g is the tangent spin to the horizon curve, r_g is the tangent radius of the bundle curve to the horizon in the extended plane, ω is the horizon frequency and bundle characteristic frequency, a_{\pm} are the horizon curves in the

extended plane, r_{\pm} are the outer and inner **BH** horizon, respectively, $\mathcal{A}_0 = a_0\sqrt{\sigma}$ is the origin spin of the bundle, where $\sigma \equiv \sin^2 \theta$, M_{irr} is the irreducible mass of the **BH**, $M_{irr(\mu)}$ is defined in Eq. (26)

Tangent radius:	$\int r_g(\omega)d\omega = \arctan(2\omega); \int_{\omega_H^+} r_g(\omega)d\omega = \int_{\omega_H^-} r_g(\omega)d\omega = \frac{\pi}{4}$
Horizons curves:	$\int a_{\pm}\delta r = \frac{(r-1)a_{\pm}}{2} - \arcsin\left[\frac{M_{irr(\mu)}}{M}\right]; \int_{r_{\pm}} a_{\pm}\delta r = \int_{r_{\pm}} a_{\pm}\delta r = \frac{\pi}{4}$
Tangent spin a_g :	$\int a_g(\omega)d\omega = \frac{\log(4\omega^2+1)}{2}; \int_{\omega_H^+} a_g(\omega)d\omega = \frac{\log(2)}{2}; \int a_g(r_g)dr_g = \frac{2\pi}{\sqrt{\sigma}}$ $\int a_g(\mathcal{A}_0)d\mathcal{A}_0 = 2 \log(\mathcal{A}_0^2 + 4); \int_0^2 a_g(\mathcal{A}_0)d\mathcal{A}_0 = \log(4)$
Areas:	$\int A_{area}(a_{\pm})\delta r = 4\pi r^2; \int_0^M A_{area}(a_{\pm})\delta r = 4\pi; \int_M^{2M} A_{area}(a_{\pm})\delta r = 12\pi$ $\int A_{area}(a_g, r_g)\delta\omega = 8\pi \int r_g(\omega)d\omega = 8\pi \arctan(2\omega)$ $\int_{\omega_H^+} A_{area}(a_g, r_g)\delta\omega = \int_{\omega_H^-} A_{area}(a_g, r_g)\delta\omega = 2\pi^2$ $\int A_{area}(a_g, r_g)\delta\mathcal{A}_0 = 16\pi \left[\mathcal{A}_0 - 2 \arctan\left(\frac{\mathcal{A}_0}{2}\right)\right]; \int_0^{2M} A_{area}(a_g, r_g)\delta\mathcal{A}_0 = 8(4 - \pi)\pi$ $\int A_{area}(r_{\pm})\delta a = \pi \pm 4 \left[a \left(\sqrt{1 - a^2} \pm 2\right) + \arcsin(a)\right]; \int_0^M A_{area}(r_{\pm})\delta a = \pi \pm 2(\pi \pm 4)$

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