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Addendum

Addendum to: Vacuum stability conditions and potential minima for a matrix representation in lightcone orbit space

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Abstract We derive the vacuum stability conditions for a left-right-symmetric potential more general than originally considered.

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In the original text, the left-right-symmetric potential considered was not the most general one allowed by the symmetry. While correct bounded-from-below conditions were derived for the considered potential, the most general potential is more complicated and only necessary or sufficient conditions can be derived with reasonable ease.

The quadratic and cubic parts of the potential are unchanged (and irrelevant to deriving vacuum stability conditions in the limit of large field values). The most general quartic potential[1] reads

$$\begin{aligned} V = {} & \lambda_1 (\operatorname{tr} \Phi^\dagger \Phi)^2 + \lambda_2 [(\operatorname{tr} \tilde{\Phi} \Phi^\dagger)^2 + (\operatorname{tr} \tilde{\Phi}^\dagger \Phi)^2] \\ & + \lambda_3 (\operatorname{tr} \tilde{\Phi} \Phi^\dagger)(\operatorname{tr} \tilde{\Phi}^\dagger \Phi) + \lambda_4 (\operatorname{tr} \Phi^\dagger \Phi)[\operatorname{tr} \tilde{\Phi} \Phi^\dagger \\ & + \operatorname{tr} \tilde{\Phi}^\dagger \Phi] + \lambda_L |H_L|^4 + \lambda_R |H_R|^4 \\ & + \lambda_{LR} |H_L|^2 |H_R|^2 + \lambda_{\Phi L} \operatorname{tr} \Phi^\dagger \Phi |H_L|^2 \\ & + \tilde{\lambda}_{\Phi L} (\operatorname{tr} \tilde{\Phi} \Phi^\dagger + \operatorname{tr} \tilde{\Phi}^\dagger \Phi) |H_L|^2 + \lambda_{\Phi R} \operatorname{tr} \Phi^\dagger \Phi |H_R|^2 \\ & + \tilde{\lambda}_{\Phi R} (\operatorname{tr} \tilde{\Phi} \Phi^\dagger + \operatorname{tr} \tilde{\Phi}^\dagger \Phi) |H_R|^2 + \lambda'_{\Phi L} H_L^\dagger \Phi \Phi^\dagger H_L \\ & + \lambda'_{\Phi R} H_R^\dagger \Phi^\dagger \Phi H_R + \tilde{\lambda}'_{\Phi L} H_L^\dagger \tilde{\Phi} \tilde{\Phi}^\dagger H_L \\ & + \tilde{\lambda}'_{\Phi R} H_R^\dagger \tilde{\Phi}^\dagger \tilde{\Phi} H_R. \end{aligned} \tag{1}$$

[1] Notice that we have changed $\lambda'_{\Phi L,R} \to \tilde{\lambda}_{\Phi L,R}$ with respect to the original in order to have a more uniform and logical notation with the added interaction terms.

Altogether, the quartic potential (1) is then, in terms of the lightcone variables, given by

$$\begin{aligned} V = {} & r^\mu \lambda_{\mu\nu} r^\nu + \lambda_{L\mu} |H_L|^2 r^\mu + \lambda_{R\mu} |H_R|^2 r^\mu \\ & + \lambda_L |H_L|^4 + \lambda_R |H_R|^4 + \lambda_{LR} |H_L|^2 |H_R|^2, \end{aligned} \tag{2}$$

where the tensor $\lambda_{\mu\nu}$ is given by Eq. (70) of the original and

$$\lambda_{L\mu} = (\lambda_{\Phi L} + \rho_{\Phi L} \lambda'_{\Phi L} + \tilde{\rho}_{\Phi L} \tilde{\lambda}'_{\Phi L}, 2\tilde{\lambda}_{\Phi L}, 0), \tag{3}$$
$$\lambda_{R\mu} = (\lambda_{\Phi R} + \rho_{\Phi R} \lambda'_{\Phi R} + \tilde{\rho}_{\Phi R} \tilde{\lambda}'_{\Phi R}, 2\tilde{\lambda}_{\Phi R}, 0). \tag{4}$$

We use the parametrisation $H_L^\dagger \Phi \Phi^\dagger H_L = \rho_{\Phi L} r^0 |H_L|^2$ and $H_R^\dagger \Phi^\dagger \Phi H_R = \rho_{\Phi R} r^0 |H_R|^2$ with $0 \leqslant \rho_{\phi L}, \rho_{\Phi R} \leqslant 1$ for the $\lambda'_{\Phi L}$ and $\lambda'_{\Phi R}$ terms, and a similar parametrisation with $\tilde{\rho}_{\Phi L}$ and $\tilde{\rho}_{\Phi R}$ for the $\tilde{\lambda}'_{\Phi L}$ and $\tilde{\lambda}'_{\Phi R}$ terms. In fact, it is easy to see, considering a basis where the bidoublet Φ is diagonal, that

$$\rho_{\Phi L} + \tilde{\rho}_{\Phi L} = 1, \qquad \rho_{\Phi R} + \tilde{\rho}_{\Phi R} = 1, \tag{5}$$

while $\rho_{\Phi L}$ and $\rho_{\Phi R}$ are independent of each other. In addition, physical values for $\rho_{\Phi L,R}$ are within the ellipse given by

$$\left(\frac{r_1}{r_0} \right)^2 + (2\rho_{\Phi L,R} - 1)^2 = 1 \tag{6}$$

illustrated in Fig. 1.

In order to derive vacuum stability conditions for the potential (2), we follow the procedure of Section 3.2 of the original. First of all, we minimise the potential with respect to r^2, in effect substituting $\lambda_{00} = \lambda_1 \to \lambda_1 + (\lambda_3 - 2\lambda_2)\theta(2\lambda_2 - \lambda_3)$ and $\lambda_{11} = 2\lambda_2 + \lambda_3 \to 2\lambda_2 + \lambda_3 - (\lambda_3 - 2\lambda_2)\theta(2\lambda_2 - \lambda_3)$.

Due to the non-trivial dependency on $\rho_{\Phi L}$ and $\rho_{\Phi R}$ in Eq. (6), derivation of the full necessary and sufficient vacuum stability conditions becomes very complicated. It is straightforward, however, to write down some necessary *or* sufficient conditions. Because the potential depends on $\rho_{\Phi L,R}$ linearly, it is minimised when these parameters take extremal values on the boundary of the ellipse (6).

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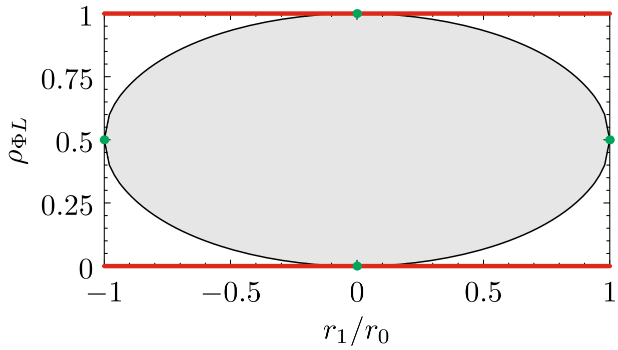

Fig. 1 Orbit space relation between $\rho_{\Phi L}$ and r_1/r_0. A similar parameter space is available for $\rho_{\Phi R}$ vs. r_1/r_0. Considering the green points yields simplest necessary conditions and considering the red lines sufficient conditions for vacuum stability

First of all, if we set r^1 to zero, then $\rho_{\Phi L,R}$ can vary in their whole ranges. Since $r^0 \geqslant 0$, we can immediately use copositivity constraints in the $(r^0, |H_L|^2, |H_R|^2)$ basis. Similarly, if we set $r^1 = \pm r^0$, we can set $\rho_{\Phi L} = \rho_{\Phi R} = 1/2$ and do the same. Together, these choices correspond to the green points at the ends of the semiaxes of the ellipse in Fig. 1. In fact, we can set $r^1 = \pm k r^0$ to get a necessary condition for $\rho_{\Phi L}$ and $\rho_{\Phi R}$ on the boundaries of the ellipse for a constant k. The above two choices correspond to $k = 0$ and $k = 1$, respectively. The coupling matrix in the $(r^0, |H_L|^2, |H_R|^2)$ basis is given by

$$\lambda_k = \begin{pmatrix} \lambda_{00} & \frac{1}{2}\lambda_{L0} & \frac{1}{2}\lambda_{R0} \\ \frac{1}{2}\lambda_{L0} & \lambda_L & \frac{1}{2}\lambda_{LR} \\ \frac{1}{2}\lambda_{R0} & \frac{1}{2}\lambda_{LR} & \lambda_R \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} \lambda_{00} &= \lambda_1 + (1-k^2)(\lambda_3 - 2\lambda_2)\theta(2\lambda_2 - \lambda_3) \\ &\quad + k^2(2\lambda_2 + \lambda_3) \pm 2k\lambda_4, \end{aligned} \tag{8}$$

$$\lambda_{L0} = \lambda_{\Phi L} + \lambda'_{\Phi L}\rho_{\Phi L} + \tilde{\lambda}'_{\Phi L}(1-\rho_{\Phi L}) \pm 2k\tilde{\lambda}_{\Phi L}, \tag{9}$$

$$\lambda_{R0} = \lambda_{\Phi R} + \lambda'_{\Phi R}\rho_{\Phi R} + \tilde{\lambda}'_{\Phi R}(1-\rho_{\Phi R}) \pm 2k\tilde{\lambda}_{\Phi R}, \tag{10}$$

where we have taken into account the relations (5) to substitute for $\tilde{\rho}_{\Phi L}$ and $\tilde{\rho}_{\Phi R}$. We must separately consider the four combinations of signs in the solution to Eq. (6),

$$\rho_{\Phi L} = \frac{1}{2}(1 \pm \sqrt{1-k^2}), \quad \rho_{\Phi R} = \frac{1}{2}(1 \pm \sqrt{1-k^2}), \tag{11}$$

in addition to the two signs of $\pm k$.

The resulting necessary conditions for the left-right symmetric scalar potential with a bidoublet and left and right doublets, for given k, are

$$\begin{aligned} \lambda_L &> 0, \\ \lambda_R &> 0, \\ \lambda_{00} &> 0, \\ \bar{\lambda}_{LR} = \frac{1}{2}\lambda_{LR} + \sqrt{\lambda_L\lambda_R} &> 0, \\ \bar{\lambda}_{L0} = \frac{1}{2}\lambda_{L0} + \sqrt{\lambda_L\lambda_{00}} &> 0, \\ \bar{\lambda}_{R0} = \frac{1}{2}\lambda_{R0} + \sqrt{\lambda_R\lambda_{00}} &> 0, \\ \sqrt{\lambda_L\lambda_R\lambda_{00}} + \lambda_{LR}\sqrt{\lambda_{00}} + \lambda_{L0}\sqrt{\lambda_R} & \\ +\lambda_{R0}\sqrt{\lambda_L} + \sqrt{2\bar{\lambda}_{LR}\bar{\lambda}_{L0}\bar{\lambda}_{R0}} &> 0. \end{aligned} \tag{12}$$

By considering the conditions (12) for several values for $k \in [0, 1]$ (in addition to those given by the green points), we can approximate the true necessary and sufficient conditions to arbitrary precision.

Alternatively, we write down the quartic couplings in the basis $(r^0, r^1, |H_L|^2, |H_R|^2)$ and rotate the 1+1 forward light-cone LC^+ into the non-negative quadrant $\mathbb{R}^2_+$ of the r^0r^1-plane. The resulting orbit space is the non-negative orthant $\mathbb{R}^4_+$ and therefore we can apply copositivity (Ref. [29] of the original) to the obtained quartic coupling matrix, given by

$$\lambda = \begin{pmatrix} \lambda_- & \frac{1}{2}\lambda_\mp & \frac{1}{2}\lambda_{L-} & \frac{1}{2}\lambda_{R-} \\ \frac{1}{2}\lambda_\mp & \lambda_+ & \frac{1}{2}\lambda_{L+} & \frac{1}{2}\lambda_{R+} \\ \frac{1}{2}\lambda_{L-} & \frac{1}{2}\lambda_{L+} & \lambda_L & \frac{1}{2}\lambda_{LR} \\ \frac{1}{2}\lambda_{R-} & \frac{1}{2}\lambda_{R+} & \frac{1}{2}\lambda_{LR} & \lambda_R \end{pmatrix}, \tag{13}$$

where

$$\lambda_- = \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3 - 2\lambda_4), \tag{14}$$

$$\lambda_+ = \frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3 + 2\lambda_4), \tag{15}$$

$$\lambda_\mp = \lambda_1 - 2\lambda_2 - \lambda_3 + 2(\lambda_3 - 2\lambda_2)\theta(2\lambda_2 - \lambda_3), \tag{16}$$

$$\begin{aligned} \lambda_{L-} = \frac{1}{\sqrt{2}}[&\lambda_{\Phi L} + \rho_{\Phi L}\lambda'_{\Phi L} + (1-\rho_{\Phi L})\tilde{\lambda}'_{\Phi L} \\ &- 2\tilde{\lambda}_{\Phi L}], \end{aligned} \tag{17}$$

$$\begin{aligned} \lambda_{R-} = \frac{1}{\sqrt{2}}[&\lambda_{\Phi R} + \rho_{\Phi R}\lambda'_{\Phi R} + (1-\rho_{\Phi R})\tilde{\lambda}'_{\Phi R} \\ &- 2\tilde{\lambda}_{\Phi R}], \end{aligned} \tag{18}$$

$$\begin{aligned} \lambda_{L+} = \frac{1}{\sqrt{2}}[&\lambda_{\Phi L} + \rho_{\Phi L}\lambda'_{\Phi L} + (1-\rho_{\Phi L})\tilde{\lambda}'_{\Phi L} \\ &+ 2\tilde{\lambda}_{\Phi L}], \end{aligned} \tag{19}$$

$$\begin{aligned} \lambda_{R+} = \frac{1}{\sqrt{2}}[&\lambda_{\Phi R} + \rho_{\Phi R}\lambda'_{\Phi R} + (1-\rho_{\Phi R})\tilde{\lambda}'_{\Phi R} \\ &+ 2\tilde{\lambda}_{\Phi R}], \end{aligned} \tag{20}$$

where we have taken into account the relations (5) to substitute for $\tilde{\rho}_{\Phi L}$ and $\tilde{\rho}_{\Phi R}$. We must consider all combinations of values $(\rho_{\Phi L}, \rho_{\Phi R}) = (0,0), (1,0), (0,1), (1,1)$. The resulting sufficient vacuum stability conditions for the left-right symmetric scalar potential with a bidoublet and left and right doublets are given by

$$
\begin{aligned}
&\lambda_L > 0, \quad \lambda_R > 0, \quad \lambda_- > 0, \quad \lambda_+ > 0,\\
&\bar{\lambda}_{LR} = \frac{1}{2}\lambda_{LR} + \sqrt{\lambda_L \lambda_R} > 0,\\
&\bar{\lambda}_{L-} = \frac{1}{2}\lambda_{L-} + \sqrt{\lambda_L \lambda_-} > 0,\\
&\bar{\lambda}_{L+} = \frac{1}{2}\lambda_{L+} + \sqrt{\lambda_L \lambda_+} > 0,\\
&\bar{\lambda}_{R-} = \frac{1}{2}\lambda_{R-} + \sqrt{\lambda_R \lambda_-} > 0,\\
&\bar{\lambda}_{R+} = \frac{1}{2}\lambda_{R+} + \sqrt{\lambda_R \lambda_+} > 0,\\
&\bar{\lambda}_{\mp} = \frac{1}{2}\lambda_{\mp} + \sqrt{\lambda_- \lambda_+} > 0,\\
&\sqrt{\lambda_L \lambda_R \lambda_-} + \lambda_{LR}\sqrt{\lambda_-} + \lambda_{L-}\sqrt{\lambda_R}\\
&\quad + \lambda_{R-}\sqrt{\lambda_L} + \sqrt{2\bar{\lambda}_{LR}\bar{\lambda}_{L-}\bar{\lambda}_{R-}} > 0,\\
&\sqrt{\lambda_L \lambda_R \lambda_+} + \lambda_{LR}\sqrt{\lambda_+} + \lambda_{L+}\sqrt{\lambda_R}\\
&\quad + \lambda_{R+}\sqrt{\lambda_L} + \sqrt{2\bar{\lambda}_{LR}\bar{\lambda}_{L+}\bar{\lambda}_{R+}} > 0,\\
&\sqrt{\lambda_L \lambda_- \lambda_+} + \lambda_{L-}\sqrt{\lambda_+} + \lambda_{L+}\sqrt{\lambda_-}\\
&\quad + \lambda_{\mp}\sqrt{\lambda_L} + \sqrt{2\bar{\lambda}_{L-}\bar{\lambda}_{L+}\bar{\lambda}_{\mp}} > 0,\\
&\sqrt{\lambda_R \lambda_- \lambda_+} + \lambda_{R-}\sqrt{\lambda_+} + \lambda_{R+}\sqrt{\lambda_-}\\
&\quad + \lambda_{\mp}\sqrt{\lambda_R} + \sqrt{2\bar{\lambda}_{R-}\bar{\lambda}_{R+}\bar{\lambda}_{\mp}} > 0,\\
&\det(\lambda) > 0 \vee \text{some element(s) of } \operatorname{adj}(\lambda) < 0,
\end{aligned}
\tag{21}
$$

where the last condition, obtained from the Cottle–Habetler–Lemke theorem (Ref. [31] of the original), is not given in full. The adjugate $\operatorname{adj}(A)$ of a matrix A is the transpose of the cofactor matrix of A. It is defined through the relation $A \operatorname{adj}(A) = \det(A)\, I$.

Since the sufficient conditions (21) may exclude an unnecessarily large part of parameter space, it may be preferable to use the conditions (12) for a number of different values for $k \in [0, 1]$.

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