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Cosmological "constant" in a universe born in the metastable false vacuum state

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Abstract The cosmological constant Λ is a measure of the energy density of the vacuum. Therefore properties of the energy of the system in the metastable vacuum state reflect properties of $\Lambda = \Lambda(t)$. We analyze properties of the energy, $E(t)$, of a general quantum system in the metastable state in various phases of the decay process: In the exponential phase, in the transition phase between the exponential decay and the later phase, where decay law as a function of time t is in the form of powers of $1/t$, and also in this last phase. We found that this energy having an approximate value resulting from the Weisskopf–Wigner theory in the exponential decay phase is reduced very fast in the transition phase to its asymptotic value $E(t) \simeq E_{min} + \alpha_2/t^2 + \ldots$ in the late last phase of the decay process. (Here E_{min} is the minimal energy of the system). This quantum mechanism reduces the energy of the system in the unstable state by a dozen or even several dozen orders or more. We show that if to assume that a universe was born in metastable false vacuum state then according to this quantum mechanism the cosmological constant Λ can have a very great value resulting from the quantum field theory calculations in the early universe in the inflationary era, $\Lambda \simeq \Lambda_{qft}$, and then it can later be quickly reduced to the very, very small values.

1 Introduction

Many physical processes including some cosmological processes are quantum decay processes. Attempts to solve the problem of a description of the evolution in time and decay of quantum unstable (or metastable) states were made practically since the birth of the Quantum Theory. Difficulties with this problem are caused by the fact that unstable states are not eigenvectors of the self-adjoint Hamiltonian $\mathfrak{H}$ governing the time evolution in the system containing such states. The problem is important because one can meet unstable (or metastable) states in many quantum processes: Starting from the spontaneous emission of electromagnetic radiation by excited quantum levels of molecules or atoms [1,2], through the radioactive decay of radioactive elements (e.g., α-decay [3]), decays of almost all known elementary particles, to the problem of the false vacuum decay, which is a quantum process [4,5]. Therefore if one wants to search for properties of the universe born in the false vacuum state, one must know how to describe the quantum decay process of such a state.

In fact, the problem of describing the decay process appeared as early as in the pre-quantum theory era, when it was necessary to quantify the changes in time of a number of decaying radioactive elements. The radioactive decay law formulated by Rutherford and Sody in the nineteenth century [6–9] allowed to determine the number $N(t)$ of atoms of the radioactive element at the instant t knowing the initial number $N_0 = N(0)$ of them at initial instant of time $t_0^{init} = 0$ and had the exponential form: $N(t) = N_0 \exp[-\lambda t]$, where where $\lambda > 0$ is a constant. Since then, the belief that the decay law has the exponential form has become common. This conviction was upheld by Wesisskopf–Wigner theory of spontaneous emission [1,2]: They found that to a good approximation the quantum mechanical non-decay probability of the exited levels is a decreasing function of time having exponential form. Further studies of the quantum decay process showed that basic principles of the quantum theory does not allow it to be described by an exponential decay law at very late times [10,11] and at initial stage of the decay process (see e.g. [11] and references therein). Theoretical analysis shows that at late times the survival probability (i. e. the decay law) should tends to zero as $t \to \infty$ much more slowly than any exponential function of time and that as function of time it has the inverse power-like form at this regime of time [10,11]. All these results caused that there is rather widespread belief that a universal feature of the quantum decay process is the presence of three time regimes of

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the decay process: the early time (initial), exponential (or "canonical"), and late time having inverse-power law form [12]. This belief is reinforced by a numerous presentations in the literature of decay curves obtained for quantum models of unstable systems.

The theoretical studies of unstable states mentioned resulted in discovery some new quantum effects, such as the Quantum Zeno (and anti-Zeno) effects [13–17] resulting from early time properties of the time evolution of the unstable state, and which were confirmed experimentally [18–23], or a reduction of the energy of the system in the unstable state at late times [24–28], which is connected with the asymptotic late time behavior of the survival probability.

As it was already mentioned, the theory of quantum decay processes has found its applications in cosmology: e.g. in the studies of cosmological models in which there is a metastable false vacuum and, consequently, the decaying dark energy. Coleman et al. in seminal papers [4,5] discussed the instability of a physical system, which is not at an absolute energy minimum, and which is separated from the absolute minimum by an effective potential barrier. They showed that if the early Universe is too cold to activate the energy transition to the minimum energy state then a quantum decay, from the false vacuum to the true vacuum, is still possible through a barrier penetration via the macroscopic quantum tunneling. In other words they showed that quantum decay processes can play an important role in the early Universe. What is more, it appears that asymptotic late time properties of of quantum decay processes can be responsible for some cosmological effects. This idea was formulated by Krauss and Dent [29,30]. They analyzing a false vacuum decay pointed out that in eternal inflation, many false vacuum regions can survive up to the times much later than times when the exponential decay law holds. Krauss and Dent gave a simple explanation of this effect: It may occur even though regions of false vacua by assumption should decay exponentially, gravitational effects force space in a region that has not decayed yet to grow exponentially fast. In general, the space grows exponentially fast only in the inflationary phase of the evolution of the Universe (see, e.g. [31,32]). Therefore a realization of Krauss and Dent's hypothesis in our Universe is possible only if the lifetime of the false vacuum is much shorter than the duration of the inflationary epoch because only then the quantum decay of the false vacuum takes place in its canonical regime. The mentioned Krauss and Dent's idea was used in [27,28] to show that when analyzing cosmological processes not only late time properties of the survival probability of decaying false vacua should be considered but also, what seems even more important, the late time properties of the energy in the false vacuum state of the system. In cosmology, when we study the decay of a false vacuum, the Universe is the quantum system under consideration.

The late time effect considered by Krauss and Dent [29] is impossible within the standard approach of calculations of decay rate Γ for decaying vacuum state (see e.g., [4,5] and many other papers). Calculations performed within this standard approach cannot lead to a correct description of the evolution of the Universe with false vacuum in all cases when the lifetime of the false vacuum state is such short that its survival probability exhibits an inverse power-law behavior at times which are of the order of the age Universe or shorter. This conclusion is valid not only when the dark energy density and its late time properties are related to the transition of the Universe from the false vacuum state to the true vacuum but also when the dark energy is formed by unstable "dark particles". In both cases the decay of the dark energy density is the quantum decay process and only the formalism based on the Fock–Krylov theory of quantum unstable states and used by Krauss and Dent [29] is able to describe correctly such a situation.

In the analysis performed in this paper we will assume the model of the dark anergy close to that considered by Landim and Abdalla, in which the observed vacuum energy is the value of the scalar potential at the false vacuum [33]. (Similar idea was used in many papers – see eg. [34,35]). In other words, we will assume that the current stage of accelerated expansion of the universe will be described by a canonical scalar field Φ such that its potential, $V(\Phi)$, has a local and true minimums. So, the field at the false vacuum will represent the darkenergy. In such a situation, the quantum state of the system in the local minimum is described by a state vector corresponding to the false vacuum state whereas the quantum state of the system in the true minimum corresponds to the state of the lowest energy of the system and it is a true vacuum. This means that the density of the energy of the system in the false vacuum state, ρ^{F}_{vac}, will be identified with the density of the dark energy, $\rho^{\mathrm{F}}_{vac} \equiv \rho_{de}$, (or, equivalently, as the cosmological term Λ) in the Einstein equations [31,32]. Implications of the assumption that, $\rho^{\mathrm{F}}_{vac} = \rho_{de} = \rho_{de}(t)$, behaves as the asymptotically late form of the energy of the system in the false vacuum state, i.e. cosmologies with decaying dark energy were studied, e.g., in [36–39]. In these studies, the problem of the possible reaction of the system to energy changes during the transition from the time epoch of exponential decay to the epoch with the decay law of the form of powers of $1/t$ was not analyzed. In particular, it was not analyzed how fast the energy tends to its asymptotic form in the era in which the decay law is proportional to the powers of $1/t$. The aim of this paper is to investigate this problem and also to analyze the possible influence of this effect on the currently observed properties of the system (i.e., in the case under consideration, the Universe): Here we show that there exists a mechanism that reduces the energy of the system in the unstable state by a dozen or even several dozen orders or

more, which can help to explain the cosmological constant problem.

The paper has the following structure. Section 2 contains preliminaries: a brief introduction in the Fock–Krylov approach to the description of quantum unstable states, a brief derivation of the effective Hamiltonian governing the time evolution of the unstable state, and short discussion of properties of the instantaneous energy of the system in the unstable state as well as of the instantaneous decay rate is presented here for readers convenience. Necessary model calculations and numerical results in a graphical form are presented in Sect. 3. Section 4 contains a discussion of possible cosmological implications results presented in previous sections. Section 5 contains final remarks.

2 Preliminaries

Understanding the basic features of an unstable system requires isolating this system from the influence of the environment on the decay process, including the possible distortion of these features by repeatedly interactions, at different times, with measuring instruments. These conditions are met in the case of quantum decay processes occurring in a vacuum. Therefore, further considerations will only cover decays that occur in a vacuum. The standard approach to study properties of quantum unstable systems decaying in the vacuum and evolving in time is to analyze their decay law (survival probability) $\mathcal{P}(t)$, which describes the probability od finding the system at the instant of time t in the metastable state $|\phi\rangle \in \mathcal{H}$ prepared at the initial instant $t_0^{init} < t$:

$$\mathcal{P}(t) = |\mathcal{A}(t)|^2, \tag{1}$$

where

$$\mathcal{A}(t) = \langle\phi|\phi(t)\rangle \tag{2}$$

is the survival amplitude and $|\phi(t)\rangle$ is the solution of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle = \mathfrak{H}|\phi(t)\rangle. \tag{3}$$

Here $\mathfrak{H}$ denotes the complete (full), self-adjoint Hamiltonian of the system acting in the Hilbert space $\mathcal{H}$ of states of this system $|\phi\rangle, |\phi(t)\rangle \in \mathcal{H}$, $\langle\phi|\phi\rangle = \langle\phi(t)|\phi(t)\rangle = 1$ The initial condition for Eq. (3) in the case considered is usually assumed to be

$$|\phi(t = t_0^{init} \equiv 0)\rangle \stackrel{\text{def}}{=} |\phi\rangle, \quad \text{or equivalently,} \quad \mathcal{A}(0) = 1. \tag{4}$$

Using the basis in $\mathcal{H}$ build from normalized eigenvectors $|E\rangle$, $E \in \sigma_c(\mathfrak{H}) = [E_{min}, \infty)$ (where $\sigma_c(\mathfrak{H})$ is the continuous part of the spectrum of $\mathfrak{H}$) of $\mathfrak{H}$ and using the expansion of $|\phi\rangle$ in this basis one can express the amplitude $\mathcal{A}(t)$ as the following Fourier integral

$$\mathcal{A}(t) \equiv \mathcal{A}(t - t_0^{init}) = \int_{E_{min}}^{\infty} \omega(E)\, e^{-\frac{i}{\hbar}E\,(t - t_0^{init})}\, dE, \tag{5}$$

where $\omega(E) = \omega(E)^*$ and $\omega(E) > 0$ is the probability to find the energy of the system in the state $|\phi\rangle$ between E and $E + dE$ and E_{min} is the minimal energy of the system. The last relation (5) means that the survival amplitude $\mathcal{A}(t)$ is a Fourier transform of an absolute integrable function $\omega(E)$. If we apply the Riemann-Lebesgue lemma to the integral (5) then one concludes that there must be $\mathcal{A}(t) \to 0$ as $t \to \infty$. This property and the relation (5) are an essence of the Fock–Krylov theory of unstable states [40,41].

So, within this approach the amplitude $\mathcal{A}(t)$, and thus the decay law $\mathcal{P}(t)$ of the metastable state $|\phi\rangle$, are determined completely by the density of the energy distribution $\omega(E)$ for the system in this state [40,41] (see also [11,42], and so on. (This approach is also applicable to models in quantum field theory [43,44]).

In [10] assuming that the spectrum of $\mathfrak{H}$ must be bounded from below and using the Paley–Wiener Theorem [45] it was proved that in the case of unstable states there must be

$$|\mathcal{A}(t)| \;\geq\; A\, e^{-b\, t^q}, \tag{6}$$

for $|t| \to \infty$. Here $A > 0$, $b > 0$ and $0 < q < 1$. This means that the decay law $\mathcal{P}(t)$ of metastable states decaying in the vacuum, (1), can not be described by an exponential function of time t if time t is suitably long, $t \to \infty$, and that for these lengths of time $\mathcal{P}(t)$ tends to zero as $t \to \infty$ more slowly than any exponential function of t. The analysis of the models of the decay processes shows that $\mathcal{P}(t) \simeq e^{-\frac{\Gamma_0 t}{\hbar}}$, (where Γ_0 is the decay rate of the considered state $|\phi\rangle$), to a very high accuracy for a wide time range t: At *canonical decay times*, i.e., from t suitably greater than some $T_0 \simeq t_0^{init} = 0$ but $T_0 > t_0^{init} = 0$ ($\mathcal{P}(t)$ has nonexponential power-like form for short times $t \in (t_0^{init}, T_0)$ – see, e.g. [10,11,46]) up to $t \gg \tau_0 = \frac{\hbar}{\Gamma_0}$ and smaller than $t = T_1$, where τ_0 is a lifetime and $t = T_1$ denotes the time t for which the long time nonexponential deviations of $\mathcal{A}(t)$ begin to dominate (see e.g., [10,11,47]). So, a notion *canonical decay times* denotes such times t that $t \in (T_0, T_1)$. From a more detailed analysis it follows that in the general case there is time $T_2 \gg T_1$ such that the decay law $\mathcal{P}(t)$ takes the inverse power-like form $t^{-\lambda}$, (where $\lambda > 0$), for suitably large $t \geq T_2 \gg T_1 \gg \tau_0$ [10,11,47,48]. This effect is in agreement with the general result (6). Effects of this type are sometimes called the "Khalfin effect" (see eg. [49]).

The problem how to detect possible deviations from the exponential form of $\mathcal{P}(t)$ in the long time region has been attracting attention of physicists since the first theoretical

predictions of such an effect [50–52]. The tests that have been performed over many years to examine the form of the decay laws for $t \gg \tau_0$ have not indicated any deviations from the exponential form of $\mathcal{P}(t)$ in the long time region. Nevertheless, conditions leading to the nonexponential behavior of the amplitude $\mathcal{A}(t)$ at long times were studied theoretically [53–61]. Conclusions following from these studies were applied successfully in experiment described in [62], where the experimental evidence of deviations from the exponential decay law at long times was reported. This result gives rise to another problem which now becomes important: if and how the long time deviations from the exponential decay law depend on the model considered (that is, on the form of $\omega(E)$, and if (and how) these deviations affect the energy of the metastable state and its decay rate in the long time region.

Note that in fact the amplitude $\mathcal{A}(t) = \langle\phi|\phi(t)\rangle$ contains information about the decay law $\mathcal{P}(t)$ of the state $|\phi\rangle$, that is about the decay rate Γ_0 of this state, as well as the energy of the system in this state. This information can be extracted from $\mathcal{A}(t)$. Using Schrödinger equation (3) one finds that within the problem considered

$$i\hbar\frac{\partial}{\partial t}\langle\phi|\phi(t)\rangle = \langle\phi|\mathfrak{H}|\phi(t)\rangle. \tag{7}$$

From this relation one can conclude that the amplitude $A(t)$ satisfies the following equation

$$i\hbar\frac{\partial\mathcal{A}(t)}{\partial t} = h(t)\,\mathcal{A}(t), \tag{8}$$

where

$$h(t) = \frac{\langle\phi|\mathfrak{H}|\phi(t)\rangle}{a(t)} \equiv \frac{\langle\phi|\mathfrak{H}|\phi(t)\rangle}{\langle\phi|\phi(t)\rangle}, \tag{9}$$

or equivalently

$$h(t) \equiv \frac{i\hbar}{\mathcal{A}(t)}\frac{\partial\mathcal{A}(t)}{\partial t}, \tag{10}$$

The effective Hamiltonian $h(t)$ governs the time evolution in the subspace of unstable states $\mathcal{H}_{\|} = \mathbb{P}\mathcal{H}$, where $\mathbb{P} = |\phi\rangle\langle\phi|$ (see [63] and also [24–26] and references therein). The subspace $\mathcal{H}\ominus\mathcal{H}_{\|} = \mathcal{H}_{\perp} \equiv \mathbb{Q}\mathcal{H}$ is the subspace of decay products. Here $\mathbb{Q} = \mathbb{I} - \mathbb{P}$. One meets the effective Hamiltonian $h(t)$ when one starts from the Schrödinger equation for the total state space $\mathcal{H}$ and looks for the rigorous evolution equation for a distinguished subspace of states $\mathcal{H}_{||} \subset \mathcal{H}$ [63–65]. In general $h(t)$ is a complex function of time and in the case of $\mathcal{H}_{\|}$ of two or more dimensions the effective Hamiltonian governing the time evolution in such a subspace is a non-hermitian matrix $H_{\|}$ or non-hermitian operator. There is

$$h(t) = E(t) - \frac{i}{2}\Gamma(t), \tag{11}$$

where $E(t) = \Re\,[h(t)]$, $\Gamma(t) = -2\,\Im\,[h(t)]$, are the instantaneous energy (mass) $E(t)$ and the instantaneous decay rate, $\Gamma(t)$. (Here $\Re\,(z)$ and $\Im\,(z)$ denote the real and imaginary parts of z, respectively).

The quantity $\Gamma(t) = -2\,\Im\,[h(t)]$ is interpreted as the decay rate, because it satisfies the definition of the decay rate used in quantum theory. Simply, using (10) it is easy to check that

$$\begin{aligned}\frac{\Gamma(t)}{\hbar} &\stackrel{\text{def}}{=} -\frac{1}{\mathcal{P}(t)}\frac{\partial\mathcal{P}(t)}{\partial t}\\ &= -\frac{1}{|\mathcal{A}(t)|^2}\frac{\partial|\mathcal{A}(t)|^2}{\partial t} \equiv -\frac{2}{\hbar}\,\Im\,[h(t)].\end{aligned} \tag{12}$$

The formula (9) for $h(t)$ can be used to show that $h(t)$ can not be constant in time. Indeed, if to rewrite the numerator of the righthand side of (9) as follows,

$$\langle\phi|\mathfrak{H}|\phi(t)\rangle \equiv \langle\phi|\mathfrak{H}|\phi\rangle\,a(t) + \langle\phi|\mathfrak{H}|\phi(t)\rangle_{\perp}, \tag{13}$$

where $|\phi(t)\rangle_{\perp} = \mathbb{Q}|\phi(t)\rangle$, and $\langle\phi|\phi(t)\rangle_{\perp} = 0$, then one can see that there is a permanent contribution of decay products described by $|\phi(t)\rangle_{\perp}$ to the energy of the metastable state considered. The intensity of this contribution depends on time t. This contribution into the instantaneous energy is practically very small and constant in time to a very good approximation at canonical decay times, whereas at the transition times, when $t > T_1$ (but $t < T_2$, it is fluctuating function of time and the amplitude of these fluctuations may be significant. What is more relations (9) and (13) allow one to proof that in the case of metastable states $\Re\,[h(t)] \neq const$ for $t > 0$. Namely, using these relations one obtains that

$$h(t) = E_{\phi} + \frac{\langle\phi|\mathfrak{H}|\phi(t)\rangle_{\perp}}{\mathcal{A}(t)}, \tag{14}$$

where E_{ϕ} is the expectation value of $\mathfrak{H}$: $E_{\phi} = \langle\phi|\mathfrak{H}|\phi\rangle$. From this relation one can see that $h(0) = E_{\phi}$ if the matrix elements $\langle\phi|\mathfrak{H}|\phi\rangle$ exists. It is because $|\phi(t=0)\rangle_{\perp} = 0$ and $\mathcal{A}(t=0) = 1$.

Note now that from (9) and (10) it follows that $h(t)$ must be a continuous function of time t for $t \geq 0$. So, if to assume *a contrario* that $h(t) = const.$ for all $t \geq 0$ then using (9) and (13) one immediately infers that it is possible only if for all $t \geq 0$ there is $\langle\phi|\mathfrak{H}\mathbb{Q}|\phi(t)\rangle \equiv c_h\langle\phi|\phi(t)\rangle$, where $c_h = const.$ From the definition of $\mathbb{P}$ and $\mathbb{Q}$ it results that in such a case there must be $\langle\phi|\mathfrak{H}\mathbb{Q}|\phi(t)\rangle|_{t=0} = \langle\phi|\mathfrak{H}\mathbb{Q}|\phi\rangle = 0$, but at the same time there is, $\langle\phi|\mathfrak{H}\mathbb{Q}|\phi(t)\rangle|_{t=0} = c_h\langle\phi|\phi\rangle$ at $t = 0$. So only solution is $c_h \equiv 0$. Now because of the continuity of $h(t)$ the solution $c_h = 0$ is valid also for all $t > 0$. Thus the case $h(t) = const$ for all $t \geq 0$ occurs only if $\langle\phi||\mathfrak{H}\mathbb{Q}|\phi(t)\rangle = 0$ for all $t \geq 0$. It is possible only if $[\mathbb{Q}, \mathfrak{H}] \equiv [\mathbb{P}, \mathfrak{H}] = 0$ but then the vector $|\phi\rangle$ defining the projector $\mathbb{P} = |\phi\rangle\langle\phi|$ can not describe a metastable state. In a result there must be $h(t) \neq const$ for a metastable state $|\phi\rangle$.

Using projectors $\mathbb{P}$, $\mathbb{Q}$, Eq. (10) can be rewritten as follows (see, eg. [28,63])

$$h(t) \equiv E_\phi + v(t), \tag{15}$$

and

$$v(t) = \frac{\langle\phi|\mathfrak{H}\mathbb{Q}|\phi(t)\rangle}{\mathcal{A}(t)} = \frac{\langle\phi|\mathfrak{H}|\phi(t)\rangle_\perp}{\mathcal{A}(t)}. \tag{16}$$

From the definition of $\mathbb{P}$ it follows that $|\phi(t=0)\rangle_\perp = 0$, which means that $v(0) = 0$ and $h(0) = E_\phi$ and thus

$$E(0) = E(t=0) = E_\phi = \langle\phi|H|\phi\rangle, \tag{17}$$

(if the matrix element $\langle\phi|H|\phi\rangle$ exists), and

$$E(t) \simeq E_\phi = \langle\phi|H|\phi\rangle, \quad \text{for} \quad t \to 0. \tag{18}$$

So, in a general case, at canonical decay times $t < T_1$, there is (see [28,63])

$$E(t) \simeq E_0 \stackrel{\text{def}}{=} E_\phi - \Delta_\phi^{(1)} \equiv E(0) - \Delta_\phi^{(1)}, \tag{19}$$

where $\Delta_\phi^{(1)} = -\Re\left[v(t)\right]$, wherein $v(t) \simeq const.$ at canonical decay times, and $|\Delta_\phi^{(1)}| \ll |E_\phi|$.

The representation of the survival amplitude $\mathcal{A}(t)$ as the Fourier transform (5) can be used to find the late time asymptotic form of $\mathcal{A}(t)$, $\mathcal{P}(t)$ and the instantaneous energy $E(t)$ and decay rate $\Gamma(t)$ (see [24,25]). There is,

$$E(t) \underset{t\to\infty}{\sim} \stackrel{\text{def}}{=} E_{lt}(t) = E_{min} + \frac{\alpha_2}{t^2} + \frac{\alpha_4}{t^4} + \cdots, \tag{20}$$

and

$$\Gamma(t) \underset{t\to\infty}{\sim} \stackrel{\text{def}}{=} \Gamma_{lt}(t) = \frac{\alpha_1}{t} + \frac{\alpha_3}{t^3} + \cdots, \tag{21}$$

where α_k, are real numbers for $k = 1, 2, \ldots$ and $\alpha_1 > 0$ and the sign of α_k for $k > 1$ depends on the model considered (see [25]).

An important property of $h(t)$ can be found using the relation $|\phi(t)\rangle = \exp\left[-\frac{i}{\hbar}t\mathfrak{H}\right]|\phi\rangle$, which means that the amplitude $\mathcal{A}(t)$ can be written as follows: $\mathcal{A}(t) \equiv \langle\phi|\exp\left[-\frac{i}{\hbar}t\mathfrak{H}\right]|\phi\rangle$. It is not difficult to see that this form of $\mathcal{A}(t)$ and hermiticity of $\mathfrak{H}$ imply that [11]

$$(\mathcal{A}(t))^* = \mathcal{A}(-t). \tag{22}$$

The conclusion resulting from (22) and from the relation (10) is that

$$h(-t) = (h(t))^*. \tag{23}$$

Therefore there must be

$$E(-t) = E(t) \quad \text{and} \quad \Gamma(-t) = -\Gamma(t), \tag{24}$$

That is, the instantaneous energy $E(t) = \Re\left[h(t)\right]$ is an even function of time t and the instantaneous decay rate $\Gamma(t) = -2\,\Im\left[h(t)\right]$ an odd function of t.

3 Calculations and results

As it was said in the previous Section in order to calculate the survival amplitude $\mathcal{A}(t)$ within the Fock–Krylov theory of unstable states we need the energy density distribution function $\omega(E)$. From an analysis of general properties of the energy (mass) distribution functions $\omega(E)$ of real unstable systems it follows that $\omega(E)$ has properties analogous to the scattering amplitude, i.e., it can be decomposed into a threshold factor, a pole-function $P(E)$ with a simple pole and a smooth form factor $f(E)$: There is

$$\omega(E) = \Theta(E - E_{\min})\,(E - E_{\min})^{\alpha_l}\,P(E)\,f(E), \tag{25}$$

where α_l depends on the angular momentum l through $\alpha_l = \alpha + l$, [11] (see equation (6.1) in [11]), $0 \le \alpha < 1$) and $\Theta(E)$ is a step function: $\Theta(E) = 0$ for $E \le 0$ and $\Theta(E) = 1$ for $E > 0$ and $f(E)$ is such a function that $P(E)\,f(E) \to 0$ as $E \to \infty$. The simplest choice is to take $\alpha = 0, l = 0$, $f(E) = 1$ and to assume that $P(E)$ has a Breit–Wigner (BW) form of the energy distribution density. (The mentioned Breit–Wigner distribution was found when the cross-section of slow neutrons was analyzed [66].) It turns out that the decay curves obtained in this simplest case are very similar in form to the curves calculated for the above described more general $\omega(E)$, (see [67,68] and analysis in [11]). So to find the most typical properties of the decay process it is sufficient to make the relevant calculations for $\omega(E)$ modeled by the the Breit–Wigner distribution of the energy density:

$$\omega(E) \equiv \omega_{BW}(E) \stackrel{\text{def}}{=} \frac{N}{2\pi}\,\Theta(E - E_{min})\,\frac{\Gamma_0}{(E - E_0)^2 + (\frac{\Gamma_0}{2})^2}, \tag{26}$$

where N is a normalization constant. The parameters E_0 and Γ_0 correspond to the energy of the system in the metastable state and its decay rate at the exponential (or canonical) regime of the decay process. E_{min} is the minimal (the lowest) energy of the system. For $\omega(E) = \omega_{BW}(E)$ one can find relatively easy an analytical form of $\mathcal{A}(t)$ at very late times as well as an asymptotic analytical form of $h(t)$, $E(t)$ and $\Gamma(t)$ for such times. In previous Section it was stated that $\omega(E)$ contains information characterizing the given metastable state: In the case $\omega(E) = \omega_{BW}(E)$ quantities E_0, Γ_0 and E_{min} are exactly the parameters characterizing the metastable state considered. The different values of these parameters correspond to different metastable states.

Inserting $\omega_{BW}(E)$ into formula (5) for the amplitude $\mathcal{A}(t)$ and assuming for simplicity that $t_0^{init} = 0$, after some algebra one finds that

$$\mathcal{A}(t) = \frac{N}{2\pi}\,e^{-\frac{i}{\hbar}E_0 t}\,\mathcal{I}_\beta\left(\frac{\Gamma_0 t}{\hbar}\right) \tag{27}$$

where

$$\mathcal{I}_\beta(\tau) \stackrel{\text{def}}{=} \int_{-\beta}^{\infty} \frac{1}{\eta^2+\frac{1}{4}}\, e^{-i\eta\tau}\, d\eta. \tag{28}$$

Here $\tau = \frac{\Gamma_0 t}{\hbar} \equiv \frac{t}{\tau_0}$, τ_0 is the lifetime, $\tau_0 = \frac{\hbar}{\Gamma_0}$, and $\beta = \frac{E_0 - E_{min}}{\Gamma_0} > 0$.

Having the amplitude $\mathcal{A}(t)$ we can use it to analyze properties of the instantaneous energy $E(t)$ and instantaneous decay rate $\Gamma(t)$. These quantities are defined using the effective Hamiltonian $h(t)$ which is build from $\mathcal{A}(t)$. In order to find $h(t)$ we need the quantity $i\,\hbar\,\frac{\partial \mathcal{A}(t)}{\partial t}$ (see (10)). From Eq. (27) one finds that

$$i\hbar\frac{\partial \mathcal{A}(t)}{\partial t} = E_0\,\mathcal{A}(t) + \Gamma_0\,\frac{N}{2\pi}\,e^{-\frac{i}{\hbar}E_0 t}\,\mathcal{J}_\beta(\tau(t)), \tag{29}$$

where

$$\mathcal{J}_\beta(\tau) = \int_{-\beta}^{\infty} \frac{x}{x^2+\frac{1}{4}}\, e^{-ix\tau}\, dx, \tag{30}$$

or simply (see (28)),

$$\mathcal{J}_\beta(\tau) \equiv i\frac{\partial \mathcal{I}_\beta(\tau)}{\partial \tau}. \tag{31}$$

Now the use of (27), (29) and (10) leads to the conclusion that within the model considered there is,

$$h(t) = i\hbar\frac{1}{\mathcal{A}(t)}\,\frac{\partial \mathcal{A}(t)}{\partial t} = E_0 + \Gamma_0\,\frac{\mathcal{J}_\beta(\tau(t))}{\mathcal{I}_\beta(\tau(t))}, \tag{32}$$

which means that

$$E(t) = \Re\,[h(t)] = E_0 + \Gamma_0\,\Re\left[\frac{\mathcal{J}_\beta(\tau(t))}{\mathcal{I}_\beta(\tau(t))}\right], \tag{33}$$

and

$$\Gamma(t) = -2\,\Im[h(t)] = -2\,\Gamma_0\,\Im\left[\frac{\mathcal{J}_\beta(\tau(t))}{\mathcal{I}_\beta(\tau(t))}\right]. \tag{34}$$

Using (27)–(34) one can find analytically within the model considered late time asymptotic forms of $E(t)$ and $\Gamma(t)$. There is for $t \to \infty$ (see [39,69]):

$$\begin{aligned} E(t)|_{t\to\infty} &= \Re\,[h(t)]|_{t\to\infty} \\ &\simeq E_{min} \;-\; 2\,\frac{\beta}{\Gamma_0\,(\beta^2+\frac{1}{4})}\,\left(\frac{\hbar}{t}\right)^2 + \ldots, \end{aligned} \tag{35}$$

and,

$$\Gamma(t)|_{t\to\infty} = -2\Im\,[h(t)] \simeq 2\,\frac{\hbar}{t} + \ldots. \tag{36}$$

In order to visualize properties of $E(t)$ it is convenient to use the following function

$$\kappa(t) \stackrel{\text{def}}{=} \frac{E(t) - E_{\min}}{E_0 - E_{\min}}. \tag{37}$$

Using (33) one finds that

$$E(t) - E_{min} = E_0 - E_{min} + \Gamma_0\,\Re\left[\frac{\mathcal{J}_\beta(\tau)}{\mathcal{I}_\beta(\tau)}\right], \tag{38}$$

Now, it to divide two sides of Eq. (38) by $(E_0 - E_{min})$ then one obtains the function $\kappa(t)$ (see (37)) we are looking for

$$\kappa(\tau(t)) = 1 + \frac{1}{\beta}\,\Re\left[\frac{\mathcal{J}_\beta(\tau(t))}{\mathcal{I}_\beta(\tau(t))}\right]. \tag{39}$$

Using the above derived formulae one can find numerically within the model considered the survival probability $\mathcal{P}(t)$ and $\kappa(\tau(t))$ describing the behavior of the instantaneous energy $E(t)$. Results of these calculations performed for chosen β are presented in Fig. 1a, b.

From the results of the previous section and those presented above and also in Fig. 1b it follows that a behavior of the instantaneous energy $E(t)$ and the instantaneous decay rate $\Gamma(t)$ differ depending on the time domain in which we examine their values. At canonical decay times $0 \ll t < T_1$ they are close to a good approximation to the values resulting from the Weisskopf–Wigner theory of spontaneous emission. For times $t > T_2$ their behavior is described by late time asymptotic formulae (20) and (21). At transition time region of times $T_1 < t < T_2$ the instantaneous energy $E(t)$ and the instantaneous decay rate $\Gamma(t)$ decrease to their late time asymptotic forms. Unfortunately the information of how fast $E(t)$ tends to its asymptotic form (20) at times $t > T_1$ is invisible in $\kappa(\tau(t))$ – see Fig. 1b. In order to remove this deficiency one should come back to the Eq. (37). Namely by manipulating Eqs. (37)–(39) we get the desired result,

$$\frac{E(\tau(t))}{E_{min}} = 1 + \left(\frac{E_0}{E_{min}} - 1\right)\,\kappa(\tau(t)). \tag{40}$$

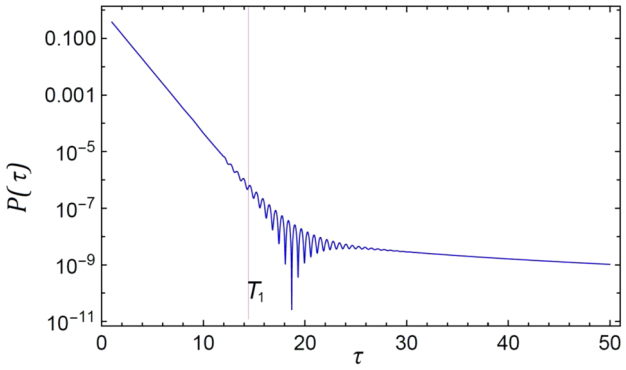
This last equation can be used to show how fast $E(t)$ tends to its asymptotic form (20) at times $t > T_1$ for assumed β and the ratio $\frac{E_0}{E_{min}}$. Results obtained for $\omega(E) = \omega_{BW}(E)$ are presented in graphical form in Figs. 1, 2, 3 and 4.

We show now that similar results can be obtained not only in the approximate case $\omega_{BW}(E)$ of the density of the energy (mass) distribution $\omega(E)$ but also when one considers a more general forms of $\omega(E)$. As it was mentioned $\omega(E)$ should have a form given by Eq. (25), where the simple pole contribution, $P(E)$, is often modeled by $\omega_{BW}(E)$.

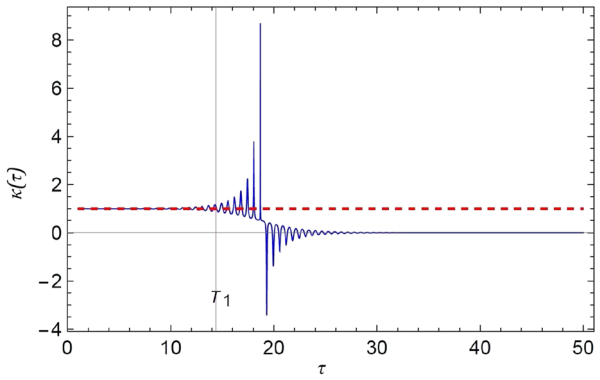
Guided by this observation we follow [67,68] and assume that

$$\begin{aligned} \omega_\eta(E) &= N\,\sqrt{E - E_{min}} \\ &\times \frac{\sqrt{\Gamma_0}}{(E-E_0)^2 + (\Gamma_0/2)^2}\, e^{-\eta\,\frac{E - E_0 + E_{min}}{\Gamma_0}}, \end{aligned} \tag{41}$$

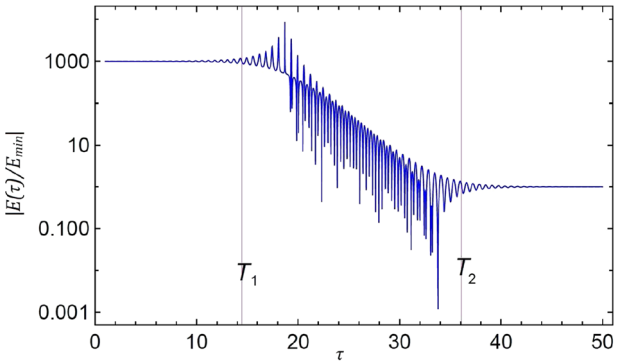
with $\eta > 0$. Inserting this $\omega_\eta(E)$ into (5) we can calculate the survival amplitude $\mathcal{A}(t)$ for this case and then the effective Hamiltonian $h(t)$ and thus $E(t)$ that we want to study.



(a) The survival probability $\mathcal{P}(\tau)$



(b) $\kappa(\tau)$

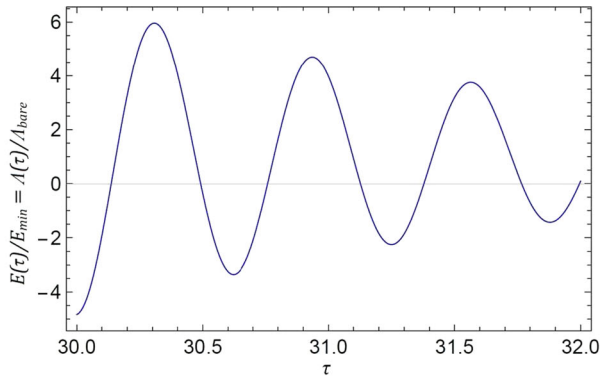


(c) The ratio $E(\tau)/E_{min}$

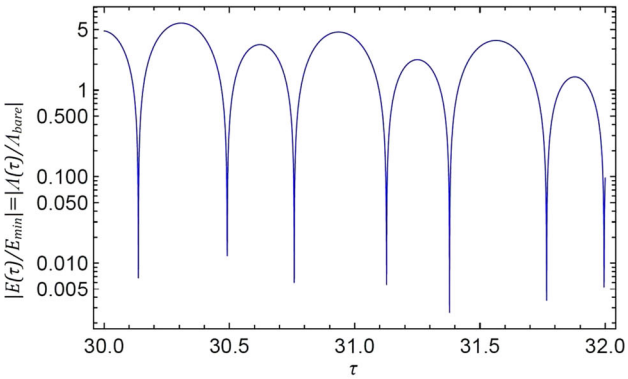
Fig. 1 Results obtained for $\omega_{BW}(E)$ given by Eq. (26). The case $\beta = 10$. **a** A decay curve $\mathcal{P}(\tau) = |\mathcal{A}(\tau)|^2$; **b** an illustration of the typical behavior of energy $E(\tau)$: The solid line – $\kappa(\tau) \equiv (E(\tau) - E_{min})/(E_0 - E_{min})$, The dashed line – $E(\tau) = E_0 = \text{const}$ ($\kappa(\tau) = 1$); **c** the modulus of the ratio $E(\tau)/E_{min}$, the case $E_0/E_{min} = 1000$. In all figures the time, t, is measured in lifetimes τ_0: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$ is the lifetime

Analogously to the above analyzed case of the Breit–Wigner energy density distribution, $\omega_{BW}(E)$, we get

$$\frac{E(\tau) - E_{min}}{E_0 - E_{min}} = \kappa_\eta(\tau), \tag{42}$$



(a) y–axis – Normal scale



(b) y–axis – Logarithmic scale: Modulus of the ratio $E(\tau)/E_{min}$

Fig. 2 The ratio $E(\tau)/E_{min}$ (an enlarged part of Fig 1c). The case $\beta = 10$ and $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 1000$. In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$

where

$$\kappa_\eta(\tau) = 1 + \ + \frac{1}{\beta}\,\Re\left[\frac{\mathcal{J}^\eta_\beta(\tau)}{\mathcal{I}^\eta_\beta(\tau)}\right]. \tag{43}$$

and

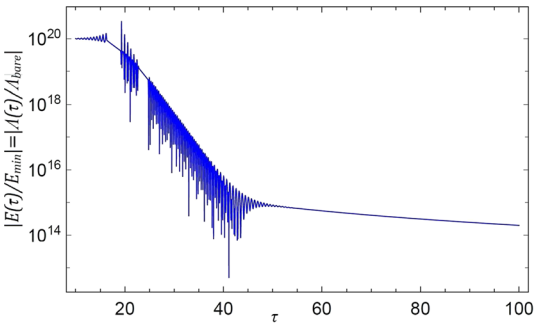
$$\mathcal{J}^\eta_\beta = \int_{-\beta}^{\infty} \frac{x\sqrt{x+\beta}}{x^2+\frac{1}{4}}\, e^{-x\,\eta}\, e^{-ix\tau}\, dx, \tag{44}$$

$$\mathcal{I}^\eta_\beta = \int_{-\beta}^{\infty} \frac{\sqrt{x+\beta}}{x^2+\frac{1}{4}}\, e^{-x\,\eta}\, e^{-ix\tau}\, dx. \tag{45}$$

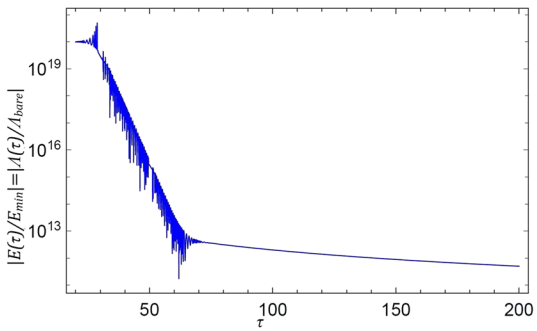
After some algebra, analogously to the result (40), we obtain the following relation in the considered here case:

$$\frac{E(\tau(t))}{E_{min}} = 1 + \left(\frac{E_0}{E_{min}} - 1\right)\kappa_\eta(\tau(t)). \tag{46}$$

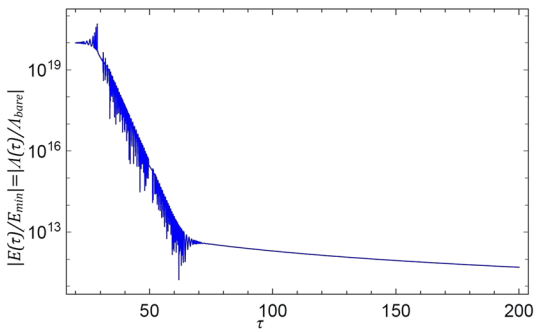
This equation, similarly to the Eq. (40), can be used to show how fast $E(t)$ tends to its asymptotic form (20) at times $t > T_1$ for assumed $\omega(E)$, and for chosen β, η and the ratio $\frac{E_0}{E_{min}}$. Results obtained for $\omega(E) = \omega_\eta(E)$ are presented graphically in Figs. 5, 6, 7, 8 and 9.



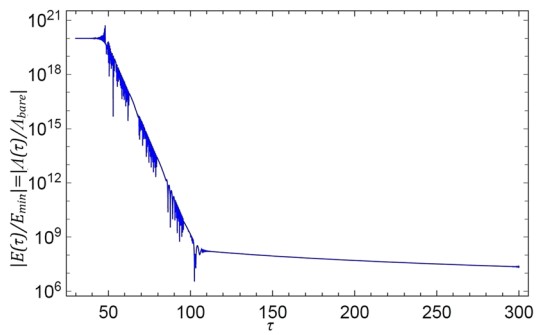
(a) The case $\beta = 10$.



(b) The case $\beta = 100$.

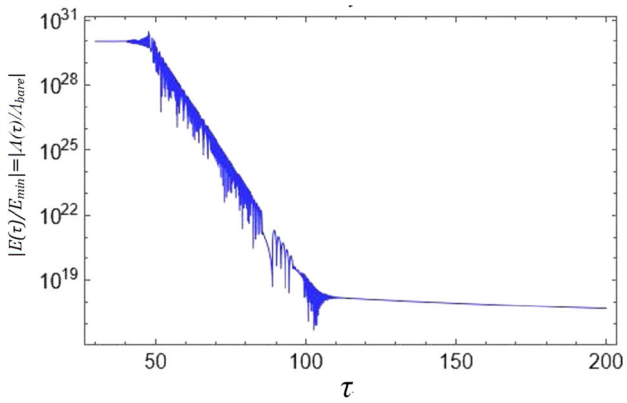


(c) The case $\beta = 1000$.

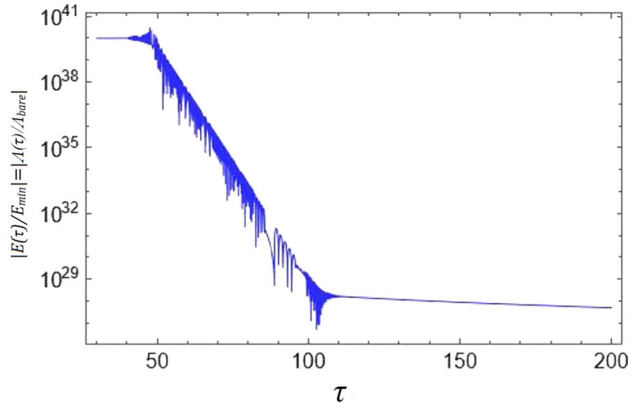


(d) The case $\beta = 10000$.

Fig. 3 The ratio $E(\tau)/E_{min} \equiv \Lambda(\tau)/\Lambda_{bare}$ obtained for a $\omega_{BW}(E)$ given by formula (26). In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$. The case $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 10^{20}$



(a) The case $E_0/E_{min} = \Lambda_0/\Lambda_{min} = 10^{30}$.



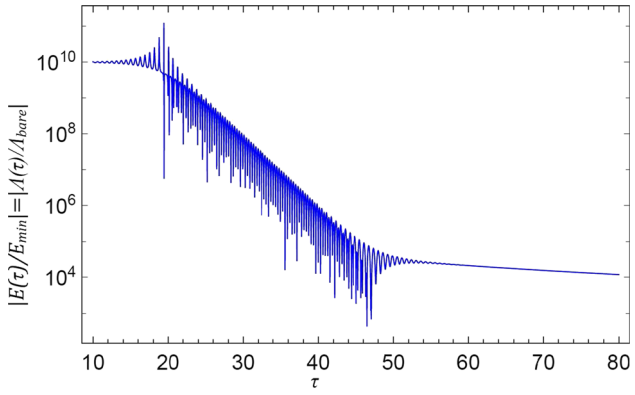
(b) The case $E_0/E_{min} = \Lambda_0/\Lambda_{min} = 10^{40}$

Fig. 4 The ratio $E(\tau)/E_{min} \equiv \Lambda(\tau)/\Lambda_{bare}$ obtained for a $\omega_{BW}(E)$ given by formula (26). In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$. The case $\beta = 10000$

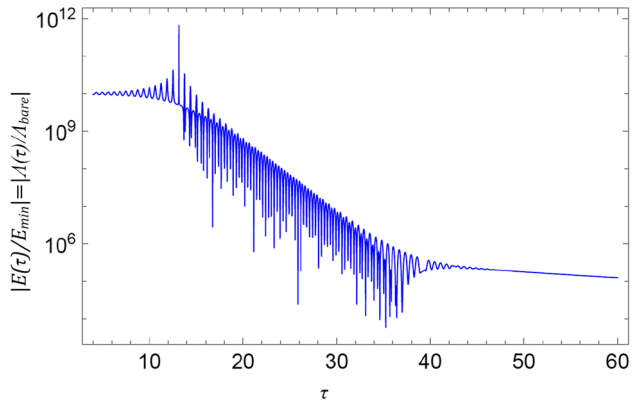
4 Discussion: possible cosmological applications

In previous sections we used the Krylov–Fock theory of unstable states to find late time properties of the survival amplitude and instantaneous energy. Similar estimations of the late time behavior of the survival amplitude can be also found by means another method, e.g. methods of the quantum scattering theory. This method was used in [70], where the long time deviations from the exponential form of the decay law were also studied and where one can find attempts to estimate the value of T_1. Newton on page 624 in Chap. 19 of [70] writes:

Let us now take some simple examples. Consider the case of a nuclear deexcitation by γ-ray emission. In a typical instance the energy may be $E_0 \sim 200$ keV and the lifetime $\tau \sim 10^{-8}$ sec so that $\Gamma/E_0 \sim 3 \times 10^{-13}$. ....

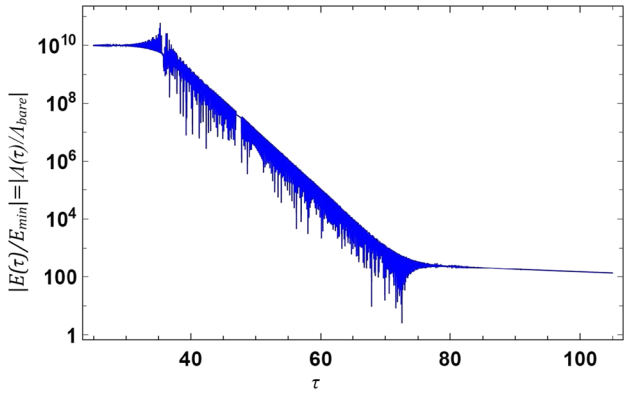


(a) The case $\beta = 10$, $\eta = 0.25$.

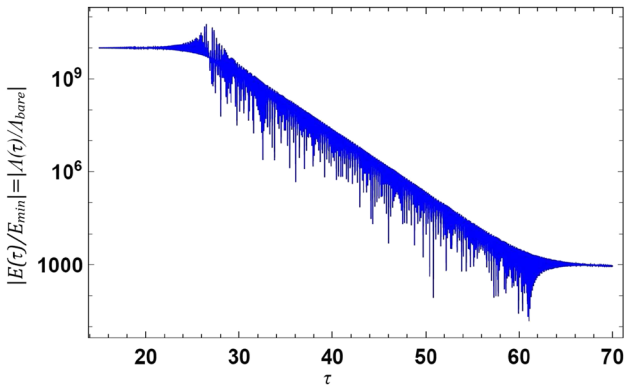


(b) The case $\beta = 10$, $\eta = 0.5$

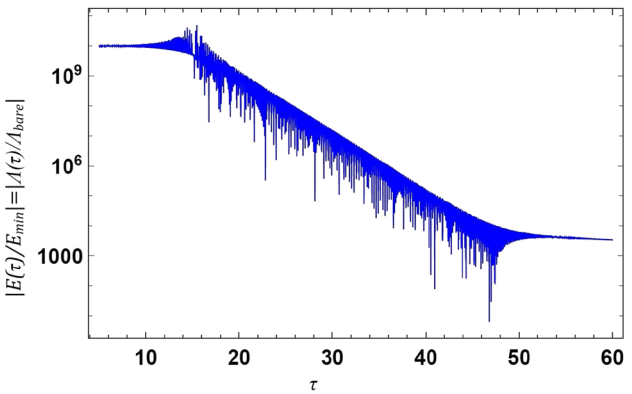
Fig. 5 A ratio $E(\tau)/E_{min} \equiv \Lambda(\tau)/\Lambda_{bare}$ obtained for a $\omega_{BW}(E)$ given by formula (26). In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$. The case $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 10^{10}$



(a) The case $\beta = 100$, $\eta = 0.01$.



(b) The case $\beta = 100$, $\eta = 0.05$.

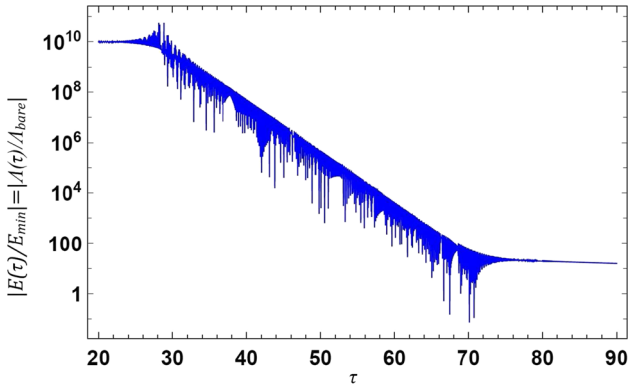


(c) The case $\beta = 100$, $\eta = 0.1$

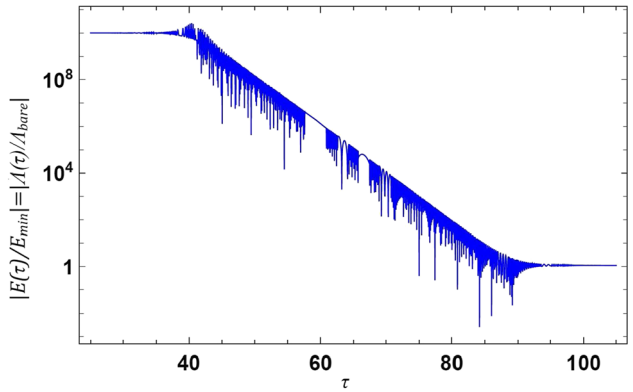
Fig. 6 A ratio $E(\tau)/E_{min} \equiv \Lambda(\tau)/\Lambda_{bare}$ obtained for a $\omega_{BW}(E)$ given by formula (26). The case $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 10^{10}$. In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$

The decay curve should be roughly exponential after 1/8 of a mean life from the peak and excellently exponential after 2. It then remains exponential for 10^{17} lifetimes! In order to destroy most of the exponential-decay curve, one would have to move the detector away to a distance of about 10^{22} miles. .....

Note that 10^{22} miles equals 1.7×10^9 light years approximately, so the effect described by Newton and in other papers should be visible when analyzing the spectrum of the electromagnetic radiation emitted by cosmic objects at a distance of 1.7×10^9 light years or more from the Earth's observer. Of course in this case it is practically impossible to restore the late time form of the decay curve by means measurements but as it has already been shown for times $t > T_1$ the following quantum effect should take place during the late time phase of the quantum decay process: The energy of the system in the initial metastable state, which is approximately equal E_0 at canonical decay times, is forced to decrease as $1/t^2$ to the minimal energy $E_{min} < E_0$. This effect was described in previous sections and earlier in [24–28,71]. In the case of the emission of the electromagnetic radiation the excited atomic level can be considered as an the initial metastable state of the system. (In Newton's example the excited state of a nucleus is the initial state). So, in the case of very distant cosmic objects emitting electromagnetic radiation this effect should contribute into the redshift making it apparently larger than it really is. Such a possibility seems to be very impor-

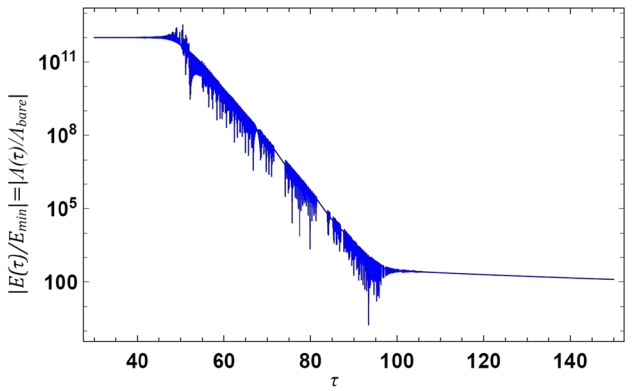


(a) The case $\beta = 1000$, $\eta = 0.01$.

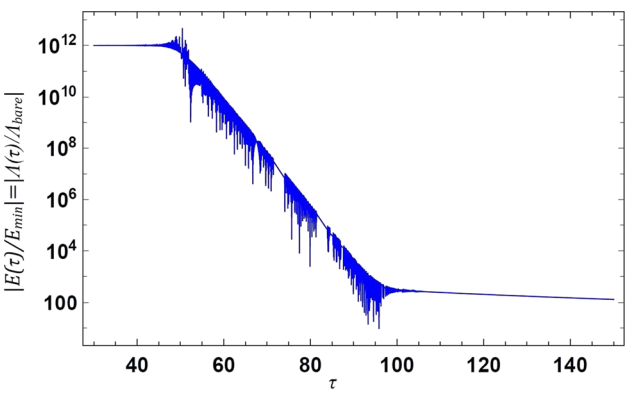


(b) The case $\beta = 10000$, $\eta = 0.001$.

Fig. 7 A ratio $E(\tau)/E_{min} \equiv \Lambda(\tau)/\Lambda_{bare}$ obtained for a $\omega_(E)$ given by formula (41). The case $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 10^{10}$. In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$



(a) The case $\beta = 1000$, $\eta = 0.0001$.



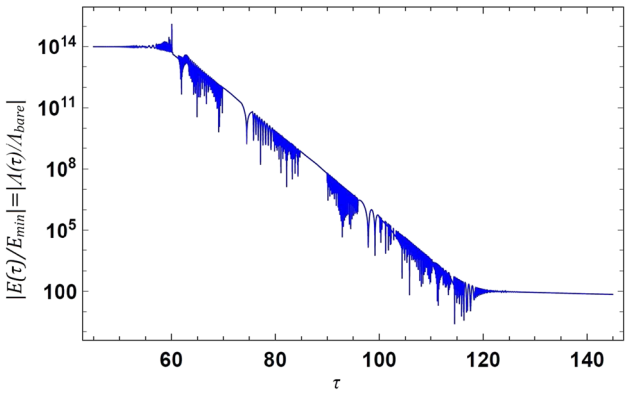
(b) The case $\beta = 1000$, $\eta = 0.00001$.

Fig. 8 A ratio $E(\tau)/E_{min} \equiv \Lambda(t)/\Lambda_{bare}$ obtained for a $\omega(E)$ given by formula (41). The case $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 10^{12}$. In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$

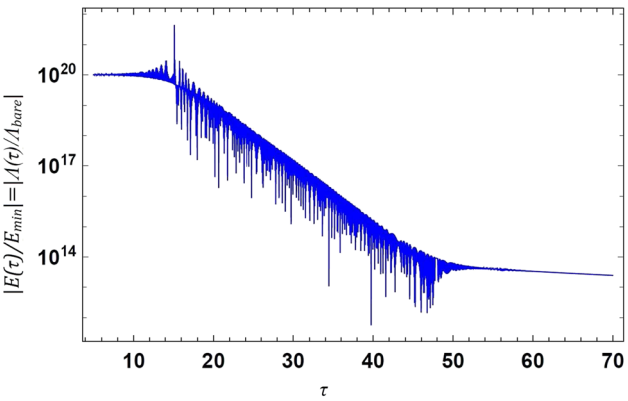
tant as this effect may distort the observational results (red shift, luminosity, and so on) and thus lead to wrong conclusions (For example estimations of a tension of the Hubble parameter are based on the measurements of the red shift of distant astrophysical objects, etc.). Possible changes of the red shift caused by this possible effect are described in details in [24] (see also [71]), where an influence of this property on measured values of possible deviations of the fine structure constant α as well as other astrophysical and cosmological parameters were studied and this is why this is only signalized here.

Let us analyze now results presented in Figs. 1c, 2, 3, 4, 5, 6, 7, 8 and 9. In Fig. 1c, in contrast to Fig. 1b, it can be seen how quickly in the transition time region, $T_1 < t < T_2$, the energy $E(t)$ is reduced to its late time asymptotic form, $E_{lt}(t) = E_{min} + \alpha_2/t^2 + \ldots$. Namely, if to compare Figs. 1c, 3, 4, 5, 6, 7, 8 and 9 and Fig. 2a a conclusion can be drawn that at transition times $T_1 < t < T_2$ the instantaneous energy $E(t)$, (where $E(t) \simeq E_0$ for times $t < T_1$ and $t > T_0$), decreases like an oscillatory modulated exponential function until reaching its asymptotic form $E_{lt}(t)$. This quantum effect is very strong and efficient: as can be seen from Figs. 1c, 3, 4, 5, 6, 7, 8 and 9, the reduction of energy $E(t) \simeq E_0$ (for $t < T_1$) depending on the values of parameters β and η, may be more than 10 orders: It can be $E(T_1)/E_{lt}(T_2) \sim 10^{12}$ – see Figs. 3d, 9a, and more. By selecting the appropriate parameters β in $\omega_{BW}(E)$, or β and η in $\omega_\eta(E)$, or the other, more appropriate, energy density distribution function $\omega(E)$, even much greater reduction of the energy $E(t) \simeq E_0$ can be achieved when time t runs from $t = T_1$ to $t = T_2$. Potentially, this effect can reduce the energy by tens of orders or more. Taking this property into account it seems to be reasonable to hypothesize that this quantum effect can help to explain the cosmological constant problem, which is consequence of the interpretation of the dark energy as the vacuum energy: the observed present value of the cosmological constant is 120 orders of magnitude smaller than we expect from quantum physics calculations.

One can meet in the large literature cosmological models with metastable vacuum (see, eg. [72–75] and many others). Some of these models admit the lifetime of the Universe to be very small [73] or even smaller than the Planck time (see



(a) The case $\beta = 10000$, $\eta = 0.0001$, $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 10^{14}$.



(b) The case $\beta = 100$, $\eta = 0.1$, $E_0/E_{min} = \Lambda_0/\Lambda_{bare} = 10^{20}$.

Fig. 9 A ratio $E(\tau)/E_{min} \equiv \Lambda(t)/\Lambda_{bare}$ obtained for a $\omega(E)$ given by formula (41). In all figures the time t is measured in lifetimes: $\tau = t/\tau_0$: $\tau = t/\tau_0$ and $\tau_0 = \hbar/\Gamma_0$

[74,75]). Of course this decaying vacuum is described by the quantum state corresponding to a local minimum of the energy density which is not the absolute minimum of the energy density of the system considered. In such a case the formalism described in this paper is fully applicable. Let us consider now a cosmological scenario in which the lifetime of the false vacuum is shorter than the duration of the inflation phase and its decay process began before (or just before) the beginning of the inflation phase and then it is continued during the inflationary epoch and later. This scenario corresponds with the hypothesis analyzed by Krauss and Dent [29,30]. Their hypothesis suggests that some false vacuum regions do survive well up to the time T_1 or later. So, let $|\phi\rangle = |0\rangle^{\mathrm{F}}$ be a false and $|0\rangle^{\mathrm{T}}$ true vacuum states, respectively, and $E_0 = E_0^{\mathrm{F}}$ be the energy of a state corresponding to the false vacuum measured at the canonical decay times, which leads to the vacuum energy density calculated using quantum field theory methods. Let $E^{\mathrm{T}} = E_{min}$ be the energy of true vacuum (i.e., the true ground state of the system). The fact that the decay of the false vacuum is the quantum decay process [4,5,29,30,76] means that state vector corresponding to the false vacuum is a quantum unstable (or metastable) state. Therefore all the general properties of quantum unstable systems must also occur in the case of such a quantum unstable state as the false vacuum. This applies in particular to such properties as late time deviations from the exponential decay law and properties of the energy $E(t) = E^{\mathrm{F}}(t)$ of the system in the quantum false vacuum state. In [27] it was pointed out the energy of those false vacuum regions which survived up to T_1 and much later differs from E_0^{F}.

If one wants to generalize the above results obtained on the basis of quantum mechanics to quantum field theory one should take into account among others a volume factors so that survival probabilities per unit volume should be considered and similarly the energies and the decay rate: $E \mapsto \rho(E) = \frac{E}{V_0}$, $\Gamma_0 \mapsto \gamma = \frac{\Gamma_0}{V_0}$, where $V_0 = V(t_0^{init})$ is the volume of the considered system at the initial instant t_0^{init}, when the time evolution starts. The volume V_0 is used in these considerations because the initial unstable state $|\phi\rangle \equiv |0\rangle^{\mathrm{F}}$ at $t = t_0^{init} = 0$ is expanded into eigenvectors $|E\rangle$ of $\mathfrak{H}$, (where $E \in \sigma_c(\mathfrak{H})$), and then this expansion is used to find the density of the energy distribution $\omega(E)$ at this initial instant t_0^{init}. Now, if we identify $\rho_{de}(t_0^{init})$ with the energy E_0^{F} of the unstable system divided by the volume V_0: $\rho_{de}(t_0^{init}) \equiv \rho_0^{\mathrm{F}} \equiv \rho_0^{qft} \stackrel{\mathrm{def}}{=} \rho_{de}^0 = \frac{E_0^{\mathrm{F}}}{V_0}$ and $\rho_{bare} = \frac{E_{min}}{V_0}$, (where ρ_0^{qft} is the vacuum energy density calculated using quantum field theory methods) then it is easy to see that the mentioned changes $E \mapsto \frac{E}{V_0}$ and $\Gamma_0 \mapsto \frac{\Gamma_0}{V_0}$ do not changes the parameter β:

$$\beta = \frac{E_0^{\mathrm{F}} - E_{min}}{\Gamma_0} \equiv \frac{\rho_{de}^0 - \rho_{bare}}{\gamma_0} > 0, \tag{47}$$

(where $\gamma_0 = \Gamma_0/V_0$, or equivalently, $\Gamma_0/V_0 \equiv \frac{\rho_{de}^0 - \rho_{bare}}{\beta}$). This means that the relations (20), (35), (37), (40), (42), (43), (46) can be replaced by corresponding relations for the densities ρ_{de} or Λ (see, eg., [28,39,77,78]). Simply, within this approach $E(t) = E^{\mathrm{F}}(t)$ corresponds to the running cosmological constant $\Lambda(t)$ and E_{min} to the Λ_{bare}. For example, we have

$$\begin{aligned}\kappa(t) &= \frac{E^{\mathrm{F}}(t) - E_{min}}{E_0^{\mathrm{F}} - E_{min}} \equiv \frac{\frac{E^{\mathrm{F}}(t)}{V_0} - \frac{E_{min}}{V_0}}{\frac{E_0^{\mathrm{F}}}{V_0} - \frac{E_{min}}{V_0}} \\ &= \frac{\rho^F(t) - \rho_{bare}}{\rho_0^F - \rho_{bare}} = \frac{\rho_{de}(t) - \rho_{bare}}{\rho_{de}^0 - \rho_{bare}} \\ &= \frac{\Lambda(t) - \Lambda_{bare}}{\Lambda_0 - \Lambda_{bare}}.\end{aligned} \tag{48}$$

and similarly,

$$\frac{E^{\mathrm{F}}(t)}{E_{min}} = \frac{\Lambda(t)}{\Lambda_{bare}}, \quad \frac{E_0^{\mathrm{F}}}{E_{min}} = \frac{\Lambda_0}{\Lambda_{bare}}, \tag{49}$$

etc. Here $\rho^{F}(t) = \frac{E^{\mathrm{F}}(t)}{V_0} \equiv \rho_{de}(t)$, $\Lambda(t) = \frac{8\pi G}{c^2}\,\rho_{de}(t)$, (or $\Lambda(t) = 8\pi G\,\rho_{de}(t)$ in $\hbar = c = 1$ units), etc. Equivalently, $\rho_{de}(t) = \frac{c^2}{8\pi G}\Lambda(t)$.

Taking into account these relations and analyzing results presented in Figs. 1c, 2, 3, 4, 5, 6, 7, 8 and 9 one can conclude that within the assumed scenario there should be,

$$\Lambda(t) \simeq \Lambda_0 \simeq \frac{8\pi G}{c^2}\,\frac{E_\phi}{V_0} \equiv \frac{8\pi G}{c^2}\,\frac{{}^{\mathrm{F}}\langle 0|\mathfrak{H}|0\rangle^{\mathrm{F}}}{V_0}, \quad t \in (T_0, T_1), \tag{50}$$

at canonical decay times $t < T_1$. (Here we used the relation (19) and the property that $|\Delta_\phi^{(1)}| \ll |E_\phi|$ from which it follows that in our analysis it is enough to assume that $E_0^{\mathrm{F}} \simeq E_\phi$, i.e., that $E_0^{\mathrm{F}} \simeq {}^{\mathrm{F}}\langle 0|\mathfrak{H}|0\rangle^{\mathrm{F}}$). In other words there should be $\Lambda(t) \simeq \Lambda_0 \equiv \Lambda_{qft} = \frac{8\pi G}{c^2}\,\rho_{de}^{qft}$ at times $t < T_1$. Then latter, when time t runs from $t = T_1$ to $t = T_2$ the quantum effect discussed above forces this $\Lambda(t) \simeq \Lambda_0$ to reduce its value for $(t > T_2)$ to the following one:

$$\Lambda(t) \simeq \Lambda_{eff}(t) = \Lambda_{\mathrm{bare}} + \frac{\alpha_2^\Lambda}{t^2} + \frac{\alpha_4^\Lambda}{t^4} + \cdots \ll \Lambda_0. \tag{51}$$

Of course in order to reproduce the current value of $\Lambda_{eff}(t)$ from Λ_{qft} within the model considered above one should find suitable $\omega(E)$, maybe more complicated that $\omega_{BW}(E)$ or $\omega_\eta(E)$ considered in this paper. Such a scenario means, similarly to the idea presented by Krauss and Dent [29] and described in Sec. 1, that the inflation epoch takes during canonical decay times, $T_0 < t \le T_1$ when $\rho_{de}(t) \simeq \rho_{de}^0$, and thus $\Lambda(t) \simeq \Lambda_0$, are extremely large, then the post-inflationary epoch begins. At the beginning of the the post-inflationary epoch $\Lambda(t)$ is still very large, but is starting to decrease. It decreases at times $T_1 \le t < T_2$ as an oscillatory modulated exponential function to the value $\Lambda_{eff}(t)$ given by Eq. (51). Then, at times $t > T_2$, $\Lambda(t)$ evolves in time as $\Lambda_{eff}(t)$ and tends to Λ_{are} as $t \to \infty$.

Einstein's equations with the Robertson–Walker metric in the standard form of Friedmann equations [31,79] look as follows: The first one,

$$\frac{\dot a^2(t)}{a^2(t)} + \frac{kc^2}{R_0^2\,a^2(t)} = \frac{8\pi G_N}{3}\,\rho + \frac{\Lambda c^2}{3}, \tag{52}$$

and the second one,

$$\frac{\ddot a(t)}{a(t)} = -\frac{4\pi G_N}{3}\left(\frac{3p}{c^2} + \rho\right) + \frac{\Lambda c^2}{3}. \tag{53}$$

where "dot" denotes the derivative with respect to time t, $\dot a(t) = \frac{da(t)}{dt}$, ρ and p are mass (energy) density and pressure respectively, k denote the curvature signature, and $a(t) = R(t)/R_0$ is the scale factor, $R(t)$ is the proper distance at epoch t, $R_0 = R(t_0)$ is the distance at the reference time t_0, (it can be also interpreted as the radius of the Universe now) and here t_0 denotes the present epoch. The pressure p and the density ρ are are related to each other through the equation of state, $p = w\rho\,c^2$, where w is constant [31]. There is $w = 0$ for a dust, $w = 1/3$ for a radiation and $w = -1$ for a vacuum energy.

Now if the system is in the false vacuum state, $|0\rangle^F$, then at canonical decay times, $T_0 < t < T_1$, the energy of the system in this state equals $E^{\mathrm{F}}(t) = E_0^{\mathrm{F}} \approx {}^F\langle 0|\mathfrak{H}|0\rangle^F$ to a very good approximation and then $\Lambda(t) \simeq \Lambda_0 = \frac{8\pi G}{c^2}\,\frac{{}^{\mathrm{F}}\langle 0|\mathfrak{H}|0\rangle^{\mathrm{F}}}{V_0} > 0$ is very large. We assume now that the lifetime of the false vacuum state is much shorter then duration of the inflationary epoch. In such a situation the vacuum energy dominates. Therefore in such a case we can ignore the matter density ρ in Eq. (52). In this situation the behavior of expansion rate $\dot a(t)$ at times $T_0 < t < T_1$ is such that the curvature signature in Eq. (52) can be always approximated as $k \approx 0$ (see, e.g. [31]) and then Eq. (52) simplifies to

$$\frac{\dot a^2(t)}{a^2(t)} \simeq \frac{\Lambda_0\,c^2}{3}, \quad (T_0 < t < T_1), \tag{54}$$

The solution of this equation is

$$a(t_2) = a(t_1)\,e^{(t_2 - t_1)\sqrt{\frac{\Lambda_0 c^2}{3}}}, \quad (t_1, t_2 \in (T_0, T_1)), \tag{55}$$

which shows that within the considered scenario the scale factor $a(t)$ grows exponentially fast at times, $T_0 < t < T_1$ as it should be at the inflationary epoch. So, in these times the universe considered, that is the universe born in a metastable false vacuum state behaves like de Sitter universe.

Note that if to use the identity $\frac{\ddot a(t)}{a(t)} \equiv \frac{\ddot R(t)}{R(t)}$ and replace $\frac{\ddot a(t)}{a(t)}$ on the left side of Eq. (53) by $\frac{\ddot R(t)}{R(t)}$, and then multiply this equation by the product $m_p\,R(t)$, instead of (53) we get an equation that looks like Newton's equation of motion (see, eg. [31,79]),

$$m_p\,\ddot R(t) = -G\,\frac{m_p\,M_{eff}}{R^2(t)} + m_p\,\frac{\Lambda c^2}{3}\,R(t), \tag{56}$$

for the point mass m_p lying a the sphere with the radius $R(t)$. Here $M_{eff} = \frac{4\pi}{3}\,R^3(t)\,(\rho + \frac{3p}{c^2})$ is the effective total mass of the sphere of the radius $R(t) \equiv a(t)\,R_0$, $p = w\rho$. This equation is completely equivalent to Eq. (53). In order to see that it is enough to put $m_p = 1$ (or to divide it by m_p) and then to use $R(t) = a(t)\,R_0$. Analyzing (56) one can see that the term $m_p\frac{\Lambda c^2}{3}\,R(t)$ in this equation plays the same role as a force in Newton's equations of motion [79]. And now if there is $\Lambda > 0$ then the force $F = m_p\,\frac{\Lambda c^2}{3}\,R(t) \stackrel{\mathrm{def}}{=} F_{rep}$ is a repulsive force and grows with increasing $a(t)$, (or $R(t) = a(t)\,R_0$), whereas for $\Lambda < 0$ the force $F = \frac{\Lambda c^2}{3}\,R(t) = F_{att}$ is a force of an attraction [31,79].

Now basing on these properties of Λ we can analyze consequences of the behavior of the energy $E(t) = E^{\mathrm{F}}(t)$, and thus $\Lambda(t)$, at times $T_1 < t < T_2$. From Figs. 1b and 2a one can see that for times $T_1 < t < T_2$ there are such time intervals shorter than (T_1, T_2), that $E^{\mathrm{F}}(t)$ is positive at some of

them and negative for the others. In general $E^{\mathrm{F}}(t)$ is oscillatory modulated at this time region by changing its value smoothly over time from positive to negative and vice versa. These properties of of $E^{\mathrm{F}}(t)$ are reflected in corresponding, analogous behavior of $\Lambda(t)$ on these time intervals: $\Lambda(t)$ is oscillatory modulated for $t \in (T_1, T_2)$. As a result, acceleration $\ddot{R}(t) \equiv \ddot{a}(t)\, R_0$ increases or decreases depending on whether time t runs over the interval with a positive $\Lambda(t)$ or a negative $\Lambda(t)$ and thus the radius $R(t) = a(t)\, R_0$ of the sphere increases slower or faster: Simply our hypothetical sphere under consideration with the radius $R(t)$ is vibrating. In other words, within the considered scenario the Universe in the decaying false vacuum state is pulsating for $t \in (T_1, T_2)$. This means that the Universe (i.e. the sphere with the radius $R(t) = a(t)\, R_0$) evolving in time and behaving in this way should generate gravitational waves in this phase of its time evolution when time t runs from $t = T_1$ to $t = T_2$. From the point of view of a today's observer, there is a chance that these relic gravitational waves can be recorded and this is potential observational effect of the scenario analyzed in this paper.

From Eq. (52), one more conclusion follows: If to consider the time interval $t \in (T_1, T_2)$ only and insert the oscillatory modulated $\Lambda(t)$ into this equation than one can conclude that the Hubble parameter $H(t) = \frac{\dot{a}(t)}{a(t)}$ should also be oscillatory modulated at this time region. So, in general, properties of $E^{\mathrm{F}}(t)$ and thus $\Lambda(t)$ at times $t \in (T_1, T_2)$ generated by quantum mechanism considered in this paper correspond with some properties of Early Dark Energy (EDE) or New Early Dark Energy (NEDE) recently discussed in many papers [80–85]. The effects predicted by EDE are obtained using the potential $V(\Phi)$ having an oscillating form, which leads to an energy density $\rho_{de}(t)$ having a similar property at early times. The advantage of the above-described quantum mechanism over the EDE or NEDE theories lies in the fact that this mechanism requires neither additional fields generating EDE nor oscillating potentials (see [80–85]).

Properties of $\Lambda(t)$ for times $t > T_2$ are described by Eq. (51): There is $\Lambda(t) \simeq \Lambda_{eff}(t)$ for $t > T_2$, where $\Lambda_{eff}(t)$ is defined by Eq (51). Detailed analysis and discussion of this case can be found in separate papers – see: [36–39,78].

5 Final remarks

The results and conclusions presented in the previous section were obtained on the basis of the following assumptions: (i) a transition from the false vacuum state to the true vacuum state (i.e. the decay of the false vacuum) is the quantum decay process, (ii) the Universe was born in the false vacuum state, (iii) the lifetime of the false vacuum state, $\tau_F = \frac{\hbar}{\Gamma_F}$, (where Γ_F is the decay rate, or decay width, of the false vacuum state $|0\rangle^F$), is shorter than the duration of the inflationary phase of the evolution of the Universe. The picture of the time evolution of the Universe discussed in Sect. 4 and resulting from these assumptions seems to be selfconsistent.

Potentially the effect described and discussed in Sects. 3 and 4 may be considered as a candidate to explain and the problem of the cosmological constant [31,32,86–88]. Such a conclusion may be substantiated, for example, by the following analysis: namely, based on the the results presented in Sect. 3 we can estimate the time needed for the value of $\Lambda(t)$ to decrease from the value of Λ_0 to a value close to Λ_{bare}. So, let us analyze results presented in Fig. 3d. There was assumed that $\Lambda_0/\Lambda_{bare} = 10^{20}$. From Fig. 3d we can conclude that $\Lambda(T_2)/\Lambda_{bare} \simeq 10^8$ and that $\tau = T_1 \simeq 50$ and $\tau = T_2 \simeq 100$. Next using Eq. (51) one finds that there is,

$$\frac{\Lambda(\tau)}{\Lambda_{bare}} - 1 = \left(\frac{T_2}{\tau}\right)^2 \left[\frac{\Lambda(T_2)}{\Lambda_{bare}} - 1\right], \tag{57}$$

for $\tau > T_2$. Now using the values of T_2 and of the ratio $\Lambda(T_2)/\Lambda_{bare}$ deduced from Fig. 3d one concludes that

$$\frac{\Lambda(\tau)}{\Lambda_{bare}} - 1 \approx \left(\frac{10^6}{\tau}\right)^2. \tag{58}$$

This means, e.g. that $\Lambda(\tau)/\Lambda_{bare} \approx 2$ for $\tau \equiv t/\tau_F = 10^6$. Hence, e.g. if $\tau_F \simeq 10^{-36}$ [s] then the value $\Lambda_0/\Lambda_{bare} = 10^{20}$ taken by $\Lambda(\tau(t))$ for $\tau < T_1 \simeq 50$, (in this case $\tau = T_1 = 50$ corresponds to $t = 50\tau_F = 5 \times 10^{-35}$ [s]) reduces to the value $\Lambda(\tau(t))/\Lambda_{bare} \approx 2$ at time $t \approx 10^6\tau_F = 10^{-30}$ [s]. Analogously, if to assume that $\tau_F \simeq 10^{-38}$ [s] (see, e.g. [31], Chap. 9) then the value $\Lambda(\tau(t))/\Lambda_{bare} \approx 2$ can be reached in $t \approx 10^{-32}$ [s]. Analyzing results presented in Figs. 3 and 4 one can conclude that the degree of reduction of the energy $E(t)$ (or $\Lambda(t)$) does not depend on a the ratio Λ_0/Λ_{bare} but on the magnitude of the coefficient β. Therefore one can expect that for $\beta = 10^4$ (as it is presented in Fig. 3d, 4) there should be $\Lambda(T_1)/\Lambda(T_2) \sim 10^{12}$ for, e.g., $\Lambda_0/\Lambda_{bare} \sim 10^{100}$ too and $T_2 \sim 100\tau_F$. In such a case there is $\Lambda(T_2)/\Lambda_{bare} \sim 10^{88}$ and thereafter ratio $\Lambda(\tau(t))/\Lambda_{bare}$ can reach its value $\Lambda(\tau(t))/\Lambda_{bare} \simeq 2$ in time $t \simeq 10^8$ [s] if $\tau_F \simeq 10^{-38}$ [s] and in time $t \simeq 10^{10}$ [s] if $\tau_F \simeq 10^{-36}$ [s]. This shows that quantum mechanism discussed in this paper is very effective. So, one can expect that for a suitable $\omega(E)$ the degree of the reduction of $\Lambda(\tau)$ can be much greater. It seems to be possible that assuming $\tau_F \sim 10^{-36}$ [s] (or of a similar order) this mechanism would be able to reduce even the value of $\Lambda_0/\Lambda_{bare} \sim 10^{120}$ to the value $\Lambda(\tau)/\Lambda_{bare} \sim 2$ no later than at time $t \sim 5 \times 10^5$ years.

Similarly it seems that this quantum effect can also help to explain the H-tension problem [80–84,89,90]. The only problem is to find a suitable model with the required lifetime, τ_F, of the false vacuum state and thus a suitable energy density distribution $\omega(E)$, what requires further studies. Note that cosmological models with lifetime of the false vacuum

state even shorter than the Planck time were considered in the literature (see e.g. [74,75]) but such a lifetime seems to be too short in order that canonical decay times could coincide with the inflationary phase.

Some hints concerning values of the basic parameters of the model we are looking for can be found by analyzing the results obtained numerically and presented in Sect. 3. For example, let us analyze the results presented in Fig. 3d. From the Eq. (47) one finds that $\Gamma_F = \frac{1}{\beta}(E_0^{\mathrm{F}} - E_{min})$. The results presented in Fig. 3d were obtained with the assumption that $\beta = 10^4$ and $\frac{E_0}{E_{min}} \equiv \frac{E_0^{\mathrm{F}}}{E_{min}} = 10^{20}$. Hence $E_{min} = 10^{-20}\, E_0^{\mathrm{F}} \equiv 10^{-20}\, E_0$, and $\Gamma_F = \Gamma_0 = 10^{-4} \times (1-10^{-20})\, E_0 \simeq 10^{-4} E_0 \approx 10^{-4} \langle\phi|\mathfrak{H}|\phi\rangle$. Here the approximation $E_0 \approx \langle\phi|\mathfrak{H}|\phi\rangle$ was used, which is sufficient in order to find approximate suitable values of parameters of the model – see explanations below Eq. (50). This means that there should be $E_{min} \approx 10^{-20}\, \langle\phi|\mathfrak{H}|\phi\rangle$, where $|\phi\rangle$ is the decaying metastable state. (There is $|\phi\rangle = |0\rangle^F$ in the considered case.) Thus, it can be expected that the quantum field theory model with the Hamiltonian $\mathfrak{H}$, in which the following approximate relations will take place, $E_0 \approx \langle\phi|\mathfrak{H}|\phi\rangle$, $\Gamma_0 \approx 10^{-4}\, \langle\phi|\mathfrak{H}|\phi\rangle$, and $E_{min} \approx 10^{-20}\, \langle\phi|\mathfrak{H}|\phi\rangle$, will adequately reflect the process presented in Fig. 3d.

There are observational data that can be used to constrain possible parameters of the theoretical model that implements the scenario described in this paper, i.e. the form of the potential $V(\Phi)$ and thus the Lagrangian and the Hamiltonian $\mathfrak{H}$, which we are looking for. Namely, from cosmological observations we have estimations of the time at which inflationary process begins and ends. These times can be used to limit the length of the lifetime, τ_F, of the false vacuum: The scenario described above can be realized only if the lifetime, τ_F, is comparable to the time at which the inflationary process begins and it is significantly shorter than the time at which the inflation ends. Having constraints on the lifetime we have a constrain on the decay rate $\Gamma_F = \Gamma_0 = \hbar/\tau_F$.

Of course, apart from these conditions, $\omega(E)$ resulting from the properties of such Hamiltonian $\mathfrak{H}$ must lead to $\Lambda(t)$ satisfying the constraints imposed by Eq. (52). Namely the amplitude $|\Lambda(t)|$ of possible variations of $\Lambda(t)$ at times $t \in (T_1, T_2)$ must be such that the condition $H^2(t) >$ is satisfied.

One more remark concerning Eqs. (54), (55) resulting from properties of the energy of very short living metastable false vacuum and possible connection of them with the inflationary process. From discussion presented in the previous section it follows that at times $t < T_1$ the energy density $\rho^F(t) \sim \rho_0^F \simeq \frac{^{\mathrm{F}}\langle 0|\mathfrak{H}|0\rangle^{\mathrm{F}}}{V_0}$ can be very large. Using equation of state $p(t) = -\rho^F(t)\, c^2$ one finds that in this epoch, when $t < T_1$, the pressure $p(t) = p = -\rho_0^F\, c^2$ can take a huge negative values. As a result which the scale factor $a(t)$ grows exponentially fast for $t < T_1$, which is reflected in the solution (55) of Eq. (54). This effect is exactly what is needed in the inflation process (see, e.g. [31,98,99]). Next, at times $t \in (T_1, T_2)$ the density $\rho^F(t)$ has an oscillatory form and it is still large but decreases to the values $\rho^F(t) - \rho_{bare} \sim \alpha_2^\rho / t^2$ for times $t > T_2$. This means that the scale factor $a(t)$ continues to rise rapidly, but slower and slower when time t runs form $t = T_1$ to $t = T_2$ and the time $t = T_2$ can be considered as the end of the rapid expansion process. So, there is potential possibility that this effect can drive the inflation process. Concluding: If the lifetime, τ_F, is suitably small then potentially contribution of this effect into the inflation process can be significant or even it can be responsible for this process. The answer to the question whether it is so and what problems this mechanism solves depends on finding the appropriate potential $V(\Phi)$ and thus the Hamiltonian $\mathfrak{H}$, which requires further researches.

Details concerning the case $\rho^F(t) - \rho_{bare} \sim \alpha_2^\rho / t^2$ can be found in [36–39].

Some time one may ask if a transition from the metastable false vacuum state to the true vacuum state, i.e. to the state corresponding with to the absolute minimum of the energy of the system considered, which is realized as a quantum tunneling process, can be correctly described within the Fock–Krylov theory of quantum decays. The answer is yes, it can be. In general within the quantum theory the quantum tunneling is used to model some quantum decay processes, e.g. the process of α-decay (see, e.g. [3]), and these decay processes can be also described using the Fock–Krylov theory (see e.g. [60,91–95]). Strictly speaking in the case of the quantum tunneling used to model the quantum decay process the survival probability can be also expressed in the form of the Fourier transform as it was done in Eq. (5). This means that the general formalism based on the Fock–Krylov theory is fully suitable for describing the properties of a decaying false vacuum and a running dark energy. This is why the Fock–Krylov theory was used by Krauss and Dent to analyze late time behavior of false vacuum decay [29,30], which papers were an inspiration for our studies.

It should be emphasized here that the conclusions resulting from the formalism used in this paper apply not only to the quantum metastable state of the system prepared at the initial moment at the local minimum of the potential $V(\Phi)$, but also to other types of metastable false vacuum states, e.g. false vacuum states considered in [98,99].

Cosmological model with running Λ having the form $\Lambda = \Lambda(t) = \Lambda_{eff}(t)$, where $\Lambda_{eff}(t)$ is given by Eq. (51), was discussed in details in [39], where it was shown that this $\Lambda_{eff}(t)$ should approximate well $\Lambda(t)$ for times $t > T_2$. (In [39] the time T_2 is denoted as T_{q-c}, and, among others, different time scales in the process of decaying metastable dark energy are discussed). It is also shown in [39] that the good approximation of Eq. (51) valid for times $t > T_2$ is to replace the cosmological time t in it with the Hubble cosmological scale time $t_{\mathrm{H}} = \frac{1}{H}$. As the result, instead of (51) one gets

$$\Lambda(t)=\Lambda(H(t))\simeq\Lambda_{\rm bare}+\alpha_2\,(H(t))^2+\alpha_4\,(H(t))^4+\cdots\,, \tag{59}$$

that is exactly the parameterization considered in [96,97] and in many papers of these and other authors.

Generally, cosmological models with decaying (or running) $\Lambda=\Lambda(t)$ were considered by many authors (see e.g. [100] and references therein, [101–106], and also [79]) but the use of the decaying $\Lambda(t)$ by them was not motivated by the properties of the false vacuum as a quantum unstable state. The advantage of the method used in this paper on other methods is that we do not assume the form of running $\Lambda(t)$, but we derive properties of $\Lambda(t)$ and its form, e.g. like $\Lambda_{ef}(t)$ for $t>T_2$, from the basic assumptions of quantum theory using the assumption that the transition from a false vacuum to a true vacuum is the quantum decay process.

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References

1. V.F. Weisskopf, E.T. Wigner, Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie. Z. Phys. **63**, 54 (1930)
2. V.F. Weisskopf, E.T. Wigner, Über die natürliche Linienbreite in der Strahlung des harmonischen Oszillators. Zeit. f. Phys. **65**, 18 (1930)
3. G. Gamow, Zur Quantentheorie des Atomkernes. Zeit. f. Phys. **51**, 204–212 (1928)
4. S.R. Coleman, The fate of the false vacuum. 1. Semiclassical theory. Phys. Rev. D **15**, 2929 (1977)
5. C.G. Callan Jr., S.R. Coleman, The fate of the false vacuum. 2. First quantum corrections. Phys. Rev. D **16**, 1762 (1977)
6. E. Ruthheford, Radioactivity produced in substances by the action of thorium compounds. Philos. Mag. XLIX **1**, 161 (1900)
7. E. Rutherford, F. Soddy, The cause and nature of radioactivity. Part I. Philos. Mag. **IV**, 370–396 (1902)
8. E. Rutherford, F. Soddy, The cause and nature of radioactivity. Part II. Philos. Mag. **IV**, 569–585 (1902)
9. E. Rutherford, *Radioactive Substances and Their Radiations* (Cambridge Unversity Press, Cambridge, 1913)
10. L.A. Khalfin, Contribution to the decay theory of a quasi-stationary state. Zh. Exp. Teor. Fiz. **33**, 1371 (1957)
11. L. Fonda, G.C. Ghirardi, A. Rimini, Decay theory of unstable quantum systems. Rep. Prog. Phys. **41**, 587 (1978)
12. M. Peshkin, A. Volya, V. Zelevinsky, Non-exponential and oscillatory decays in quantum mechanics. Europhys. Lett. **107**, 40001 (2014)
13. B. Misra, E.C.G. Sudarshan, The Zeno's paradox in quantum theory. J. Math. Phys. **18**, 756–763 (1977)
14. A. Beige, G.C. Hegerfeldt, Projection postulate and atomic quantum Zeno effect. Phys. Rev. A **53**, 53 (1996)
15. A. Beige, G.C. Hegerfeldt, D.G. Sondermann, Atomic quantum Zeno effect for ensembles and single systems. Found. Phys. **27**, 1671–1688 (1997)
16. K. Koshinoa, A. Shimizuc, Quantum Zeno effect by general measurements. Phys. Rep. **412**, 191–275 (2005)
17. P. Facchi, S. Pascazio, Quantum Zeno phenomena: pulsed versus continuous measurement. Fortschr. Phys. **49**, 941–947 (2001)
18. W.M. Itano et al., Quantum Zeno effect. Phys. Rev. A **41**, 2295–2300 (1990)
19. M.C. Fischer, B. Gutiérrez-Medina, M.G. Raizen, Observation of the quantum Zeno and anti-Zeno effects in an unstable system. Phys. Rev. Lett. **87**, 040402 (2001)
20. D.Z.X. Wenqiang Zheng, X. Peng, X. Zhou, D. Jiangfeng, C.P. Sun, Experimental demonstration of the quantum Zeno effect in NMR with entanglement-based measurements. Phys. Rev. A **87**, 032112 (2013)
21. E.W. Streed et al., Continuous and pulsed quantum Zeno effect. Phys. Rev. Lett. **97**, 260402 (2006)
22. Y.S. Patil, S. Chakram, M. Vengalattore, Measurement-induced localization of an ultracold lattice gas. Phys. Rev. Lett. **115**, 140402 (2015)
23. A. Signoles et al., Confined quantum Zeno dynamics of a watched atomic arrow. Nat. Phys. **10**, 715–719 (2014)
24. K. Urbanowski, A quantum long time energy red shift: a contribution to varying alpha theories. Eur. Phys. J. C **58**, 151 (2008)
25. K. Urbanowski, General properties of the evolution of unstable states at long times. Eur. Phys. J. D **54**, 25 (2009)
26. K. Urbanowski, Long time properties of the evolution of an unstable state. Cent. Eur. J. Phys. **7**, 696 (2009)
27. K. Urbanowski, Comment on 'Late time behavior of false vacuum decay: possible implications for cosmology and metastable inflating states'. Phys. Rev. Lett. **107**, 209001 (2011)
28. K. Urbanowski, Properties of the false vacuum as the quantum unstable state. Theor. Math. Phys. **190**, 458 (2017)
29. L.M. Krauss, J. Dent, The late time behavior of false vacuum decay: possible implications for cosmology and metastable inflating states. Phys. Rev. Lett. **100**, 171301 (2008)
30. L.M. Krauss, J. Dent, G.D. Starkman, Late time decay of the false vacuum, measurement, and quantum cosmology. Int. J. Mod. Phys. D **17**, 2501 (2009)
31. T.-P. Cheng, *Relativity, Gravitation, and Cosmology: A Basic Introduction* (Oxford University Press, Oxford, 2005)
32. S. Weinberg, *Cosmology* (Oxford University Press, Oxford, 2008)
33. R.G. Landim, E. Abdalla, Metastable dark energy. Phys. Lett. B **764**, 271–276 (2017)

34. D. Stojkovic, G.D. Starkman, R. Matsuo, Dark energy, colored anti-de Sitter vacuum, and the CERN Large Hadron Collider phenomenology. Phys. Rev. D **77**, 063006 (2008)
35. J.A.S. Lima, Extended metastable dark energy. Phys. Dark Universe **30**, 100713 (2020)
36. A. Stachowski, M. Szydłowski, K. Urbanowski, Cosmological implications of the transition from the false vacuum to the true vacuum state. Eur. Phys. J. C **77**, 357 (2017)
37. M. Szydlowski, A. Stachowski, K. Urbanowski, Quantum mechanical look at the radioactive-like decay of metastable dark energy. Eur. Phys. J. C **77**, 902 (2017)
38. M. Szydlowski, A. Stachowski, Cosmological models with running cosmological term and decaying dark matter. Phys. Dark Universe **15**, 96 (2017)
39. M. Szydłowski, A. Stachowski, K. Urbanowski, The evolution of the FRW universe with decaying metastable dark energy—a dynamical system analysis. J. Cosmol. Astropart. Phys. **04**, 029 (2020)
40. N.S. Krylov, V.A. Fock, On two main interpretations of energy–time uncertainty. Zh. Eksp. Teor. Fiz. **17**, 93 (1947)
41. V.A. Fock, *Fundamentals of Quantum Mechanics* (Mir Publishers, Moscow, 1978)
42. N.G. Kelkar, M. Nowakowski, No classical limit of quantum decay for broad states. J. Phys. A **43**, 385308 (2010)
43. F. Giacosa, Non-exponential decay in quantum field theory and in quantum mechanics: the case of two (or more) decay channels. Found. Phys. **42**, 1262 (2012)
44. F. Giacosa, QFT derivation of the decay law of an unstable particle with nonzero momentum. Adv. High Energy Phys. **2018**, 4672051 (2018)
45. R.E.A.C. Paley, N. Wiener, *Fourier Transforms in the Complex Domain* (American Mathematical Society, New York, 1934)
46. A. Peres, Nonexponential decay law. Ann. Phys. **129**, 33 (1980)
47. K.M. Sluis, E.A. Gislason, Decay of a quantum-mechanical state described by a truncated Lorentzian energy distribution. Phys. Rev. A **43**, 4581 (1991)
48. M.L. Goldberger, K.M. Watson, *Collision Theory* (Willey, New York, 1964)
49. D.G. Arbo, M.A. Castagnino, F.H. Gaioli, S. Iguri, Minimal irreversible quantum mechanics. The decay of unstable states. Phys. A **227**, 469–495 (2000)
50. J.M. Wessner, D.K. Andreson, R.T. Robiscoe, Radiative decay of the 2P state of atomic hydrogen: a test of the exponential decay law. Phys. Rev. Lett. **29**, 1126–1128 (1972)
51. E.B. Norman, S.B. Gazes, S.C. Crane, D.A. Bennet, Tests of the exponential decay law at short and long times. Phys. Rev. Lett. **60**, 2246–2249 (1988)
52. P.T. Greenland, Seeking non-exponential decay. Nature **335**, 298 (1988)
53. J. Seke, W.N. Herfort, Deviations from exponential decay in the case of spontaneous emission from a two-level atom. Phys. Rev. A **38**, 833 (1988)
54. R.E. Parrot, J. Lawrence, Persistence of exponential decay for metastable quantum states at long times. Europhys. Lett. **57**, 632–638 (2002)
55. J. Lawrence, Nonexponential decay at late times and a different Zeno paradox. J. Opt. B Quantum Semiclass. Opt. **4**(4), S446–S449 (2002)
56. I. Joichi, S. Matsumoto, M. Yoshimura, Time evolution of unstable particle decay seen with finite resolution. Phys. Rev. D **58**, 045004 (1998)
57. N.G. Kelkar, M. Nowakowski, K.P. Khemchandani, Hidden evidence of nonexponential nuclear decay. Phys. Rev. C **70**, 024601 (2004)
58. M. Nowakowski, N.G. Kelkar, Long Tail of quantum decay from scattering data. AIP Conf. Proc. **1030**, 250–255 (2008). arXiv:0807.5103
59. R. Santra, J.M. Shainline, C.H. Greene, Siegert pseudostates: completeness and time evolution. Phys. Rev. A **71**, 032703 (2005)
60. R.G. Winter, Evolution of a quasi-stationary state. Phys. Rev. **123**, 1503 (1961)
61. T. Jiitoh, S. Matsumoto, J. Sato, Y. Sato, K. Takeda, Phys. Rev. A **71**, 012109 (2005)
62. C. Rothe, S.I. Hintschich, A.P. Monkman, Violation of the exponential-decay law at long times. Phys. Rev. Lett. **96**, 163601 (2006)
63. K. Urbanowski, Early-time properties of quantum evolution. Phys. Rev. A **50**, 2847 (1994)
64. F. Giraldi, Logarithmic decays of unstable states. Eur. Phys. J. D **69**, 5 (2015)
65. F. Giraldi, Logarithmic decays of unstable states II. Eur. Phys. J. D **70**, 229 (2016)
66. G. Breit, E. Wigner, Capture of slow neutrons. Phys. Rev. **49**, 519–531 (1936)
67. N.G. Kelkar, M. Nowakowski, No classical limit of quantum decay for broad states. J. Phys. A Math. Theor. **43**, 385308 (2010)
68. D.F. Ramírez Jiménez, N.G. Kelkar, Formal aspects of quantum decay. Phys. Rev. A **104**, 022214 (2021)
69. K. Raczyńska, K. Urbanowski, Survival amplitude, instantaneous energy and decay rate of an unstable system: analytical results. Acta Phys. Polon. B **49**, 1683 (2018)
70. R.G. Newton, *Scattering Theory of Waves and Particles* (Springer, New York, 1982)
71. K. Urbanowski, K. Raczyńska, Possible emission of cosmic X- and ă-rays by unstable particles at late times. Phys. Lett. B **731**, 236–241 (2014)
72. J. Rubio, Higgs inflation and vacuum stability. J. Phys. Conf. Ser. **631**, 012032 (2015)
73. A. Kennedy, G. Lazarides, Q. Shafi, Decay of the false vacuum in the very early universe. Phys. Lett. B **99**, 38 (1981)
74. V. Branchina, E. Messina, Stability, Higgs boson mass and new physics. Phys. Rev. Lett. **111**, 241801 (2013)
75. V. Branchina, E. Messina, M. Sher, Lifetime of the electroweak vacuum and sensitivity to Planck scale physics. Phys. Rev. D **91**, 013003 (2015)
76. S. Winitzki, Age-dependent decay in the landscape. Phys. Rev. D **77**, 063508 (2008)
77. K. Urbanowski, M. Szydlowski, Cosmology with a decaying vacuum. AIP Conf. Proc. **1514**, 143 (2012)
78. M. Szydlowski, A. Stachowski, K. Urbanowski, Cosmology with a decaying vacuum energy parametrization derived from quantum mechanics. J. Phys. Conf. Ser. **626**, 012033 (2015)
79. V. Sahni, A. Starobinsky, The Case for a Positive Cosmological Λ-term. International Journal of Modern Physics D **09**, 373–443 (2000)
80. V. Poulin, T.L. Smith, T. Karwal, M. Kamionkowski, Early dark energy can resolve the Hubble tension. Phys. Rev. Lett. **122**, 221301 (2019)
81. T.L. Smith, V. Poulin, M.A. Amin, Oscillating scalar fields and the Hubble tension: a resolution with novel signatures. Phys. Rev. D **101**, 063523 (2020)
82. F. Niedermann, M.S. Sloth, New early dark energy. Phys. Rev. D **103**, L041303 (2021)
83. V. Poulin, T.L. Smith, A. Bartlett, Dark energy at early times and ACT: a larger Hubble constant without latetime priors. (2021). arXiv:2109.06229v1 [astro-ph.CO]
84. E. Di Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri, D.F. Mota, A.G. Riess, J. Silk, in the realm of the Hubble tension. A review of solutions. Class. Quantum Gravity **38**, 153001 (2021)

85. S. Vagnozzi, Consistency tests of ΛCDM from the early integrated Sachs–Wolfe effect: implications for early-time new physics and the Hubble tension. Phys. Rev. D **104**, 063524 (2021)
86. S. Weinberg, The cosmological constant problem. Rev. Mod. Phys. **61**, 1 (1989)
87. S. Weinberg, The cosmological constant problems, in *Sources and Detection of Dark Matter and Dark Energy in the Universe. Proceedings, 4th International Symposium, DM 2000, Marina del Rey, USA, February 23–25, 2000* (2000), pp. 18–26
88. S.M. Carroll, The Cosmological Constant. Living Rev. Relativ. **4**, 1 (2001). arXiv:astro-ph/0004075v2
89. E. Di Valentino, E.V. Linder, A. Melchiorri, Vacuum phase transition solves the H_0 tension. Phys. Rev. D **97**, 043528 (2018)
90. F. Niedermann, M.S. Sloth, Resolving the Hubble tension with new early dark energy. Phys. Rev. D **102**, 063527 (2020)
91. W. van Dijk, Y. Nogami, Novel expression for the wave function of a decaying quantum system. Phys. Rev. Lett. **83**, 2867 (1999)
92. W. van Dijk, Y. Nogami, Analytical approach to the wave function of a decaying quantum system. Phys. Rev. C **65**, 024608 (2002)
93. D.A. Dicus, W.W. Rebko, R.F. Schwitters, T.M. Tinsley, Time development of a quasistationary state. Phys. Rev. A **65**, 032116 (2002)
94. J. Martorell, J.G. Muga, D.W.L. Sprung, Quantum post-exponential decay, in *Time in Quantum Mechanics, volume 789 of Lecture Notes in Physics (Berlin)*, vol. 2, ed. by G. Muga, A. Ruschhaupt, A. del Campo (Springer, Berlin, 2009), pp. 239–275
95. W. van Dijk, F.M. Toyama, Decay of a quasistable quantum system and quantum backflow. Phys. Rev. A **100**, 052101 (2019)
96. I.L. Shapiro, J. Sola, C. Espana-Bonet, P. Ruiz-Lapuente, Variable cosmological constant as a Planck scale effect. Phys. Lett. B **574**, 149 (2003)
97. C. Espana-Bonet, P. Ruiz-Lapuente, I.L. Shapiro, J. Sola, Testing the running of the cosmological constant with type Ia supernovae at high z. JCAP **0402**, 006 (2004)
98. A.H. Guth, Inflation. Proc. Natl. Acad. Sci. USA **90**, 4871–4877 (1993)
99. A.H. Guth, Eternal inflation and its implications. J. Phys. A Math. Theor. **40**, 6811–6826 (2007)
100. J.M. Overduin, F.I. Cooperstock, Evolution of the scale factor with a variable cosmological term. Phys. Rev. D **58**, 043506 (1998)
101. I.G. Dymnikova, M.Y. Khlopov, Self-consistent initial conditions in inflationary cosmology. Gravit. Cosmol. **4**(Supplement), 50–55 (1988)
102. I.G. Dymnikova, M.Y. Khlopov, Decay of cosmological constant as Bose condensate evaporation. Mod. Phys. Lett. A **2000**(15), 2305–2314 (2000). arXiv:astro-ph/0102094
103. I.G. Dymnikova, M.Y. Khlopov, Decay of cosmological constant in self-consistent inflation. Eur. Phys. J. C **20**, 139–146 (2001)
104. S. Ray, M.Y. Khlopov, P.P. Ghosh, U. Mukhopadhyay, Phenomenology of Λ-CDM model: a possibility of accelerating Universe with positive pressure. Int. J. Theor. Phys. **2011**(50), 939–951 (2011). arXiv:0711.0686 [gr-qc]
105. K. Bamba, S. Capozziello, S. Nojiri, S.D. Odintsov, Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. Astrophys. Space Sci. **342**, 155 (2012). arXiv:1205.3421 [gr-qc]
106. E. Abdalla, L.L. Graef, B. Wang, A model for dark energy decay. Phys. Lett. B **726**, 786–790 (2013)