



On well-posedness and algebraic type of the five-dimensional charged rotating black hole with two equal-magnitude angular momenta

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Abstract We study various mathematical aspects of the charged rotating black hole with two equal-magnitude angular momenta in five dimensions. We introduce a coordinate system that is regular on the horizon and in which Einstein–Maxwell equations reduce to an autonomous system of ODEs. Employing Bondi and Kruskal-like coordinates, we analyze the geometric regularity of the black hole metric at infinity and the horizon, respectively, and the well-posedness of the corresponding boundary value problem. We also study the algebraic types of the electromagnetic and curvature tensors. While outside the horizon the electromagnetic and Ricci tensors are of type D, the Weyl tensor is algebraically general. The Weyl tensor simplifies to type II on the horizon and type D on the bifurcation sphere. These results imply inconsistency of the metric with the Kerr–Schild form with a geodesic Kerr–Schild vector. This feature is shared by the four-dimensional Kerr–Newman metric and the vacuum Myers–Perry or charged Schwarzschild–Tangherlini geometries in arbitrary dimension, but hence not by the black hole we have considered here.

1 Introduction

Black holes represent the most basic objects of general relativity in four and higher dimensions. It is relatively straightforward to find static, spherically symmetric black hole solutions to vacuum Einstein or Einstein–Maxwell equations.

However, it is much more difficult to generalize such black hole solutions to the rotating case and usually, an additional simplifying assumption about the metric has to be employed on top of stationarity and axial symmetry.

In particular, in constructing the Kerr metric in [16], the metric was assumed to be of Petrov type D. For the construction of a vacuum higher-dimensional rotating black hole metric [23], the essential assumption was that it can be cast in the Kerr–Schild form, similarly as in the four-dimensional case:

$$g_{\mu\nu} = \eta_{\mu\nu} - 2\mathcal{H}k_{\mu}k_{\nu}, \quad (1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric, \mathcal{H} a scalar function and k is a null vector with respect to $\eta_{\mu\nu}$ ¹.

The above two assumptions – Petrov type D and the Kerr–Schild form – in fact also hold for the four-dimensional charged rotating black hole described by the Kerr–Newman metric [25,26] and for charged Schwarzschild–Tangherlini black holes [38] in arbitrary dimension. Furthermore, in these cases, the null eigenvectors of the electromagnetic field F coincide with the principal null directions of the Weyl tensor [27]. Note also that for Kerr, Kerr–Newman, Myers–Perry, and Schwarzschild–Tangherlini metrics, the Kerr–Schild vector k is geodesic.

In contrast with the four-dimensional case, where the Kerr–Newman solution was constructed in less than a year after the publication of the Kerr metric, the exact solution representing a higher-dimensional charged rotating black hole is now, three and a half decades after the publication of the Myers–Perry solution [23], still unknown.² Being a 5-dimensional object, such a black hole is not of direct rel-

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¹ It follows that k is also null with respect to the full metric $g_{\mu\nu}$.

² Note that various approximate solutions, for example for small charge [3,24], slow rotation [1,2] or large dimensions [4], are known.

evance to the emerging field of gravitational wave astrophysics, which has accompanied the development of LIGO, LISA and other gravitational wave observatories. However, it may be of interest to more detailed future phenomenological investigations of new physics scenarios like brane world models, where 5-dimensional gravity interacts with matter confined to a 4-dimensional brane [11]. Independent of their phenomenological significance, charged rotating black holes are a good theoretical testing ground for methods of interpolating between known charged static and uncharged rotating solutions, which besides higher dimensions may become useful also in modified theories of gravity in 4 dimensions.

In this paper, we focus on the charged rotating black hole in five dimensions with two equal-magnitude angular momenta, starting with the metric ansatz previously employed in the numerical studies of this black hole, initiated in [18, 19] and continued in [9, 24]. These works have demonstrated the existence (or mathematically speaking, provided very strong evidence thereof) of a solution to the Einstein–Maxwell equations with this ansatz, by numerically solving an ordinary boundary value problem in the region between the horizon and infinity. Their focus was on studying the numerical relationships between various black hole parameters that can only be determined from the knowledge of the global solution, rather than from only local approximations. For instance, while the horizon radius, surface gravity, rotation speed, and surface electric potential at the horizon are all quantities local to the horizon, and while the total mass, angular momentum, charge, and gyromagnetic ratio are local to infinity, definite relations between these two groups of parameters must be globally determined. However, these previous works have not discussed or have left implicit certain structural or algebraic properties of these black holes. It is to these matters that we turn our attention in this work.

Namely, we focus on two main aspects: the geometric regularity of the black hole solution at infinity and at the horizon, and on the algebraic type of the metric in the region between these two extremes. Indeed, we confirm that the black hole metric has a regular extension across the horizon and that it possesses a regular null infinity without incoming or outgoing radiation. On the other hand, we also show that in the bulk the metric is *algebraically general*³ and incompatible

with Kerr–Schild form (1) with geodesic k , so it could not have appeared in the lists of exact solutions obtained under these simplifying assumptions. On the other hand, employing Kruskal-like coordinates, we show that the metric does become algebraically special at the horizon, in agreement with the geometric horizon conjecture of [7] (cf. also [20]).

It is worth noting that while most of our results hold for equal-magnitude angular momenta, the results on the algebraically general Weyl tensor and non-compatibility of the metric with the Kerr–Schild form (1) with geodesic k can be obviously extended to the case of generic angular momenta (see [18] for the corresponding metric ansatz).

The paper is structured as follows: In Sect. 2, we recall the metric ansatz from [19] and show why it is not optimal for studying the regularity of the metric at the horizon by matching it to the known exact uncharged (Myers–Perry) and non-rotating (charged Schwarzschild–Tangherlini) solutions. In Sect. 3, we introduce a different radial coordinate and a modified metric ansatz, which is better adapted to studying the regularity of the horizon. We then obtain the corresponding autonomous system of Einstein–Maxwell equations in the form of a constrained ordinary boundary value problem with singular end-points. Then a careful comparison of the solution at the singular end-points with the geometric regularity conditions at infinity and at the horizon shows that the boundary value problem is indeed well-posed. This analysis also suggests how to ensure numerical accuracy in future investigations of these solutions.

In Sect. 4, we then study the simplifying assumptions holding for Kerr, Kerr–Newman, Schwarzschild–Tangherlini, and Myers–Perry metrics and we show that none of these hold for the five-dimensional charged rotating black hole with two equal-magnitude angular momenta. We show that this black hole is algebraically general. It also turns out that while the Ricci tensor is of type D and aligned with a type D Maxwell tensor F , the geometric properties of the common null aligned vector imply that the five-dimensional charged rotating black hole is not compatible with the Kerr–Schild ansatz (1) with geodesic k . In five dimensions the necessary condition for such compatibility boils down to $\hat{Q}\hat{J} = 0$ (see Equation (60)) which explains why this ansatz holds only for rotating vacuum or charged non-rotating five-dimensional black holes.

We should note that the previous work [9] also attempted to put the charged rotating black hole metric into Kerr–Schild form (1), but they only succeeded by replacing the reference Minkowski metric η_{ab} by a different metric with unclear algebraic or geometric properties. Moreover, the authors claim that by using some differential identities, they have simplified the corresponding Einstein–Maxwell equations to a system on only two unknown functions, while also giving an asymptotic solution to the resulting equations. Unfortunately, we have identified a mistake in a key identity that allowed their

³ Here we refer to the generalization of the standard Petrov classification of the Weyl tensor in four dimensions to other tensors and to higher dimensions [8, 22] (see [29] for a review). *Algebraically general* means that the Weyl tensor does not admit a multiple Weyl aligned null direction (multiple WAND). In contrast, type II and D spacetime admit one and (at least) two multiple WANDs respectively.

simplification and have also found that their asymptotic solution is not accurate. A more detailed discussion can be found in Appendix A.

2 Metric ansatz

Specializing the metric and vector potential ansatz for equal-magnitude angular momenta black holes in Einstein–Maxwell theory from [19] to five dimensions, gives

$$\begin{aligned}
 ds^2 = g_{\mu\nu} dx^\mu dx^\nu = & -f dt^2 + \frac{m}{f} (dr^2 + r^2 d\theta^2) \\
 & + \frac{n}{f} r^2 \left[\sin^2 \theta \left(d\phi - \frac{\omega}{r} dt \right)^2 + \cos^2 \theta \left(d\psi - \frac{\omega}{r} dt \right)^2 \right] \\
 & + \frac{m-n}{f} r^2 \sin^2 \theta \cos^2 \theta (d\phi - d\psi)^2, \tag{2}
 \end{aligned}$$

and

$$A = A_\mu dx^\mu = a_0 dt + a_\phi \left(\sin^2 \theta d\phi + \cos^2 \theta d\psi \right), \tag{3}$$

where the six functions f, m, n, ω, a_0 and a_ϕ only depend on the radial coordinate r . Ordering the coordinates as t, r, θ, ϕ, ψ , the metric in matrix form is

$$g_{\mu\nu} = \frac{1}{f} \begin{pmatrix} -f^2 + n\omega^2 & 0 & 0 & -n\omega r \sin^2 \theta & -n\omega r \cos^2 \theta \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & mr^2 & 0 & 0 \\ -n\omega r \sin^2 \theta & 0 & 0 & (m \cos^2 \theta + n \sin^2 \theta) r^2 \sin^2 \theta & (n-m)r^2 \sin^2 \theta \cos^2 \theta \\ -n\omega r \cos^2 \theta & 0 & 0 & (n-m)r^2 \sin^2 \theta \cos^2 \theta & (m \sin^2 \theta + n \cos^2 \theta) r^2 \cos^2 \theta \end{pmatrix}, \tag{4}$$

The outer horizon, located at $r = r_H$, is determined by the condition $f(r_H) = 0$, with the value of r_H closest to $r = \infty$. Two sets of boundary conditions are imposed, one at infinity and one at the horizon. At infinity ($r \rightarrow \infty$) we require asymptotic flatness, namely

$$\begin{aligned}
 ds^2 \sim & -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\
 & + r^2 \cos^2 \theta d\psi^2, \tag{5}
 \end{aligned}$$

$$A \sim 0, \tag{6}$$

where we have followed [19] by not specifying further details about what higher dimensional asymptotic flatness means (we give a more detailed definition of asymptotic flatness later in Sect. 3.1). In [19], the corresponding boundary condition is stated as the following asymptotic limits:

$$f, m, n \sim 1, \tag{7}$$

$$\omega, a_0, a_\phi \sim 0. \tag{8}$$

At the horizon ($r = r_H$), we require regularity, which means the existence of local coordinates in which the metric smoothly extends through the horizon. In [19], the corresponding boundary condition is stated as follows:

$$f(r_H), m(r_H), n(r_H) = 0, \tag{9}$$

$$|\omega(r_H)|, |a_0|, |a_\phi| < \infty, \tag{10}$$

$$a'_\phi(r_H) = 0. \tag{11}$$

Following [19, Sec. 3], provided the function f, ω and a_0 have asymptotic expansions at spatial infinity ($r \rightarrow \infty$) of the form

$$f = 1 - \frac{\hat{M}}{r^2} + \dots, \quad \omega = \frac{\hat{J}}{r^3} + \dots, \quad a_0 = \frac{\hat{Q}}{r^2} + \dots, \tag{12}$$

the physical mass, angular momentum and electric charges are respectively given by

$$M = \frac{(d-2)A(S^{d-2})}{16G_d} \hat{M} = \frac{3\pi^2}{8G_5} \hat{M},$$

$$J = \frac{A(S^{d-2})}{8G_d} \hat{J} = \frac{\pi^2}{4G_5} \hat{J},$$

$$Q = \frac{(d-3)A(S^{d-2})}{4G_d} \hat{Q} = \frac{\pi^2}{G_5} \hat{Q}, \tag{13}$$

where $d = 5$ is the spacetime dimension, G_d is the d -dimensional Newton’s constant, and $A(S^{d-2}) = A(S^3) = 2\pi^2$ is the area of the unit 3-sphere. The physical charges M and J are obtained by integrating the usual Komar forms at spatial infinity.

2.1 Special exact solutions

As we shall see in Sect. 3, the radial coordinate implied by the ansatz (2) has non-differentiable behavior (a square root singularity) at the horizon $r = r_H$. This phenomenon is most efficiently exhibited by matching the ansatz to the known exact solutions without rotation (5-dimensional charged Schwarzschild–Tangherlini metric) or without charge (5-dimensional Myers–Perry metric), which we do next. Since we are interested in making the comparison in the region exterior to the black hole, we restrict ourselves to the subex-

tremal ranges of the charge, mass, and angular momentum parameters of the exact solutions.

Before moving on, let us introduce a new radial coordinate $r = r(r)$, which will play the role of the more familiar radial coordinate in these exact solutions that regularly extends through the horizon. It so happens that in all cases considered in this paper, this new radial coordinate will obey the relationship

$$\frac{(dr)^2}{r^2} = \frac{r^2(dr)^2}{(r^2 - r_+^2)(r^2 - r_-^2)} \iff r = r_H \sqrt{\frac{\sqrt{r^2 - r_-^2} + \sqrt{r^2 - r_+^2}}{\sqrt{r^2 - r_-^2} - \sqrt{r^2 - r_+^2}}}, \tag{14}$$

where r_{\pm} are some constants and the differential relation has been solved to match $r(r_H) = r_+$. The square root singularity is clearly visible at the horizon in the expansion $r = r_H + \mathcal{O}(\sqrt{r - r_+})$, while at infinity $r = \mathcal{O}(r)$ and has a regular expansion in integer powers of r^{-1} .

The 5-dimensional *charged Schwarzschild–Tangherlini* (ST) solution (sometimes also called the 5-dimensional *Reissner–Nordström* solution) is given by

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2),$$

$$A = \left(\frac{\hat{Q}}{r^2} - \hat{C}\right) dt, \quad f = 1 - \frac{\hat{M}}{r^2} + \frac{4}{3} \frac{\hat{Q}^2}{r^4}$$

$$= \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4},$$

$$\hat{M} = r_+^2 + r_-^2, \quad \text{with } \hat{Q} = \frac{\sqrt{3}}{2} r_+ r_-, \tag{15}$$

where the mass and charge are parametrized by the constants (\hat{M}, \hat{Q}) or (r_+, r_-) , $r_+ > r_-$. Matching the ST solution with the ansatz (2) gives the radial coordinate relation (14) and

$$f = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4},$$

$$m = \frac{r^2}{r^2} f,$$

$$n = \frac{r^2}{r^2} f,$$

$$\omega = 0,$$

$$a_0 = \frac{\sqrt{3}}{2} \frac{r_+ r_-}{r^2} - \hat{C},$$

$$a_\phi = 0. \tag{16}$$

The 5-dimensional *Myers–Perry* (MP) solution with equal-magnitude angular momenta is given by

$$ds^2 = -dt^2 + \frac{\Sigma r^2 dr^2}{\Sigma^2 - \hat{M}r^2}$$

$$+ \Sigma r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2)$$

$$+ \frac{\hat{M}}{\Sigma} \left(dt + a \sin^2\theta d\phi + a \cos^2\theta d\psi\right)^2,$$

$$A = 0, \quad \Sigma = r^2 + a^2, \quad \Sigma^2 - \hat{M}r^2 = (r^2 - r_+^2)(r^2 - r_-^2),$$

$$\text{with } \hat{M} = (r_+ + r_-)^2, \quad a = \hat{J}/\hat{M} = \sqrt{r_+ r_-}, \tag{17}$$

where the mass and the angular momentum are parametrized by the constants (\hat{M}, \hat{J}) or (r_+, r_-) , $r_+ > r_-$. Matching the MP solution with the ansatz (2) gives

$$f = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{(r^2 + r_+ r_-)^2 + r_+ r_- (r_- + r_+)^2} = \frac{\Sigma^2 - \hat{M}r^2}{\Sigma^2 + a^2 \hat{M}},$$

$$m = \frac{(r^2 + r_+ r_-)}{r^2} f = \Sigma f,$$

$$n = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2(r^2 + r_+ r_-)} = \frac{\Sigma^2 - \hat{M}r^2}{\Sigma},$$

$$\omega = r \frac{\sqrt{r_+ r_-} (r_+ + r_-)^2}{(r^2 + r_+ r_-)^2 + r_+ r_- (r_- + r_+)^2} = \frac{a \hat{M}}{\Sigma^2 + a^2 \hat{M}},$$

$$a_0 = 0,$$

$$a_\phi = 0. \tag{18}$$

3 Regularity at infinity and at the horizon

Since the radial coordinate r used in the ansatz (2) has been found to be non-smooth at the horizons of known exact solutions, we must reparametrize the ansatz using our new regular coordinate r . In addition, we have found that removing most of the explicit dependence on r in the metric makes the corresponding Einstein equations (given below) autonomous, which is technically convenient.

Our new *regular* ansatz for the metric and vector potential is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f dt^2 + \frac{m}{f} \left(\frac{(r dr)^2}{N} + d\theta^2\right)$$

$$+ \frac{N}{m} \left[\sin^2\theta (d\phi - \varpi dt)^2 + \cos^2\theta (d\psi - \varpi dt)^2\right]$$

$$+ \frac{m^2 - fN}{mf} \sin^2\theta \cos^2\theta (d\phi - d\psi)^2 \tag{19}$$

and

$$A = A_\mu dx^\mu = a_0 dt + a_\phi (\sin^2\theta d\phi + \cos^2\theta d\psi). \tag{20}$$

The functions f, m, N, a_0 and a_ϕ all depend only on the new radial coordinate r . In matrix form, ordering the coordinates as $(t, r, \theta, \phi, \psi)$, the metric reads

$$g_{\mu\nu} = \begin{pmatrix} -f + \frac{N}{m}\varpi^2 & 0 & 0 & -\frac{N}{m}\varpi \sin^2 \theta & -\frac{N}{m}\varpi \cos^2 \theta \\ 0 & \frac{r^2 m}{fN} & 0 & 0 & 0 \\ 0 & 0 & \frac{m}{f} & 0 & 0 \\ -\frac{N}{m}\varpi \sin^2 \theta & 0 & 0 & \frac{(m^2 \cos^2 \theta + fN \sin^2 \theta)}{fm} \sin^2 \theta & \frac{(fN - m^2)}{fm} \sin^2 \theta \cos^2 \theta \\ -\frac{N}{m}\varpi \cos^2 \theta & 0 & 0 & \frac{(fN - m^2)}{fm} \sin^2 \theta \cos^2 \theta & \frac{(m^2 \sin^2 \theta + fN \cos^2 \theta)}{fm} \cos^2 \theta \end{pmatrix}. \tag{21}$$

Obviously, compared to (2), only the metric part has been reparametrized. We are using a new radial coordinate r and the old parametrization is recovered via the translation formulas

$$m = mr^2, \tag{22a}$$

$$\varpi = \frac{\omega}{r}, \tag{22b}$$

$$N = \frac{mnr^4}{f}, \tag{22c}$$

$$\frac{(r dr)^2}{N} = \frac{(dr)^2}{r^2}. \tag{22d}$$

Note that the radial coordinate r appears explicitly in the new ansatz (19) only through the combination $r dr = \frac{1}{2} dR$, where $R = r^2$. Hence, it is possible and useful to write the resulting Einstein equations as an autonomous system of ODEs with respect to the squared radial coordinate R . In particular, we will see that the way N appears in the ansatz gives it a very simple equation of motion. As long as the location of the horizon satisfies $r = r_H \neq 0$, if r is a regular coordinate at the horizon, then so is R . Again, we are interested in the interval from infinity $r \sim \infty$ to the horizon $f(r = r_H) = 0$, the root closest to infinity.

Plugging (19) directly into the Einstein equations gives somewhat complicated expressions. It is unenlightening to present them directly. Instead, inspired by the simplifications that have already been noted in [9, 19, 24], we will give them in equivalent but more structured form. In what follows, we use the notation $(-)' = \frac{d}{dR}(-)$ and all functions are treated as functions of R . First, we have the set of conservation laws

$$N'' = 2 \implies N = (R - r_+^2)(R - r_-^2), \tag{23a}$$

$$\left(\frac{N}{f}(a'_0 + \varpi a'_\phi)\right)' = 0 \implies \frac{N}{f}(a'_0 + \varpi a'_\phi) = -\hat{Q}, \tag{23b}$$

$$\left(\frac{N^2}{fm}\varpi' + 4\hat{Q}a_\phi\right)' = 0 \implies \frac{N^2}{fm}\varpi' + 4\hat{Q}a_\phi = -2\hat{J}, \tag{23c}$$

where $r_+, r_-, \hat{Q}, \hat{J}$ have been introduced as integration constants. Then we have the boundary value problem (BVP)

$$\begin{aligned} f^2 m \left(\frac{m}{f^2} f'\right)' - mm' \left(\frac{m'}{m} - 2\frac{N'}{N}\right) f - \left(4\frac{m^2}{N} - f\right) f \\ = \frac{8}{3} \frac{fm}{N} \left(2m^2(a'_\phi)^2 - fa_\phi^2\right) \\ - \frac{4\hat{Q}^2 f^2 m^2}{3 N^2}, \end{aligned} \tag{24a}$$

$$\begin{aligned} fm \left(\frac{N}{m} m'\right)' + \left(\frac{f^2 N}{m^2} - 2f\right) m = \frac{4}{3} f \left(2m^2(a'_\phi)^2 - fa_\phi^2\right) \\ + \frac{4\hat{Q}^2 f^2 m}{3 N} + 4\frac{f^2 m^2}{N^2} (\hat{J} + 2\hat{Q}a_\phi)^2, \end{aligned} \tag{24b}$$

$$m(ma'_\phi)' - fa_\phi = 2\hat{Q} \frac{fm^2}{N^2} (\hat{J} + 2\hat{Q}a_\phi), \tag{24c}$$

and finally we have the constraint

$$\begin{aligned} m^2(a'_\phi)^2 = fa_\phi^2 - m + \frac{fN}{4m} + \hat{Q}^2 \frac{fm}{N} + \frac{fm^2}{N^2} (\hat{J} + 2\hat{Q}a_\phi)^2 \\ + \frac{Nm'}{8} \left(\frac{f'}{f} - 2\frac{m'}{m} + 4\frac{N'}{N}\right) \\ - \frac{Nm f'}{8f} \left(2\frac{f'}{f} - \frac{m'}{m} + 2\frac{N'}{N}\right). \end{aligned} \tag{25}$$

This last constraint is compatible with the BVP system because, letting C be the difference between the left- and right-hand sides of (25),

$$(C/f^2)' = 0 \quad \text{mod (24)}, \tag{26}$$

which means that, if the boundary conditions for the BVP system are chosen such that $C/f^2 \rightarrow 0$ either at infinity or at the horizon, then the constraint $C = 0$ is guaranteed for any BVP solution.

Direct substitution shows that when our system of equations is satisfied, then so are the unsimplified Einstein equations. Conversely, starting from the unsimplified Einstein equations and eliminating all variables except N gives the first conservation law from (23). Using it to eliminate N' , as well as every variable and its derivatives that do not appear

in the second conservation law, gives that same conservation law. Repeating this idea once more, gives the third conservation law. Upon eliminating N, ϖ, a_0 and their derivatives from the unsimplified equations, solving for f'', m'', a''_ϕ gives the BVP system (24). Finally eliminating these and all other second derivatives leaves the constraint equation (25). The derivation of the unsimplified equations and the above elimination steps are most easily carried out using computer algebra; we have relied in part on the xTensor and xCoba packages from [21].

In the following sections, we will check how the geometric conditions of asymptotic flatness and regularity at the horizon translate to boundary conditions for the BVP system (24). Under these conditions, both endpoints become *regular* (or *Fuchsian*) *singular points*, where a solution can be constructed by the Frobenius method (as a power series in R). At each endpoint, we perform a Fuchsian analysis (see [17] for a brief exposition of the standard methods from [40], together with references to the larger literature), which counts the free parameters in the space of solutions compatible with the boundary condition. This analysis will help us confirm that the Einstein equations restricted to our ansatz are indeed well-posed. Well-posedness is of course a prerequisite property for trying to construct either numerical or asymptotic approximations to high accuracy. Unfortunately, the original papers [19, 24] did not report such a well-posedness analysis, nor specify in what way their numerical methods were compatible with the expected asymptotic structure of the solution at the horizon and infinity, which in principle may have led to spurious numerical artifacts near those points.

3.1 Asymptotic flatness

There are different notions of asymptotic flatness both in four and higher dimensions. We will refer specifically to the existence of a regular null infinity, where both asymptotic Poincaré symmetries and their charges are well defined. Specifically in five dimensions, this notion was analyzed in [37] and translated to precise component asymptotics for the metric in Bondi coordinates, which is what we will refer to below. Later, the same analysis was generalized to arbitrary higher dimension in [35]. Earlier work treated asymptotic flatness at null infinity in even dimensions [15] and at spatial infinity [36] using conformal methods. A unified treatment of the higher dimensional asymptotics of scalar, electromagnetic and gravitational fields at null infinity was given in [34], which is the main reference that we will follow. Along similar lines, [14] treated higher dimensional asymptotics for electromagnetic fields at both null and spatial infinities. Independently, various asymptotic behaviors for electromagnetic fields in higher dimensions were also analyzed in [31].

Effectively summarizing the results of the above references (most clearly presented in [34, Sec. II B] for elec-

tromagnetic and in [34, Sec. II D–E] for gravitational fields), the conditions of having an asymptotically flat regular (past/future) null infinity, with the absence of (incoming/outgoing) radiation (both gravitational and electromagnetic), but with non-vanishing contributions to total mass and electric charge, is equivalent to the existence of an asymptotic coordinate system $(u, r, \theta, \phi, \psi)$ and corresponding electromagnetic gauge where

$$g_{\mu\nu} = \begin{pmatrix} -1 \pm 1 & 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & 0 & r^2 \cos^2 \theta \end{pmatrix} + \begin{pmatrix} \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-1}) & \mathcal{O}(r^{-1}) & \mathcal{O}(r^{-1}) \\ \mathcal{O}(r^{-2}) & 0 & 0 & 0 & 0 \\ \mathcal{O}(r^{-1}) & 0 & \mathcal{O}(r^0) & \mathcal{O}(r^0) & \mathcal{O}(r^0) \\ \mathcal{O}(r^{-1}) & 0 & \mathcal{O}(r^0) & \mathcal{O}(r^0) & \mathcal{O}(r^0) \\ \mathcal{O}(r^{-1}) & 0 & \mathcal{O}(r^0) & \mathcal{O}(r^0) & \mathcal{O}(r^0) \end{pmatrix} \tag{27a}$$

and

$$A_\mu = (\mathcal{O}(r^{-2}) \ 0 \ \mathcal{O}(r^{-1}) \ \mathcal{O}(r^{-1}) \ \mathcal{O}(r^{-1})) . \tag{27b}$$

Some explanation is in order. The \pm choice of sign in the metric distinguishes the past and future null infinities. The exact zero entries in $g_{\mu\nu}$ and A_μ are part of the gauge fixing conditions. The Bondi gauge [37] usually also requires that the angular submatrix g_{IJ} , where I, J run through θ, ϕ, ψ , has determinant exactly equal to $r^6 \sin^2 \theta \cos^2 \theta$ (where $\sin^2 \theta \cos^2 \theta$ is the determinant of the unit round 3-sphere metric in the same coordinates). However, it is also sufficient that this subdeterminant has this limit only for $r \rightarrow \infty$, with subleading terms given by an asymptotic expansion in integer powers of r , because then the r coordinate can be redefined to obtain exact equality, without affecting the structure of the asymptotics. Below, when adapting our ansatz to the Bondi gauge, we will not perform this extra redefinition, in favor of our more convenient r coordinate. However, such a redefinition would of course be possible. Note also that the Bondi r coordinate is logically distinct from the coordinate denoted by the same symbol in the metric (2). Since the Bondi radial coordinate is only used in this section, there should be no confusion between the two.

In dimension $d = 5$, in the absence of gravitational radiation, the leading radial *Coulombic* asymptotic for both gravitational and electromagnetic perturbations is $\mathcal{O}(r^{-d+3}) = \mathcal{O}(r^{-2})$ in an orthonormal basis [34],⁴ which explains different decay rates in different coordinate tensor components

⁴ Reference [34] uses harmonic instead of Bondi gauge. But from their discussion, it is clear that one can transform from one gauge to the other without changing the leading asymptotics.

above. Coulombic terms are those that can contribute to finite and non-vanishing mass, angular momentum and electric charges. In the presence of gravitational and electromagnetic radiation, the perturbations would decay only as $\mathcal{O}(r^{-d/2+1}) = \mathcal{O}(r^{-3/2})$ in an orthonormal basis [34]. It should also be mentioned that the leading asymptotic terms will be further constrained by the Einstein equations themselves, which may lead to faster decay for some components, as we shall see in our case.

We can define the desired Bondi (which could also be called Eddington–Finkelstein) coordinates as

$$\begin{aligned} du_{\pm} &= dt \pm \sqrt{\frac{m}{N}} \frac{dR}{2f}, \\ r &= r, \\ \theta &= \theta, \\ d\phi_{\pm} &= d\phi \pm \varpi \sqrt{\frac{m}{N}} \frac{dR}{2f}, \\ d\psi_{\pm} &= d\psi \pm \varpi \sqrt{\frac{m}{N}} \frac{dR}{2f}. \end{aligned} \tag{28}$$

In the coordinates $(u_{\pm}, r, \theta, \phi_{\pm}, \psi_{\pm})$, the matrix form of the metric is

$$g_{\mu\nu} = \begin{pmatrix} -f + \frac{N}{m}\varpi^2 & \pm\sqrt{\frac{r^2m}{N}} & 0 & -\frac{N}{m}\varpi \sin^2\theta & -\frac{N}{m}\varpi \cos^2\theta \\ \pm\sqrt{\frac{r^2m}{N}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{m}{f} & 0 & 0 \\ -\frac{N}{m}\varpi \sin^2\theta & 0 & 0 & \frac{(m^2 \cos^2\theta + fN \sin^2\theta)}{fm} \sin^2\theta & \frac{(fN - m^2)}{fm} \sin^2\theta \cos^2\theta \\ -\frac{N}{m}\varpi \cos^2\theta & 0 & 0 & \frac{(fN - m^2)}{fm} \sin^2\theta \cos^2\theta & \frac{(m^2 \sin^2\theta + fN \cos^2\theta)}{fm} \cos^2\theta \end{pmatrix}, \tag{29a}$$

while the vector potential takes on the form

$$A_{\mu} - (d\alpha)_{\mu} = (a_0 \ 0 \ 0 \ a_{\phi} \ \sin^2\theta \ a_{\psi} \ \cos^2\theta), \tag{29b}$$

with the gauge transformation parameter defined by $\alpha'(r) = -\sqrt{r^2m/N}(a_0 + \varpi a_{\phi})/f$.

Hence, comparing (29) with (27), both the metric and the vector potential are in desired Bondi form, provided we require the following leading asymptotics (recalling that $R = r^2$)

$$\begin{aligned} f &= 1 + \mathcal{O}(R^{-1}), \\ m &= R + \mathcal{O}(R^0), \\ a_{\phi} &= \mathcal{O}(R^{-1/2}), \\ N &= R^2 + \mathcal{O}(R^1), \\ \varpi &= \mathcal{O}(R^{-3/2}), \\ a_0 &= \mathcal{O}(R^{-1}). \end{aligned} \tag{30}$$

Note that comparing with (27) has allowed us to determine not only the leading but also the subleading terms of f, m and N . We have not yet specified the structure of the full asymptotic expansion in (30), since in principle we can allow subleading terms with arbitrary fractional powers of R . However, we are also free to restrict the subleading terms as we see fit, with the only justification necessary the a posteriori well-posedness of the BVP part of the Einstein Eqs. (24) and (23).

So we may assume that the asymptotic expansion is in integer powers of R , which is consistent because we are solving equations that are autonomous with respect to R , and perform a Fuchsian analysis of the BVP system at the singular point $R = \infty$. For linear equations, this is the same as applying the method of Frobenius. For a non-linear equation, the first step is take the leading terms from (30) and check that they are consistent with the BVP system (24). Namely, the leading order terms are collected and it is checked that they cancel among themselves. We have specifically written (24) in a way that all leading terms appear on the left-hand side. The cancellation, which is verified by direct calculation, could have imposed some conditions on the undetermined leading coefficient in a_{ϕ} , but it turns out that it does not.

The second step is to linearize (24) and apply the method of Frobenius to the linear equation whose coefficients have asymptotics determined by the leading terms (30). If $R = \infty$ turns out to be a regular singular point of this linear equation, we can extract the corresponding indicial equation and determine the remaining free coefficients in the expansion. For systems of ODEs, determining whether a singular point is regular is not entirely trivial. The procedure must take into account that the leading terms in different components of the system of unknowns or of the system of equations may be of different orders, let alone that the components of the equations and of the unknowns are allowed to mix.

There is a convenient criterion to check when a system of k equations of order p has a regular singular point (again, see [17] for a brief exposition). Let $E[v] = 0$ be such a system of linear equations, whose coefficients have asymptotic expansions in powers of R . Let S and T be $k \times k$ matrices such that all of S, T, S^{-1}, T^{-1} have components that are Laurent polynomials in R (for that, it is necessary and sufficient that $\det S$ and $\det T$ are non-vanishing monomials).

The system has a regular singular point at $R = \infty$ if $T^{-1}E[Sv] = E_0(R \partial_R)[v] +$ lower order terms, where the components of $E_0(x)$ are constant coefficient polynomials in x , $\det E_0(x) \neq 0$, and for the purposes of collecting lower order terms $R^n (R \partial_R)^m$ has order n (which will typically be negative). We call matrices S and T *leading multipliers*. The integer solutions of the *indicial equation* $\det E_0(n) = 0$ are the critical exponents (or indices) of R^n whose coefficients in the expansion of the solution may be a free parameter. The eigenvectors of $E_0(n)$ corresponding to the critical exponents determine how these free parameters enter the expansion. At a true real regular singular point, the polynomial degree of $\det E_0(n)$ should be pk , determined by the size and order of the system. If the degree is lower, then only a subset of the solutions exhibit a Frobenius expansion in powers of R (with possible logarithmic contributions) and there will exist other solutions that grow or decay faster than simple powers of R .

The third and last step counts the number of free parameters in the asymptotic expansion of a solution. If $v = S [\sum_{k>n} v_k R^k + v_n R^n + O(R)^{n-1}]$, then knowing that the $v_{k>n}$ coefficients solve the ODE system to appropriate order we can solve for the next coefficient, provided that $E_0(n)$ is invertible, via

$$E_0(n)v_n = e_n(v_{k>n}), \tag{31}$$

where $e_n(-)$ collects all the remaining terms of the original non-linear ODE system at order n . The invertibility of $E_n(n)$ fails precisely when n is a critical exponent. Then two things happen: because $E_0(n)$ does not have full row rank, the consistency of the equation imposes constraints on $e_n(v_{k>n})$ and hence on any free parameters appearing among the $v_{k>n}$ coefficients; also, because $E_0(n)$ does not have full column rank, new free parameters appear in the solution for the v_n coefficients. In general the number of free parameters gained and lost at a time need not be the same, so at each critical exponent the total number of free parameters in the expansion may increase, decrease, or stay the same; it stabilizes after the last critical exponent.

In general, leading multiplier matrices may be quite complicated, but we have specifically written the BVP system (24) to make them simple. Namely, the resulting $E_0(R \partial_R)[v]$ operator is

$$\begin{bmatrix} (R \partial_R)^2 + 1 & 2(R \partial_R - 1) & 0 \\ 1 & (R \partial_R + 2)(R \partial_R - 1) & 0 \\ 0 & 0 & (R \partial_R + 1)(R \partial_R - 1) \end{bmatrix} \times \begin{bmatrix} \delta f \\ \delta m \\ \delta a_\phi \end{bmatrix}, \tag{32}$$

where S multiplies δm by R^{-1} and T^{-1} multiplies the middle equation by R , while acting as the identity on other components. The critical exponents are $k = 1$ (2), 0 (1), -1 (3), with multiplicities indicated in parentheses. Because of the

constraints on the leading terms from (30), the coefficients corresponding to the $k = 1, 0$ exponents must vanish. At $k = -1$, one free parameter is allowed in front of the leading term of a_ϕ and two more free parameters appear in the subleading terms of f and m , with algebraic multiplicity 2. As is usual in applying the Frobenius method, the algebraic multiplicity 2 implies that the two independent coefficients appear in front of R^{-1} and $R^{-1} \log R$ in the expansions of f and m/R . However, the logarithmic terms are incompatible with the asymptotics (30) corresponding to a regular null infinity. Thus, only one free coefficient at order R^{-1} remains. The remaining two free parameters remain free at all further orders of the expansion.

Once the asymptotic expansions for f, m, a_ϕ have been determined, we can also asymptotically integrate the first order conservation laws (23) for ϖ and a_0 . For each of them, there is a free parameter that appears as an additive constant.

In summary, the following leading terms uniquely fix the most general asymptotic expansion of a solution in powers of R that is compatible with the asymptotic form (30):

$$\begin{aligned} f &= 1 + \frac{f^{(-1)}}{R} + \mathcal{O}(R^{-2}), \\ m &= R + m^{(0)} + \mathcal{O}(R^{-1}), \\ a_\phi &= \frac{a_\phi^{(-1)}}{R} + \mathcal{O}(R^{-2}), \\ N &= R^2 - (r_+^2 + r_-^2)R + r_+^2 r_-^2, \\ \varpi &= \varpi^{(0)} + \frac{\hat{J}}{R^2} + \mathcal{O}(R^{-3}), \\ a_0 &= a_0^{(0)} + \frac{\hat{Q}}{R} + \mathcal{O}(R^{-2}), \end{aligned} \tag{33}$$

which has exactly two free parameters ($a_\phi^{(-1)}$ and a linear combination of $f^{(-1)}$ and $m^{(0)}$), since the yet undetermined coefficients are constrained by

$$\begin{aligned} f^{(-1)} - 2m^{(0)} - (r_+^2 + r_-^2) &= 0, \\ \varpi^{(0)} &= 0, \\ a_0^{(0)} &= 0, \end{aligned} \tag{34}$$

after enforcing compatibility with the prescribed asymptotic limits of f, m, ϖ and a_0 . Comparing with (12) justifies the notation \hat{J} and \hat{Q} for the coefficients in the expansion of ω and a_0 and also allows us to identify

$$f^{(-1)} = -\hat{M}, \tag{35}$$

with \hat{M}, \hat{J} and \hat{Q} being proportional to the physical mass, angular momentum and electric charges, respectively, as specified in (13). Finally, there are no further constraints coming from enforcing the limit $C/f^2 \rightarrow 0$, which is needed to ensure that the constraint $C = 0$ (23) is enforced by the compatibility equation (26). Recall that r_+^2, r_-^2 are integration

constants that appear in (23). Note that, after we have carefully taken into account the Einstein–Maxwell equations, the final leading asymptotics $a_\phi = \mathcal{O}(R^{-1})$ and $\varpi = \mathcal{O}(R^{-2})$ decay faster than strictly required by asymptotic flatness in (30).

Having shown that at infinity the series expansion for the solution is uniquely fixed as a function of $r_\pm, \hat{J}, \hat{Q}, a_\phi^{(-1)}$ and say \hat{M} , we report here the first few terms of $f, m,$ and a_ϕ extending (33) (defining $r_H^2 = r_+^2 - r_-^2$):

$$\begin{aligned}
 f &= 1 - \frac{\hat{M}}{(R - r_-^2)} + \frac{\frac{1}{2}\hat{M}(\hat{M} - r_H^2) + \frac{4}{3}\hat{Q}^2}{(R - r_-^2)^2} + \frac{\frac{4}{9}(a_\phi^{(-1)})^2 - \frac{1}{6}\hat{M}(\hat{M}^2 - 3\hat{M}r_H^2 + 2r_H^4) + \frac{4}{9}\hat{Q}^2(-4\hat{M} + 3r_H^2) + \frac{2}{3}\hat{J}^2}{(R - r_-^2)^3} \\
 &\quad + \frac{\frac{2}{9}(a_\phi^{(-1)})^2(-2\hat{M} + 3r_H^2) + \frac{16}{9}a_\phi^{(-1)}\hat{J}\hat{Q} + \frac{1}{9}\hat{Q}^2(11(\hat{M} - r_H^2)^2 - 2\hat{M}r_H^2)}{(R - r_-^2)^4} \\
 &\quad + \frac{\frac{1}{24}\hat{M}(\hat{M} - r_H^2)(\hat{M} - 2r_H^2)(\hat{M} - 3r_H^2) + \hat{J}^2(-\frac{7}{6}\hat{M} + r_H^2) + \frac{32}{27}\hat{Q}^4}{(R - r_-^2)^4} + \mathcal{O}(R^{-5}), \\
 m &= (R - r_-^2) - \frac{1}{2}(\hat{M} + r_H^2) + \frac{\frac{4}{3}\hat{Q}^2}{(R - r_-^2)} + \frac{\frac{1}{72}(16(a_\phi^{(-1)})^2 + 3\hat{M}^3 - 3\hat{M}r_H^4) + \frac{2}{9}\hat{Q}^2(-4\hat{M} + 3r_H^2) + \frac{5}{6}\hat{J}^2}{(R - r_-^2)^2} \\
 &\quad + \frac{\frac{1}{360}(-\hat{M} + 5r_H^2)(16(a_\phi^{(-1)})^2 + 3\hat{M}^3 - 3\hat{M}r_H^4) + \frac{88}{45}a_\phi^{(-1)}\hat{J}\hat{Q}}{(R - r_-^2)^3} \\
 &\quad + \frac{\frac{1}{45}\hat{Q}^2(9\hat{M}^2 - 40\hat{M}r_H^2 + 23r_H^4) + \hat{J}^2(-\frac{29}{30}\hat{M} + \frac{5}{6}r_H^2) + \frac{128}{135}\hat{Q}^4}{(R - r_-^2)^3} + \mathcal{O}(R^{-4}), \\
 a_\phi &= \frac{a_\phi^{(-1)}}{(R - r_-^2)} + \frac{\frac{2}{3}\hat{J}\hat{Q} + \frac{1}{6}a_\phi^{(-1)}(\hat{M} + 3r_H^2)}{(R - r_-^2)^2} + \frac{\frac{1}{12}a_\phi^{(-1)}(\hat{M}^2 + 2\hat{M}r_H^2 + 3r_H^4) + \frac{1}{6}\hat{J}\hat{Q}(-\hat{M} + 4r_H^2)}{(R - r_-^2)^3} \\
 &\quad + \frac{-\frac{1}{360}a_\phi^{(-1)}(16(a_\phi^{(-1)})^2 - 6\hat{M}^3 - 45\hat{M}^2r_H^2 - 48\hat{M}r_H^4 - 45r_H^6)}{(R - r_-^2)^4} \\
 &\quad + \frac{-\frac{34}{135}a_\phi^{(-1)}\hat{M}\hat{Q}^2 - \frac{7}{30}a_\phi^{(-1)}\hat{J}^2 + \frac{1}{60}\hat{J}\hat{Q}(5\hat{M}^2 - 15\hat{M}r_H^2 + 34r_H^4) + \frac{8}{135}\hat{J}\hat{Q}^3}{(R - r_-^2)^4} + \mathcal{O}(R^{-5}). \tag{36}
 \end{aligned}$$

As discussed in more detail in Sect. 3.3, our reduced Einstein–Maxwell equations possess the shift symmetry $R \mapsto R + R_0$, which also shifts the constants $r_\pm^2 \rightarrow r_\pm^2 + R_0$. The above expansion was obtained under the simplifying assumption $r_-^2 = 0$ and then converted to a form that is manifestly invariant under this shift symmetry. Elsewhere in the text, we will use r_H to denote the location of the outer horizon in general. So, strictly speaking, the numerical value of r_H in the above expansion should only be interpreted as the location of the horizon under the assumption $r_-^2 = 0$.

This expansion is used several times in Sect. 4 to test whether some differential constraints forced by special algebraic types would be satisfied on-shell, that is, satisfying the

Einstein–Maxwell equations and our boundary conditions. For instance, deducing the condition (60) required an expansion at least to the order given above.

3.2 Regularity at the horizon

The vanishing $f(r_H)$ on the horizon signals a coordinate singularity there, as is expected for coordinates adapted to a

timelike Killing vector when it becomes null. The regularity of the metric (2) and the corresponding vector potential at the horizon must then be checked in other coordinates that penetrate the horizon, where the only requirement for a tensor field to be regular is for it to have smooth components in that coordinate system.

Traditionally, the simplest horizon penetrating coordinates on a black hole are of Eddington–Finkelstein type. The coordinates (28) that we have introduced to play the role of the Bondi frame at null infinity are actually also of Eddington–Finkelstein type at the horizon. The $(u_+, R, \theta, \phi_+, \psi_+)$ coordinates regular at the future horizon and $(u_-, R, \theta, \phi_-, \psi_-)$ ones regular at the past horizon.

We will show below the horizon regularity of our improved ansatz (19) with respect to these coordinates. Unfortunately, neither of these coordinate systems is regular at the bifurcation sphere where the two horizons intersect. Thus we will also introduce Kruskal-like coordinates that cover a neighborhood of the bifurcation sphere and show that our improved ansatz is regular there as well.

To reproduce the horizon behavior (9) imposed in the original work [19], we must set $f = m = N = 0$ at $R = r_H^2$. As a consequence of the integration of the equation of motion (23) for N , we must set $r_H = r_+^2$, where we have required $r_+ \neq 0$ and ordered the integration constants as $r_+^2 > r_-^2$. Combining these requirements with regularity at the horizon, via smoothness of the tensor components (29) in Eddington–Finkelstein coordinates, gives the horizon asymptotics

$$f = \mathcal{O}(R - r_H^2), \tag{37a}$$

$$m = \mathcal{O}(R - r_H^2), \tag{37b}$$

$$a_\phi = \mathcal{O}(1), \tag{37c}$$

$$N = (r_+^2 - r_-^2)(R - r_H^2) + \mathcal{O}((R - r_H^2)^2), \tag{37d}$$

$$\varpi = \mathcal{O}(1), \tag{37e}$$

$$a_0 = \mathcal{O}(1), \tag{37f}$$

with each function also smooth in R at the horizon and where the leading coefficients of f and m are non-vanishing. Again, the asymptotics of N , ϖ and a_0 are obtained by integrating the conservation laws (23), which remain consistent with the regularity of the tensor coefficients in (29).

Kruskal (or Kruskal–Szekeres) coordinates for Schwarzschild spacetime are well-known. It is not as easy to find analogous coordinates constructed for the Kerr and related black holes. The original construction by Carter [6] was restricted to the (t, r) plane along the rotation axis. Pretorius and Israel [33] seem to have been the first to construct

global double-null coordinates regular on the Kerr bifurcation sphere. Motivated by applications in the global non-linear stability of Kerr and related black holes, analogous double-null coordinates have been constructed also for a wider class of geometries, see [5, 10] and references therein.

For our purposes, since we are interested only in a neighborhood of the bifurcation sphere, it is easiest to follow the idea from the much simpler construction by Hayward [13]. Namely, consider a constant κ , which we will choose appropriately, and the coordinates $(U, V, \theta, \Phi, \Psi)$ defined by

$$\begin{aligned} \frac{dU}{U} - \frac{dV}{V} &= \kappa dt, & dt &= \frac{1}{\kappa} \left(\frac{dU}{U} - \frac{dV}{V} \right), \\ \frac{dU}{U} + \frac{dV}{V} &= \kappa \sqrt{\frac{m}{N}} \frac{dR}{2f}, \\ \sqrt{\frac{m}{N}} \frac{dR}{2f} &= \frac{1}{\kappa} \left(\frac{dU}{U} + \frac{dV}{V} \right), \\ d\Phi &= d\phi - \varpi(r_H^2) dt, \\ d\phi &= d\Phi + \frac{\varpi(r_H^2)}{\kappa} \left(\frac{dU}{U} - \frac{dV}{V} \right), \\ d\Psi &= d\psi - \varpi(r_H^2) dt, \\ d\psi &= d\Psi + \frac{\varpi(r_H^2)}{\kappa} \left(\frac{dU}{U} - \frac{dV}{V} \right). \end{aligned} \tag{38}$$

Integrating the second equation, we find

$$\begin{aligned} d \ln(UV) &= \frac{\kappa}{2f'(r_H^2)} \sqrt{\frac{m'(r_H^2)}{N'(r_H^2)}} \frac{dR}{R - r_H^2} + \mathcal{O}((R - r_H^2)^0) \\ &\iff UV = (R/r_H^2 - 1)h(R), & h(R) &= 1 \\ & & & + \mathcal{O}(R - r_H^2), \end{aligned} \tag{39}$$

provided we choose the overall integration constant appropriately and set

$$\kappa = 2f'(r_H^2) \sqrt{\frac{N'(r_H^2)}{m'(r_H^2)}}. \tag{40}$$

In these new Kruskal coordinates, denoting $\Delta\varpi = \varpi - \varpi(r_H^2)$, the metric takes the form

$$g_{\mu\nu} = \begin{pmatrix} V^2 \frac{(\Delta\varpi)^2 N}{\kappa^2 U^2 V^2 m} & \frac{2f}{\kappa^2 UV} - UV \frac{(\Delta\varpi)^2 N}{\kappa^2 U^2 V^2 m} & 0 & -V \frac{(\Delta\varpi) N}{\kappa UV m} \sin^2 \theta & -V \frac{(\Delta\varpi) N}{\kappa UV m} \cos^2 \theta \\ \frac{2f}{\kappa^2 UV} - UV \frac{(\Delta\varpi)^2 N}{\kappa^2 U^2 V^2 m} & U^2 \frac{(\Delta\varpi)^2 N}{\kappa^2 U^2 V^2 m} & 0 & U \frac{(\Delta\varpi) N}{\kappa UV m} \sin^2 \theta & U \frac{(\Delta\varpi) N}{\kappa UV m} \cos^2 \theta \\ 0 & 0 & \frac{m}{f} & 0 & 0 \\ -V \frac{(\Delta\varpi) N}{\kappa UV m} \sin^2 \theta & U \frac{(\Delta\varpi) N}{\kappa UV m} \sin^2 \theta & 0 & \frac{(m^2 \cos^2 \theta + fN \sin^2 \theta)}{fm} \sin^2 \theta & \frac{(fN - m^2)}{fm} \sin^2 \theta \cos^2 \theta \\ -V \frac{(\Delta\varpi) N}{\kappa UV m} \cos^2 \theta & U \frac{(\Delta\varpi) N}{\kappa UV m} \cos^2 \theta & 0 & \frac{(fN - m^2)}{fm} \sin^2 \theta \cos^2 \theta & \frac{(m^2 \sin^2 \theta + fN \cos^2 \theta)}{fm} \cos^2 \theta \end{pmatrix}. \tag{41}$$

Recalling that $\Delta\varpi, f, \mathbf{m}, \mathbf{N} = \mathcal{O}(\mathbf{R} - r_{\text{H}}^2) = \mathcal{O}(UV)$, with the leading terms of f and \mathbf{m} non-vanishing, all the components and also

$$\det g = -\frac{4\mathbf{m}\mathbf{N}}{\kappa^4 U^2 V^2} \cos^2 \theta \rightarrow -\frac{4r_{\text{H}}^4 \mathbf{m}'(r_{\text{H}}^2) \mathbf{N}'(r_{\text{H}}^2)}{\kappa^4} \cos^2 \theta \tag{42}$$

remain finite as $\mathbf{R} \rightarrow r_{\text{H}}^2$. Hence, under the conditions already imposed by (37), the metric is regular in the Kruskal coordinates $(U, V, \theta, \Phi, \Psi)$ in a neighborhood of the bifurcation sphere $U = V = 0$, where the future and past horizons $\mathbf{R} = r_{\text{H}}^2$ intersect. The vector potential transforms to

$$A = \frac{\Delta(a_0 + \varpi a_\phi)}{\kappa UV} (V dU - U dV) + a_\phi (\sin^2 \theta d\Phi + \cos^2 \theta d\Psi) + d \left[\frac{a_0(r_{\text{H}}^2) + \varpi(r_{\text{H}}^2)a_\phi(r_{\text{H}}^2)}{\kappa} \ln \frac{U}{V} \right], \tag{43}$$

where we have used the shortcut notation

$$\Delta(a_0 + \varpi a_\phi) = \left(a_0 + \varpi(r_{\text{H}}^2)a_\phi \right) - \left(a_0(r_{\text{H}}^2) + \varpi(r_{\text{H}}^2)a_\phi(r_{\text{H}}^2) \right). \tag{44}$$

As discussed in the case of Eddington–Finkelstein-like coordinates, the last term of the vector potential is singular but pure gauge (it is exact), while the rest of the terms are manifestly regular.

Explicitly solving the coordinate transformation, we find

$$\begin{aligned} U &= \sqrt{\mathbf{R}/r_{\text{H}}^2 - 1} e^{\kappa t/2} \sqrt{h}, \\ dU &= \frac{\kappa}{2} \sqrt{\mathbf{R}/r_{\text{H}}^2 - 1} e^{\kappa t/2} \sqrt{h} \left(\sqrt{\frac{\mathbf{m}}{\mathbf{N}}} \frac{d\mathbf{R}}{2f} + dt \right), \\ V &= \sqrt{\mathbf{R}/r_{\text{H}}^2 - 1} e^{-\kappa t/2} \sqrt{h}, \\ dV &= \frac{\kappa}{2} \sqrt{\mathbf{R}/r_{\text{H}}^2 - 1} e^{-\kappa t/2} \sqrt{h} \left(\sqrt{\frac{\mathbf{m}}{\mathbf{N}}} \frac{d\mathbf{R}}{2f} - dt \right), \\ \Phi &= \phi - \varpi(r_{\text{H}}^2)t, \\ d\Phi &= d\phi - \varpi(r_{\text{H}}^2) dt \\ \Psi &= \psi - \varpi(r_{\text{H}}^2)t, \\ d\Psi &= d\psi - \varpi(r_{\text{H}}^2) dt. \end{aligned} \tag{45}$$

Next, we repeat the Fuchsian analysis of $\mathbf{R} = r_{\text{H}}^2$ as a regular singular point of the BVP part (24) of the Einstein–Maxwell equations restricted to our ansatz, to determine the remaining free parameters in the Taylor expansion of the solution. Since the steps of the procedure were already explained in Sect. 3.1, we merely summarize the results. For convenience, we will denote $\rho \equiv \mathbf{R} - r_{\text{H}}^2$.

Plugging the leading terms (37) into the BVP system (24), a direct calculation shows that they are compatible, without

any new restrictions on the undetermined coefficients. Again, we have written the BVP system in such a way that all cancelling leading terms appear on the left-hand side. Upon linearization, we find that the leading multiplier matrices S and T may just be taken to be identity. The resulting $E_0(\rho\partial_\rho)[v]$ operator is

$$\begin{aligned} &\begin{bmatrix} (\rho\partial_\rho - 3)(\rho\partial_\rho - 1) & \frac{f^{(1)}}{\mathbf{m}^{(1)}}(\rho\partial_\rho - 1) & 0 \\ 0 & (\rho\partial_\rho - 1)^2 & 0 \\ -\frac{a_\phi^{(0)}}{(\mathbf{m}^{(1)})^2} - \frac{2\hat{Q}(\hat{J} + 2\hat{Q}a_\phi^{(0)})}{(r_+^2 - r_-^2)^2} & \frac{4f^{(1)}\hat{Q}(\hat{J} + 2\hat{Q}a_\phi^{(0)})}{\mathbf{m}^{(1)}(r_+^2 - r_-^2)^2} & (\rho\partial_\rho)^2 \end{bmatrix} \\ &\times \begin{bmatrix} \delta f \\ \delta \mathbf{m} \\ \delta a_\phi \end{bmatrix}. \end{aligned} \tag{46}$$

Exchanging the first two rows and columns, we see that the matrix is lower triangular, so that we can immediately read off its eigenvalues and determinant (the indicial equation). The critical ρ^k exponents are $k = 0$ (2), 1 (3), 3 (1), with multiplicities indicated in parentheses. Note that regularity at the horizon excludes any logarithmic terms like $\rho^k (\log \rho)^l$ in the expansion. Multiplicities in the critical exponents that generate the same eigen-vectors do not generate more free parameters. At $k = 0$ we already have the free leading coefficient of a_ϕ . At $k = 1$, we have the free leading coefficients of f and \mathbf{m} . At $k = 3$ we have one new free coefficient in the expansion of f and no restrictions on the other free coefficients. Finally, enforcing the limit $\mathbf{C}/f^2 \rightarrow 0$ needed to ensure that the constraint $\mathbf{C} = 0$ (25) is enforced by the compatibility equation (26) does not bring in any new restrictions on the free parameters. Once the Taylor series for f, \mathbf{m}, a_ϕ have been determined, we can also integrate the first order conservation laws (23) for ϖ and a_0 . For each of them, there is a free parameter that appears as an additive constant. In summary, the following leading terms uniquely fix the Taylor series of a solution in powers of $\rho = \mathbf{R} - r_{\text{H}}^2$, with all free parameters explicitly shown:

$$\begin{aligned} f &= f^{(1)}\rho + \dots + f^{(3)}\rho^3 + \mathcal{O}(\rho^4), \\ \mathbf{m} &= \mathbf{m}^{(1)}\rho + \mathcal{O}(\rho^2), \\ a_\phi &= a_\phi^{(0)} + \mathcal{O}(\rho), \\ \mathbf{N} &= (r_+^2 - r_-^2)\rho + \rho^2, \\ \varpi &= \varpi^{(0)} + \mathcal{O}(\rho), \\ a_0 &= a_0^{(0)} + \mathcal{O}(\rho). \end{aligned} \tag{47}$$

The additive constants $\varpi^{(0)}$ and $a_0^{(0)}$ are not fixed by the requirements of regularity at the horizon (37). Instead, they must be fixed by enforcing compatibility with the prescribed limits (30) at infinity, as was discussed at the end of Sect. 3.1. Thus, the parameters $\varpi^{(0)}$ and $a_0^{(0)}$ are uniquely fixed, but their values cannot be obtained locally at the horizon.

3.3 Well-posedness

In this section we discuss the well-posedness of the BVP system (24) with the boundary conditions prescribed by asymptotic flatness (Section 3.1) and regularity at the horizon (Sect. 3.2), as well as the overall uniqueness of the corresponding charged rotating black hole solution.

The BVP system (24) is a second order ODE system for three unknowns, with external parameters r_+^2 , r_-^2 , \hat{J} and \hat{Q} . With the external parameters fixed, its solution space is hence 6-dimensional (counting the free initial data at any regular point, for instance). The conclusion of Sect. 3.1 is that there is a 2-dimensional space of solutions compatible with asymptotic flatness (parametrized by the constants $f^{(-1)}$, $m^{(0)}$, $a_\phi^{(-1)}$ (33), with one constraint (34) among them). The conclusion of Sect. 3.2 is that there is a 4-dimensional space of solutions compatible with regularity at the horizon, parametrized by the constants $f^{(1)}$, $f^{(3)}$, $m^{(1)}$, $a_\phi^{(0)}$ in Equation (47). Hence, within the overall 6-dimensional solution space, the intersection of these 2- and 4-dimensional families will generically be 0-dimensional, meaning that the solution if it exists is expected to be locally unique (under small perturbations of free parameters at either boundary). In other words, the boundary conditions prescribed by Sects. 3.1 and 3.2 define a well-posed BVP for the non-linear system (24).

The BVP system (24) itself depends on 4 external parameters, the integration constants r_+^2 , r_-^2 , \hat{J} , \hat{Q} from the conservation laws (23). Two of them, \hat{J} and \hat{Q} , have direct physical interpretations as being proportional to the total angular momentum and total charge, respectively (13). Because the total Einstein–Maxwell system of equations restricted to the ansatz (19) is autonomous with respect to \mathbf{R} , the coordinate shift $\mathbf{R} \rightarrow \mathbf{R} + \mathbf{R}_0$ is a symmetry, since it is also compatible with both the horizon and asymptotic boundary conditions. However, being a simple coordinate transformation it does not change the isometry class of the solution. Note that this shift affects the integration constants as $r_\pm^2 \rightarrow r_\pm^2 + \mathbf{R}_0$. We can use the shift symmetry to place the horizon at any convenient location, for example by setting $r_-^2 = 0$, which we will assume from now on. This choice places the horizon at $r_H = r_+$. Having decided the location of the horizon, by the well-posedness of the BVP discussed in the previous paragraph, the solution is (locally) uniquely fixed by the three remaining parameters. In particular, the total mass can be written as $\hat{M} = \hat{M}(r_H^2, \hat{J}, \hat{Q})$, where we can use the relation (35) between $f^{(-1)}$ and \hat{M} . Physically, it is more convenient to dynamically invert this relationship to $r_H^2 = r_H^2(\hat{M}, \hat{J}, \hat{Q})$.

In summary, we see that the isometry class of a charged rotating black hole with ansatz (19) and satisfying the Einstein–Maxwell equations is (locally) uniquely deter-

mined by its total mass \hat{M} , total angular momentum \hat{J} , and total charge \hat{Q} .

As proof of concept, taking the above well-posedness discussion into account, we have implemented a numerical solver for the BVP system (24) that takes the external \hat{M} , \hat{J} and \hat{Q} parameters as input and constructs the corresponding solution of the Einstein–Maxwell equations both outside and inside the horizon. On the exterior region, we use BVP solver in a basis of Chebyshev polynomials that are weighted at infinity and the horizon by the expected asymptotics and boundary conditions. Once the solution on the exterior region has converged, its values and derivatives near the horizon are used to feed into a standard ODE solver for the interior region. Figure 1 illustrates the resulting solution for generic values of the external parameters. In particular, we see that even a naive extrapolation of the exterior solution inside the horizon shows rather good agreement with the interior solution on a sizable neighborhood of the horizon.

4 Algebraic classification

In this section, we study algebraic types of curvature tensors and of the electromagnetic tensor in the charged Myers–Perry metric.

Recall that four-dimensional Kerr and Kerr–Newman black holes are of Petrov type D and both metrics admit a Kerr–Schild form (1) with geodesic \mathbf{k} . Furthermore, for the Kerr–Newman black hole, the null eigenvectors of the electromagnetic tensor \mathbf{F} coincide with the principal null directions of the Weyl tensor [27] (*i.e.*, Weyl tensor and \mathbf{F} are aligned).

Although the vacuum Myers–Perry black hole is of type D [12,32] and admits Kerr–Schild form with geodesic \mathbf{k} , it has been pointed out already in [23], that its charged generalization does not admit the Kerr–Schild form assuming that the vector potential \mathbf{A} coincides (up to normalization) with \mathbf{k} . As a consequence of the algebraically special form of the Ricci tensor in the charged Myers–Perry black hole, we will also arrive at the incompatibility of the charged Myers–Perry black hole metric with Kerr–Schild ansatz (1) with geodesic \mathbf{k} , but without employing the strong constraint $\mathbf{A} \propto \mathbf{k}$.

For determining the compatibility of the charged Myers–Perry metric with the Kerr–Schild form with geodesic \mathbf{k} , we will employ the following proposition from [28]:

Proposition 1 (Geodesicity of the Kerr–Schild vector \mathbf{k} [28]) *The null vector \mathbf{k} in the Kerr–Schild metric (1) is geodesic if, and only if, it is an aligned null direction of the Ricci tensor, i.e., if $R_{ab}k^a k^b = 0$.*

Note (see [28]) that if \mathbf{k} in the Kerr–Schild metric (1) is geodesic, then \mathbf{k} is also a multiple aligned Weyl direction

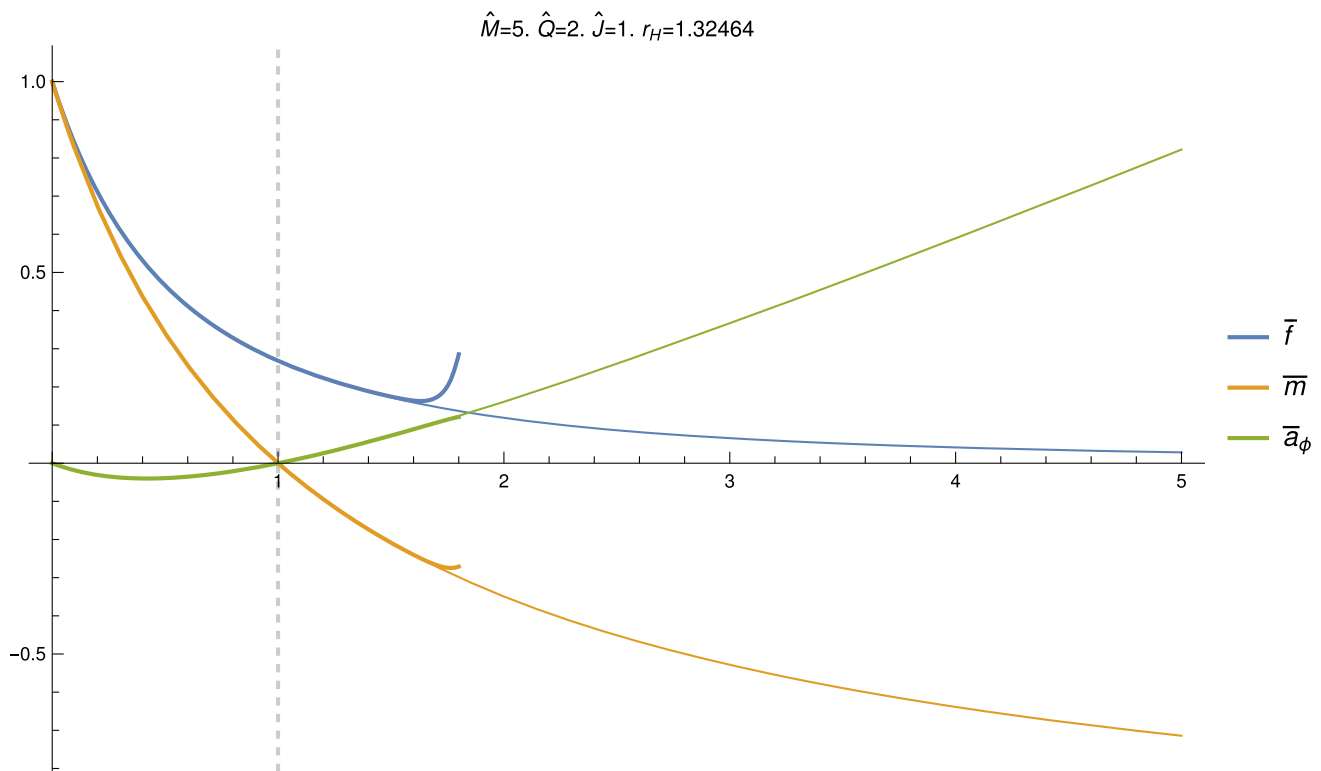


Fig. 1 Numerical solution for the black hole with external parameters $\hat{M} = 5$, $\hat{J} = 1$, $\hat{Q} = 2$. The horizon radius is dynamically determined to be $r_H \approx 1.32464$. The horizontal variable is $\bar{R} = r_H^2/R$, with $\bar{R} = 0$ corresponding to infinity and $\bar{R} = 1$ to the horizon, indicated

by the dashed vertical line. The barred functions have been rescaled as $\bar{f} = f/(1 - \bar{R})$, $\bar{m} = m\bar{R}/r_H^2$ and $\bar{a}_\phi = a(1 - \bar{R})$. The thick curves show the exterior solution and its extrapolation some distance inside the horizon. The thin curves show the interior solution

(WAND) of the spacetime, and thus the spacetime is algebraically special. Since we also show in this section that the charged Myers–Perry metric is not algebraically special, this could be used as another argument against compatibility with the Kerr–Schild form with geodesic k .

We also discuss the algebraic properties of the test electromagnetic field in the Myers–Perry background and compare this field with the weak field limit of the charged Myers–Perry metric.

4.1 Algebraic type of Maxwell and Ricci tensors

The Maxwell tensor obtained from the potential (20) reads

$$F_{\mu\nu} = \begin{pmatrix} 0 & -a'_0 & 0 & 0 & 0 \\ a'_0 & 0 & 0 & a'_\phi \sin^2 \theta & a'_\phi \cos^2 \theta \\ 0 & 0 & 0 & a_\phi \sin(2\theta) & -a_\phi \sin(2\theta) \\ 0 & -a'_\phi \sin^2 \theta & -a_\phi \sin(2\theta) & 0 & 0 \\ 0 & -a'_\phi \cos^2 \theta & a_\phi \sin(2\theta) & 0 & 0 \end{pmatrix}. \tag{48}$$

The eigenvalues and eigenvectors of $F^\mu{}_\nu$ are given by

1. Eigenvalue 0, eigenvector $k^\mu_{(0)} = \frac{2}{\alpha}(a'_\phi, 0, 0, -a'_0, -a'_0)$.

2. Eigenvalues $\pm\alpha$, where

$$\alpha = 2\sqrt{-fa'_\phi{}^2 + \frac{N}{m}a'^2}, \tag{49}$$

and $a' = a'_0 + \varpi a'_\phi$, eigenvectors

$$k^\mu_{(\pm)} = \sqrt{\frac{2}{f m N}} \alpha \begin{pmatrix} -Na' & \mp f N \alpha & 0 & f m a'_\phi \\ -N \omega a' & f m a'_\phi - N \omega a' \end{pmatrix}. \tag{50}$$

3. Eigenvalues $\pm i\beta$, where $\beta = \frac{2fa_\phi}{m}$ with eigenvectors

$$\ell^\mu_{(\pm)} = \sqrt{f/2m} (0, 0, \pm i, -\cot \theta, \tan \theta). \tag{51}$$

The above eigenvectors can be used to construct a null basis

$$\begin{aligned} \ell &= m_{(0)} = k^{(+)}, & n &= m_{(1)} = -k^{(-)}, & m_{(2)} &= k^{(0)}, \\ m_{(3)} &= \sqrt{2} \Re \ell^{(+)}, & m_{(4)} &= \sqrt{2} \Im \ell^{(+)}. \end{aligned} \tag{52}$$

The only non-vanishing scalar products are then $\ell \cdot n = 1$ and $m_{(i)} \cdot m_{(j)} = \delta_{ij}$ for $i, j = 2, 3, 4$ and the metric reads

$$g_{\mu\nu} = 2\ell_{(\mu}n_{\nu)} + \delta_{ij}m_{\mu}^{(i)}m_{\nu}^{(j)}. \tag{53}$$

In this basis, the Maxwell tensor reads

$$F_{\mu\nu} = 2\alpha\ell_{[\mu}n_{\nu]} - 2\beta m_{[\mu}^{(3)}m_{\nu]}^{(4)}, \tag{54}$$

and the stress-energy tensor has the form

$$\begin{aligned} T_{\mu\nu} &= F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \\ &= \frac{1}{2}(\alpha^2 - \beta^2)m_{\mu}^{(2)}m_{\nu}^{(2)} + (\alpha^2 + \beta^2)(m_{\mu}^{(3)}m_{\nu}^{(3)} \\ &\quad + m_{\mu}^{(4)}m_{\nu}^{(4)} - 2\ell_{(\mu}n_{\nu)}) \end{aligned} \tag{55}$$

Since both tensors F and T in (54) and (55) contain only boost weight zero terms, they are both algebraically special, more precisely of type D in the algebraic classification of tensors [22]. Note that this property holds off-shell (*i.e.*, without imposing the field equations).

The Einstein equations (with $4\pi G_5 = 1$)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2T_{\mu\nu}, \quad R_{\mu\nu} = 2T_{\mu\nu} - \frac{2}{3}g^{\rho\sigma}T_{\rho\sigma}g_{\mu\nu}, \tag{56}$$

take the form

$$\begin{aligned} R_{\mu\nu} &= -\frac{4}{3}(2\alpha^2 + \beta^2)\ell_{(\mu}n_{\nu)} + \frac{2}{3}(\alpha^2 - \beta^2)m_{\mu}^{(2)}m_{\nu}^{(2)} \\ &\quad + \frac{2}{3}(\alpha^2 + 2\beta^2)m_{\mu}^{(3)}m_{\nu}^{(3)} + \frac{2}{3}(\alpha^2 + 2\beta^2)m_{\mu}^{(4)}m_{\nu}^{(4)}. \end{aligned} \tag{57}$$

The Ricci tensor is thus of type D on-shell (*i.e.*, assuming that the Einstein equations hold).

Consider a general vector $v = v_0n + v_1\ell + v_2m_{(2)} + v_3m_{(3)} + v_4m_{(4)}$ in the above null basis. We want to identify all null vectors v obeying $R_{ab}v^av^b = 0$. Provided $\alpha \neq 0$, from the null condition and Equation (57) it follows that $v_2 = v_3 = v_4 = 0$ and $v_0v_1 = 0$. Up to a factor, ℓ and n are thus the only null vectors obeying $R_{ab}v^av^b = 0$. The exceptional case $\alpha = 0$, after using the conservation laws (23), corresponds to

$$-fa_{\phi}^{\prime 2} + \frac{\hat{Q}^2 f^2}{Nm} = 0. \tag{58}$$

This condition looks independent from the reduction of the equations of motion to the constrained BVP (24) and (25), so it is likely inconsistent with them. Formally, this can be checked by substituting the leading order asymptotic expansion (33). Setting to zero the resulting leading $\mathcal{O}(R^{-3})$ coefficient gives $\hat{Q} = 0$, meaning that only the uncharged case allows $\alpha = 0$.

The null congruence ℓ (and n) is geodesic if and only if

$$a'_{\phi}a''_0 - a'_0a''_{\phi} = 0. \tag{59}$$

This condition also looks independent from the reduction of the equations of motion to the constrained BVP (24) and (25), so it is likely inconsistent with them. Formally, this can be checked by continuing the expansion (33) by two more subleading orders (as we have done in (36)) and plugging it into the desired geodesic condition (59). Setting to zero any non-vanishing terms in the resulting asymptotic series then imposes non-trivial constraints on the parameters which determine the general solution (namely \hat{M} , $a_{\phi}^{(-1)}$ and r_{\pm} , \hat{J} , \hat{Q} , as discussed in Sect. 3.1). The first non-trivial term in the expansion of (59) appears at order $\mathcal{O}(R^{-6})$. Setting to zero the coefficients of R^{-6} and R^{-7} determines the \hat{M} and $a_{\phi}^{(-1)}$ parameters, while plugging these values into the coefficient of R^{-8} and setting it to zero results in the simple constraint

$$\hat{J}^3\hat{Q}^5 = 0. \tag{60}$$

In other words, the geodesic condition (59) on ℓ cannot hold whenever both $\hat{J} \neq 0$ and $\hat{Q} \neq 0$.

This result, together with Proposition 1, clearly explains why the metric for a charged rotating black hole in five dimensions cannot be written in the Kerr–Schild form (1) with a geodesic k , while the limiting cases with either $\hat{Q} = 0$ or $\hat{J} = 0$ are compatible with this form.

4.1.1 Electromagnetic test field

It is of also of interest to compare the results of the previous section with the cases of a test field and the weak-field limit. It is well known that in a vacuum spacetime, Killing vectors can be used to construct electromagnetic test fields obeying vacuum Maxwell equations in the background spacetime [39]. The contribution of the test field to the energy-momentum tensor is neglected and thus test fields have a physical meaning only in the weak-field limit. Electromagnetic test fields for the five-dimensional Myers–Perry black hole were analyzed in [3]. Additional properties of test fields in various five-dimensional black hole/ring spacetimes were studied in [27]. In particular, it has been shown in [27] that for the Myers–Perry black hole, null eigenvectors of the test field are aligned with the WANDs of the background vacuum Myers–Perry metric. The Weyl tensor and the Maxwell tensor are in this case both of type D and aligned.

The Maxwell tensor corresponding to the test field potential $A^{\mu} = \hat{\alpha}\delta_t^{\mu}$ reads

$${}^!F_{\mu\nu} = \frac{\hat{\alpha}\hat{M}}{(\mathbf{R} + a^2)^2} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & a \sin^2 \theta & a \cos^2 \theta & 0 \\ 0 & 0 & 0 & -a(\mathbf{R} + a^2) \sin(2\theta) & a(\mathbf{R} + a^2) \sin(2\theta) & 0 \\ 0 & -a \sin^2 \theta & a(\mathbf{R} + a^2) \sin(2\theta) & 0 & 0 & 0 \\ 0 & -a \cos^2 \theta & -a(\mathbf{R} + a^2) \sin(2\theta) & 0 & 0 & 0 \end{pmatrix}, \tag{61}$$

and the eigenvalues and corresponding null eigenvectors of $F^\mu{}_\nu$ are given by

$$\pm \alpha_T = \pm \frac{2\hat{\alpha}\hat{M}\sqrt{\mathbf{R}}}{(\mathbf{R} + a^2)^2}, \tag{62}$$

$$k_{T(\pm)}^\mu = \left(a + \frac{\mathbf{R}}{a}, \mp \frac{2\sqrt{\mathbf{R}}}{a(a^2 + \mathbf{R})} (a^4 + 2a^2\mathbf{R} + \mathbf{R}(\mathbf{R} - \hat{M})) , 0, 1, 1 \right), \tag{63}$$

where the subscript T stands for ‘‘test field’’.

4.1.2 Behaviour for small charge and comparison with the test field

Let us introduce the function b by

$$a_\phi = \hat{Q}b. \tag{64}$$

Then the field equations (24) and constraint (25) contain only even powers of \hat{Q} , and for small charge we can expand the metric and potential functions in terms of \hat{Q}^2 :

$$f = \sum_{k=0}^\infty f_{(k)} \frac{\hat{Q}^{2k}}{k!}, \quad m = \sum_{k=0}^\infty m_{(k)} \frac{\hat{Q}^{2k}}{k!}, \quad b = \sum_{k=0}^\infty b_{(k)} \frac{\hat{Q}^{2k}}{k!}. \tag{65}$$

By neglecting second and higher powers of \hat{Q} , the field equations (24) and constraint (25) for the leading terms $f_{(0)}$, $m_{(0)}$ and $b_{(0)}$ reduce to the vacuum field equations for the uncharged Myers–Perry black hole, plus one additional equation for $b_{(0)}$:

$$m_{(0)}(b'_{(0)}m_{(0)})' - b_{(0)}f_{(0)} = 2\hat{J} \frac{f_{(0)}m_{(0)}^2}{N^2}. \tag{66}$$

The Functions $f_{(0)}$ and $m_{(0)}$ thus correspond to the vacuum Myers–Perry metric (18), and Eq. (66) is solved by

$$b_{(0)} = \frac{\hat{J}}{\hat{M}(a^2 + R)}. \tag{67}$$

Setting $\hat{\alpha} = \hat{Q}/\hat{M}$, the eigenvalue α (49) of the Maxwell tensor reduces to the eigenvalue α_T (62) of the test field and the eigenvectors (50) reduce to the test field eigenvectors (63).

Therefore, in the weak field limit the Maxwell tensor (48) reduces to the test field tensor (61) and is aligned with the (vacuum) background Weyl tensor. As we will see in the next

section, this alignment of the Weyl and electromagnetic tensors does not survive beyond the weak field approximation.

4.2 Algebraic type of the Weyl tensor

4.2.1 Weyl type of the metric ansatz

It is well-known that the Kerr–Newman black hole in four dimensions is an algebraically special solution of Weyl type D. As we will point out below, for the five-dimensional charged rotating black hole this does not hold anymore. The algebraic classification of the Weyl tensor in higher dimensions is based on the alignment of a null vector, the so-called Weyl aligned null direction (WAND), where the definition of the Weyl types is linked with the multiplicity of such a WAND (see for example [29] for a review). The standard approach to determine the algebraic type of a given Weyl tensor is to find an aligned null frame in which the vanishing components of the Weyl tensor then indicate its type. Here, we employ the higher-dimensional extension of the Bel–Debever criteria [30], another simpler and frame-independent method to answer the question whether the five-dimensional charged rotating black hole solution is algebraically special. The Bel–Debever criterion for a specific algebraic type is a polynomial equation involving the Weyl tensor and an unknown null vector ℓ , which is the respective WAND of given multiplicity. For instance, the Weyl tensor is of algebraic type II if and only if there exists a null vector ℓ satisfying

$$C_{\mu\nu\rho\sigma}^{\text{II}} \equiv \ell_{[\sigma} C_{\mu]\lambda[\nu\rho} \ell_{\tau]} \ell^\lambda = 0. \tag{68}$$

Taking the regular metric ansatz (19) in terms of the radial coordinate \mathbf{R} into account, it is convenient to start with a general ℓ of the form

$$\ell^\mu = \left(-\frac{\hat{\ell}^t}{f}, 2\sqrt{\frac{\mathbf{N}}{\mathbf{m}}}\hat{\ell}^R, \frac{\hat{\ell}^\theta}{\sqrt{\mathbf{m}}}, \sqrt{\frac{\mathbf{m}}{f\mathbf{N}}}\hat{\ell}^\phi, -\frac{\hat{\ell}^t \varpi}{f}, \sqrt{\frac{\mathbf{m}}{f\mathbf{N}}}\hat{\ell}^\psi - \frac{\hat{\ell}^t \varpi}{f} \right), \tag{69}$$

which simplifies the condition for ℓ being null:

$$8 \left[(\hat{\ell}^t)^2 - (\hat{\ell}^R)^2 - (\hat{\ell}^\theta)^2 \right] - \left[3(\hat{\ell}^\phi)^2 + 2\hat{\ell}^\phi \hat{\ell}^\psi + 3(\hat{\ell}^\psi)^2 \right] + 4 \left[(\hat{\ell}^\phi)^2 - (\hat{\ell}^\psi)^2 \right] \cos(2\theta) - (\hat{\ell}^\phi - \hat{\ell}^\psi)^2 \times \left[\cos(4\theta) + \frac{2m^2}{fN} \sin^2(2\theta) \right] = 0. \tag{70}$$

The Bel–Debever criterion (68) for the Weyl tensor of our ansatz (19) is considerably complicated. It turns out that the simplest component of the rank-5 tensor $C_{Rt\phi\theta\psi}^{\text{II}}$ factorizes and yields the condition

$$\left[(\hat{\ell}^\theta)^2 + (\hat{\ell}^R)^2 \right] \left[\hat{\ell}^t \left(Nf' + fN' - 2\frac{fN}{m}m' \right) + \sqrt{\frac{fN^3}{m}} (\hat{\ell}^\phi \sin^2 \theta + \hat{\ell}^\psi \cos^2 \theta) \varpi' \right] = 0, \tag{71}$$

where a prime denotes a derivative with respect to R . We thus obtain two branches: either the first bracket vanishes, which entails $\ell^\theta = \ell^R = 0$, or the second bracket vanishes. Both conditions ensure that the component $C_{Rt\phi\theta\psi}^{\text{II}}$ vanishes, and for each branch we can simplify the criterion (68) by substituting back the corresponding condition. We then repeat this procedure: picking up the remaining simplest component which factorizes, and plugging the condition in each new subbranch into the criterion (68), until all the components of the criterion are satisfied. In this way, we obtain at the end 109 sets of conditions on the metric functions and the components of the WAND ℓ under which the Bel–Debever criterion for type II (68) is satisfied. However, all these sets are contained in the following 9 disjunct cases:

$$1 : (\hat{\ell}^t)^2 = (\hat{\ell}^R)^2 + (\hat{\ell}^\theta)^2 + (\hat{\ell}^\phi)^2 \sin^2 \theta + (\hat{\ell}^\psi)^2 \cos^2 \theta, \quad fN = m^2, \quad \varpi' = 0, \quad f'' = \frac{(f')^2}{4f} + f \frac{4N(N'' + 1) - 3(N')^2}{12N^2}, \tag{72a}$$

$$2 : \hat{\ell}^\theta = 0, \quad \hat{\ell}^\psi = \hat{\ell}^\phi, \quad (\hat{\ell}^t)^2 = (\hat{\ell}^R)^2 + (\hat{\ell}^\phi)^2, \quad \left(\frac{fN}{m^2} \right)' = \varpi' = 0, \quad f'' = \frac{f^2}{3m^2} + \frac{(f')^2}{4f} + f \frac{4NN'' - 3(N')^2}{12N^2}, \tag{72b}$$

$$3 : \hat{\ell}^\theta = 0, \quad \hat{\ell}^\phi = 0, \quad \hat{\ell}^\psi = 0, \quad (\hat{\ell}^t)^2 = (\hat{\ell}^R)^2, \quad \left(\frac{fN}{m^2} \right)' = \varpi' = 0, \quad f'' = \frac{5f^2}{3m^2} + \frac{(f')^2}{4f} + f \frac{4N(N'' - 4) - 3(N')^2}{12N^2}, \tag{72c}$$

$$4 : \hat{\ell}^\theta = 0, \quad \hat{\ell}^\phi = 0, \quad \hat{\ell}^\psi = 0, \quad (\hat{\ell}^t)^2 = (\hat{\ell}^R)^2,$$

$$\left(\frac{fN}{m^2} \right)' = \varpi' = 0, \quad f'' = -\frac{f^2}{3m^2} + \frac{(f')^2}{4f} + f \frac{4N(N'' + 2) - 3(N')^2}{12N^2}, \tag{72d}$$

$$5 : \hat{\ell}^\theta = 0, \quad \hat{\ell}^\phi = 0, \quad \hat{\ell}^\psi = 0, \quad (\hat{\ell}^t)^2 = (\hat{\ell}^R)^2, \quad \left(\frac{fN}{m^2} \right)' = 0, \quad \varpi'' = \frac{3(Nf' - fN')\varpi'}{4fN}, \tag{72e}$$

$$6 : \hat{\ell}^t = \hat{\ell}^\phi \sqrt{\frac{N}{fN}} \varpi, \quad \hat{\ell}^\theta = 0, \quad \hat{\ell}^\psi = \hat{\ell}^\phi, \quad (\hat{\ell}^R)^2 = (\hat{\ell}^\phi)^2 \left(\frac{N}{fN} \varpi^2 - 1 \right), \quad \left(\frac{fN\varpi}{m^2} \right)' = 0, \quad 4m^2 f'' = f^2 + \frac{mf'(2mf' - 3N\varpi\varpi')}{f} - f((m')^2 - 2mm'') + 3N\varpi m' \varpi' - m(f'm' - 2N\varpi\varpi''), \tag{72f}$$

$$7 : \hat{\ell}^t = \hat{\ell}^\phi \sqrt{\frac{N}{fN}} \varpi, \quad \hat{\ell}^\theta = 0, \quad \hat{\ell}^\psi = \hat{\ell}^\phi, \quad (\hat{\ell}^R)^2 = (\hat{\ell}^\phi)^2 \left(\frac{N}{fN} \varpi^2 - 1 \right), \quad \left(\frac{fN\varpi}{m^2} \right)' = 0, \quad (N\varpi^2 - fN)\varpi'' = \frac{8f^3\varpi + 3mN\varpi^2 f' \varpi'}{4fN} - fN \frac{2\varpi^2 - \varpi N' \varpi' + N(\varpi')^2}{N\varpi} - \varpi' \frac{6mf' + \varpi(3\varpi N' - 7N\varpi')}{4}, \quad 12f'' = \frac{20f^2}{m^2} + \frac{3(f')^2}{f} - 4\varpi' \left(\frac{3f'}{\varpi} - \frac{5N\varpi'}{m} \right) - \frac{3f(N')^2}{N^2} + 2f \frac{5N'\varpi' - 2\varpi(4 - N'')}{N\varpi} - f \frac{11(\varpi')^2 - 12\varpi\varpi''}{\varpi^2}, \tag{72g}$$

$$8 : \hat{\ell}^R = 0, \quad \hat{\ell}^\theta = 0, \quad \hat{\ell}^\psi = \hat{\ell}^\phi, \quad (\hat{\ell}^t)^2 = (\hat{\ell}^\phi)^2, \quad \varpi' = \frac{\hat{\ell}^\phi}{\hat{\ell}^t} \sqrt{\frac{f^3 N}{m^3}} \left(\frac{m^2}{fN} \right)', \quad 6f'' = \frac{f^2}{m^2} + 3\frac{(f')^2}{f} - \frac{f'N'}{N} + \frac{2f(N')^2}{N^2} - 3f \frac{(m')^2 - 2mm''}{m^2} - f \frac{m'N' + 2mN''}{mN}, \tag{72h}$$

$$9 : \hat{\ell}^t = -\hat{\ell}^\phi \sqrt{\frac{fN^3}{m}} \frac{\varpi'}{Nf' + fN' - 2\frac{fN}{m}m'}, \quad \hat{\ell}^\theta = 0, \quad \hat{\ell}^\psi = \hat{\ell}^\phi, \quad (\hat{\ell}^R)^2 + (\hat{\ell}^\phi)^2 = (\hat{\ell}^t)^2,$$

$$\begin{aligned}
 \frac{f''}{2f} &= \frac{(m')^2}{m^2} - \frac{2m'N'}{mN} + \frac{m''}{m} + \left(\frac{2f'}{f} + \frac{\varpi''}{\varpi'} \right) \\
 \left(\frac{f'}{2f} + \frac{N'}{2N} - \frac{m'}{m} \right) &+ \frac{(N')^2}{N^2} - \frac{N''}{2N}, \\
 2(2fNm' - m(Nf' + fN'))m'' &= \frac{2m^2N(f')^3}{f^2} \\
 + \frac{2fN(m')^3}{m} &+ f(Nf' - (m')^2N') \\
 - f^2 \frac{2Nm' - mN'}{m} & \\
 - m f' \frac{-2mf'N' - 3N^2(\varpi')^2 + N(5f'm' + 4mf'')}{f} & \\
 - mN'(f'm' + 4mf'') & \\
 + Nm'(f'm' + 8mf'') - N^2\varpi'(3m'\varpi' + 2m\varpi'') &. \tag{72i}
 \end{aligned}$$

In other words, the metric ansatz (19) is of Weyl type II if and only if the conditions of any of these 9 cases are met. Nevertheless, it is not clear so far whether these conditions could be satisfied on-shell. First, we recall that the uncharged and non-rotating limits of the charged rotating solution lead to known exact solutions, namely the 5-dimensional Myers–Perry black hole and the 5-dimensional charged Schwarzschild–Tangherlini black hole, respectively, which are given in Sect. 2.1. Although both special solutions are of Weyl type D, they satisfy the type II conditions from *distinct* cases which suggests that the charged rotating solution is of more general algebraic type. Specifically, the Myers–Perry black hole fulfills the conditions of case 9, whereas the charged Schwarzschild–Tangherlini black hole meets the conditions of case 5.

$${}^!g_{\mu\nu} = \begin{pmatrix} V^2NW^2 & F - UVNW^2 & 0 & VNW \sin^2\theta & VNW \cos^2\theta \\ F - UVNW^2 & U^2NW^2 & 0 & -UNW \sin^2\theta & -UNW \cos^2\theta \\ 0 & 0 & M & 0 & 0 \\ VNW \sin^2\theta & -UNW \sin^2\theta & 0 & (N \sin^2\theta + M \cos^2\theta) \sin^2\theta & (N - M) \sin^2\theta \cos^2\theta \\ VNW \cos^2\theta & -UNW \cos^2\theta & 0 & (N - M) \sin^2\theta \cos^2\theta & (N \cos^2\theta + M \sin^2\theta) \cos^2\theta \end{pmatrix}. \tag{74}$$

Now, let us discuss the compatibility of the conditions (72) with a general charged rotating solution of the Einstein–Maxwell equations. The conditions of cases 1, 2, 3 and 4 are satisfied only in the non-rotating limit, since for $\varpi' = 0$ we obtain $\hat{J} = 0$ from the conservation law (23c) together with the BVP (24c). To analyze the remaining cases, we make use of the on-shell asymptotic expansions (36) of the functions f , m and a_ϕ , where without loss of generality we set $r_-^2 = 0$. In the case 5, plugging the asymptotic expansions into $(Nf/m^2)' = 0$, the numerator given by $Nf = R^2 - (\hat{M} + r_H^2)R + \frac{1}{2}\hat{M}(\hat{M} + r_H^2) + \frac{4}{3}\hat{Q}^2 + \mathcal{O}(R^{-1})$ and the denominator given by $m^2 = R^2 - (\hat{M} + r_H^2)R + \frac{1}{4}(\hat{M} + r_H^2)^2 + \frac{8}{3}\hat{Q}^2 +$

$\mathcal{O}(R^{-1})$ differ at order R^0 . Therefore, Nf/m^2 in general depends on R and the conditions of case 5 are not fulfilled. Analogously, in cases 6 and 7, from the first terms of the asymptotic expansion of $(Nf\varpi/m^2)' = 0$ it is obvious that the fraction of $Nf\varpi = \hat{J} + \mathcal{O}(R^{-1})$ and $m^2 = R^2 + \mathcal{O}(R)$ is not independent of R , such that also this condition is not met for a general solution. Substituting the asymptotic expansion (36) in the condition $\varpi' = \pm\sqrt{Nf^3/m^3}[m^2/(Nf)]'$ of case 8 we obtain $2\hat{J}/R^3 = \pm(16\hat{Q}^2 - 3\hat{M}^2 + 3r_H^4)/(6R^{7/2}) + \mathcal{O}(R^{-4})$, requiring $\hat{J} = 0$. Lastly, in case 9 the leading order term in its condition reads $-4\hat{J}R + \mathcal{O}(R^0) = 0$, and again requires that $\hat{J} = 0$, such that none of the conditions can be fulfilled for a general charged rotating solution.

4.2.2 Weyl type on the horizon

We have shown above that in the bulk the charged Myers–Perry metric is algebraically general. In this section, we determine the algebraic type of the metric on the horizon. The form (41) is manifestly regular at the horizons and the bifurcation sphere, but it is more cumbersome than necessary. In Kruskal-like coordinates, an a priori regular ansatz may be parameterized as follows, where any function with implicit coordinate dependence depends only on the product UV and have non-vanishing leading terms of order zero, e.g., $F = F(UV) = F(0) + \mathcal{O}(UV)$:

$$\begin{aligned}
 ds^2 &= 2F dU dV + N[W(U dV - V dU) \\
 &\quad - \sin^2\theta d\phi - \cos^2\theta d\psi]^2 \\
 &\quad + M d\theta^2 + M \sin^2\theta \cos^2\theta (d\phi - d\psi)^2, \tag{73}
 \end{aligned}$$

To determine the type of the Weyl tensor on the horizon, we employ the null coframe $\ell = dU$, $n = dV$, $m^{(2)} = d\theta$, $m^{(3)} = d\phi - d\psi$, $m^{(4)} = W(V dU - U dV) + d\phi$ (up to normalization).

Calculating all boost-weight +2 components of the Weyl tensor $C^{\alpha\beta\gamma\delta}\ell_\alpha m^{(i)}_\beta \ell_\gamma m^{(j)}_\delta$, we find that they all vanish for $U = 0$. Furthermore, all boost-weight +1 components vanish there as well. It follows that ℓ is a multiple WAND at $U = 0$. Similarly, n is a multiple WAND at $V = 0$. Therefore, the spacetime is at least of Weyl type II at either of the horizons ($V = 0$ or $U = 0$) and type D at the bifurcation sphere ($U = V = 0$). The same results also hold for the

Riemann tensor. Note that these observations hold even off-shell, without imposing the Einstein–Maxwell equations.

5 Discussion

We have studied various mathematical aspects of a charged rotating black hole with two equal-magnitude angular momenta in five dimensions.

In Sect. 3, we proposed a metric ansatz (19) which is regular on the horizon. Furthermore, for this metric ansatz the Einstein–Maxwell system (24) and (25) is autonomous, which is technically convenient. We studied geometric regularity conditions for the Einstein–Maxwell system (24) following from asymptotic flatness with the absence of incoming and outgoing radiation, as well as regularity at the horizon. By comparing the solution at the singular end-points with these conditions, we showed that the boundary value problem is well-posed.

In Sect. 4, we performed the algebraic classification of the curvature tensors and the electromagnetic tensor F . We found that the Weyl tensor is algebraically general in the bulk, type II on the horizon, and type D on the bifurcation sphere. The electromagnetic field strength tensor F and the stress-energy tensor T are both of type D off-shell, while the Ricci tensor is type D on-shell. These algebraic properties are inconsistent with the Kerr–Schild form (1) of the metric with geodesic k , such that the charged rotating black hole with two equal-magnitude angular momenta in five dimensions cannot be described by such a Kerr–Schild metric. On the other hand, changing to coordinates appropriate for studying the on-horizon properties of the spacetime, the metric (73) takes a form somewhat reminiscent of the Kerr–Schild form. In this case, however, the background metric is a four-dimensional type D warped product metric, and the “Kerr–Schild” vector is not null.

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Appendix A: Comparison with the work of Fan, Liang, Mei

In [9], Fan et al. looked at the same metric ansatz, introduced by Kunz et al. [19, 24], for a 5-dimensional charged rotating black hole that we are considering in this work. They claim to have reduced the corresponding Einstein equations to only two scalar variables (their Z and A_ϕ). In contrast, in Sect. 3, the most we could reduce our system to is the three scalar variables f , m , a_ϕ , in Eqs. (24) and (25). Unfortunately, a closer comparison reveals some inconsistencies in their formulas.

In more detail, the parametrization of the ansatz used in [9] is

$$\begin{aligned} ds^2 = & -F dt^2 \\ & + \frac{dR^2}{W} + R^2 \left(\frac{dx^2}{1-x^2} + (1-x^2) d\phi^2 + x^2 d\psi^2 \right) \\ & + N \left[(1-x^2) d\phi + x^2 d\psi \right]^2 \\ & - 2B \left[(1-x^2) d\phi + x^2 d\psi \right] dt, \end{aligned} \quad (75a)$$

$$A = A_t dt + A_\phi \left[(1-x^2) d\phi + x^2 d\psi \right]. \quad (75b)$$

where a comparison with our parametrization (19) reveals the dictionary

$$x = \cos \theta, \quad (76a)$$

$$r = \sqrt{m/f}, \quad (76b)$$

$$F = f - \frac{N}{m} \varpi^2, \quad (76c)$$

$$W = N \left(\frac{m'}{m} - \frac{f'}{f} \right)^2, \quad (76d)$$

$$N = \frac{N}{m} - \frac{m}{f}, \quad (76e)$$

$$B = \frac{N}{m} \varpi, \quad (76f)$$

$$A_t = a_0, \quad (76g)$$

$$A_\phi = a_\phi, \quad (76h)$$

where all derivatives on the right-hand side are with respect to our coordinate $R = r^2$. In the two equations,

$$W = \frac{16Z(c_0+Z)}{r^2(\partial_r Z)^2}, \quad \text{Eq. (3) of [9]}$$

$$N = \frac{Z}{r^2 F} - \frac{B^2}{F} - r^2, \quad \text{Eq. (4) of [9]}$$

the first one introduces a new variable $Z = Z(r)$ and a constant c_0 , while the second one follows from this definition in conjunction with the Einstein equations. Using our dictionary and simple algebraic manipulation, we find that Z and c_0 are uniquely determined by the identities

$$\begin{aligned} Z = N &= (R - r_+^2)(R - r_-^2), \quad c_0 + Z = (N'/2)^2 \\ &= \left(R - \frac{r_+^2 + r_-^2}{2} \right)^2, \end{aligned} \quad (77)$$

where we have used the form of N (23) imposed by the Einstein equations. Next, after introducing the variable \tilde{A}_ϕ in their Equation (7), their formula (6) for F is equivalent to a combination of our (23c) and (24c). However, their final complicated formula (8) for B turns out to be inconsistent. We have verified the inconsistency by plugging in our general asymptotic solution (36), for any non-trivial values of the free parameters \hat{M} and $a_\phi^{(-1)}$. Unfortunately, the claimed reduction by [9] of the Einstein equations to just the two variables Z and A_ϕ cannot be correct without the validity of their Equation (8), which we have also verified cannot be saved by undoing possible sign errors or other minor typos. Finally, the asymptotic solution at infinity (9) for Z and A_ϕ in [9] cannot be matched up to our asymptotic solution (36) by any correspondence between their free parameters (m, a, s) and ours ($\hat{M}, a_\phi^{(-1)}$), with an inconsistency appearing already at the next subleading order.

References

1. A.N. Aliev, Rotating black holes in higher dimensional Einstein–Maxwell gravity. *Phys. Rev. D* **74**, 024011 (2006). <https://doi.org/10.1103/PhysRevD.74.024011> arXiv:hep-th/0604207
2. A.N. Aliev, A slowly rotating charged black hole in five dimensions. *Mod. Phys. Lett. A* **21**, 751–757 (2006). <https://doi.org/10.1142/S0217732306019281> arXiv:gr-qc/0505003
3. A.N. Aliev, V.P. Frolov, Five-dimensional rotating black hole in a uniform magnetic field: the gyromagnetic ratio. *Phys. Rev. D* **69**, 084022 (2004). <https://doi.org/10.1103/PhysRevD.69.084022> arXiv:hep-th/0401095
4. T. Andrade, R. Emparan, D. Licht, Charged rotating black holes in higher dimensions. *JHEP* **2**, 076 (2019). [https://doi.org/10.1007/JHEP02\(2019\)076](https://doi.org/10.1007/JHEP02(2019)076) arXiv:1810.06993
5. M.A. Argañaraz, O.M. Moreschi, Double null coordinates for Kerr spacetime. *Phys. Rev. D* **104**, 024049 (2021). <https://doi.org/10.1103/PhysRevD.104.024049>
6. B. Carter, Global structure of the Kerr family of gravitational fields. *Phys. Rev.* **174**, 1559–1571 (1968). <https://doi.org/10.1103/physrev.174.1559>
7. A. Coley, D. McNutt, A. Shoom, Geometric horizons. *Phys. Lett. B* **771**, 131–135 (2017). <https://doi.org/10.1016/j.physletb.2017.05.004> arXiv:1710.08457
8. A. Coley, R. Milson, V. Pravda, A. Pravdová, Classification of the Weyl tensor in higher dimensions. *Class. Quantum Gravity* **21**, L35–L41 (2004). <https://doi.org/10.1088/0264-9381/21/7/L01> arXiv:gr-qc/0401008
9. H.-M. Fan, Z.-C. Liang, J. Mei, Note on the Kerr–Newman metric in five dimensions. *Phys. Rev. D* **93**, 064045 (2016). <https://doi.org/10.1103/PhysRevD.93.064045> arXiv:1512.07066
10. A.T. Franzen, P.M. Girão, Double null coordinates for Kerr–Newman–de Sitter spacetimes (2020). arXiv:2008.13513
11. R. Gregory, Braneworld black holes, in *Physics of black holes: a guided tour*, ed. by E. Papantonopoulos, *Lecture Notes in Physics*, vol. 769, pp. 259–298. Springer, Berlin (2009). https://doi.org/10.1007/978-3-540-88460-6_7. arXiv:0804.2595
12. N. Hamamoto, T. Houri, T. Oota, Y. Yasui, Kerr–NUT–de Sitter curvature in all dimensions. *J. Phys. A* **40**, F177–F184 (2007). <https://doi.org/10.1088/1751-8113/40/7/F01> arXiv:hep-th/0611285
13. S.A. Hayward, Kerr black holes in horizon-generating form. *Phys. Rev. Lett.* **92**, 191101 (2004). <https://doi.org/10.1103/PhysRevLett.92.191101> arXiv:gr-qc/0401111
14. M. Henneaux, C. Troessaert, Asymptotic structure of electromagnetism in higher spacetime dimensions. *Phys. Rev. D* **99**, 125006 (2019). <https://doi.org/10.1103/PhysRevD.99.125006> arXiv:1903.04437
15. S. Hollands, A. Ishibashi, Asymptotic flatness and Bondi energy in higher dimensional gravity. *J. Math. Phys.* **46**, 022503 (2005). <https://doi.org/10.1063/1.1829152> arXiv:gr-qc/0304054
16. R.P. Kerr, Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.* **11**, 237–238 (1963). <https://doi.org/10.1103/PhysRevLett.11.237>
17. I. Khavkine, Reducing triangular systems of ODEs with rational coefficients, with applications to coupled Regge–Wheeler equations. *Differ. Geom. Appl.* **70**, 101632 (2020). <https://doi.org/10.1016/j.difgeo.2020.101632> arXiv:1801.09800
18. J. Kunz, F. Navarro-Lérida, A.K. Petersen, Five-dimensional charged rotating black holes. *PLB* **614**, 104–112 (2005). <https://doi.org/10.1016/j.physletb.2005.03.056> arXiv:gr-qc/0503010
19. J. Kunz, F. Navarro-Lérida, J. Viebahn, Charged rotating black holes in odd dimensions. *Phys. Lett. B* **639**, 362–367 (2006). <https://doi.org/10.1016/j.physletb.2006.06.066> arXiv:hep-th/0605075
20. J. Lewandowski, T. Pawłowski, Quasi-local rotating black holes in higher dimension: geometry. *Class. Quantum Gravity* **22**, 1573–1598 (2005). <https://doi.org/10.1088/0264-9381/22/9/007> arXiv:gr-qc/0410146
21. J.M. Martín-García, A. García-Parrado, A. Stecchina, B. Wardell, C. Pitrou, D. Brizuela, D. Yllanes, G. Faye, L. Stein, R. Portugal, T. Nutma, T. Bäckdahl, xAct: efficient tensor computer algebra for the Wolfram Language (2021). <http://www.xact.es>
22. R. Milson, A. Coley, V. Pravda, A. Pravdová, Alignment and algebraically special tensors in Lorentzian geometry. *Int. J. Geom. Methods Mod. Phys.* **2**, 41–61 (2005). <https://doi.org/10.1142/S0219887805000491> arXiv:gr-qc/0401010
23. R.C. Myers, M.J. Perry, Black holes in higher dimensional spacetimes. *Ann. Phys. (N.Y.)* **172**, 304–347 (1986). [https://doi.org/10.1016/0003-4916\(86\)90186-7](https://doi.org/10.1016/0003-4916(86)90186-7)
24. F. Navarro-Lérida, Perturbative charged rotating 5D Einstein–Maxwell black holes. *Gen. Relativ. Gravit.* **42**, 2891–2900 (2010). <https://doi.org/10.1007/s10714-010-1033-1> arXiv:0706.0591
25. E. Newman et al., Metric of a rotating, charged mass. *J. Math. Phys.* **6**, 918–919 (1965). <https://doi.org/10.1063/1.1704351>
26. E. Newman, A. Janis, Note on the Kerr spinning-particle metric. *J. Math. Phys.* **6**, 915–917 (1965). <https://doi.org/10.1063/1.1704350>

27. M. Ortaggio, V. Pravda, Black rings with a small electric charge: gyromagnetic ratios and algebraic alignment. *JHEP* **12**, 054 (2006). <https://doi.org/10.1088/1126-6708/2006/12/054> [arXiv:gr-qc/0609049](https://arxiv.org/abs/gr-qc/0609049)
28. M. Ortaggio, V. Pravda, A. Pravdová, Higher dimensional Kerr–Schild spacetimes. *Class. Quantum Gravity* **26**, 025008 (2009). <https://doi.org/10.1088/0264-9381/26/2/025008> [arXiv:0808.2165](https://arxiv.org/abs/0808.2165)
29. M. Ortaggio, V. Pravda, A. Pravdová, Algebraic classification of higher dimensional spacetimes based on null alignment. *Class. Quantum Gravity* **30**, 013001 (2013). <https://doi.org/10.1088/0264-9381/30/1/013001> [arXiv:1211.7289](https://arxiv.org/abs/1211.7289)
30. M. Ortaggio, Bel–Debever criteria for the classification of the Weyl tensors in higher dimensions. *Class. Quantum Gravity* **26**, 195015 (2009). <https://doi.org/10.1088/0264-9381/26/19/195015> [arXiv:0906.3818](https://arxiv.org/abs/0906.3818)
31. M. Ortaggio, Asymptotic behavior of Maxwell fields in higher dimensions. *Phys. Rev. D* **90**, 124020 (2014). <https://doi.org/10.1103/PhysRevD.90.124020> [arXiv:1406.3186](https://arxiv.org/abs/1406.3186)
32. V. Pravda, A. Pravdová, A. Coley, R. Milson, Bianchi identities in higher dimensions. *Class. Quantum Gravity* **21**, 2873–2897 (2004). <https://doi.org/10.1088/0264-9381/21/12/007> [arXiv:gr-qc/0401013](https://arxiv.org/abs/gr-qc/0401013)
33. F. Pretorius, W. Israel, Quasi-spherical light cones of the Kerr geometry. *Class. Quantum Gravity* **15**, 2289–2301 (1998). <https://doi.org/10.1088/0264-9381/15/8/012> [arXiv:gr-qc/9803080](https://arxiv.org/abs/gr-qc/9803080)
34. G. Satishchandran, R.M. Wald, Asymptotic behavior of massless fields and the memory effect. *Phys. Rev. D* **99**, 084007 (2019). <https://doi.org/10.1103/PhysRevD.99.084007> [arXiv:1901.05942](https://arxiv.org/abs/1901.05942)
35. K. Tanabe, S. Kinoshita, T. Shiromizu, Asymptotic flatness at null infinity in arbitrary dimensions. *Phys. Rev. D* **84**, 044055 (2011). <https://doi.org/10.1103/PhysRevD.84.044055> [arXiv:1104.0303](https://arxiv.org/abs/1104.0303)
36. K. Tanabe, N. Tanahashi, T. Shiromizu, Asymptotic flatness at spatial infinity in higher dimensions. *J. Math. Phys.* **50**, 072502 (2009). <https://doi.org/10.1063/1.3166141> [arXiv:0902.1583](https://arxiv.org/abs/0902.1583)
37. K. Tanabe, N. Tanahashi, T. Shiromizu, On asymptotic structure at null infinity in five dimensions. *J. Math. Phys.* **51**, 062502 (2010). <https://doi.org/10.1063/1.3429580> [arXiv:0909.0426](https://arxiv.org/abs/0909.0426)
38. F.R. Tangherlini, Schwarzschild field in n dimensions and the dimensionality of space problem. *Il Nuovo Cimento* **27**, 636–651 (1963). <https://doi.org/10.1007/BF02784569>
39. R.M. Wald, Black hole in a uniform magnetic field. *Phys. Rev. D* **10**, 1680–1685 (1974). <https://doi.org/10.1103/PhysRevD.10.1680>
40. W. Wasow, *Asymptotic expansions for ordinary differential equations*, vol. XIV, Pure and Applied Mathematics. Interscience, New York (1965)