Thermodynamics of Kerr-AdS black holes in the restricted phase space

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Abstract The thermodynamics for Kerr-AdS black hole in four dimensions is revisited using the recently proposed restricted phase space formalism, which includes the central charge C of the dual CFT and the chemical potential μ , but excludes the pressure and the conjugate volume, as thermodynamic variables. The Euler relation holds automatically, and the first order homogeneity of the mass and the zeroth order homogeneity of the intensive variables are made explicit. Thermodynamic processes involving each pair of conjugate variables are studied in some detail, with emphasis on the scaling properties of the equations of states. It turns out that the thermodynamic behavior of the Kerr-AdS black hole is very similar to that of the RN-AdS black hole studied earlier. In particular, it is found that, there is a first order supercritical phase equilibrium in the $T - S$ processes at fixed nonvanishing angular momentum, while at vanishing angular momentum or at fixed angular velocities, there is always a non-equilibrium transition from a small unstable black hole state to a large stable black hole state. Moreover, there is a Hawking–Page phase transition in the $\mu - C$ processes. Due to the complicatedness of the Kerr metric, the exact critical point and the Hawking–Page temperature are worked out explicitly only in the slow rotating limit, however the characteristic thermodynamic properties do not rely on the slow rotating approximation.

1 Introduction

Black hole thermodynamics has been an important and active subject of study ever since the pioneering works of Bekenstein [1,2] and Bardeen, Carter, Hawking [3,4]. For both historical and technical reasons, the study of black hole thermodynamics can be subdivided into two major stages or for-

malisms, i.e. the *traditional* and the *extended phase space* (EPS) stages/formalisms. The traditional formalism concerns mainly with the establishment of thermodynamic relations and the evaluation of thermodynamic quantities in various black hole solutions. Among other things, the Smarr relation [5] and the Wald method [6] played some important roles. The Hawking–Page transition [7] has also been found in the development of the traditional formalism. The extended phase space formalism initiated in [8] opened a new era for the thermodynamics of black hole in AdS spacetime by introducing and extra (P, V) pair of state variables, where P is related to the cosmological constant via $P = -\Lambda/8\pi G$. This triggered a great number of subsequent works, see [9–17] for examples. These developments are mainly concentrated in the thermodynamic behaviors, especially the $P - v$ criticalities. A most recent variant of the extended phase space formalism is developed by Visser [18], with the aid of certain considerations about the AdS/CFT correspondence [19]. The major innovation in Visser’s formalism is the inclusion of the CFT central charge C and the conjugate chemical potential μ as novel thermodynamic parameters. Meanwhile, the volume and the pressure are changed into that of the CFT, i.e. $\mathcal{V} \sim L^{d-2}$ with L denoting the AdS radius, and $\mathcal{P}$ determined via the CFT equation of states (EOS) $E = (d-2)\mathcal{P}\mathcal{V}$ where d is the dimension of the bulk spacetime. Please note that the idea of introducing the chemical potential and central charge (or the square of the number of colors in the CFT) as new thermodynamic variables has been explored earlier in [20–24], however, Visser’s work surpasses the previous works in that the (P, V) variables are changed into $(\mathcal{P}, \mathcal{V})$, so that, for charged rotating AdS black holes, the first law takes the form

$$dE = TdS - \mathcal{P}d\mathcal{V} + \tilde{\Phi}d\tilde{Q} + \Omega dJ + \mu dC,$$

which is accompanied with an Euler-like relation

$$E = TS + \tilde{\Phi}\tilde{Q} + \Omega J + \mu C, \quad (1)$$

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where $\tilde{\Phi}$ and $\tilde{Q}$ are the properly rescaled electric potential and electric charge.

Visser's formalism is a grand framework in the sense that it captures thermodynamic properties of both the black hole in the bulk and of the CFT on the boundary. By replacing (P, V) with $(\mathcal{P}, \mathcal{V})$, the total mass of the spacetime restores its original interpretation as internal energy, which conforms to the traditional formalism of black hole thermodynamics. Meanwhile there is no need to introduce the thermodynamic volume V .¹ By fixing the central charge C , Visser's formalism leads to a thermodynamic description for the CFT which is holographically dual to the AdS black hole in the bulk [27, 28].

Equation (1) implies that E is a first order homogeneous function in $(S, \tilde{Q}, J, C)$ but not in $\mathcal{V}$. However, if black hole thermodynamics is indeed thermodynamics in the standard sense as described in any textbooks, e.g. [29], the internal energy needs to be a first order homogeneous function in *all* extensive variables. Moreover, both the original EPS formalism and Visser's variant suffer from the "ensemble of theories" issue in the sense that varying cosmological constant (or AdS radius) implies changing the underlying gravity theory.

To avoid the above mentioned issues, an alternative restricted version of Visser's formalism called the *restricted phase space* (RPS) formalism for AdS black holes is proposed by us in [30], which is studied in detail in the example case of four dimensional RN-AdS black hole in Einstein-Maxwell theory, and some interesting thermodynamic behaviors such as the first order phase equilibrium at supercritical temperatures (i.e. *supercritical phase equilibrium*) and the Hawking-Page transitions are revealed. The present work is a continuation of [30] to the case of Kerr-AdS black hole in Einstein gravity. It will be shown that, in spite of the different geometries, the thermodynamic behaviors for the Kerr-AdS black hole is very similar to that of the RN-AdS in the RPS formalism. This leads us to guess that there might be some universality lying behind the RPS formalism. Moreover, the study of black hole thermodynamics in the RPS formalism may also be helpful in further understanding the AdS/CFT correspondence, as will be briefly discussed at the end of this paper. Please note that the thermodynamics and phase structures for Kerr-AdS black hole in the extended phase space formalism is studied in [31]. A comparison between the results of [31] and ours may help to understand the differences between the extended and the restricted phase space formalisms.

¹ In contrast, V is required in the EPS formalism in order that the first law holds. Moreover, V is also connected with the Killing potential [8] and is known to obey the so-called reverse isoperimetric inequality [25, 26]. Even though, since V is not connected with the geometric volume, it is still puzzling why V plays some role in black hole thermodynamics.

Since the RPS formalism allows for a variable Newton constant, it seems necessary to clarify the differences between the variable cosmological constant and Newton constant. In essence, the cosmological constant is a part of the Lagrangian density in the Einstein-Hilbert action, while the Newton constant is simply an overall factor in the total action. This is true not only for pure Einstein gravity, but also for Einstein-Maxwell theory, provided the electric charge Q is appropriately rescaled (as did in [18]), which is allowed because Q is an integration constant. Therefore, varying cosmological constant implies changing the corresponding field equation, while varying Newton constant does not have this effect. One can think of a variable cosmological constant as being corresponding to different gravity theories, while a variable Newton constant to the same underlying theory.

This paper is organized as follows. In Sect. 2 we briefly introduce the Kerr-AdS spacetime with emphasis on the values of thermodynamic quantities and the proof of validity for the first law and Euler relation. In Sect. 3, we rewrite the black hole mass as a macro state function in extensive variables and represent the intensive variables in terms of the equations of states. In this process, the correct homogeneity behaviors become evident. Section 4 is devoted to the study of three typical types of thermodynamic processes, which constitute the major context of this work. In this section, the thermodynamic behaviors very similar to the RN-AdS case are rediscovered. Finally, in Sect. 5, we present a brief summary of the results and make some further discussions.

2 Kerr-AdS black hole in 4-dimensions and the RPS formalism

The metric for Kerr-AdS black hole in 4-dimensions is given as follows,

$$ds^2 = -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi} d\phi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\sin^2 \theta \Delta_\theta}{\rho^2} \left(a dt - \frac{r^2 + a^2}{\Xi} d\phi \right)^2,$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta_r = (r^2 + a^2) \left(1 + \frac{r^2}{\ell^2} \right) - 2Gmr, \\ \Xi = 1 - \frac{a^2}{\ell^2}, \quad \Delta_\theta = 1 - \frac{a^2}{\ell^2} \cos^2 \theta.$$

The mass M and angular momentum J are related to the parameters m, a via

$$M = \frac{m}{\Xi^2}, \quad J = \frac{am}{\Xi^2},$$

where m can be solved from the equation $\Delta_r(r_+) = 0$, with r_+ representing the radius of the event horizon. This enables us to write M and J as a function of the parameters r_+ , a , G ,

$$\begin{aligned} M &= \frac{1}{2\Xi^2 r_+ G} \left(r_+^2 + a^2 + \frac{r_+^4}{\ell^2} + \frac{a^2 r_+^2}{\ell^2} \right), \\ J &= \frac{a}{2\Xi^2 r_+ G} \left(r_+^2 + a^2 + \frac{r_+^4}{\ell^2} + \frac{a^2 r_+^2}{\ell^2} \right). \end{aligned} \quad (2)$$

The other thermodynamic parameters are given as

$$\begin{aligned} T &= \frac{r_+}{4\pi(r_+^2 + a^2)} \left(1 + \frac{a^2}{\ell^2} + \frac{3r_+^2}{\ell^2} - \frac{a^2}{r_+^2} \right), \\ S &= \frac{\pi(r_+^2 + a^2)}{G\Xi}, \quad \Omega = \frac{a\Xi}{r_+^2 + a^2} + \frac{a}{\ell^2}. \end{aligned}$$

Moreover, the RPS formalism introduces an extra pair of thermodynamic variables, i.e.

$$C = \frac{\ell^2}{G}, \quad \mu = \frac{M - TS - \Omega J}{C}. \quad (3)$$

It can be seen that, except for μ , all other thermodynamic quantities must be non-negative. Let us mention that the chemical potential can also be defined independent of the other thermodynamic quantities using the AdS/CFT dictionary $Z_{\text{CFT}} = Z_{\text{Gravity}}$, because the Gibbs free energy $W = \mu C = -T \log Z_{\text{CFT}} = -T \log Z_{\text{Gravity}}$, wherein $Z_{\text{Gravity}} = \exp(-\mathcal{A}_E)$, $\mathcal{A}_E$ represents the Euclidean action evaluated at the black hole configuration. For details, see [32–34].

The RPS formalism differs from Visser's formalism in that the AdS radius, ℓ , is fixed as a constant. Using the above quantities (which are all viewed as functions of r_+ , a and G), it can be checked straightforwardly that the first law of thermodynamics holds,

$$dM = TdS + \Omega dJ + \mu dC, \quad (4)$$

and, from Eq. (3), the Euler relation is also satisfied explicitly,

$$M = TS + \Omega J + \mu C. \quad (5)$$

These two equations are fundamental to the RPS formalism.

Please be reminded that, although the variables μ , C are borrowed from the dual CFT, they can be actually understood as the chemical potential and the effective number N_{bulk} of microscopic degrees of freedom of the black hole in the bulk. In other words, one may introduce the new rules $\mu_{\text{CFT}} = \mu_{\text{bulk}}$, $C = N_{\text{bulk}}$ into the holographic dictionary. For simplicity of symbols we shall keep the original (μ, C)

notations rather than replacing them with $(\mu_{\text{bulk}}, N_{\text{bulk}})$, but please always bear in mind that what we are studying is the thermodynamics of the black holes in the bulk, rather than that of the dual CFT. Let us also stress that, the bulk variables μ , C are only determined up to an arbitrary but opposite constant rescaling: the changes $\mu \rightarrow \lambda^{-1}\mu$, $C \rightarrow \lambda C$ with constant λ will not bring any harm to Eqs. (4) and (5).²

3 Equations of states and homogeneity

In order to analyze the thermodynamic properties, we need to rewrite the mass as well as the variables T , Ω , μ as functions of the extensive variables S , J , C . This is accomplished in the following steps. First, let us rewrite a , G in terms of (J, M, C) ,

$$a = \frac{J}{M}, \quad G = \frac{\ell^2}{C}. \quad (6)$$

Next, we can solve r_+ from the expression for S , yielding

$$r_+^2 = \frac{S(\ell^2 M^2 - J^2)}{\pi C M^2} - \frac{J^2}{M^2}. \quad (7)$$

Finally, inserting Eqs. (6) and (7) into Eq. (2), we get an algebraic equation for M , whose solution reads

$$M = \frac{\sqrt{\pi C + S} \sqrt{S^3 + \pi C (4\pi^2 J^2 + S^2)}}{2\pi^{3/2} \ell \sqrt{C} \sqrt{S}}. \quad (8)$$

Using Eq. (8) and the first law, we can easily obtain the equations of states (EOS)

$$\begin{aligned} T &= \left(\frac{\partial M}{\partial S} \right)_{J,C} \\ &= \frac{4\pi C S^3 + 3S^4 + \pi^2 C^2 (S^2 - 4\pi^2 J^2)}{4\pi^2 \ell S^{3/2} \sqrt{C} (\pi C + S) \cdot \sqrt{S^3 + \pi C (4\pi^2 J^2 + S^2)}}, \end{aligned} \quad (9)$$

$$\begin{aligned} \Omega &= \left(\frac{\partial M}{\partial J} \right)_{S,C} \\ &= \frac{2\pi^{3/2} J \sqrt{C} (\pi C + S)}{\ell \sqrt{S} \cdot \sqrt{S^3 + \pi C (4\pi^2 J^2 + S^2)}}, \end{aligned} \quad (10)$$

² Further studies indicate that our formalism also works for non-AdS black holes without a holographic dual. In those cases, the conjugate variables (μ, N) are defined in a way similar to Eq. (3) where N takes the place of C , with ℓ replaced by an arbitrarily chosen constant length scale. For details, see [35, 36].

$$\mu = \left(\frac{\partial M}{\partial C} \right)_{S,J} = \frac{\pi^2 C^2 (4\pi^2 J^2 + S^2) - S^4}{4\pi^{3/2} \ell C^{3/2} \sqrt{S(\pi C + S)} \cdot \sqrt{S^3 + \pi C (4\pi^2 J^2 + S^2)}}. \quad (11)$$

We can see that M and all the intensive variables are now given as functions of the extensive variables S, J, C . It is evident that M scales as $M \rightarrow \lambda M$, while T, Ω, μ are not rescaled, if the independent variables scale as $S \rightarrow \lambda S, J \rightarrow \lambda J, C \rightarrow \lambda C$. This proves the first order homogeneity of M and zeroth order homogeneity of T, Ω, μ in S, J, C . Moreover, in accordance with the first law (4) and the Euler relation (5), we can write down the Gibbs-Duhem equation

$$d\mu = -SdT - Jd\Omega,$$

where $\mathcal{S} = S/C, \mathcal{J} = J/C$ are both zeroth order homogeneous functions in S, J, C . The Euler and Gibbs-Duhem relations and the various homogeneity properties for the internal energy and intensive variables are of essential importance in standard thermodynamics [29]. However, these relations and the corresponding homogeneity behaviors are absent in other formalisms for black hole thermodynamics.

Equations (9) to (11) provide three algebraic relations for six state parameters. Therefore, a macro state for the Kerr-AdS black hole can be determined by only three of the six variables $(T, S), (\Omega, J), (\mu, C)$. Moreover, each of the EOS involves only four of the state variables. A thermodynamic process is characterized by the continuous change of any of the state variables appearing in the EOS.

Before dwelling into the study of the thermodynamic processes, let us remind that there is an upper bound for J ,

$$J \leq J_{\max} = \frac{S\sqrt{\pi^2 C^2 + 4\pi CS + 3S^2}}{2\pi^2 C},$$

due to the requirement of non-negativity for T . This is the famous Kerr bound rewritten in terms of the macro state variables.

4 Thermodynamic processes

In this section we study the thermodynamic processes for Kerr-AdS black holes in the RPS formalism. Since the EOS (9)–(11) are quite complicated, a generic thermodynamic process may correspond to an arbitrary curve on a three-dimensional hypersurface in the space of macro states parametrized by the extensive variables (S, J, C) . Therefore, it looks impossible to make a complete analysis for all admissible processes.

In the following, we shall restrict ourselves to only three kinds of specific processes, i.e. $T - S, \Omega - J$ and $\mu - C$ processes. These are the simplest processes involving only one pair of conjugate intensive-extensive variables. Even for these simple cases, the exact result is still very involved and unillustrative. To proceed, we have two choices, i.e. (1) by resorting to numeric approaches which is more precise but less illustrative, and (2) by making some reasonable approximation, e.g. consider only the black hole states in the slow rotating limit (i.e. a very small) and present analytic results in this approximation. We choose to proceed in the second choice when necessary.

4.1 $T - S$ processes in the slow rotating limit

One of the characteristic properties for black hole thermodynamics is encoded in the $T - S$ curves at fixed J , which contains a first order phase transition which becomes second order at the critical point. This is true in either the EPST or the RPS formalisms, the latter has only been tested in the special example case of RN-AdS black hole in four space-time dimensions. The present work will contribute a second example case for the RPS formalism.

The critical point on the $T - S$ curve at fixed J is characterized by the inflection point which obeys the following equations,

$$\left(\frac{\partial T}{\partial S} \right)_{J,C} = 0, \quad \left(\frac{\partial^2 T}{\partial S^2} \right)_{J,C} = 0. \quad (12)$$

In the full form of the EOS (9), these equations are so complicated that it takes pages to write down the solution which is still in implicit form. Therefore it is better to resort to the slow rotating approximation.

By expanding $M = M(S, J, C)$ and all the EOS into power series in J and keep terms up to order J^2 , we get the following equations,

$$M(S, J, C) = \frac{C [\pi C (2\pi^2 J^2 + S^2) + S^3]}{2\pi^{3/2} \ell (CS)^{3/2}}, \quad (13)$$

$$T(S, J, C) = \frac{\pi C S^2 + 3S^3 - 6\pi^3 C J^2}{4\pi^{3/2} \ell \sqrt{CS^5}}, \quad (14)$$

$$\Omega(S, J, C) = \frac{2\pi^{3/2} J}{\ell} \sqrt{\frac{C}{S^3}}, \quad (15)$$

$$\mu(S, J, C) = \frac{\pi C (2\pi^2 J^2 + S^2) - S^3}{4\pi^{3/2} \ell (CS)^{3/2}}. \quad (16)$$

It is remarkable that, at this order of approximation, the first law (4) still holds exactly, and the homogeneity behaviors are still correct.

Fig. 1 $T - S$ and $F - T$ curves at fixed $J > 0$

Using Eq. (14), we can solve the approximate critical point equations (12) analytically, yielding the critical parameters

$$\begin{aligned} S_c &= \frac{\pi C}{96} (\sqrt{2689} - 31), \\ J_c &= \frac{C}{768} \sqrt{\frac{1}{30} (3683\sqrt{2689} - 180701)}. \end{aligned} \quad (17)$$

Accordingly, the critical temperature reads

$$T_c = \frac{1}{\pi \ell} \sqrt{\frac{3}{2}} \left(\frac{32197\sqrt{2689} - 1449979}{160 (\sqrt{2689} - 31)^{5/2}} \right). \quad (18)$$

We can also introduce the Helmholtz free energy

$$F(T, J, C) = M(S, J, C) - TS,$$

which, in the slow rotating limit, takes the value

$$F(T, J, C) = \frac{\pi C^{3/2} S^{5/2} (10\pi^2 J^2 + S^2) - C^{1/2} S^{11/2}}{4\pi^{3/2} \ell C S^4}, \quad (19)$$

where S should be understood as being implicitly determined by Eq. (14). At the critical point, we have

$$F_c = \frac{5 (71989 - 1291\sqrt{2689})}{1024\sqrt{6} (\sqrt{2689} - 31)^{3/2}} \frac{C}{\ell}. \quad (20)$$

Equations (17), (18) and (20) contain complicated expressions involving big integers which do not seem to make too much sense. It is better to rewrite these relations in terms of the approximate float point numbers, giving

$$\begin{aligned} S_c &\approx 0.68250 C, & J_c &\approx 0.02411 C, \\ T_c &\approx 0.26939 \ell^{-1}, & F_c &\approx 0.10556 \ell^{-1} C. \end{aligned}$$

Now we can re-express the EOS (14) and the free energy in terms of the relative parameters

$$s = \frac{S}{S_c}, \quad j = \frac{J}{J_c}, \quad \tau = \frac{T}{T_c}, \quad f = \frac{F}{F_c}.$$

Using approximate float point coefficients, the results read

$$\tau(s, j) = \left[(0.41305s + 0.63377)s^2 - 0.04682j^2 \right] s^{-5/2}, \quad (21)$$

$$f(t, j) = \left[(1.10389 - 0.23982s)s^2 + 0.13593j^2 \right] s^{-3/2}, \quad (22)$$

where s in Eq. (22) is determined implicitly via Eq. (21). Let us emphasize that, in terms of the relative parameters, the EOS and the expression for the free energy is completely independent of the central charge C . This implies that the same thermodynamic behavior is shared for black holes with any central charge. The same phenomenon also appears in standard thermodynamics for ordinary matter and is known as *the law of corresponding states*, in which the role of central charge is replaced by the number of particles.

Figure 1 presents the plots of the $T - S$ and $F - T$ curves at fixed $J > 0$. From the swallow tail structure on the $F - T$ curves we can see that there is a first order phase transition for $0 < J < J_c$, $T > T_c$. The transition temperature T_{trans} at which the first order phase transition occurs corresponds to the root (crossing point) of the swallow tail. The $T - S$ curves indicate that, when the first order phase transition occurs, there are three black hole states with the same temperature, angular momentum and central charge, but with different entropies. Among these, the states with the smallest and the largest entropies/event horizon radii are both stable, while the medium sized black hole state is unstable. Thus the phase equilibrium is basically an equilibrium between the stable small and large black states. When $T_{\text{trans}} = T_c$, the

Fig. 2 $T - S$ and $F - T$ curves at $J = 0$

swallow tail disappears and the phase transition becomes second order. The same characteristic properties were also seen in the case of RN-AdS black hole in the RPS formalism.

Notice that, in plotting the above curves, there is no need to worry about the upper bound for J , because the figures only cover a small portion of the positive T values.

Notice also that, the above discussion applies only to the cases $J > 0$. When $J = 0$, the black hole falls back to Schwarzschild-AdS, and the $T - S$ and $F - T$ curves becomes quite different. In essence, the stable small black hole states disappear completely, and the $T - S$ curve contains a single minimum. Using Eq. (14), we can easily find that the minimum is located at

$$S_{\min} = \frac{\pi C}{3}, \quad T_{\min} = \frac{\sqrt{3}}{2\pi\ell}.$$

Then the EOS (14) and the free energy equation (19) can be rescaled into

$$\hat{\tau} = \frac{\hat{s}^2(\hat{s} + 1)}{2\hat{s}^{5/2}}, \quad \hat{f} = \frac{(3 - \hat{s})\hat{s}^{1/2}}{2}, \quad (23)$$

where $\hat{s} = S/S_{\min}$, $\hat{\tau} = T/T_{\min}$ and $\hat{f} = F/F_{\min}$ with $F_{\min} = C/(6\sqrt{3}\ell)$. The corresponding $T - S$ and $F - T$ curves are depicted in Fig. 2. For any $T > T_{\min}$, there are two black hole states with the same T, C but different S , of which the small one is unstable and the large one stable. The transitions from small to large black holes should occur at any $T > T_{\min}$ without any equilibrium condition. This is in sharp contrast to the first order equilibrium phase transition which occur only at some specific temperature for each fixed nonvanishing J , as described previously.

We can also consider the $T - S$ process at fixed Ω . To do so, we need to insert Eq. (15) into (14), obtaining an expression $T = T(S, \Omega, C)$. It turns out that there is a single extremum for T (which is a minimum) in the resulting equation located

at

$$S_{\text{ex}} = \frac{2\pi C}{3\ell(2 - \ell^2\Omega^2)}, \quad T_{\text{ex}} = \frac{1}{2\pi\ell} \sqrt{\frac{3}{2}} (2 - \ell^2\Omega^2)^{1/2}.$$

Introducing the new relative variables $\tilde{s} = \frac{S}{S_{\text{ex}}}$, $\tilde{\tau} = \frac{T}{T_{\text{ex}}}$, the $T - S$ EOS can be rewritten in the form

$$\tilde{\tau} = \frac{\tilde{s}^2(\tilde{s} + 1)}{2\tilde{s}^{5/2}}, \quad (24)$$

which is free of both C and Ω . It is interesting that the $\tilde{\tau} - \tilde{s}$ relation given in (24) is identical to the $\hat{\tau} - \hat{s}$ relation given in (23).

For better understanding of the $T - S$ processes at fixed Ω , it is necessary to look at the behavior of the $\mu - T$ relation $\mu = \mu(T, \Omega)$, where μ is to be viewed as Gibbs free energy divided by the central charge. To get the $\mu - T$ relation, we can insert eq. (15) into (16) to replace J with Ω , but the dependent on T is still implicit in the resulting relation. However, it can be easily find that μ is peaked precisely at $S = S_{\text{ex}}$ with the peak value given by

$$\mu_{\text{ex}} = \frac{1}{3\sqrt{6}\ell} (2 - \ell^2\Omega^2)^{-1/2}.$$

Introducing the new variable

$$\tilde{m} = \frac{\mu}{\mu_{\text{ex}}},$$

the desired $\mu - T$ relation can be written in the form

$$\tilde{m} = \frac{\tilde{s}(3 - \tilde{s})}{2\tilde{s}^{1/2}}, \quad (25)$$

where $\tilde{s}$ is implicitly given by Eq. (24) in terms of $\tilde{\tau}$. It is remarkable that both Eqs. (24) and (25) are not explicitly dependent on Ω . This may be viewed as a law of corresponding states at an enhanced level.

Fig. 3 $T - S$ and $\mu - T$ curves at fixed Ω

Using Eqs. (24) and (25), we present the $T - S$ and $\mu - T$ curves in Fig. 3. Besides the non-equilibrium transitions from the unstable small black hole state to the stable large black hole state at $T > T_{\text{ex}}$ which is quite similar to the $J = 0$ case discussed above, there is also a Hawking–Page transition occurring at some $T = T_{\text{HP}} > T_{\text{ex}}$ signified by $\mu(T_{\text{HP}}, \Omega) = 0$. The Hawking–Page temperature T_{HP} corresponds to $\tilde{s} = 3$, which, by use of Eq. (24), reads

$$T_{\text{HP}} = \frac{2\sqrt{3}}{3} T_{\text{ex}}.$$

Before moving on to the next subsection, let us stress that the purpose for using the slow rotating limit is solely due to the demand for explicit analytical expressions for the critical point parameters and the Hawking–Page temperature. The same thermodynamic behaviors persists without this approximation, but the corresponding analysis can only be worked out using numeric method. Since the numerics are not as illustrative as the analytic expressions in the slow rotating limit, we omit the details completely.

4.2 $\Omega - J$ processes at fixed S

Next let us look at the $\Omega - J$ processes at fixed S . According to Eq. (15), Ω is simply proportional to J at fixed (S, C) in the slow rotating limit, the corresponding curve is just a segment of a straight line beginning from the origin of the $\Omega - J$ plane and ending at $J = J_{\text{max}}$. In order to see more features of the $\Omega - J$ processes, it is better to go back to the full form of the EOS (10). Recall that, as extensive variables, S and J must be both proportional to C , thus we can substitute $S = SC$ and $J = JC$ into Eq. (10), yielding

$$\Omega = \frac{2\pi^{3/2}}{\ell} \left(\frac{S + \pi}{S} \right)^{1/2} \frac{J}{[4\pi^3 J^2 + S^2(S + \pi)]^{1/2}},$$

Fig. 4 $\Omega - J$ curves at fixed S

where S, J are both intensive quantities which are independent of C .

In terms of the variable J , the upper bound for J becomes

$$J_{\text{max}} = \frac{S(3S^2 + 4\pi S + \pi^2)^{1/2}}{2\pi^2}.$$

This equation determines the end point of each $\Omega - J$ curve.

Figure 4 presents the $\Omega - J$ curves at different values of S . Each curve corresponds to a different value of S , and they all start from the origin and end at $J = J_{\text{max}}$. There is neither inflection point nor extremum on the $\Omega - J$ curves, which indicate that there can be no phase transition on the $\Omega - J$ plane in the space of macro states. Notice that for J very small, each $\Omega - J$ curve can be approximated by a segment of a straight line starting from the origin. This is the main feature captured by the slow rotating limit expression (15).

4.3 $\mu - C$ processes at fixed (S, J)

Unlike the $T - S$ and $\Omega - J$ processes studied in the previous two subsections, the $\mu - C$ processes call for the variation of the absolute value of C , therefore it seems that the laws

Fig. 5 $\mu - C$ curve at fixed (S, J)

of corresponding states will not apply here. However, as will be seen below, the study of $\mu - C$ process can still be carried out in a manner in which certain scaling behavior plays a prominent role.

Although we may choose to proceed with the full EOS (11), the corresponding analytical analysis is too cumbersome and unillustrative. Therefore, we resort to the slow rotating limit again by studying only the approximate EOS (16). It can be seen that there is a single extremum on each $\mu - C$ curve at fixed (S, J) , which corresponds to a maximum of μ . The extremal point is located at

$$C_{\max} = \frac{3S^3}{\pi(2\pi^2 J^2 + S^2)}, \quad \mu_{\max} = \frac{(2\pi^2 J^2 + S^2)^{3/2}}{6\sqrt{3}\ell S^3}.$$

Notice that $\mu_{\max}$ is positive for any nonvanishing choice of S, J . Introducing the dimensionless variables

$$c = \frac{C}{C_{\max}}, \quad m = \frac{\mu}{\mu_{\max}},$$

the EOS (16) becomes

$$m = \frac{3c - 1}{2c^{3/2}}. \quad (26)$$

Two important remarks are in due here:

1. The single equation (26) describes the $\mu - C$ processes at any fixed values of S and J , which clearly indicates some scaling properties;
2. The very same equation (26) has also appeared in the description of $\mu - C$ processes for RN-AdS black holes in the RPS formalism [30]. The appearance of the same reduced $\mu - C$ EOS in both RN-AdS and Kerr-AdS cases may not be a coincidence. There seems to be some universality lying behind.

For completeness, we reproduce the $\mu - C$ curve for the Kerr-AdS case as presented in Fig. 5. Let us remind that at

$C = \frac{1}{3}C_{\max}$, μ becomes zero, which signifies a Hawking–Page transition. A better description for the Hawking–Page transition has already been presented in Fig. 3.

5 Concluding remarks

The study of Kerr-AdS black hole in the RPS formalism has revealed several remarkable features which are common to the case of RN-AdS case studied in our previous work. In particular, we would like to summarize the following features:

1. The thermodynamics in the RPS formalism conforms to the standard description of traditional extensive thermodynamics. The mass of the black hole plays the role of internal energy and is a homogeneous function of the first order in all extensive variables. Consequently, the Euler relation and Gibbs–Duhem equation hold perfectly. Notice that, the absence of pressure–volume variables leaves no room for defining the enthalpy – the variation of which, by definition, equals the *isobaric* exchange of heat [29] – for black holes in our formalism, as opposed to the EPS formalism.
2. Each $T - S$ process at fixed nonvanishing angular momentum contains a first order equilibrium phase transition at certain supercritical temperature. At the critical temperature the phase transition becomes second order. Further, at subcritical temperatures, the phase transition disappears completely. On the contrary, at vanishing angular momentum or at fixed angular velocities, there is always a non-equilibrium transition from a small unstable black hole state to a large stable black hole state. The same behaviors are also found in the case of RN-AdS with the variables (Ω, J) replaced by $(\hat{\Phi}, \hat{Q})$ [30], in spite of the fact that the underlying geometries are very different.
3. The $\Omega - J$ processes in Kerr-AdS (just like the $\hat{\Phi} - \hat{Q}$ processes in RN-AdS) at fixed S are trivial in the sense that there are no phase transitions in such processes.
4. The Hawking–Page transition always appear for AdS black holes in the RPS formalism, and there seems to be some universality in the $\mu - C$ processes which needs some further exploration.
5. For those who felt uncomfortable with variable Newton constant, let us mention that C can be kept fixed, in which case the first law in our formalism falls back to that of the traditional formalism for black hole thermodynamics, which is analogous to the ordinary thermodynamics for closed systems. In such cases, the thermodynamic processes described in Sects. 4.1, 4.2 are not affected, but the $\mu - C$ processes described in Sect. 4.3 will be forbidden. Let us stress that, even when C is kept fixed, the chemical

potential μ is still meaningful, and the Euler relation (5) still holds. Moreover, the Euler relation (5) and the Smarr relation

$$M = 2(TS + \Omega J)$$

are two distinct mass formulae [36]. One can even find a third mass formula

$$M = 2\mu C$$

in our formalism, and all these mass formulae can be easily generalized to higher dimensional rotating black holes with arbitrarily many permissible rotation parameters [36].

It should be emphasized that, although some of the features for Kerr-AdS are described only in the slow rotating limit, the same features are actually maintained in the full exact cases. The only complexity in the full cases is in the exact solution of the critical point and/or characteristic parameters, however the characteristic features of the thermodynamic processes have been verified by use numerical techniques.

The close similarity between the thermodynamic behaviors of the RN-AdS and Kerr-AdS black holes forces us to make the conjecture that there might be some hidden universality properties lying behind the RPS formalism. Therefore, making further explorations about the behaviors of other black holes or black holes from other gravity models are of utmost importance.

Among the various thermodynamic behaviors discussed so far, particular attentions need to be paid toward the existence of the phase transitions at supercritical temperatures. In ordinary non-gravitational thermodynamic systems, higher temperature means more intensive random motions of individual particles which disfavors the formation of symmetry breaking order. Therefore phase transitions appear mostly at subcritical temperatures. However, in gravitational systems, higher temperature also means higher thermal energy which in turn produces gravity. Therefore, in gravitational systems, higher temperature also implies stronger gravitation which may favor the formation of order. The supercritical phase transitions observed in the RPS formalism of AdS black hole thermodynamics may simply be signifying that thermal gravitation overcomes random motion.

The last point to be noticed is the correspondence to the CFT side. As we have done in Sect. 4.3, the thermodynamic process with varying C is a normal thermodynamic process on the black hole side, however, it implies changing the dual CFT. This reveals a hidden property of the AdS/CFT correspondence. On the macroscopic level, the AdS/CFT correspondence is a correspondence between a given macro state

of the AdS black hole and one of the dual CFT. If the macro state of the AdS black hole changes, the corresponding macro state might need to be found in a different dual CFT. The same is true if one considers thermodynamic processes on the CFT side, as did in Ref. [28]. There, a thermodynamic process with varying volume corresponds to black hole states with variable cosmological constant, which implies an ensemble of gravitational theories, of which each member has a different cosmological constant. It is clear that such processes are meaningful on the CFT side but not on the gravity side. We end the paper by stressing once again that the AdS/CFT correspondence is only a correspondence between macro states but not between macro processes if the theories on both sides are fixed.

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