On the “Einstein–Gauss–Bonnet gravity in four dimension”

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Abstract To ensure the existence of a well defined linearized gravitational wave equation, we show that the spacetimes in the so-called “Einstein–Gauss–Bonnet gravity in four dimension” have to be locally conformally flat.

1 Introduction

Recently, a novel Einstein–Gauss–Bonnet gravity in four dimension has been proposed [1]. The theory is based on the action

$$S = \int d^D x \sqrt{-g} \left(R - 2\Lambda + \frac{\hat{\alpha}}{D-4} L_{\text{GB}} \right), \quad (1.1)$$

where R is the D -dimensional Ricci scalar, and Λ is the cosmological constant. The Gauss–Bonnet term is given by

$$L_{\text{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.$$

Here, the usual Gauss–Bonnet coupling constant α has been replaced by $\hat{\alpha}/(D-4)$. Then the so-called four dimensional Einstein–Gauss–Bonnet theory (4D EDB) is defined by considering the limit $D \rightarrow 4$. In Ref. [1], it is also claimed that under this limit the Gauss–Bonnet invariant gives rise to non-trivial contributions to gravitational dynamics. This is quite different from the content of the well-known Lovelock theorem [2].

However, the 4D EDB gravity cannot exist on a conceptual level. Since the decomposition of the Lovelock tensor which is required to make this gravity theory work leads to a violation of the Bianchi identity, in which case, gravity cannot be coupled to a conserved source as proved impressively by Gurses et al. [3,4]. Similiar conclusions also be found in the work by Arrechea et al. [5,6]. In addition, several ambiguities on the limit $D \rightarrow 4$ for this gravity were mentioned by several authors. A necessary condition for this procedure

to work is that there is an embedding of the four-dimensional spacetime into the higher D -dimensional spacetime so that the equations in the latter can be properly interpreted after taking the limit in the Ref. [7]. In the Ref. [8], by using the ADM decomposition, the authors proposed that the limit $D \rightarrow 4$ depends on the way how to regularize the Hamiltonian or/and the equations of motion. These papers all pointed out the non-viability of 4D EDB gravity.

Based on these works, in our manuscript, we further study the restriction on the theory, and point out another unreliability of 4D EGB gravity by studying the behavior of the principal symbol of the linearized perturbation equation under the limit $D \rightarrow 4$. Consider the equations of motion linearized around a background solution, we get

$$P^{\mu\nu\rho\sigma\alpha\beta} \partial_\alpha \partial_\beta \delta g_{\rho\sigma} + \dots = 0, \quad (1.2)$$

where the ellipsis denotes the terms with lower than 2-derivatives acting on $\delta g_{\mu\nu}$. Let ξ_μ be an arbitrary covector, then $P^{\mu\nu\rho\sigma\alpha\beta} \xi_\alpha \xi_\beta$ is the so-called principal symbol. Hyperbolicity and causality of the gravitational theory is determined by the principal symbol [9–12], so it is natural to require that the principal symbol should be well behaved in this novel Einstein–Gauss–Bonnet gravity. Otherwise, we have no well defined linearized gravitational wave equations.

This paper is organized as follows. In Sect. 2, by requiring the principal symbol to be well defined under the limit $D \rightarrow 4$, it is shown that spacetimes in such novel Einstein–Gauss–Bonnet gravity have to be locally conformally flat. The tensor perturbation equations for a general spherically symmetric spacetime are considered in Sect. 3, and we get the detailed form of the principal symbol. Section 4 is conclusion and discussion.

2 Principal symbol

Suppose we have a D -dimensional spacetime. Based on the above ideas, the principal symbol should be well defined

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under the limit $D \rightarrow 4$. For a diffeomorphism invariant theory, based on the discussion of the symmetry of the tensor $P^{\mu\nu\rho\sigma\alpha\beta}$, Reall has proved that the principal symbol has a form [12],

$$P^{\mu\nu\rho\sigma}(\xi) = P^{\mu(\rho|\alpha\nu|\sigma)\beta}\xi_\alpha\xi_\beta, \quad (2.1)$$

where for Einstein–Gauss–Bonnet gravity in D dimension ($D > 4$), we have

$$P_{\mu\rho\alpha}{}^{\nu\sigma\beta} = \frac{1}{2}\delta_{\mu\rho\alpha}^{\nu\sigma\beta} + \hat{\alpha}\left[\frac{D+1}{(D-1)(D-2)}\delta_{\mu\rho\alpha}^{\nu\sigma\beta}R - \frac{2}{D-2}A_{\mu\rho\alpha}{}^{\nu\sigma\beta} + \frac{1}{2(D-4)}W_{\mu\rho\alpha}{}^{\nu\sigma\beta}\right]. \quad (2.2)$$

In the above Eq. (2.2), $\delta_{\mu\rho\alpha}^{\nu\sigma\beta}$ is a generalized Kronecker-delta tensor, while $A_{\mu\rho\alpha}{}^{\nu\sigma\beta}$ and $W_{\mu\rho\alpha}{}^{\nu\sigma\beta}$ are defined as follows

$$A_{\mu\rho\alpha}{}^{\nu\sigma\beta} = \begin{vmatrix} R_{\mu}^{\nu} & R_{\mu}^{\sigma} & R_{\mu}^{\beta} \\ \delta_{\rho}^{\mu} & \delta_{\rho}^{\sigma} & \delta_{\rho}^{\beta} \\ \delta_{\alpha}^{\nu} & \delta_{\alpha}^{\sigma} & \delta_{\alpha}^{\beta} \end{vmatrix} + \begin{vmatrix} \delta_{\mu}^{\nu} & \delta_{\mu}^{\sigma} & \delta_{\mu}^{\beta} \\ R_{\rho}^{\mu} & R_{\rho}^{\sigma} & R_{\rho}^{\beta} \\ \delta_{\alpha}^{\nu} & \delta_{\alpha}^{\sigma} & \delta_{\alpha}^{\beta} \end{vmatrix} + \begin{vmatrix} \delta_{\mu}^{\nu} & \delta_{\mu}^{\sigma} & \delta_{\mu}^{\beta} \\ \delta_{\rho}^{\mu} & \delta_{\rho}^{\sigma} & \delta_{\rho}^{\beta} \\ R_{\alpha}^{\nu} & R_{\alpha}^{\sigma} & R_{\alpha}^{\beta} \end{vmatrix}, \quad (2.3)$$

and

$$W_{\mu\rho\alpha}{}^{\nu\sigma\beta} = 36\delta_{[\mu}^{\nu}\delta_{\rho}^{\sigma}\delta_{\alpha]}^{\beta]}, \quad (2.4)$$

where $R_{\mu\nu}$ and $C_{\rho\alpha\sigma\beta}$ are the Ricci tensor and Weyl tensor of the spacetime respectively. It is not hard to find that $W_{\mu(\rho|\alpha\nu|\sigma)\beta}$ vanishes when $D = 4$. So an indeterminate form of type 0/0 comes out from the last term on the right hand side of Eq. (2.2), i.e.,

$$\lim_{D \rightarrow 4} \frac{W_{\mu(\rho|\alpha\nu|\sigma)\beta}}{2(D-4)}. \quad (2.5)$$

To ensure the existence of $P^{\mu(\rho|\alpha\nu|\sigma)\beta}$ under the limit $D \rightarrow 4$, a suitable condition, i.e.,

$$W_{\mu(\rho|\alpha\nu|\sigma)\beta} = 0, \quad (2.6)$$

is imposed for all $D > 4$. Furthermore, in the case with $D > 4$, by contracting the two indices ρ and σ in Eq. (2.6), we obtain

$$C_{\mu\alpha\nu\beta} = 0. \quad (2.7)$$

This means the spacetimes of the theory have to be locally conformally flat. In fact, recently, based on the equations of motion, people have realized that the theory is well defined only when

$$\mathcal{L}_{\mu\nu} = C_{\mu\alpha\beta\gamma}C_{\nu}{}^{\alpha\beta\gamma} - \frac{1}{4}g_{\mu\nu}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \quad (2.8)$$

is vanished for $D > 4$, and a well defined four dimensional Einstein–Gauss–Bonnet theory can be generated as $D \rightarrow 4$

[3–6]. Here, with a similar logic, we obtained a more restrictive condition, i.e., Eq. (2.7), on the metrics of the theory.

3 Examples

To make the problem more transparent, let us consider a spacetime manifold $\mathcal{M}^D \cong M^m \times N^n$ with a metric

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(y)dy^a dy^b + r^2(y)\gamma_{ij}(z)dz^i dz^j, \quad (3.1)$$

where $D = m + n$, and the coordinate system $\{x^\mu\}$ is given by $\{y^1, \dots, y^m; z^1, \dots, z^n\}$. The tuple (M^m, g_{ab}) forms a m -dimensional Lorentzian manifold if $m > 1$, and (N^n, γ_{ij}) is an n -dimensional Riemann manifold. This Riemann manifold (N^n, γ_{ij}) is assumed to be an Einstein manifold, i.e.,

$$\hat{R}_{ij} = (n-1)K\gamma_{ij}, \quad (3.2)$$

where $\hat{R}_{ij}$ is the Ricci tensor of (N^n, γ_{ij}) , and K is the sectional curvature of the space. The metric compatible covariant derivatives associated with g_{ab} and γ_{ij} are denoted by D_a and $\hat{D}_i$, respectively.

For the metric (3.1), the tensor perturbation can be put into a form [13]

$$\left(P^{ab}{}_{ij}{}^{kl}D_a D_b + P^{mn}{}_{ij}{}^{kl}\hat{D}_m \hat{D}_n + P^a{}_{ij}{}^{kl}D_a + V_{ij}{}^{kl}\right) \left(\frac{h_{kl}}{r^2}\right) = -\frac{2\kappa_D^2}{r^2}\delta T_{ij}, \quad (3.3)$$

where

$$P^{ab}{}_{ij}{}^{kl} = P^{ab}\delta_i{}^k\delta_j{}^l - \frac{4\alpha}{r^2}g^{ab}\hat{C}_i{}^k{}_{j}{}^l, \quad (3.4)$$

$$P^a{}_{ij}{}^{kl} = P^a\delta_i{}^k\delta_j{}^l - 4\alpha(n-2)\frac{D^a r}{r}\frac{\hat{C}_i{}^k{}_{j}{}^l}{r^2}, \quad (3.5)$$

$$P^{mn}{}_{ij}{}^{kl} = P^{mn}\delta_i{}^k\delta_j{}^l + \frac{4\alpha}{r^2}\left(\hat{C}_j{}^{knl}\delta_i{}^m + \hat{C}_i{}^{knl}\delta_j{}^m + \hat{C}_j{}^{mln}\delta_i{}^k + \hat{C}_i{}^{mln}\delta_j{}^k - \hat{C}^{mkn}l\gamma_{ij} - \hat{C}_i{}^k{}_{j}{}^l\gamma^{mn}\right), \quad (3.6)$$

and

$$V_{ij}{}^{kl} = V\delta_i{}^k\delta_j{}^l + \frac{2\hat{C}_i{}^k{}_{j}{}^l}{r^2} + \alpha\left\{4\left[{}^mR - 2(n-3)\frac{{}^m\Box r}{r} + (n^2 - 7n + 16)\frac{K}{r^2} - (n-3)(n-4)\frac{(Dr)^2}{r^2}\right]\frac{\hat{C}_i{}^k{}_{j}{}^l}{r^2} - \frac{8}{r^4}\hat{C}_{imjn}\hat{C}^{mkn}l + \frac{4}{r^4}\hat{C}_{mnj}{}^k\hat{C}^{mn}{}_i{}^l - \frac{2}{r^4}\hat{C}^{mnp}l\hat{C}_{mnp}{}^k\gamma_{ij} + \frac{\hat{C}^{mnpq}\hat{C}_{mnpq}}{r^4}\delta_i{}^k\delta_j{}^l\right\}. \quad (3.7)$$

In the above equations, $\alpha = \hat{\alpha}/(D-4)$, and we have defined

$$P^{ab} = g^{ab} + 2(n-2)\alpha \left\{ 2 \frac{D^a D^b r}{r} + \left[(n-3) \frac{K - (Dr)^2}{r^2} - 2 \frac{m \square r}{r} \right] g^{ab} \right\} - 4\alpha \cdot {}^m G^{ab}, \quad (3.8)$$

$$P^{mn} = \left\{ 1 + 2\alpha \left[{}^m R - \frac{2(n-3)m \square r}{r} + (n-3)(n-4) \frac{K - (Dr)^2}{r^2} \right] \right\} \frac{\gamma^{mn}}{r^2}, \quad (3.9)$$

$$P^a = n \frac{D^a r}{r} + 2(n-2)\alpha \left\{ 4 \frac{D^a D^b r}{r} + \left[{}^m R - 2(n-1) \frac{m \square r}{r} + (n-2)(n-3) \frac{K - (Dr)^2}{r^2} \right] g^{ab} \right\} \frac{D_b r}{r} - 8\alpha \cdot {}^m G^{ab} \frac{D_b r}{r}, \quad (3.10)$$

and

$$\begin{aligned} V = & {}^m R - 2(n-1) \frac{m \square r}{r} + \frac{n(n-3)K}{r^2} \\ & - \frac{(n-1)(n-2)(Dr)^2}{r^2} - 2\Lambda \\ & + \alpha \left\{ {}^m L_{GB} + 8(n-1) \cdot {}^m G^{ab} \frac{D_a D_b r}{r} \right. \\ & - 4(n-1)(n-2) \frac{(D^a D^b r)(D_a D_b r)}{r^2} \\ & + 4(n-1)(n-2) \left(\frac{m \square r}{r} \right)^2 + 2n(n-3) \frac{K \cdot {}^m R}{r^2} \\ & - 2(n-1)(n-2) \frac{(Dr)^2 \cdot {}^m R}{r^2} \\ & - 4n(n-3)^2 \frac{K \cdot m \square r}{r^3} + 4(n-1)(n-2)(n-3) \\ & \times \frac{(Dr)^2 \cdot m \square r}{r^3} - 2n(n-3)^2(n-4) \frac{K \cdot (Dr)^2}{r^4} \\ & + (n-3)(n-4)(n^2 - 3n - 2) \frac{K^2}{r^4} \\ & \left. + (n-1)(n-2)(n-3)(n-4) \left[\frac{(Dr)^2}{r^2} \right]^2 \right\}. \quad (3.11) \end{aligned}$$

In these equations, $m \square = g^{ab} D_a D_b$ is the Laplace–Beltrami operator on (M^m, g_{ab}) . ${}^m R$ and ${}^m G^{ab}$ are the Ricci scalar and Einstein tensor of (M^m, g_{ab}) respectively, while ${}^m L_{GB}$ is the Gauss–Bonnet term which is nontrivial only when $m > 3$. The symbol $\hat{C}_{ijkl}$ represents the Weyl tensor of the Riemann manifold (N^n, γ_{ij}) .

First, consider the case $m = 1$, i.e., $D = 1 + n$, the metric (3.1) becomes

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + r^2(t) \gamma_{ij} dz^i dz^j.$$

In this case, the non-trivial components of the Weyl tensor of this spacetime can be written as

$$C_{ijkl} = r^2 \hat{C}_{ijkl}. \quad (3.12)$$

Obviously, from Eqs. (3.4)–(3.7), the limit $D \rightarrow 4$ or $n \rightarrow 3$, can be done only in the case $\hat{C}_{ijkl} = 0$. This means the Weyl tensor C_{ijkl} is also vanished according to the relation (3.12). Under this limit, from Eqs. (3.8)–(3.11), P^{tt} , P^{mn} , P^t , V have following forms

$$P^{tt} = - \left[1 + 2\hat{\alpha} \left(\frac{K}{r^2} + H^2 \right) \right], \quad (3.13)$$

$$P^{mn} = \left[1 + 2\hat{\alpha} \left(2\dot{H} + H^2 - \frac{K}{r^2} \right) \right] \frac{\gamma^{mn}}{r^2}, \quad (3.14)$$

$$P^t = -3H + 2\hat{\alpha} \left[-2\dot{H} - 3H^2 - \frac{K}{r^2} \right] H, \quad (3.15)$$

and

$$V = 4\dot{H} + 6H^2 - 2\Lambda + 2\hat{\alpha} \left[4H^2 \dot{H} + 3H^4 + \frac{K^2}{r^4} \right]. \quad (3.16)$$

where “ $\cdot$ ” denotes the derivative with respect to the coordinate t , and $H = \dot{r}/r$ is Hubble parameter. In the case $K = 0$, from the above results, Eq. (3.3) exactly reduces to the relevant part in Ref. [1]. When the matter field is absent, Eq. (3.3) gives the gravitational wave equations on vacuum. For example, the gravitational wave equations on Minkowski spacetime, de Sitter spacetime, and anti de Sitter spacetime.

Second, we consider the case of $m = 2$. The components of the Weyl tensor of the spacetime can be expressed as [14]

$$\begin{aligned} C_{abcd} &= 2c_1 w g_{a[c} g_{d]b}, \\ C_{iajb} &= -c_2 w r^2 g_{ab} \gamma_{ij}, \\ C_{ijmn} &= 2c_3 w r^4 \gamma_{i[m} \gamma_{n]j} + r^2 \hat{C}_{ijmn}, \end{aligned} \quad (3.17)$$

where

$$w = {}^2 R + 2 \frac{{}^2 \square r}{r} + 2 \frac{K - (Dr)^2}{r^2}, \quad (3.18)$$

and

$$c_1 = \frac{n-1}{2(n+1)}, \quad c_2 = \frac{n-1}{2n(n+1)}, \quad c_3 = \frac{1}{n(n+1)}. \quad (3.19)$$

For the metric (3.1) with $\hat{C}_{ijkl} = 0$ (this is also a necessary condition of the existence of the limit $D \rightarrow 4$), the tensor perturbation can be put into a form

$$\begin{aligned} & \left(P^{ab} D_a D_b + P^{mn} \hat{D}_m \hat{D}_n + P^a D_a + V \right) \left(\frac{h_{ij}}{r^2} \right) \\ & = - \frac{2\kappa_D^2}{r^2} \delta T_{ij}. \end{aligned} \quad (3.20)$$

Here, P^{ab} , P^{mn} , P^a , V have the same forms as in Eqs. (3.8)–(3.11) except that ${}^m G_{ab} = 0$ and ${}^m L_{GB} = 0$ when $m = 2$.

It is easy to find that P^{ab} and P^a have well defined limits under $D \rightarrow 4$ or $n \rightarrow 2$. However, generally, P^{mn} is not well defined under this limit. Actually, we have

$$P^{mn} = \left\{ 1 + 2\alpha \left[w - \frac{2(n-2)^2 \square r}{r} + (n-2)(n-5) \frac{K - (Dr)^2}{r^2} \right] \right\} \frac{\gamma^{mn}}{r^2}, \quad (3.21)$$

To ensure the regularity of “effective metric” $P^{\mu\nu} = (P^{ab}, P^{mn})$ of the tensor perturbation equation, we have to impose a condition $w = 0$. Thus, Eq. (3.17) imply that the spacetime has to be (locally) conformally flat.

Furthermore, the function V (one part of the effective potential of the theory) is rewritten as

$$\begin{aligned} V = & {}^2R - 2(n-1) \frac{{}^2\square r}{r} + \frac{n(n-3)K}{r^2} \\ & - \frac{(n-1)(n-2)(Dr)^2}{r^2} - 2\Lambda \\ & + \alpha(n-2) \left\{ -4(n-1) \frac{(D^a D^b r)(D_a D_b r)}{r^2} \right. \\ & + 4(n-1) \left(\frac{{}^2\square r}{r} \right)^2 \\ & + 2(n-1) \frac{K \cdot {}^2R}{r^2} - 2(n-1) \frac{(Dr)^2 \cdot {}^2R}{r^2} \\ & - 4(n^2 - 4n + 1) \frac{K \cdot {}^2\square r}{r^3} \\ & + 4(n-1)(n-3) \frac{(Dr)^2 \cdot {}^2\square r}{r^3} \\ & - 2(n^3 - 8n^2 + 17n - 2) \frac{K \cdot (Dr)^2}{r^4} \\ & + (n^3 - 8n^2 + 15n + 8) \frac{K^2}{r^4} \\ & \left. + (n-1)(n-3)(n-4) \left[\frac{(Dr)^2}{r^2} \right]^2 \right\} \\ & - \alpha \cdot w \cdot \frac{4K}{r^2}. \end{aligned} \quad (3.22)$$

Obviously, under the limit $D \rightarrow 4$ or $n \rightarrow 2$, the potential V is regular only in the case $w = 0$. This also requires the vanished Weyl tensor of the spacetime.

However, in this case, due to the Birkhoff type theorems in four dimension, h_{ij} is trivial [This tensor perturbation is based on the tensor decomposition on a two dimensional space (N^2, γ_{ij}) . It is very different from the gravitational perturbation based the tensor decomposition on a three dimensional space (N^3, γ_{ij}) , i.e., the case with $m = 1$]. Namely, this means the solution of the tensor perturbation equation (3.20) is trivial. Nevertheless, it can not hinder our discussions on the principal symbol of the tensor perturbation equations.

4 Conclusion and discussion

The original proposal of the 4D EGB gravity is in contradiction with the Lovelock theorems and hence came into questions. To overcome the deficiencies of the 4D EGB gravity, people have proposed new procedures for the $D \rightarrow 4$ limit and these method lead to a well defined action principle [15, 16]. They found well-defined theories which belong to a class of Horndeski theory [17]. A similar approach can be found in the earlier work by Mann and Ross [18]. The same results are deduced in the Refs. [19, 20] by adding a counter term in D -dimensions and then taking the $D \rightarrow 4$ limit.

In this paper, another perspective is proposed to point out the irrationality of the 4D EGB gravity, i.e., the principal symbol of the linearized perturbation equation has a well defined limit. By this, we have shown the spacetimes in the so-called “Einstein–Gauss–Bonnet Gravity in four dimension” have to be (locally) conformally flat. For example, the Friedmann–Lemaître–Robertson–Walker and maximally symmetric solutions found in Ref. [1] are this kind of spacetimes. For the spacetimes with non-vanishing Weyl tensor [21], Dadhich found some issues of causality – the rescaled Gauss–Bonnet coupling constant radically alters the causal structure of the black hole and it turned a black hole into a white hole! Actually, in this paper, to get a well defined theory, the spacetimes have to be locally conformally flat. So, locally, the metric always has a form $g_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$. It is concluded that conformally flat spacetimes are preferred in this theory. This point is similar to the conclusions in [3–6, 22].

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