Strong coupling constant from moments of quarkonium correlators revisited

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Abstract We revisit previous determination of the strong coupling constant from moments of quarkonium correlators in $(2 + 1)$ -flavor QCD. We use previously calculated moments obtained with Highly Improved Staggered Quark (HISQ) action for five different quark masses and several lattice spacings. We perform a careful continuum extrapolations of the moments and from the comparison of these to the perturbative result we determine the QCD Lambda parameter, $\Lambda_{\overline{MS}}^{n_f=3} = 332 \pm 17 \pm 2(\text{scale})$ MeV. This corresponds to $\alpha_s^{n_f=5}(\mu = M_Z) = 0.1177(12)$.

1 Introduction

There are many lattice QCD calculations aiming to obtain the strong coupling constant, α_s , using different methods, since it is important for the precision tests of the Standard Model, see Refs. [1,2] for recent reviews. One of the methods to determine α_s on the lattice is to consider moments of quarkonium correlators. The method was pioneered by HPQCD collaboration [3] in 2008 and used in many lattice calculations since [4–8]. One of the challenges in this method is to obtain reliable continuum results for the moments of quarkonium correlators. To deal with this problem Bayesian fits have been used in Refs. [3–5]. On the other hand the analysis of Refs. [6,8] uses many lattice spacings, utilizing a large set of gauge ensembles from HotQCD collaboration [9,10] as well as additional gauge configurations on very fine lattices with $a < 0.04$ fm [11]. However, even with eleven different lattice spacings the continuum extrapolation of the moments of quarkonium correlators turned out to be challenging. The continuum extrapolations in Refs. [8] were reasonably stable for $m_h = m_c$ and $m_h = 1.5m_c$. How-

ever, for larger heavy-quark masses, namely $m_h = 2m_c, 3m_c$ and $4m_c$ they turned out to be problematic. As the result the strong coupling constant determined from the lattice results for $m_h = 2m_c$ was significantly lower, and inconsistent with the results obtained at the lowest two quark masses. The α_s value from the combined analysis at different m_h therefore turned out to be lower than other determinations from the moments of quarkonium correlators and had large errors. The aim of this paper is to improve the analysis of Ref. [8] at larger quark masses through the combined fits of the lattice results at different quark masses and resolve the above discrepancy. The other problem in the determination of the strong coupling constant is the proper estimate of the perturbative error. This is closely related to the choice of the renormalization scale, which should be proportional to m_h . In Refs. [3–5,7] $\mu = 3m_h$ was used, while in Refs. [6,8] the renormalization scale μ was set to be equal to m_h . In this paper we will use many choices of the renormalization scale μ and demonstrate the consistency of the corresponding α_s determinations ensuring that the estimate of the perturbative uncertainties is reliable.

The rest of the paper is organized as follows. In Sect. 2 we review the general aspects of α_s determination from the moments of quarkonium correlators. In Sect. 3 we discuss the continuum extrapolations of the moments. Section 4 contains the determination of the strong coupling constant. Finally our conclusions are presented in Sect. 5.

2 Moments of quarkonium correlators and the strong coupling constant

In this paper we consider moments of the pseudo-scalar quarkonium correlator to determine the strong coupling constant along the lines presented in Ref. [8]. We will discuss the main points of this procedure for completeness. The pseudo-

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scalar quarkonium correlator is defined as

$$G_n = \sum_t t^n G(t), \quad G(t) = a^6 \sum_{\mathbf{x}} (am_{h0})^2 \langle j_5(\mathbf{x}, t) j_5(0, 0) \rangle. \quad (1)$$

Here $j_5 = \bar{\psi} \gamma_5 \psi$ is the pseudo-scalar current and m_{h0} is the lattice heavy-quark mass. One can also consider the vector current, however, the lattice calculations of the corresponding correlation function turned out to be less precise [3]. For a lattice with temporal size N_t the above definition of the moments can be generalized as follows:

$$G_n = \sum_t t^n (G(t) + G(N_t - t)). \quad (2)$$

In the continuum the moments G_n are finite only for $n \geq 4$ (n even), since the correlation function diverges as t^{-4} for small t . Furthermore, the moments G_n do not need renormalization because the explicit factors of the quark mass are included in their definition [3]. The moments can be calculated in perturbation theory in $\overline{MS}$ scheme

$$G_n = \frac{g_n(\alpha_s(\mu), \mu/m_h)}{am_h^{n-4}(\mu_m)}. \quad (3)$$

Here μ is the $\overline{MS}$ renormalization scale. The scale, μ_m at which the $\overline{MS}$ heavy-quark mass is defined could be different from μ in general [12]. The coefficient $g_n(\alpha_s(\mu), \mu/m_h)$ is calculated up to 4-loop, i.e. including the term of order α_s^3 [13–15].

In lattice calculations it is more practical to consider the reduced moments [3]

$$R_n = \begin{cases} G_n/G_n^{(0)} & (n=4) \\ (G_n/G_n^{(0)})^{1/(n-4)} & (n \geq 6) \end{cases}, \quad (4)$$

where $G_n^{(0)}$ is the moment calculated from the free correlation function. The lattice artifacts largely cancel out in these reduced moments.

It is straightforward to write down the perturbative expansion for R_n :

$$R_n = \begin{cases} r_4 & (n=4) \\ r_n \cdot (m_{h0}/m_h(\mu_m)) & (n \geq 6) \end{cases}, \quad (5)$$

$$r_n = 1 + \sum_{j=1}^3 r_{nj}(\mu/m_h) \left(\frac{\alpha_s(\mu)}{\pi} \right)^j. \quad (6)$$

From the above equations it is clear that R_4 is suitable for the extraction of the strong coupling constant $\alpha_s(\mu)$ at scale proportional to the heavy-quark mass, m_h , while the ratios

R_n/m_{h0} with $n \geq 6$ are suitable for extracting the heavy-quark mass once $\alpha_s(\mu)$ is determined. One can also use the ratios of the reduced moments, namely R_6/R_8 and R_8/R_{10} to determine α_s . We will discuss these ratios in the Appendix.

3 Continuum extrapolations of the reduced moments of quarkonium correlators

In our analysis we used previously published lattice QCD results for the reduced moments in (2+1)-flavor QCD obtained for heavy-quark masses $m_h = m_c, 1.5m_c, 2m_c, 3m_c$ and $4m_c$, with m_c being the charm-quark mass [8]. The lattice calculations have been performed using HISQ action at several values of the lattice spacing [8]. The lattice spacing has been fixed through the r_1 parameter from the static quark-antiquark potential [9–11], and the value

$$r_1 = 0.3106(18) \text{ fm} \quad (7)$$

obtained from the pion decay constant was used [16]. Furthermore, the calculations have been performed at two values of the light quark masses corresponding to the pion mass of 161 MeV and 320 MeV in the continuum limit, and no dependence on the light quark mass of the reduced moments was found within errors [8]. The lattice results on R_n , $n = 4, 6, 8, 10$ are found in Tables VII–XI of Ref. [8] for different lattice spacings. The errors in the tables include statistical errors, errors related to mistuning of the charm-quark mass and finite volume errors. All the errors have been added in quadrature. The bare charm-quark masses are found in Table I of [8].

Because the tree-level lattice artifacts cancel out for the reduced moments the lattice spacing dependence can be parameterized as

$$R_4(m_h) = R_4^{cont}(m_h) + \sum_{i=1}^N \sum_{j=1}^{M_i} b_{ij} (\alpha_s^b)^i [1 + \sum_{k=1}^i d_{ijk} \ln^k(am_{h0})] (am_{h0})^{2j} \quad (8)$$

$$\frac{R_n(m_h)}{m_{h0}} = \left(\frac{R_n(m_h)}{m_{h0}} \right)^{cont} + \sum_{i=1}^N \sum_{j=1}^{M_i} c_{ij}^{(n)} (\alpha_s^b)^i [1 + \sum_{k=1}^i e_{ijk} \ln^k(am_{h0})] (am_{h0})^{2j}, \quad n \geq 6, \quad (9)$$

where $\alpha_s^b = g_0^2/(4\pi u_0^4)$, $g_0^2 = 10/\beta$ is the boosted gauge coupling. We performed joint fits of the lattice results on R_4 and R_n/m_{h0} obtained at different quark masses to Eqs. (8) and (9) setting $d_{ijk} = e_{ijk} = 0$. The reason for setting the coefficients of the log terms to zero was to avoid having too many poorly constrained parameters since the logarithmic dependence on am_{h0} is much weaker than the power-law dependence. For the continuum extrapolations of R_4 , where the lattice spacing dependence is the most prominent, we also performed fits allowing for a few terms proportional to $\log(am_{h0})$. These fits are discussed in the Appendix. Furthermore, the maximal number of terms N and M_i in Eqs. (8) and (9) should be sufficiently large so that higher order terms have negligible impact on the continuum extrapolation. The continuum extrapolations of R_4 based on the joint fits of $m_h = m_c - 4m_c$ lattice data turned out to be much more stable with respect to fit range variations than the extrapolations performed in Ref. [8] separately for each value of m_h . In particular, there was no problem incorporating $4m_c$ data in the analysis unlike in Ref. [8], where this was not possible. A sample fit of the data on R_4 is shown in Fig. 1. The joint fits capture the main feature of the data on R_4 , namely the steeper a^2 -dependence for $R_4(m_c)$ for small lattice spacing and the smaller effective slope of the lattice spacing dependence of the data at larger m_h , see Fig. 1. The latter observation is the consequence of the fact that many terms contribute to Eq. (8), and often with opposite signs.

To obtain the continuum result for R_4 we performed fits of all available data up to certain maximal value of am_{h0} , which we denote by $(am_{h0})_{max}$. An important consideration in choosing $(am_h)_{max}$ is the fact that in the free theory the expansion in am_{h0} converges for $am_{h0} < \pi/2$ [17]. To be on the safe side we choose $(am_{h0})_{max}^2 \leq 1.2$ in our analysis. The larger $(am_{h0})_{max}$ is, the more terms in Eq. (8) should be included. More terms also means more fit parameters, which, however, are difficult to constrain with limited number of data points. We find that using two powers of α_s^b , i.e. $N = 2$ is sufficient given our data for R_4 . Fits with $N > 2$ cannot constrain the parameters. We consider $(am_{h0})_{max}^2 = 0.4, 0.5, 0.6, 0.7, 0.8, 1.0$ and 1.2 and use fits with different M_i , adjusting it as $(am_{h0})_{max}$ increases. We do not find significant dependence on $(am_{h0})_{max}^2$ for the resulting continuum values of R_4 . Furthermore, for a given $(am_{h0})_{max}$ changing M_i does not lead to statistically significant differences. For our final continuum values for R_4 we use the results of fits with $(am_{h0})_{max}^2 = 0.7$, $M_1 = 8$ and $M_2 = 6$. The statistical errors of this fit are used as the final error estimates of the continuum result. The continuum results for R_4 for different quark masses are given in Table 1. For the lowest two quark masses, $m_h = m_c$ and $m_h = 1.5m_c$ the continuum results agree well with the ones obtained in Ref. [8], while for the larger quark masses our continuum results are significantly larger.

Fig. 1 The lattice spacing dependence of $R_4(m_h)$ for different values of the heavy-quark mass. The lines represent the fit with Eq. (8) with $N = 2$, $M_1 = 8$, $M_2 = 6$ and $(am_h)_{max}^2 = 0.7$, see text

As a cross-check we also use Akaike information criterion (AIC) [18, 19] to obtain continuum result for R_4 from the performed fits. First, we calculate the AIC weights for each fit and then calculate the weighted average of the fit results with the corresponding weights. Interestingly, this resulted in central values of R_4 very similar to those shown in Table 1. Furthermore, as mentioned above, we also performed the continuum extrapolations, which allow for a few terms proportional to $\log(am_{h0})$. The corresponding continuum results for R_4 are not significantly different from the ones in Table 1, see Appendix.

To obtain continuum results for R_n/m_{h0} , $n \geq 6$ it is sufficient to consider fits with $N = 1$ and $M_1 = 2$. This is because the errors on R_n/m_{h0} are much larger than for R_4 . These errors are dominated by the uncertainties in m_{h0} , which are essentially the uncertainties in m_{c0} multiplied by the corresponding constant (3/2, 2, 3 and 4). The uncertainties in m_{c0} come from the errors in tuning the charm-quark mass in the lattice calculations due to the errors of the ground state charmonium mass and the error in the lattice spacing [8]. The errors on m_{c0} are given in the sixth column of Table 1 in Ref. [8]. We used several values of $(am_{h0})_{max}$ in our fits. The differences in the central values of $(R_n/m_{h0})^{cont}$ corresponding to the fits with various $(am_{h0})_{max}$ turned out to be much smaller than the statistical errors. So one can choose any of these fit results. For the final continuum estimate we choose the fits with the smallest χ^2/df , which turned out to be the fits with $(am_{h0})_{max}^2 = 1.0$ for R_6/m_{h0} and R_8/m_{h0} , and the fit with $(am_{h0})_{max}^2 = 0.8$ for R_{10}/m_{h0} . The corresponding results are given in Table 1. The new continuum results agree very well with the previous ones but

Table 1 The continuum results for the reduced moments of quarkonium correlators at different heavy-quark masses. The last column shows the α_s values extracted from R_4 with $\mu = m_h$. The first, second, and third

m_h	R_4	R_6/m_{c0}	R_8/m_{c0}	R_{10}/m_{c0}	$\alpha_s(m_h)$
$1.0m_c$	1.2778(20)	1.0200(16)	0.9166(17)	0.8719(21)	0.3798(28)(31)(22)
$1.5m_c$	1.2303(30)	1.0792(20)	0.9860(20)	0.9462(23)	0.3151(43)(14)(4)
$2.0m_c$	1.2051(37)	1.1182(23)	1.0317(23)	0.9944(26)	0.2804(51)(9)(1)
$3.0m_c$	1.1782(44)	1.1729(27)	1.0923(26)	1.0574(31)	0.2434(61)(5)(0)
$4.0m_c$	1.1631(45)	1.2098(31)	1.1321(30)	1.0985(31)	0.2226(62)(4)(0)

have smaller errors. For some cases the error reduction is significant.

4 Determination of the strong coupling constant

Having determined the continuum limit of the reduced moments of quarkonium correlators we are in the position to obtain the strong coupling constant. As discussed in section II the value of $\alpha_s(\mu)$ can be obtained by comparing R_4 calculated in perturbation theory to the continuum extrapolated lattice result. The renormalization scale μ has to be of the order of the heavy-quark mass. We also need to fix the renormalization scale μ_m at which the heavy-quark mass is defined. The most natural choice is $\mu_m = \mu$ and will be used throughout in this paper. We consider the following choices of the renormalization scale, $\mu/m_h = 2/3, 1, 3/2, 2, 3$. With these choices of the renormalization scale we can obtain several determinations of $\alpha_s(\mu)$ which correspond to the lattice results at different values of m_h , providing important consistency checks, since the composition of the error budget changes with m_h . There are three sources of errors in $\alpha_s(\mu)$ determined from R_4 . One that comes from the error on the continuum value of R_4 given in Table 1. There is a perturbative error due to missing higher order corrections. We estimated this error in the same way as in Ref. [8], namely we added a term proportional to $r_{43}(\alpha_s/\pi)^4$ with the coefficient which was varied between -5 and $+5$, to be conservative. Finally, there is an error due to the gluon condensate contribution which was estimated in Ref. [8] by varying the poorly known gluon condensate by factor two.

For larger heavy-quark mass the perturbative error is smaller because the corresponding $\alpha_s(\mu)$ is smaller. The lattice error on the other hand is larger for larger m_h since R_4 is closer to one and the relative error on R_4 increases. The error due to the gluon condensate rapidly decreases with increasing m_h and is negligible for $m_h > 2m_c$. As an example we show the values of α_s for $\mu = m_h$ in Table 1, which clearly demonstrate these features. The $\alpha_s(\mu)$ values from the analysis for different choices of μ are shown in Fig. 2. In this figure the different sources of errors have been combined in

errors in α_s correspond to the lattice error, the perturbative error, and the error due to the gluon condensate, respectively, see text

Fig. 2 The strong coupling constant at different scales proportional to m_h . The different symbols correspond to the determinations at different values of μ/m_h

quadrature. The different symbols in the figure correspond to $\alpha_s(\mu)$ obtained from R_4 for different choices of μ/m_h and agree well within the estimated errors. This is an important consistency check for our procedure and serves as an indication that the errors have been properly estimated.

The above analysis gives values of α_s at multiple scales proportional to $m_h(\mu = nm_h)$, $n = 2/3, 1, 3/2, 2, 3$. But the value of $m_h(\mu = nm_h)$ was left undetermined. Using the above values of $\alpha_s(nm_h)$ and the continuum extrapolated lattice results on R_n/m_{h0} we can easily determine the running charm-quark mass $m_c(\mu = nm_h) = m_h(\mu = nm_h)/n$. There are several uncertainties in this determination. One is due to the error in the continuum extrapolated value of R_n/m_{h0} . The second source of the uncertainty is due to the missing higher order perturbative corrections in R_n . There is also an uncertainty due to the gluon condensate contribution. These have been estimated in the same manner as in the case of R_4 . Finally there is an uncertainty due to the error in α_s . The latter in turn is also affected by the error due to the gluon condensate contribution, which is correlated with the gluon condensate error in R_6 . This correlation should be taken into account. The perturbative error in R_4 and R_n , $n \geq 6$ can be

Fig. 3 The running charm-quark mass. The filled symbols correspond to the determination from R_6 , while the open symbols correspond to the determination from R_8 . The different symbols refer to different choices of μ for a given value of m_h

assumed to be uncorrelated. The statistical errors in R_4 and R_n , $n \geq 6$ are of course correlated. However, these errors are sub-dominant. The systematic effects due to the finite volume and mis-tuning of the charm-quark mass are quite different in R_4 and R_n , $n \geq 6$. Therefore the errors on the continuum values of R_4 and R_6/m_{h0} are treated as independent. Combining the errors along these lines we show the results on the running charm-quark mass in Fig. 3. The charm-quark mass values determined from R_6/m_{c0} and R_8/m_{c0} are shown as filled and open symbols in the figure, and agree well within errors. The charm-quark masses obtained from R_{10}/m_{c0} are not shown since they appear to be very close to the ones obtained from R_8/m_{c0} .

The values of m_c determined for different μ/m_h but same value of μ in GeV also agree with each other except for $\mu/m_h = 3$, which are about two sigma lower than the ones for $\mu/m_h = 1$, for $\mu < 5$ GeV. This fact may indicate that the above procedure of estimating the perturbative error due to missing higher order terms was not sufficiently conservative for $\mu < 5$ GeV. Since the dependence of α_s on the charm-quark mass is logarithmic the above small inconsistency in m_c determination is insignificant compared to other sources of errors. We also note that because of the uncertainty in the absolute scale we could not improve significantly the charm-quark mass determination compared to the result of Ref. [8] despite the reduced errors in the continuum values for R_n/m_{c0} , $n \geq 6$.

Having determined the charm-quark mass we are in a position to obtain the running coupling constant and the Λ -parameter for three-flavor QCD. Using the values of α_s shown in Fig. 2 and the charm-quark mass at various scales we determine $\Lambda_{\overline{MS}}^{n_f=3}$ using the RunDec package [20,21]. We utilize the 5-loop beta function in this study. We use the

explicit scheme defined by Eq. (4) of Ref. [20] to obtain $\Lambda_{\overline{MS}}^{n_f=3}$. We also use the implicit scheme defined by Eq. (5) of Ref. [20] and the difference between the results obtained in explicit and implicit schemes is considered as one of the errors in the determination of the Λ parameter. The other sources of errors include the perturbative errors in the determination of m_c and α_s for various μ , the lattice errors in these quantities as well as the errors due to the gluon condensate. Our results for $\Lambda_{\overline{MS}}^{n_f=3}$ are shown in Table 2. In this Table we show the error budget for the Λ -parameter, including the lattice error, the perturbative error, and the error due to the gluon condensate. The lattice error and the condensate error are obtained by propagating the corresponding errors in α_s and m_c taking into account the correlations. The perturbative error shown in Table 2 is obtained by propagating the perturbative error on α_s and m_c taking into account the correlations, and adding the error due to scheme dependence (explicit or implicit) in quadrature. From the table we see that the perturbative error in $\Lambda_{\overline{MS}}^{n_f=3}$ decreases with increasing m_h , following the trend observed in α_s , while the lattice error increases. Thus the determination of the Λ -parameter for different heavy-quark masses is complementary and provides an important consistency check. Furthermore, the perturbative error is the smallest for $\mu = m_h$ and increases when $\mu \neq m_h$. In Fig. 4 we show the three-flavor Λ -parameter as function of μ/m_h . The different symbols in the figure correspond to different values of m_h and the errors given in Table 2 were added in quadrature. Looking at the figure one sees that the values of the Λ -parameter obtained for different quark masses and different μ/m_h agree well within the estimated errors. On the other hand the values of $\Lambda_{\overline{MS}}^{n_f=3}$ tend to be larger for larger values of μ/m_h . In fact, for the choice $\mu = 3m_h$ the central value of the Λ parameter is very close to the HPQCD result [4]. Since the errors on $\Lambda_{\overline{MS}}^{n_f=3}$ determination vary significantly with μ and m_h it is not straightforward to quote a final number for the Λ -parameter. Given the above observations it makes sense to choose a central value for $\Lambda_{\overline{MS}}^{n_f=3}$ that corresponds to m_h and μ in the middle of the range used in this study. The continuum extrapolation for R_4 for $m_h = m_c$ and $m_h = 1.5m_c$ is quite robust since the results presented in this study and in Ref. [8], which uses a different strategy, agree with each other. The perturbative error for $\mu = 2m_h$ is also not too large. Therefore, the value of $\Lambda_{\overline{MS}}^{n_f=3} = 329(14)$ MeV obtained for $m_h = 1.5m_c$ and $\mu = 2m_h$ can be considered as a representative one. Alternatively we can average over different values to obtain the final estimate of the Λ -parameter. Since the errors of different $\Lambda_{\overline{MS}}^{n_f=3}$ determinations are correlated we use an unweighted average. This gives $\Lambda_{\overline{MS}}^{n_f=3} = 331.8$ MeV. Therefore, we take 332 MeV for the central value of the three-flavor Λ parameter. We also consider the weighted average of the result in Fig. 4 using the

Table 2 The $\Lambda_{\overline{MS}}^{n_f=3}$ obtained for different values of m_h and different choices of the renormalization scale. The first error comes from the lattice calculations, the second error is the perturbative error, and the last error is due to the gluon condensate

m_h/m_c	$\mu/m_h = 2/3$	$\mu/m_h = 1$	$\mu/m_h = 3/2$	$\mu/m_h = 2$	$\mu/m_h = 3$
1.0		323(4)(6)(3)	323(4)(7)(3)	327(4)(13)(3)	340(4)(21)(3)
1.5	314(8)(23)(1)	326(9)(4)(1)	326(8)(5)(1)	329(8)(10)(1)	341(9)(18)(1)
2.0		327(13)(3)(0)	327(13)(4)(0)	330(13)(9)(0)	341(14)(16)(0)
3.0	325(20)(20)(0)	332(21)(2)(0)	332(21)(4)(0)	335(22)(22)(0)	344(22)(14)(0)
4.0		336(26)(2)(0)	336(26)(3)(0)	339(27)(7)(0)	347(28)(17)(0)

Fig. 4 The three-flavor Lambda parameter determined for different μ/m_h . The different symbols correspond to different values of m_h and have been slightly shifted horizontally for better visibility. The horizontal line corresponds to the final estimate of $\Lambda_{\overline{MS}}^{n_f=3}$ and the band indicates its uncertainty

perturbative error and the error due to scheme choice added in quadrature, since these errors can be assumed to be uncorrelated for different m_h and μ . This results in $\Lambda_{\overline{MS}}^{n_f=3} = 331.4$ MeV, which is very close to the above value. We assign an error to $\Lambda_{\overline{MS}}^{n_f=3}$ such that all central values in Fig. 4 are covered by it. This results in 17 MeV as our error estimate for the Λ parameter. Thus, taking into account the uncertainty in the absolute scale (from r_1) our final result reads:

$$\Lambda_{\overline{MS}}^{n_f=3} = 332 \pm 17 \pm 2 \text{ (scale) MeV.} \quad (10)$$

Our error estimate for the $\Lambda_{\overline{MS}}^{n_f=3}$ parameter is quite conservative and is significantly larger than the one by HPQCD collaboration [4]. This is due to the absence of Bayesian priors and different choices of the renormalization scale μ and not just $\mu = 3m_h$. We compare our results for the Λ -parameter with other three-flavor lattice determinations, namely from the moments of quarkonium correlators [4], from the static quark anti-quark potential [22–24], from step-scaling analysis by ALPHA collaboration [25], from ghost-gluon vertex in Landau gauge [26], and from the light quark vector current

Fig. 5 Comparison of the Λ -parameter in three-flavor QCD from Refs. [4, 23–27] (from left to right) to the present determination. The result of Ref. [22] are not shown as these are superseded by Ref. [23]

correlator [27]. This comparison is shown Fig. 5. We see that our result is consistent with other lattice determinations.

For phenomenological applications it is important to know the strong coupling constant in the five-flavor theory at the scale of the Z-boson mass, M_Z . This can be obtained from the above result for $\Lambda_{\overline{MS}}^{n_f=3}$ by using the perturbative running of the coupling constant and decoupling at the charm and bottom thresholds. We use the RunDeC package [20, 21] to do this with the 5-loop beta function. There are several ways we can proceed using the RunDeC package. From the Λ parameter in Eq. (10) we can determine α_s at the decoupling scale μ_c , where we match it to the four-flavor coupling using the $\overline{MS}$ charm-quark mass $m_c(\mu_c)$ with the help of *DecAsUpMS* routine [20, 21]. A reasonable choice of scale μ_c should be between $m_c(m_c)$ and 2 GeV. Then we evolve the four-flavor coupling to scale μ_b where we perform the matching to the five-flavor theory using $\overline{MS}$ bottom quark mass $m_b(\mu_b)$ (using *DecAsUpMS* routine), and finally we evolve $\alpha_s^{n_f=5}(\mu_b)$ to $\mu = M_Z$. The scale μ_b should be around the bottom quark mass. The running charm-quark mass in $\overline{MS}$ scheme is obtained from R_8 as described above. In this study we use $\mu_b = m_b(m_b) = 4.188(10)$ obtained from averaging the available lattice results [2] and $\mu_c = m_c(\mu_c)$, $\mu_c = 1.5m_c(\mu_c)$ and $\mu_c = 2m_c(\mu_c)$. This

results in $\alpha_s^{n_f=5}(\mu = M_Z) = 0.117668, 0.117727$ and 0.117757 , respectively. The uncertainty in $m_b(m_b)$ leads to much smaller spread in the resulting $\alpha_s^{n_f=5}(\mu = M_Z)$ than above, and therefore, will be neglected. Similarly the effect of varying the scale μ_b also leads to smaller variations in $\alpha_s^{n_f=5}(\mu = M_Z)$. We can repeat the above procedure using the pole (on-shell) masses M_c and M_b of the charm and bottom quark instead of the $\overline{MS}$ masses and also set $\mu_c = M_c$ and $\mu_b = M_b$ with the help of the *DecAsUpOS* routine [20,21]. This gives $\alpha_s^{n_f=5}(\mu = M_Z) = 0.118046$ if the RunDeC default values $M_c = 1.5$ GeV and $M_b = 4.8$ GeV are used. Finally, we can match the three-flavor Λ parameter to the four-flavor one, and then the four-flavor Λ parameter to $\Lambda_{\overline{MS}}^{n_f=5}$ with the help of *DecLambdaUp* routine [20,21] and the values of $m_c(m_c)$ and $m_b(m_b)$ above. From this we obtain $\alpha_s^{n_f=5}(\mu = M_Z) = 0.117952$. Using the above considerations we choose 0.117727 as our central value for $\alpha_s^{n_f=5}(\mu = M_Z)$ and assign an uncertainty of 0.00032 for the running and decoupling procedure. Finally, the uncertainty in $\Lambda_{\overline{MS}}^{n_f=3}$ translates into an error of 0.00115 for $\alpha_s^{n_f=5}(\mu = M_Z)$. Combining this with the uncertainty of the running and the matching in quadrature we obtain

$$\alpha_s^{n_f=5}(\mu = M_Z) = 0.1177(12). \quad (11)$$

This value of $\alpha_s^{n_f=5}$ agrees with other determinations from the moments of quarkonium correlators within errors [3–5,7,28]. It also agrees with the averaged $\alpha_s^{n_f=5}$ from lattice determinations [1,2].

Before closing this section let us discuss the determination of the strong coupling constant from the ratios R_6/R_8 and R_8/R_{10} . As mentioned in section II the heavy-quark mass drops out in these ratios and therefore they are well suited for the determination of α_s . Naively, one would expect that the continuum extrapolation of these ratios is simpler than for R_4 as the higher order moments are less sensitive to the short distance physics. Using such reasonings in Ref. [7] the strong coupling constant was determined using only these ratios. The ratios R_6/R_8 and R_8/R_{10} have been also used in an attempt to determine α_s with additional cross-checks [8]. It turns out, however, that the cutoff dependence of R_6/R_8 and R_8/R_{10} is far from simple, and it is challenging to describe it quantitatively. Furthermore, the finite volume effects are also significant for these ratios. The continuum extrapolations for R_6/R_8 and R_8/R_{10} from simultaneous fits of the lattice data at different quark masses are discussed in the Appendix. We explain there why these continuum extrapolations are difficult. It turns out that in order to obtain consistent α_s determination from these ratios additional priors have to be imposed. The continuum results for R_6/R_8 at $m_h = m_c$ and $m_h = 1.5m_c$ are consistent with the previous results

[8]. However, it is not possible to obtain reliable continuum results for R_6/R_8 for $m_h > 2m_c$. In the case of R_8/R_{10} the finite volume effects are quite severe for $m_h = m_c$ and therefore, reliable continuum results can be obtained only for $m_h \geq 1.5m_c$. The continuum results for R_8/R_{10} turned out to be systematically larger than in Ref. [8]. As the result the corresponding α_s values are larger than the α_s values obtained from R_8/R_{10} in Ref. [8] and agree well with the corresponding ones obtained from R_4 . On the other side the strong coupling constant extracted from R_8/R_{10} has larger error and therefore, does not improve the precision of our α_s determination. Nevertheless, it does provide a useful cross-check of our analysis.

5 Conclusion

In this paper we revisited the determination of the strong coupling constant from the moments of quarkonium correlators. Using previously published lattice results on the reduced moments in (2+1)-flavor QCD with heavy-quark masses $m_h = m_c, 1.5m_c, 2m_c, 3m_c$ and $4m_c$ at several lattice spacings we estimated the continuum results on the fourth moment. These estimates were based on simultaneous fits of the lattice spacing dependence of the reduced moments at several quark masses, similar to the analysis of HPQCD collaboration [4,5]. The new continuum estimates turned out to be much more robust compared to the ones obtained from fits of the cutoff dependence of R_4 performed separately for each quark mass [8]. While both studies use the same form to parameterize the cutoff dependence of R_4 , there is an essential difference. The present analysis strongly relies on the specific form of the cutoff dependence given by Eqs. (8) and (12), while in Ref. [8] it is just an effective way to parameterize the lattice spacing dependence of these quantities, and is not essential for the final continuum result. In this study we constrain the lattice spacing dependence at each heavy-quark mass with the lattice spacing dependence of all other heavy-quark masses, while the previous analysis in Ref. [8] permitted independent variation of the coefficients at different heavy-quark masses. The continuum results at m_c and $1.5m_c$ are in good agreement in these approaches. This is reassuring for controlling the continuum extrapolation of these quantities at least for the two lower values of the quark masses. We also revisited the continuum extrapolations of R_6/R_8 and R_8/R_{10} using simultaneous fits of the lattice results at different quark masses. We have shown that the apparent weaker cutoff dependence of these ratios is misleading, and reliable continuum extrapolations are challenging. We were able to obtain reliable continuum extrapolations for R_6/R_8 only for $m_h \leq 2m_c$. For R_8/R_{10} reliable continuum results are available only for $m_h > m_c$ because of the finite volume effects.

Comparing the continuum extrapolated results for R_4 for $m_h = m_c - 4m_c$ and several values of the renormalization scale μ we determined the Λ parameter in three-flavor QCD and the running of the strong coupling constant in the range 1.3 GeV to 10 GeV, which is not accessible experimentally. The use of different renormalization scales provides an important consistency check of the estimated perturbative errors. Additional consistency check for α_s determination comes from the calculation of R_8/R_{10} .

The obtained value of $\Lambda_{\overline{MS}}^{n_f=3}$ agrees well with other recent lattice determinations. Performing the running of α_s to larger scales and decoupling at the charm and bottom thresholds we obtain $\alpha_s^{n_f=5}(\mu = M_Z) = 0.1177(12)$ that is in good agreement with FLAG as well as the PDG average. Finally, it is reassuring to see that previous inconsistencies in the determination of α_s from the moments of quarkonium correlators have been resolved.

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The raw data entering Figs. 1 and 6 are available through the arXiv eprint of Ref. [8]. The data entering Figs. 2–4 in the paper will be uploaded on the arXiv together with the eprint copy of the published version of the paper.]

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Appendix: Continuum extrapolation of the ratios and α_s determination

In this appendix we discuss continuum extrapolations for R_4 which allow for terms proportional to $\log(am_{h0})$. Furthermore, we discuss the continuum extrapolations of the ratios R_6/R_8 and R_8/R_{10} and the determination of α_s from these ratios.

As discussed in the main text including terms proportional to $\log(am_{h0})$ is challenging as the logarithmic dependence on am_{h0} is much weaker than the power-law dependence. Therefore, only a few logarithmic terms can be included in the fits to avoid over-fitting and the number of terms in Eq. (8) should be also reduced. We performed five different fits using different values of $(am_{h0})_{max}^2$ and non-zero values of d_{111} and d_{121} and all other d_{ijk} set to zero. We performed the following fits:

- fit 1: $(am_{h0})_{max}^2 = 0.4$, $M_1 = 3$, $M_2 = 2$, $d_{111} \neq 0$
- fit 2: $(am_{h0})_{max}^2 = 0.6$, $M_1 = 3$, $M_2 = 2$, $d_{111} \neq 0$
- fit 3: $(am_{h0})_{max}^2 = 0.8$, $M_1 = 4$, $M_2 = 2$, $d_{111} \neq 0$
- fit 4: $(am_{h0})_{max}^2 = 1.0$, $M_1 = 3$, $M_2 = 2$, $d_{111} \neq 0$, $d_{121} \neq 0$
- fit 5: $(am_{h0})_{max}^2 = 1.2$, $M_1 = 4$, $M_2 = 2$, $d_{111} \neq 0$, $d_{121} \neq 0$

These were the only possible fits that avoided over-fitting. The continuum extrapolated values of R_4 from fit 1–5 are given in Table 3. As one can see from the table the continuum R_4 values agree with the ones presented in the main text within errors but have smaller errors. Thus the inclusions of terms proportional to $\log(am_{h0})$ does not lead to significant changes in the continuum extrapolated value of R_4 . Since we want to have conservative error estimates for the continuum results on R_4 we use the values presented in the main text.

As discussed in the main text it is also possible to determine the strong coupling constant from the ratios R_6/R_8 and R_8/R_{10} as the heavy-quark mass drops out in these ratios. The apparent cutoff dependence of the ratios R_6/R_8 and R_8/R_{10} calculated on the lattice is indeed smaller than for R_4 [8]. As we have seen above, to describe the cutoff dependence of R_4 many powers of am_{h0} are needed and the coefficients often have opposite signs from one order in $(am_{h0})^2$ to the next one. Therefore, the apparent cutoff dependence of R_4 turns out to be smaller as we increase the heavy-quark mass contrary to the naive expectations. The situation could be similar for R_6/R_8 and R_8/R_{10} . Furthermore, the cutoff dependence of the numerator and denominator, while being significant, could cancel out in the ratios, thus fooling one into thinking that cutoff effects are small and can be modeled with a low order polynomial in $(am_{h0})^2$. We should keep these issues in mind when performing continuum extrapolations of the ratios.

To obtain the continuum result for R_6/R_8 we perform simultaneous fits of the lattice data at different quark masses to

$$\frac{R_n(m_h)}{R_{n+2}(m_h)} = \left(\frac{R_n(m_h)}{R_{n+2}(m_h)} \right)^{cont}$$

Table 3 The continuum extrapolated values of R_4 at different heavy quark masses, m_h from different fits which include terms proportional to $\log(am_{h0})$, see text

m_h/m_c	Fit 1	Fit 2	Fit 3	Fit 4	Fit 5
1.0	1.2805(14)	1.2782(10)	1.2817(18)	1.2810(13)	1.2806(10)
1.5	1.2308(14)	1.2286(10)	1.2318(16)	1.2309(12)	1.2302(10)
2.0	1.2045(14)	1.2024(09)	1.2053(15)	1.2044(11)	1.2038(10)
3.0	1.1763(15)	1.1743(10)	1.1770(15)	1.1759(11)	1.1754(10)
4.0	1.1608(14)	1.1582(10)	1.1610(15)	1.1601(11)	1.1595(10)

$$+ \sum_{i=1}^N \sum_{j=1}^{M_i} f_{ij}^{(n)} (\alpha_s^b)^i (am_{h0})^{2j}, \quad n \geq 6. \quad (12)$$

As in Ref. [8] we omit data on fine lattices to avoid finite volume effects when performing fits. The χ^2/df of the fit is large unless we use high order polynomials in $(am_{h0})^2$. However, using high order polynomials in the fit results in many poorly constrained parameters. Furthermore, a closer look at the lattice data reveals that the slope of the $(am_{h0})^2$ dependence is quite different for various m_h , explaining why χ^2/df is large. The apparent slope also decreases with increasing m_h , indicating that consecutive terms in Eq. (12) have opposite signs. To deal with these problems we omit lattice results for $m_h \geq 3m_c$. Including these data will require adding many more parameters in Eq. (12) which are difficult to constrain with the relatively few additional data points coming from the $m_h = 3m_c$ and $m_h = 4m_c$ data set. We also set the continuum value of R_6/R_8 at $2m_c$ to the one obtained in perturbation theory with $\alpha_s(2m_c)$ inferred from the continuum results on R_4 at $2m_c$. The continuum values of R_6/R_8 for $m_h = m_c$ and $1.5m_c$ are treated as fit parameters. Keeping terms up to $N = 2$ and $M_1 = M_2 = 3, 4$ is sufficient to obtain good fits in the interval $(0, (am_{h0})_{max}^2)$ with $(am_{h0})_{max}^2 = 0.4 - 0.6$. The fit for $(am_{h0})_{max}^2 = 0.6$ as well as the main features of the lattice data for R_6/R_8 are demonstrated in Fig. 6. No significant dependence of the continuum result on $(am_{h0})_{max}^2$ have been found. We choose results of fits with $(am_{h0})_{max}^2 = 0.6$ for the final continuum estimate, which are shown in Table 4. We also used AIC to obtain the continuum values and these were very close to the central value from the above fits. The new continuum estimate for R_6/R_8 agree with the results of Ref. [8] within errors.

From the continuum results on R_6/R_8 at m_c and $1.5m_c$ we determine the corresponding $\alpha_s(m_h)$ by comparing to the 4-loop perturbative results, which are also given in Table 4. The different sources of errors are estimated in the same way as for R_4 . The perturbative error turns out to be larger in this case. We see from the table that the α_s values agree with the ones obtained from R_4 at $\mu = m_c$ and $\mu = 1.5m_c$, c.f. Table 1. While we were not able to obtain continuum result for R_6/R_8 for $m_h = 2m_c$ we demonstrated that the lattice results on R_6/R_8 are compatible with the α_s values obtained

at smaller quark masses if the cutoff dependence is properly taken into account. Therefore, the previous inconsistencies in the determination of α_s from the lattice data at $2m_h$ [8] are now resolved.

Next we perform the continuum extrapolation for R_8/R_{10} . As for R_6/R_8 we fit the cutoff dependence of the lattice results at different quark masses with Eq. (12). The finite volume effects are the largest for R_{10} and thus for R_8/R_{10} , and it is possible that the finite volume errors in Ref. [8] were not adequate for many of the β values, especially in the case of $m_h = m_c$. This may explain why the α_s values obtained from R_8/R_{10} were systematically lower [8]. For the ratio R_8/R_{10} at $m_h = m_c$ perturbation theory would predict a value 1.0516 if $\alpha_s(m_c)$ obtained from R_4 is used. In the data, the ratio reaches the maximal value of 1.05043 at $\beta = 7.15$, and the central values monotonically decrease with increasing β , i.e. when approaching the continuum. Thus, there is a clear tension between the continuum value of R_8/R_{10} and the analysis of R_4 and R_6/R_8 at $m_h = m_c$ if the lattice results at large β are taken at a face value independently of the details of the continuum extrapolations. Furthermore, the central values of R_8/R_{10} also show non-monotonic behavior in β for $6.74 \leq \beta \leq 7.28$. We interpret this as indication that the finite volume errors are not under control for $\beta > 6.88$ and $m_h = m_c$. We note that the low central value of R_8/R_{10} is not unique to Ref. [8] but has been seen in other works [3, 6, 7] as well with the exception of Ref. [4]. The non-monotonic dependence of R_8/R_{10} with increasing β is also observed for $m_h = 1.5m_c$ and $2m_c$ but the maximum is shifted to significantly larger values of β . Finally, for $m_h = 3m_c$ and $4m_c$ this non-monotonic behavior cannot be clearly observed because of the large errors on the finest lattices. The above differences in the cutoff dependence of R_8/R_{10} at different quark masses make a simultaneous fit of the cutoff dependence very difficult. This difficulty is likely related to the finite volume effects. To solve this problem we discard data on R_8/R_{10} with small spatial extent. For $m_h = m_c$ the finite volume effects are under control for β up to $\beta = 6.88$, which corresponds to the bare charm-quark mass $am_{c0} = 0.48$ and spatial extent $N_s = 48$ (c.f. Table I in Ref. [8]). Therefore, we only include data with $L_s m_{h0} \geq 23$ in the fit. Since for $m_h = m_c$ there are only two data points satisfying this condition the con-

Fig. 6 The lattice results and the continuum extrapolations (fits) for R_6/R_8 (left) and R_8/R_{10} (right), see text

Table 4 The continuum values of the ratios R_6/R_8 and R_8/R_{10} and the corresponding coupling constants $\alpha_s(\mu = m_h)$ for different values of the heavy-quark masses m_h , see text

R_6/R_8 m_h/m_c	Continuum	$\alpha_s(m_h)$	R_8/R_{10} m_h/m_c	Continuum	$\alpha_s(m_h)$
1.0	1.10895(32)	0.3826(14)(178)(39)	1.0	—	—
1.5	1.09100(25)	0.3137(10)(76)(8)	1.5	1.04310(45)	0.3166(34)(82)(17)
2.0	—	—	2.0	1.03830(68)	0.2808(51)(50)(4)
3.0	—	—	3.0	1.03249(94)	0.2382(69)(24)(1)
4.0	—	—	4.0	1.02987(106)	0.2191(293)(17)(0)

tinuum value of $R_8(m_c)/R_{10}(m_c)$ was fixed to the perturbative result above (1.0516). With this cut and constraint the joint fits have good χ^2/df and the fit results are robust with respect to variation of the upper limit of the fit range, $(am_{h0})_{max}^2$, which was varied from 0.5 to 1.1. The number of terms in the fit Ansatz had to be adjusted accordingly. For $(am_{h0})_{max}^2 = 0.5$ we used $N = 2$, $M_1 = 3$ and $M_2 = 2$, while for $(am_{h0})_{max}^2 = 1.1$ we used $N = 2$, $M_1 = 5$ and $M_2 = 4$. For the final continuum estimate we use the fit with $(am_{h0})_{max}^2 = 0.8$, $N = 2$, $M_1 = 4$ and $M_2 = 3$. This fit is shown in Fig. 6 together with the lattice data on R_8/R_{10} . The corresponding continuum results for R_8/R_{10} are shown in Table 4. We also applied the AIC to different fit results and the resulting continuum estimates turned out to be close to the central value of the fit with $(am_{h0})_{max}^2 = 0.8$. From the continuum values for R_8/R_{10} in Table 4 we determine $\alpha_s(m_h)$ by comparing to the perturbative result for $\mu = m_h$. These values are in good agreement with the $\alpha_s(m_h)$ values obtained from R_4 within errors, see Tables 1 and 4. Thus, we have an additional cross-check for α_s determination at scales $\mu = 1.5m_c - 4m_c$.

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