



On the Schwarzschild-Anti-de Sitter black hole with an $f(R)$ global monopole

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Abstract In this work, we follow the recently revisited $f(R)$ theory of gravity for studying the interaction between quantum scalar particles and the gravitational field of a generalized black hole with an $f(R)$ global monopole. This background has a term playing the role of an effective cosmological constant, which permits us to call it as Schwarzschild-Anti-de Sitter (SAdS) black hole with an $f(R)$ global monopole. We examine the separability of the Klein–Gordon equation with a non-minimal coupling and then we discuss both the massless and massive cases for a conformal coupling. We investigate some physical phenomena related to the asymptotic behavior of the radial function, namely, the black hole radiation, the quasibound states, and the wave eigenfunctions.

1 Introducing the Schwarzschild-Anti-de Sitter black hole with an $f(R)$ global monopole

In a recent paper, Caramês et al. [1] obtained a class of exact solutions for the modified field equations in the presence of a global monopole for regions outside its core, which generalize some previous results in the $f(R)$ theory of gravity. In this section, we will give a brief review about their results and then setup the background in which we want to investigate the behavior of quantum scalar fields following the Vieira–Bezerra–Kokkotas method [2,3].

This paper has three broad goals. First, to obtain a metric which describes the Schwarzschild-Anti-de Sitter black hole with an $f(R)$ global monopole. Second, to discuss the separability of the Klein–Gordon equation in the background under consideration. Third, to compute the Hawking radiation, the quasibound states, and the wave eigenfunctions.

The action of the $f(R)$ theory of gravity is given by

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [f(R) + \mathcal{L}], \quad (1)$$

where g is the determinant of a 4×4 matrix constructed from the metric tensor, \mathcal{L} is the Lagrangian density, and $\kappa = 8\pi$. Note that we are adopting the natural units, namely, $G \equiv c \equiv \hbar \equiv 1$. Now, this action is extremized with respect to the metric tensor, which leads to the following field equations

$$\kappa T_{\mu\nu} = F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu [F(R)] + \square[F(R)]g_{\mu\nu}, \quad (2)$$

where $F(R) = df(R)/dR$. On the other hand, the global monopole spacetime model is described by the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{4}\lambda(\phi^a \phi^a - \eta^2)^2, \quad (3)$$

where λ is a positive coupling constant, η is the energy scale at which the symmetry is broken, and the Higgs field ϕ^a is given by an isotriplet of scalar fields, whose form corresponds to the well-known hedgehog *Ansatz*. Then, a spherically symmetric line element, which describes the spacetime around a static source, can be written, in general, as

$$ds^2 = -B(r) dt^2 + A(r) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (4)$$

where $A(r)$ and $B(r)$ are functions to be determined; they are related by $Y(r) \equiv A(r)B(r)$. In this model, the energy-momentum tensor has a very simple form. It is given by

$$T_\mu^\nu \approx \text{diag}\left(\frac{\eta^2}{r^2}, \frac{\eta^2}{r^2}, 0, 0\right). \quad (5)$$

Now, an alternative parametrization for $F(R)$ is adopted, namely, $F(R(r)) = \mathcal{F}(r) = 1 + \psi(r)$, where $\psi(r) = \psi_0 r$ is

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a function encoding the deviation from the Einstein’s general relativity. From this parametrization, we get

$$Y(r) = A(r)B(r) = Y_0, \tag{6}$$

where Y_0 is a constant. Thus, after some algebra, it can be found the following expression for the metric coefficient $B(r)$:

$$B(r) = Y_0(1 - 8\pi\eta^2) + \left(\frac{\psi_0}{2} - \frac{1}{3r}\right)c_1 - r\psi_0[Y_0(1 - 16\pi\eta^2) + \psi_0c_1] + \frac{r^2}{2} \left\{ \psi_0^2 Y_0(3 - 32\pi\eta^2) + 2c_2 + 2\psi_0^2[Y_0(1 - 16\pi\eta^2) + \psi_0c_1] \ln\left(\psi_0 + \frac{1}{r}\right) \right\}, \tag{7}$$

where c_1 and c_2 are constants (of integration) to be determined (or to be opportunely chosen). In fact, it is worth noticing that this solution is more general than that ones found in the literature, since it carries corrections that are absent in all other approaches (including the ones where the approximations $|\psi_0 r| \ll 1$ and the weak field limit were taken into account). Thus, Caramès *et al.* found a set of black hole solutions, which are displayed in Table 1 of Ref. [1].

Now, we will take some useful approximations into account in order to establish a new particular black hole background. This will be possible due to the suitable choice of the constants c_1 and c_2 , as follows. First of all, we set $Y_0 = 1$, which implies that $A(r) = [B(r)]^{-1}$. Then, by assuming a small correction on the Einstein’s general relativity, we can keep just the linear terms in $\psi_0 r$ by considering the constant ψ_0 very tiny, which means that $\psi_0^2 \sim 0$. Furthermore, we can throw away all the crossing terms involving ψ_0 and η^2 . Thus, we can choice $c_1 = 6M$ and $c_2 = \tilde{\Lambda}/3$, which means that they are associated to the Newtonian potential and to the effects of an effective cosmological constant, respectively (for details, see Ref. [4] and references therein). Finally, we get

$$B(r) = 1 - 8\pi\eta^2 + 3M\psi_0 - \frac{2M}{r} - \psi_0 r + \frac{\tilde{\Lambda}}{3}r^2, \tag{8}$$

where M represents the total mass centered at the origin of the system of coordinates, $\tilde{\Lambda}$ is an effective cosmological constant playing the same role as the standard cosmological constant ($3/\ell^2$) in the dynamics of the universe. Therefore, we have obtained a metric corresponding to the SAdS black hole with an f(R) global monopole. From now on, due to the choice of approximations described above, we will use the following values for the involved parameters: $\psi_0 = 0.02$, $8\pi\eta^2 \sim \eta^2 = 10^{-6}$, $\tilde{\Lambda} = 0.12$ ($\ell_{\text{ISCO}} = 5$), and $M = 1$. In fact, these are the values expected within the Grand Unified

Theories (GUT) for the potential appearance of topological defects in the early universe.

For the sake of simplicity, let us rewrite Eq. (8) as

$$B(r) = B_0 + \frac{B_1}{r} + B_2r + B_3r^2. \tag{9}$$

Thus, the event horizons are the solutions of the surface equation given by

$$B(r) = \frac{B_3}{r}(r^3 + a_2r^2 + a_1r + a_0) = 0, \tag{10}$$

where

$$a_2 = \frac{B_2}{B_3}, \tag{11}$$

$$a_1 = \frac{B_0}{B_3}, \tag{12}$$

$$a_0 = \frac{B_1}{B_3}. \tag{13}$$

Its solutions are given by [5]

$$r_1 = -\frac{1}{3}a_2 + \frac{1}{3}(I + J), \tag{14}$$

$$r_2 = -\frac{1}{3}a_2 + \frac{1}{3}(\rho I + \rho^2 J), \tag{15}$$

$$r_3 = -\frac{1}{3}a_2 + \frac{1}{3}(\rho J + \rho^2 I), \tag{16}$$

where

$$I = \sqrt[3]{-\frac{27}{2}v + \frac{3}{2}\sqrt{-3d}}, \tag{17}$$

$$J = -\frac{3y}{I}, \tag{18}$$

$$\rho = e^{2\pi i/3}, \tag{19}$$

$$\rho^2 = e^{-2\pi i/3}, \tag{20}$$

with

$$v = \frac{1}{27}(2a_2^3 - 9a_2a_1 + 27a_0), \tag{21}$$

$$d = -4y^3 - 27v^2, \tag{22}$$

$$y = \frac{1}{3}(3a_1 - a_2^2). \tag{23}$$

Therefore, we can rewrite the function $B(r)$ as

$$B(r) = \frac{B_3}{r}(r - r_1)(r - r_2)(r - r_3). \tag{24}$$

In this representation, the only positive real root is r_1 , which corresponds to the exterior event horizon. The behavior of the exterior event horizon r_1 is shown in Fig. 1. In the limit when $\tilde{\Lambda} \rightarrow 0$, the complex roots r_2 and r_3 go to infinity and hence they decouple from the general solution.

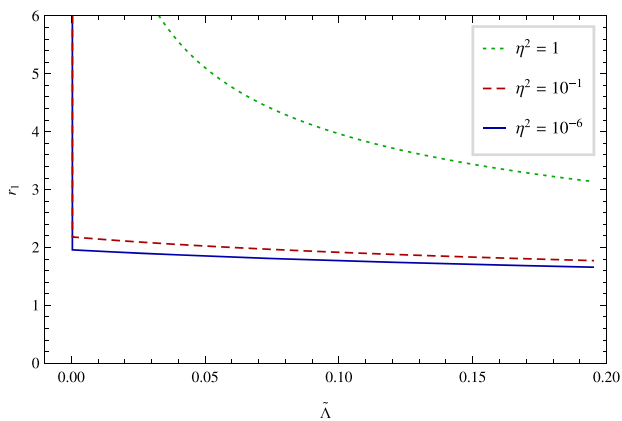


Fig. 1 The exterior event horizon r_1 as a function of $\tilde{\Lambda}$

For the chosen values, the exterior event horizon is at $r_1 = 1.74401$. The (unphysical) apparent event horizons are located at $r_2 = -0.62201 + 5.31814i$ and $r_3 = -0.62201 - 5.31814i$. It is worth noticing that the parameter B_3 must be non-zero, i.e., $\tilde{\Lambda} \neq 0$, which means that the term playing the role of an effective cosmological constant necessarily contributes to the energy density of such a spacetime.

In the next section, we will study the behavior of quantum scalar particles propagating outside the exterior event horizon of the SAdS black hole with an $f(R)$ global monopole.

The outline of this paper is the following. In Sect. 2, we discuss the separability of the Klein–Gordon equation and then solve it in terms of the Heun functions. In Sect. 3, we examine the Hawking radiation of scalar particles. In Sect. 4, we investigate the quasibound states by obtaining the spectrum of resonant frequencies. In Sect. 5, we analyze the radial wave eigenfunctions. Finally, in Sect. 6, we present our concluding remarks. In Appendix A, we present the general Heun functions, as well as its deformed case.

2 Klein–Gordon equation

In order to discuss the motion of quantum scalar particles propagating in a curved spacetime, we will consider the covariant Klein–Gordon equation with a non-minimal coupling, which is given by

$$\left\{ \frac{1}{\sqrt{-g}} \partial_\sigma (g^{\sigma\tau} \sqrt{-g} \partial_\tau) - (\mu^2 + \xi R) \right\} \Psi(\mathbf{r}) = 0, \quad (25)$$

where μ is the mass of the scalar particle, and ξ is the coupling constant. In the background under consideration, the Ricci curvature scalar R is given by

$$R = -\frac{1}{r^2} \left[r^2 \frac{d^2 B(r)}{dr^2} + 4r \frac{dB(r)}{dr} + 2B(r) - 2 \right]$$

$$= -12B_3 - \frac{2(B_0 - 1)}{r^2} - \frac{6B_2}{r}. \quad (26)$$

Thus, by substituting the metric (4) into the Klein–Gordon equation (25), we get

$$\left\{ -\frac{r^2}{B(r)} \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial r} \left[r^2 B(r) \frac{\partial}{\partial r} \right] - (\mu^2 + \xi R) r^2 + \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \frac{\partial^2}{\partial \phi^2} \right\} \Psi(t, r, \theta, \phi) = 0. \quad (27)$$

Now, we need to choose a suitable separation for the dependent variables of the scalar wave function $\Psi(t, r, \theta, \phi)$. Due to the spherical symmetry, we will write the scalar wave function as

$$\Psi(t, r, \theta, \phi) = e^{-i\omega t} u(r) Y_{lm}(\theta, \phi), \quad (28)$$

where ω is the frequency (energy) of the scalar particle, $Y_{lm}(\theta, \phi)$ is the spherical harmonic function, and $u(r) = U(r)/r$ is the radial function. Thus, Eq. (27) is separated in two parts, namely,

$$\frac{1}{\sin^2 \theta} \frac{\partial^2 Y_{lm}(\theta, \phi)}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial Y_{lm}(\theta, \phi)}{\partial \theta} \right] = 0 \quad (29)$$

and

$$\begin{aligned} & \frac{d^2 U(r)}{dr^2} + \frac{1}{B(r)} \frac{dB(r)}{dr} \frac{dU(r)}{dr} \\ & + \left\{ \frac{\omega^2}{[B(r)]^2} - \frac{1}{r^2 B(r)} \left[\lambda_{lm} + (\mu^2 + \xi R) r^2 + r \frac{dB(r)}{dr} \right] \right\} U(r) = 0, \end{aligned} \quad (30)$$

where $\lambda_{lm} = l(l+1)$ is a separation constant, with l being the azimuthal quantum number. In what follows, we will discuss and solve the radial part.

2.1 Effective potential

At this point, we would like to analyze the behavior of the effective potential, $V_{eff}(r)$. The radial equation given by Eq. (30) can be written as

$$\frac{d^2 U(r)}{dr_*^2} + [\omega^2 - V_{eff}(r)] U(r) = 0, \quad (31)$$

where

$$V_{eff}(r) = B(r) \left[\frac{\lambda_{lm}}{r^2} + \mu^2 + \xi R + \frac{1}{r} \frac{dB(r)}{dr} \right]. \quad (32)$$

As we can see, Eq. (31) looks like an one-dimensional Schrödinger equation, where we have introduced the *tortoise coordinate* r_* defined by $dr_* = dr/B(r)$. The behavior of

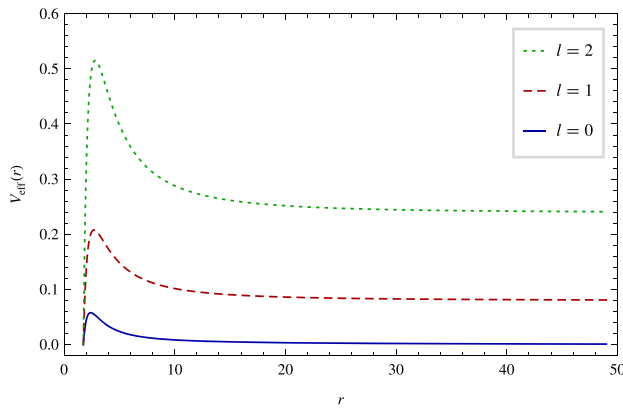


Fig. 2 The effective potentials for $\mu = 0$, $\xi = 1/6$, and different values of the azimuthal quantum number $l(= 0, 1, 2)$

the effective potential $V_{eff}(r)$ is shown in Fig. 2, for some values of the azimuthal quantum number.

From Eq. (32) and Fig. 2, we see that the effective potential approaches to zero only in the case when $l = 0$; it crosses the r -axis at $r = 100.002$. For $l > 0$, the effective potential approaches to a (finite) constant as r goes to infinity; it tends to $r = 0.079$ for $l = 1$, and to $r = 0.239$ for $l = 2$.

2.2 Radial equation

Now, let us solve the radial part of the Klein–Gordon equation. To do this, we substitute the function $B(r)$, given by Eq. (24), into the radial equation, given by Eq. (30), in order to get

$$\begin{aligned} & \frac{d^2U(r)}{dr^2} + \left(-\frac{1}{r} + \frac{1}{r-r_1} + \frac{1}{r-r_2} + \frac{1}{r-r_3} \right) \frac{dU(r)}{dr} \\ & + \left\{ \frac{1}{3r^2B_3(r-r_1)(r-r_2)(r-r_3)} [r(B_0 - 3\lambda_{lm} - 1) \right. \\ & + 3r^2[B_2 + B_3(r_1 + r_2 + r_3)] - 3r^3\mu^2 - 3B_3r_1r_2r_3] \\ & \left. + \frac{r^2\omega^2}{B_3^2(r-r_1)^2(r-r_2)^2(r-r_3)^2} \right\} U(r) = 0, \end{aligned} \tag{33}$$

where we have chosen the conformal coupling ($\xi = 1/6$), and used the Ricci curvature scalar given by Eq. (26).

Equation (33) seems to be a Fuchsian second-order equation with four finite regular singularities at the points $r = (0, r_j)$, with $j = 1, 2, 3$. Thus, it can be transformed into a kind of Heun equation. To do this, we have to define a new radial coordinate, z , by using the following homographic substitution

$$z = \frac{r - r_1}{r} \zeta, \tag{34}$$

where

$$\zeta = \frac{r_2}{r_2 - r_1}. \tag{35}$$

This transformation brings the singularities (r_1, r_2, r_3) to the points $(0, 1, b)$, where the singularity parameter b is given by

$$b = \frac{r_3 - r_1}{r_3} \zeta. \tag{36}$$

In addition, this transformation gives two important limits: when $r \rightarrow r_1$ implies that $z \rightarrow 0$, and when $r \rightarrow \infty$ implies that $z \rightarrow 1$. It means that we will obtain a solution which is analytical in the range $0 \leq z \leq 1$, that is, valid for $r_1 \leq r \leq \infty$. Therefore, it totally agrees with the fact that we are interested on the motion of scalar particles propagating outside the exterior event horizon.

Thus, by substituting Eq. (34) into Eq. (33), we obtain

$$\begin{aligned} & \frac{d^2U(z)}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-b} \right) \frac{dU(z)}{dz} \\ & + \left[\sum_{j=0}^3 \frac{L_j}{z-z_j} + \sum_{j=0}^2 \frac{Q_j}{(z-z_j)^2} \right. \\ & \left. - \frac{\mu^2}{B_3(z-\zeta)^2} \right] U(z) = 0, \end{aligned} \tag{37}$$

where $j = 0, 1, 2, 3$ labels the singularities $z = 0, 1, b, \zeta$. The parameter ζ is called an apparent singularity, since it can be removed (or have its power reduced) by performing some suitable transformations. Thus, Eq. (37) is almost a Heun-type, where we just need to remove (or reduce the power) that apparent singularity. It is easy to see that there are two ways to do this: the simplest one is set $\mu = 0$, and the other is to choose $\mu^2 \propto B_3$ together with a specific transformation of the dependent variable $U(z)$. In fact, this was first noted by Kraniotis [6], when he studied the massive Dirac equation in Kerr–Newman black hole spacetimes. Therefore, in what follows, we will solve the covariant Klein–Gordon equation with a conformal coupling in the SAdS black hole with an $f(R)$ global monopole for both massless and massive scalar particles.

2.3 Case 1: massless scalar particles

For massless scalar particles ($\mu = 0$), we define a new dependent variable U by performing the following F -homotopic transformation

$$U(z) = z^{A_1}(z-1)^{A_2}(z-b)^{A_3} Z(z), \tag{38}$$

where the exponents A_j obey to the following indicial equation

$$F(s) = s(s - 1) + s + Q_j = s^2 + Q_j = 0, \tag{39}$$

whose roots are given by

$$s_{1,2}^{z=0} = \pm i \frac{(\zeta - 1)(\zeta - b)\omega}{B_3 r_1 b} \equiv A_1, \tag{40}$$

$$s_{1,2}^{z=1} = \pm i \frac{(\zeta - 1)(\zeta - b)\omega}{B_3 r_1 (b - 1)} \equiv A_2, \tag{41}$$

$$s_{1,2}^{z=b} = \pm i \frac{(\zeta - 1)(\zeta - b)\omega}{B_3 r_1 b (b - 1)} \equiv A_3. \tag{42}$$

Thus, by substituting Eqs. (38)–(42) into Eq. (37), we get

$$\frac{d^2 Z(z)}{dz^2} + \left(\frac{1 + 2A_1}{z} + \frac{1 + 2A_2}{z - 1} + \frac{1 + 2A_3}{z - b} \right) \frac{dZ(z)}{dz} + \frac{A_4 z - A_5}{z(z - 1)(z - b)} Z(z) = 0, \tag{43}$$

where the coefficients A_4 , and A_5 are given by

$$A_4 = 2(A_1 + A_2 + A_3 + A_1 A_2 + A_1 A_3 + A_2 A_3) - \frac{B_2(b + 1 - \zeta)}{B_3 \zeta r_1} - \frac{(b + 1 - 2\zeta)[(2\zeta - 1)b - 2(\zeta - 1)\zeta]}{(\zeta - 1)\zeta(b - \zeta)} - \frac{2(\zeta - 1)^2 \omega^2 [(b - 1)b + 1](b - \zeta)^2}{B_3^2 r_1^2 (b - 1)^2 b^2}, \tag{44}$$

$$A_5 = A_1 + A_3 + A_1 A_3 + (A_1 + A_2 + 2A_1 A_2)b - 2(\zeta - 1) + \left(\frac{1}{\zeta} - 2 \right) b + \frac{(\zeta - 1)(b - \zeta)(B_0 + 3B_2 r_1 - 3\lambda_{lm} - 1)}{3B_3 \zeta r_1^2} - \frac{2(\zeta - 1)^2 \omega^2 (b + 1)(b - \zeta)^2}{B_3^2 r_1^2 b^2}. \tag{45}$$

The massless radial equation, given by Eq. (43), is similar to the general Heun equation (see Eq. (A.1) in Appendix A). Therefore, its analytical solution is given by

$$U(z) = z^{A_1} (z - 1)^{A_2} (z - b)^{A_3} \times \{ C_1 \text{HeunG}(b, q; \alpha, \beta, \gamma, \delta; z) + C_2 z^{1-\gamma} \text{HeunG}(b, q_2; \alpha_2, \beta_2, \gamma_2, \delta; z) \}, \tag{46}$$

where C_1 and C_2 are constants (to be determined). The parameters $\alpha, \beta, \gamma, \delta, \epsilon$, and q are given by

$$\alpha = 1 + A_1 + A_2 + A_3, \tag{47}$$

$$\beta = 1 + A_1 + A_2 + A_3, \tag{48}$$

$$\gamma = 1 + 2A_1, \tag{49}$$

$$\delta = 1 + 2A_2, \tag{50}$$

$$\epsilon = 1 + 2A_3, \tag{51}$$

$$q = A_5. \tag{52}$$

Furthermore, the auxiliary parameters $\alpha_2, \beta_2, \gamma_2$, and q_2 are given by

$$\alpha_2 = \alpha + 1 - \gamma, \tag{53}$$

$$\beta_2 = \beta + 1 - \gamma, \tag{54}$$

$$\gamma_2 = 2 - \gamma. \tag{55}$$

$$q_2 = q + (\alpha\delta + \epsilon)(1 - \gamma). \tag{56}$$

These are two linearly independent solutions of the general Heun equation since γ is not a positive integer, and they correspond to the exponents 0 and $1 - \gamma$ at $z = 0$. It is worth emphasizing that the final expressions for these parameters depend on the signs to be chosen for the exponents A_j , which are given by Eqs. (40)–(42).

2.4 Case 2: massive scalar particles

For massive scalar particles ($\mu = \sqrt{3B_3}/2$), the dependent variable U is now transformed as

$$U(z) = z^{A_1} (z - 1)^{A_2} (z - b)^{A_3} \frac{Z(z)}{(z - \zeta)^{\frac{1}{2}}}, \tag{57}$$

where the coefficients A_1, A_2 , and A_3 are the same as for the Case 1, that is, they are given by Eqs. (40)–(42). Thus, by substituting Eq. (57) into Eq. (37), we get

$$\frac{d^2 Z(z)}{dz^2} + \left(\frac{1 + 2A_1}{z} + \frac{1 + 2A_2}{z - 1} + \frac{1 + 2A_3}{z - b} - \frac{1}{z - \zeta} \right) \frac{dZ(z)}{dz} + \left[\frac{-D_1 - D_2 - D_3 + D_3 \zeta + D_2 b}{(z - 1)(z - b)} + \frac{(D_1 + D_3 - D_3 \zeta)b}{z(z - 1)(z - b)} + \frac{D_3(\zeta - 1)\zeta}{z(z - 1)(z - \zeta)} \right] Z(z) = 0, \tag{58}$$

where the coefficients D_1, D_2 , and D_3 are given by

$$D_1 = - \frac{2A_1 A_3 + A_1 + A_3}{b} - \frac{4A_1 A_2 \zeta + 2A_1 \zeta - 2A_1 + 2A_2 \zeta - 2\zeta L_1 - 1}{2\zeta}, \tag{59}$$

$$D_2 = L_3 + \frac{2A_2 A_3 + A_2 + A_3}{b - 1} + \frac{2A_1 A_3 + A_1 + A_3}{b} - \frac{2A_3 + 1}{2(b - \zeta)}, \tag{60}$$

$$D_3 = \frac{2A_3 + 1}{2(b - \zeta)} + \frac{-2A_1 \zeta + 2A_1 - 2A_2 \zeta - 2\zeta + 2\zeta^2 L_4}{2(\zeta - 1)\zeta} + \frac{1 - 2\zeta L_4}{2(\zeta - 1)\zeta}, \tag{61}$$

with

$$L_1 = \frac{5(\zeta - 1)\zeta - 5\zeta b + b}{4\zeta b} - \frac{(\zeta - 1)(b - \zeta)(B_0 + 3B_2r_1 - 3\lambda_{lm} - 1)}{3B_3\zeta r_1^2 b} + \frac{2(\zeta - 1)^2\omega^2(b + 1)(b - \zeta)^2}{B_3^2 r_1^2 b^3}, \tag{62}$$

$$L_2 = \frac{(5\zeta - 4)b + 4 - 5\zeta^2}{(4\zeta - 4)b + 4 - 4\zeta} + \frac{(b - \zeta)[B_0(\zeta - 1) + 3\zeta B_2r_1]}{3B_3\zeta r_1^2(b - 1)} + \frac{(1 - \zeta)(3\lambda_{lm} + 1)]}{3B_3\zeta r_1^2(b - 1)} - \frac{2(\zeta - 1)^2\omega^2(b - 2)(b - \zeta)^2}{B_3^2 r_1^2(b - 1)^3}, \tag{63}$$

$$L_3 = \frac{4(b - 1)b - 5(\zeta - 1)\zeta}{4(b - 1)b(b - \zeta)} + \frac{(\zeta - 1)\zeta(B_0 + 3B_2r_1 - 3\lambda_{lm} - 1)}{3B_3\zeta r_1^2(b - 1)b} + \frac{(\zeta - 1)b(3\lambda_{lm} + 1 - B_0)}{3B_3\zeta r_1^2(b - 1)b} - \frac{2(\zeta - 1)^2\omega^2(2b - 1)(b - \zeta)^2}{B_3^2 r_1^2(b - 1)^3 b^3}, \tag{64}$$

$$L_4 = \frac{1}{4} \left(\frac{1}{1 - \zeta} + \frac{1}{b - \zeta} - \frac{4B_2}{B_3 r_1 \zeta} - \frac{1}{\zeta} \right). \tag{65}$$

The massive radial equation, given by Eq. (58), is similar to the deformed Heun equation (see Eq. (A.26) in Appendix A), where ζ plays the role of an apparent singularity. Therefore, its analytical solution is given by

$$U(z) = z^{A_1}(z - 1)^{A_2}(z - b)^{A_3}(z - \zeta)^{-\frac{1}{2}} \times \{C_1 \text{HeunG}(b, q; \alpha, \beta, \gamma, \delta; z) + C_2 z^{1-\gamma} \text{HeunG}(b, q_2; \alpha_2, \beta_2, \gamma_2, \delta; z)\}, \tag{66}$$

where C_1 and C_2 are constants (to be determined). In this case, the parameters $\alpha, \beta, \gamma, \delta, \epsilon,$ and q are now given by

$$\alpha = 1 + A_1 + A_2 + A_3 - \frac{1}{2} \left[2(2A_1^2 + 2A_1 + 2A_2^2 + 2A_2 + 2A_3^2 + 2A_3 + 2L_1 - 2L_3b + 2L_3 - 2\zeta L_4 + 2L_4 + 5) \right]^{\frac{1}{2}}, \tag{67}$$

$$\beta = 1 + A_1 + A_2 + A_3 - \frac{1}{2} \left[2(2A_1^2 + 2A_1 + 2A_2^2 + 2A_2 + 2A_3^2 + 2A_3 + 2L_1 - 2L_3b + 2L_3 - 2\zeta L_4 + 2L_4 + 5) \right]^{\frac{1}{2}}, \tag{68}$$

$$\gamma = 2 + 2A_1, \tag{69}$$

$$\delta = 2 + 2A_2, \tag{70}$$

$$\epsilon = 1 + 2A_3, \tag{71}$$

$$q = -(D_1 + D_3 - D_3\zeta)b, \tag{72}$$

It is worth emphasizing that we have already added the unitary shifting to the parameters γ and δ , as described in Appendix A, as well as that the final expressions for these parameters also depend on the signs to be chosen for the exponents A_j , which are given by Eqs. (40)–(42). The auxiliary parameters $\alpha_2, \beta_2, \gamma_2,$ and q_2 are given by the same relations as for the Case 1, that is, they are given by Eqs. (53)–(56).

Next, we will use these analytical solutions of the radial equation, in the SAdS black hole with an f(R) global monopole, and some properties of the general Heun functions to discuss some interesting physical phenomena, namely, the Hawking radiation, the spectrum of quasibound state frequencies and its corresponding wave eigenfunctions.

3 Hawking radiation

In order to discuss the Hawking radiation, we will obtain the wave solutions describing quantum scalar particles near the exterior event horizon of a SAdS black hole with an f(R) global monopole. To do this, first we need to choose the signs of the exponents A_j given by Eqs. (40)–(42); the negative sign is the correct choice, which will be proved in the discussion of the quasibound states.

In the limit when $r \rightarrow r_1$, which implies that $z \rightarrow 0$, we can evaluate the corresponding Heun functions from the expansion given by Eq. (A.2), and hence we get $\text{HeunG}(b, q; \alpha, \beta, \gamma, \delta; 0) \sim 1$. Thus, the radial solutions for the Cases 1 and 2, which are given by Eqs. (46) and (66), respectively, have the (same) asymptotic behavior at the exterior event horizon given by

$$u(r) \sim C_1 (r - r_1)^{A_1} + C_2 (r - r_1)^{-A_1}, \tag{73}$$

where all remaining constants were included in C_1 and C_2 . In fact, this algebraic expression is the same for the Cases 1 and 2, but the constants C_1 and C_2 have different contents in each case. Note that we recovered the original radial coordinate r , as well as the original radial function $u(r)$.

Now, by taking into account the contribution of the time coordinate, on the exterior surface of the SAdS black hole with an f(R) global monopole, the full wave solution can be written as

$$\Psi(r, t) \sim C_1 \Psi_{\text{in}} + C_2 \Psi_{\text{out}}, \tag{74}$$

where the solutions describing the ingoing and outgoing scalar waves are given, respectively, by

$$\Psi_{\text{in}}(r > r_1) = e^{-i\omega t} (r - r_1)^{-\frac{i}{2\kappa_1}\omega} \tag{75}$$

and

$$\Psi_{\text{out}}(r > r_1) = e^{-i\omega t} (r - r_1)^{\frac{i}{2\kappa_1}\omega}. \tag{76}$$

The gravitational acceleration on the exterior horizon, κ_1 , is defined as

$$\kappa_1 \equiv \frac{1}{2} \left. \frac{dB(r)}{dr} \right|_{r=r_1} = \frac{B_3(r_1 - r_2)(r_1 - r_3)}{2r_1}, \tag{77}$$

such that, from Eq. (40), we get

$$A_1 = -\frac{i}{2\kappa_1}\omega. \tag{78}$$

Therefore, we follow the method described by Vieira et al. [7] to compute the relative scattering probability, Γ_1 , and the Hawking radiation spectra, \bar{N}_ω . They are given by

$$\Gamma_1 = \left| \frac{\Psi_{\text{out}}(r > r_1)}{\Psi_{\text{out}}(r < r_1)} \right|^2 = e^{-\frac{2\pi}{\kappa_1}\omega}, \tag{79}$$

and

$$\bar{N}_\omega = \frac{\Gamma_1}{1 - \Gamma_1} = (e^{\frac{2\pi}{\kappa_1}\omega} - 1)^{-1}. \tag{80}$$

From these results, we conclude that the Hawking radiation spectrum, for both massless and massive scalar particles in the SAdS black hole with an f(R) global monopole, is analogous to the black body spectrum, which has a thermal character. It is worth noticing that we used the definition of the Hawking temperature given by $k_B T_+ = \hbar\kappa_+/2\pi$, where k_B is the well know Boltzmann constant.

These results were obtained from the analytical solutions of the Klein–Gordon equation in the background under consideration. In fact, that is a semi-classical field theory approach.

4 Quasibound states

The quasibound states, also known as quasistationary levels or resonance spectra, are solutions of the equation of motion that tend to zero far from the black hole at spatial infinity. This means that they are localized in the potential well of the black hole. Thus, that is a boundary value problem with two associated boundary conditions, which gives rise to a

characteristic resonance equation for the frequency (energy) of the quantum particle.

In this physical phenomenon, the flux of quantum particles crosses into the black hole event horizon, by resulting in a spectrum that has complex frequencies, so that it is called a quasispectrum of resonant frequencies and expressed as $\omega = \omega_R + i\omega_I$, where ω_R and ω_I are the real and imaginary parts of the frequencies, respectively. The real part describes the oscillation frequency, while the imaginary part is related to the decay (if $\text{Im}[\omega] < 0$) or growth (if $\text{Im}[\omega] > 0$) rate with the time.

There are some different approaches used to derive the characteristic resonance equation [8–11]. In the present work, we will use the Vieira–Bezerra–Kokkotas method [2,3] to obtain the spectrum of quasibound state frequencies.

Thus, the first boundary condition is such that the radial solution should describe an ingoing wave at the exterior event horizon. Then, we have to impose that $C_2 = 0$ in Eq. (74), as well as in Eqs. (46) and (66). On the other hand, the second boundary condition is such that the radial solution should tend to zero far from the black hole at asymptotic infinity. In order to fully satisfy this condition, we have to take the limit $r \rightarrow \infty$ on the radial solutions given by Eqs. (46) and (66), for the Cases 1 and 2, respectively. To do this, we will write these solutions in terms of the α and β exponent solutions given by Eqs. (A.6) and (A.7). After some algebra, we get the following asymptotic behavior

$$u(r) \sim C_1 \frac{1}{r} + C_2 \frac{1}{r} \quad (\text{Case 1}), \tag{81}$$

and

$$u(r) \sim C_1 \frac{1}{\sqrt{r}} + C_2 \frac{1}{\sqrt{r}} \quad (\text{Case 2}). \tag{82}$$

However, since $C_2 = 0$ from the first boundary condition, we have that

$$u(r) \sim C_1 \frac{1}{r} \quad (\text{Case 1}), \tag{83}$$

and

$$u(r) \sim C_1 \frac{1}{\sqrt{r}} \quad (\text{Case 2}). \tag{84}$$

Thus, the radial solutions given in terms of the general and deformed Heun functions tend to zero far from the black hole at asymptotic infinity, as required by the quasibound states.

Now, the final step is to use a matching procedure in order to bring the two different asymptotic regions into their common overlap region. To do this, we will use the polynomial condition of the Heun functions as described in the Vieira–Bezerra–Kokkotas method [2,3], that is, we will obtain the

Table 1 Values of the massless scalar resonant frequencies ω_n

Case 1	
n	ω_n
0	$-0.22077 - 0.06881i$
1	$-0.44155 - 0.13763i$
10	$-2.42853 - 0.75700i$

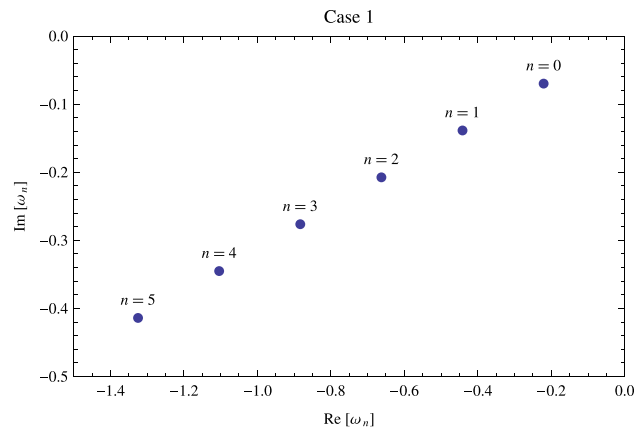


Fig. 3 The massless scalar resonant frequencies ω_n

spectrum of resonant frequencies by using the fact that the general Heun functions become a polynomial of degree n if they satisfy the so-called α -condition given by Eq. (A.8).

4.1 Case 1: massless scalar particles

In this case, the parameter α is given by Eq. (47), which can be simply written as

$$\alpha = 1 - 2i E_1 \omega, \tag{85}$$

where the coefficient E_1 is given by

$$E_1 = \frac{(b - \zeta)(\zeta - 1)}{B_3 r_1 (b - 1)} = \frac{r_2}{B_3 (r_1 - r_2)(r_2 - r_3)}. \tag{86}$$

Note that the coefficient E_1 is a complex number ($E_1 \in \mathbb{C}$). Then, by imposing the polynomial condition given by Eq. (A.8), we obtain the following expression for the massless scalar resonant frequencies

$$\omega_n = -i \frac{n + 1}{2E_1}, \tag{87}$$

where $n = 0, 1, 2, \dots$ is now the principal quantum number. Therefore, this is the spectrum of quasibound states for massless scalar particles propagating in the SAdS black hole with an f(R) global monopole. We shown some values of ω_n in Table 1, and its behavior in Fig. 3 as function of the principal quantum number n .

Table 2 Values of the massive scalar resonant frequencies $\omega_n^{(\pm)}$ for $\mu = 0.17320$

Case 2		
n	$\omega_n^{(-)}$	$\omega_n^{(+)}$
0	$-0.11038 - 0.03440i$	$-0.11038 - 0.03440i$
1	$-0.26234 - 0.32400i$	$-0.39998 + 0.11754i$
10	$-2.10050 - 1.42075i$	$-2.53575 - 0.02445i$

From Table 1 and Fig. 3, we see that the modulus of both real and imaginary parts of the massless scalar resonant frequencies increase with n , for fixed values of the parameters related to the f(R) global monopole. Therefore, the quasistationary levels consist of an infinite sequence of discrete levels, which are equally spaced. The imaginary part is always negative, which means damped oscillations and that the system may be stable.

4.2 Case 2: massive scalar particles

Now, let us analyze the case of massive scalar particles. In this case, the parameter α is given by Eq. (67), which can be simply written as

$$\alpha = 1 - 2i E_1 \omega - \frac{1}{2} \sqrt{3 - 8i E_1 \omega}. \tag{88}$$

Then, by imposing the polynomial condition given by Eq. (A.8), we obtain the following expressions for the massive scalar resonant frequencies

$$\omega_n^{(-)} = \frac{i(2n + 1) - 2\sqrt{n}}{4E_1} \tag{89}$$

and

$$\omega_n^{(+)} = \frac{i(2n + 1) + 2\sqrt{n}}{4E_1}. \tag{90}$$

This quasistationary levels are also complex, where (\pm) labels the solutions; $(-)$ is the “minus” solution, while $(+)$ is the “plus” solution. Indeed, we obtained two solutions due to the fact that the α -condition, in this case, leads to a second-order equation for ω . Therefore, this is the spectrum of quasibound states for massive scalar particles propagating in the SAdS black hole with an f(R) global monopole. We shown some values of $\omega_n^{(\pm)}$ in Table 2, and its behavior in Fig. 4 as function of the principal quantum number n .

From Table 2 and Fig. 4, we see that the modulus of both real and imaginary parts of the massive scalar resonant frequencies $\omega_n^{(-)}$ increase with n , for fixed values of the parameters related to the f(R) global monopole, and therefore we can conclude that $\omega_n^{(-)}$ are damped oscillations, and that the

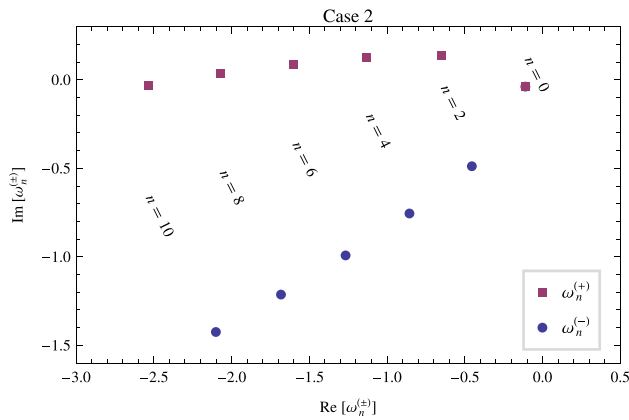


Fig. 4 The massive scalar resonant frequencies $\omega_n^{(\pm)}$ for $\mu = 0.17320$

system may be stable. On the other hand, if the particles have resonant frequencies $\omega_n^{(+)}$, the system may present instability for some excited states ($1 \leq n \leq 9$), since the imaginary part of $\omega_n^{(+)}$ change its sign.

It is worth commenting that both massless and massive scalar resonant frequencies were obtained directly from the general Heun functions, by using a polynomial condition, and, to our knowledge, there is no similar result in the literature for the SAdS black hole with an f(R) global monopole.

5 Wave eigenfunctions

In order to analyze the wave eigenfunctions related to the massless and massive scalar resonant frequencies obtained in the previous section, we will use some properties of the general Heun functions and then obtain their polynomial expressions, which are presented in Appendix A.

5.1 Case 1: massless scalar particles

For massless scalar particles, the radial function $U(z)$ is given by Eq. (38). Thus, the radial wave eigenfunctions, for massless scalar particles propagating in the SAdS black hole with an f(R) global monopole, are given by

$$U_{n;s}(z) = C_{n;s} z^{A_1} (z - 1)^{A_2} (z - b)^{A_3} \text{Hp}_{n;s}(z), \quad (91)$$

where $C_{n;s}$ is a constant (to be determined). It is worth noticing that these radial wave eigenfunctions are degenerate, since the accessory parameter $q_{n;s}$ must be properly determined for each value of s , where $0 \leq s \leq n$.

Therefore, by using Eqs. (A.17), (A.23), and (A.24), we can plot the first three squared massless radial wave eigenfunctions, which are presented in Fig. 5.

From Fig. 5, we see that the massless radial wave eigenfunctions present the desired behavior, that is, the decaying

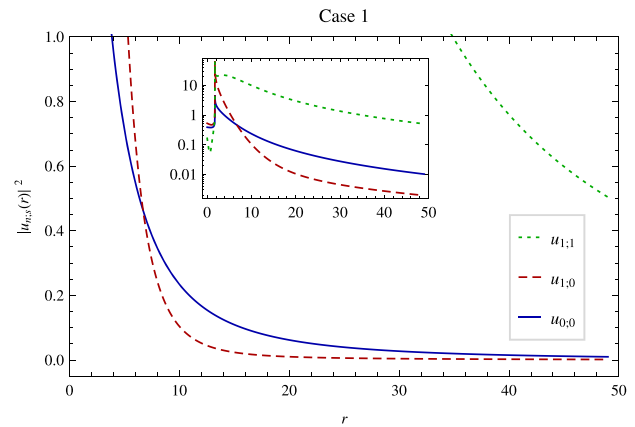


Fig. 5 The first three squared massless radial wave eigenfunctions $u_{n;s}(r) = U_{n;s}(r)/r$ related to ω_n . The units are in multiples of $C_{n;s}$

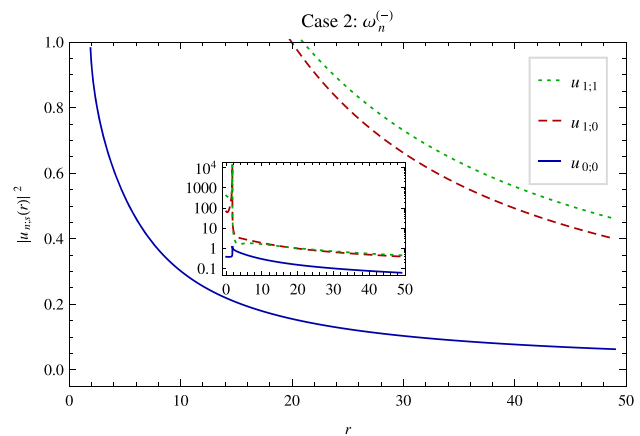


Fig. 6 The first three squared massive radial wave eigenfunctions $u_{n;s}(r) = U_{n;s}(r)/r$ related to $\omega_n^{(-)}$ for $\mu = 0.17320$. The units are in multiples of $C_{n;s}$

quasibound states (with $\text{Im}[\omega_n] < 0$) have a radial solution tending to zero at infinity and diverging at the exterior event horizon, so that it mathematically reaches a maximum value (see this in the log plot) and then crosses into the black hole.

5.2 Case 2: massive scalar particles

For massive scalar particles propagating in the SAdS black hole with an f(R) global monopole, the radial function $U(z)$ is given by Eq. (57), so that we can write their radial wave eigenfunctions as

$$U_{n;s}(z) = C_{n;s} z^{A_1} (z - 1)^{A_2} (z - b)^{A_3} \frac{\text{Hp}_{n;s}(z)}{(z - \zeta)^{\frac{1}{2}}}, \quad (92)$$

where $C_{n;s}$ is a constant (to be determined). Thus, by using Eqs. (A.17), (A.23), and (A.24), we can show the first three squared massive radial wave eigenfunctions in Figs. 6 and 7.

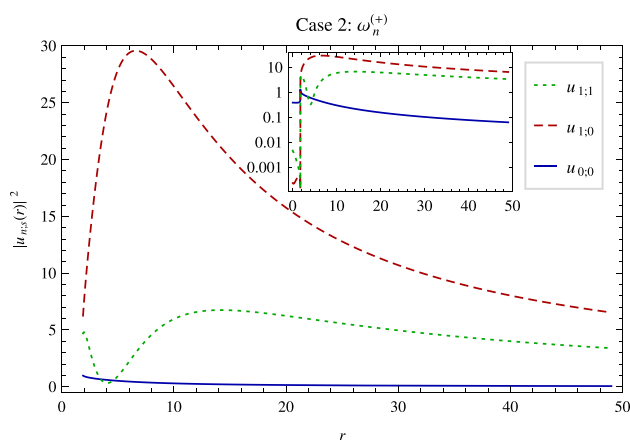


Fig. 7 The first three squared massive radial wave eigenfunctions $u_{n,s}(r) = U_{n,s}(r)/r$ related to $\omega_n^{(+)}$ for $\mu = 0.17320$. The units are in multiples of $C_{n,s}$

From Fig. 6, we conclude that the massive scalar resonant frequencies $\omega_n^{(-)}$ describes quasibound states. On the other hand, in Fig. 7 we can see that only massive scalar particles in the fundamental mode, with resonant frequencies $\omega_0^{(+)}$, are quasibound states. Otherwise, for $\omega_{n \geq 1}^{(+)}$, the radial solutions go to zero at the exterior event horizon and then they are not quasibound states.

6 Final remarks

In this work, we presented analytical solutions for both angular and radial parts of the covariant Klein–Gordon equation with a conformal coupling in the SAdS black hole with an $f(R)$ global monopole. The angular solution is given in terms of the spherical harmonic function. On the other hand, the radial solution is given in terms of the general and deformed Heun functions for massless and massive scalar fields, respectively.

We studied three very important physical phenomena related to the radial solution, namely, the Hawking radiation spectrum, which we found that is similar to the black body radiation, the resonant frequencies, where we imposed the boundary conditions related to the quasibound states, and the wave eigenfunctions, which describes the behavior of quantum scalar particles near the exterior event horizon and far from the black hole at the asymptotic infinity.

The resonant frequencies ω_n and $\omega_n^{(-)}$, which corresponds to the massless and massive scalar particles, respectively, have similar behavior, that is, their imaginary parts are always negative, do not change their signs, and hence they describes quasibound states in stable systems. On the other hand, the massive scalar resonant frequencies $\omega_n^{(+)}$ have a positive imaginary part in the fundamental mode ($n = 0$), but their first nine excited modes ($1 \leq n \leq 9$) cross down the horizon

tal axis and then become negative, which may indicate some kind of phase transition and/or instability in the system.

It is worth calling attention to the fact that this quasistationary levels are associated with the interaction of quantum scalar fields and the curvature of the spacetime under consideration and therefore it is an very interesting semi-classical phenomena, which can give us some insights in the physics of black holes, and larger astrophysical systems as well, and for this reason should be investigated from a theoretical point of view. We hope that, in a near future, it may be used to fit some astrophysical data.

More generally, there has been considerable activity in recent years in the area of quantum gravity phenomenology, which seeks to find observational signatures of the quantum nature of spacetime. These studies may shed some light on the physics of black holes, and they can indicate a possible path to the construction of a quantum theory of gravity.

Finally, it is worth commenting that, in principle, we can use this approach to study quantum scalar fields propagating in a Schwarzschild-de Sitter (SdS) black hole spacetime. However, in such a case, there exist three event horizons, which means that we have to take into account the spatial region between the exterior event horizon and the cosmological horizon; it would be interesting to extend our analysis to this context. In fact, some preliminary investigations which concern this extension were already done, and we expect to publish some results in the near future.

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Appendix A: The general and deformed Heun equations

Here we present some features about the general Heun equation and its deformed case.

Appendix A.1: The general Heun equation

The Heun equation, also called general Heun equation, is a Fuchsian type, with regular singularities at $z = 0, 1, b, \infty$. Its canonical form is given by [12]

$$\frac{d^2y(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-b}\right)\frac{dy(z)}{dz} + \frac{\alpha\beta - q}{z(z-1)(z-b)}y(z) = 0, \tag{A.1}$$

where $y(z) = \text{HeunG}(b, q; \alpha, \beta, \gamma, \delta; z)$ is the general Heun function, which is simultaneously a local Frobenius solution around two singularities s_1 and s_2 , where $s_1, s_2 \in \{0, 1, b\}$. It is analytic in some domain including both these singularities. The parameters $b, q, \alpha, \beta, \gamma, \delta$, and ϵ are generally complex and arbitrary, where b is such that $b \neq 0, 1$, and they are related by $\gamma + \delta + \epsilon = \alpha + \beta + 1$. If $\gamma \neq 0, -1, -2, \dots$, the general Heun function is analytic in the disk $|z| < 1$, and the following Maclaurin expansion applies [13]

$$\text{HeunG}(b, q; \alpha, \beta, \gamma, \delta; z) = \sum_{j=0}^{\infty} c_j z^j, \tag{A.2}$$

where

$$b\gamma c_1 - qc_0 = 0,$$

$$X_j c_{j+1} - (Q_j + q)c_j + P_j c_{j-1} = 0 \quad (\text{for } j \geq 1), \tag{A.3}$$

with $c_0 = 1$ and

$$\begin{aligned} P_j &= (j-1+\alpha)(j-1+\beta), \\ Q_j &= j[(j-1+\gamma)(1+b) + b\delta + \epsilon], \\ X_j &= b(j+1)(j+\gamma). \end{aligned} \tag{A.4}$$

Thus, these expressions leads to

$$\text{HeunG}(b, q; \alpha, \beta, \gamma, \delta; 0) \sim 1. \tag{A.5}$$

In addition, the solutions of Eq. (A.1) corresponding to the exponents α and β at $z = \infty$ are given, respectively, by

$$\begin{aligned} &z^{-\alpha} \text{HeunG}\left(\frac{1}{b}, \alpha(\beta - \epsilon) + \frac{\alpha}{b}(\beta - \delta) - \frac{q}{b}; \right. \\ &\left. \alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \delta; \frac{1}{z}\right) \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} &z^{-\beta} \text{HeunG}\left(\frac{1}{b}, \beta(\alpha - \epsilon) + \frac{\beta}{b}(\alpha - \delta) - \frac{q}{b}; \right. \\ &\left. \beta, \beta - \gamma + 1, \beta - \alpha + 1, \delta; \frac{1}{z}\right). \end{aligned} \tag{A.7}$$

On the other hand, the general Heun function becomes a polynomial of degree n if it satisfies the so-called α -condition, which is given by [12]

$$\alpha = -n, \tag{A.8}$$

where $n = 0, 1, 2, \dots$. Such polynomial solutions are denoted by $\text{Hp}_n(z) = \text{HeunG}(b, q; -n, \beta, \gamma, \delta; z)$ and can be written as

$$\text{Hp}_n(z) = \sum_{\nu=0}^{\infty} c_{\nu} z^{\nu}, \tag{A.9}$$

where the coefficients c_{ν} are given by

$$-(Q_0 + q)c_0 + X_0 c_1 = 0, \tag{A.10}$$

$$P_{\nu} c_{\nu-1} - (Q_{\nu} + q)c_{\nu} + X_{\nu} c_{\nu+1} = 0, \tag{A.11}$$

for $\nu = 1, 2, \dots, n-1$, where the parameters P_{ν}, Q_{ν} , and X_{ν} are given by Eq. (A.4). These equations are consistent if, and only if, the accessory parameter q was chosen properly, which means that there is a polynomial equation of degree $n+1$ for the determination of such a parameter. We will choose the following notation for these eigenvalues: $q_{n;m}$, where m runs from 0 to n . Thus, the corresponding general Heun polynomials are now denoted as $\text{Hp}_{n;m}(z)$.

The explicit form of the first three general Heun polynomials can be obtained as follows. For $n = 0$, we have

$$\text{Hp}_{0;m}(z) = c_0 = 1, \tag{A.12}$$

where the eigenvalues $q_{0;m}$ must obey

$$c_1 = 0, \tag{A.13}$$

where

$$-qc_0 + b\gamma c_1 = 0, \tag{A.14}$$

which implies

$$c_1 = \frac{q}{b\gamma}, \tag{A.15}$$

and then we have that

$$q_{0;0} = 0. \tag{A.16}$$

Thus, the first general Heun polynomial is given by

$$\text{Hp}_{0;0}(z) = 1. \tag{A.17}$$

Now, for $n = 1$, we have

$$\text{Hp}_{1;m}(z) = c_0 + c_1 z = 1 + \frac{q_{1;m}}{b\gamma} z, \tag{A.18}$$

where the eigenvalues $q_{1;m}$ must obey

$$c_2 = 0, \tag{A.19}$$

where

$$P_1c_0 - (Q_1 + q)c_1 + R_1c_2 = 0, \tag{A.20}$$

which implies

$$c_2 = \frac{[\gamma(1 + b) + b\delta + \epsilon + q]q - b\alpha\beta\gamma}{2b^2\gamma(1 + \gamma)}, \tag{A.21}$$

and then we have that

$$q_{1;m} = \frac{-[\gamma(1 + b) + b\delta + \epsilon] \pm \sqrt{\Delta}}{2}, \tag{A.22}$$

where $\Delta = [\gamma(1 + b) + b\delta + \epsilon]^2 + 4b\alpha\beta\gamma$. Note that the signs $-$ and $+$ stand for $m = 0$ and $m = 1$, respectively. Thus, the second and third general Heun polynomials are given, respectively, by

$$H_{p_1;0}(z) = 1 + \frac{-[\gamma(1 + b) + b\delta + \epsilon] - \sqrt{\Delta}}{2b\gamma}z \tag{A.23}$$

and

$$H_{p_1;1}(z) = 1 + \frac{-[\gamma(1 + b) + b\delta + \epsilon] + \sqrt{\Delta}}{2b\gamma}z. \tag{A.24}$$

Finally, the corresponding Hamiltonian \mathcal{H} in classical mechanics is given by

$$\begin{aligned} \mathcal{H}(q, p, t) = & \frac{-1}{t(t-1)}\{q(q-1)(q-t)p^2 \\ & + [\gamma(q-1)(q-t) + \delta q(q-t) \\ & + \epsilon q(q-1)]p + \alpha\beta q\}. \end{aligned} \tag{A.25}$$

From now on, q and p are the canonical coordinate and momentum, respectively, and t is the scaling parameter (which can be considered as time). Then, if q and p are quantum observables, we can write $\mathcal{H}(q, p, t)y = \lambda y$, where λ is the eigenvalue (which can be considered as energy).

Appendix A.2: The deformed Heun equation

Next, let us talk about the deformed Heun equation. In fact, Slavyanov and Lay [14] presented the Heun class of equations, which includes the confluent cases, in an extended form by adding an apparent singularity to each equation. However, none of these forms fits our case. Then, we will follow their ideas in order to discuss a particular case of the deformed Heun equation, which can be written as

$$\begin{aligned} \frac{d^2y(z)}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t} - \frac{1}{z-q} \right) \frac{dy(z)}{dz} \\ + \left[\frac{\alpha\beta}{(z-1)(z-t)} + \frac{ht(t-1)}{z(z-1)(z-t)} \right. \end{aligned}$$

$$\left. + \frac{pq(q-1)}{z(z-1)(z-q)} \right] y(z) = 0. \tag{A.26}$$

Note that it differs from the (general) Heun equation (A.1) by two additional terms proportional to $z - q$, which is a simple pole at $z = q$ and plays the role of an apparent singularity. The parameters h and p obey to the following relations

$$h = \text{Res}_{z=t} \frac{ht(t-1)}{z(z-1)(z-t)}, \tag{A.27}$$

$$p = \text{Res}_{z=q} \frac{pq(q-1)}{z(z-1)(z-q)}. \tag{A.28}$$

Now, an analytical solution of the deformed Heun equation given by Eq. (A.26) can be written as a Taylor expansion, namely,

$$y(z) = \sum_{k=0}^{\infty} g_k(z-q)^k, \tag{A.29}$$

where $g_0 = 1$. For simplicity, we will rewrite Eq. (A.26) as

$$\frac{d^2y(z)}{dz^2} + P(z) \frac{dy(z)}{dz} - Q(z)y(z) = 0, \tag{A.30}$$

where the coefficients $P(z)$ and $Q(z)$ are given in term of the following Laurent expansions

$$P(z) = \sum_{k=-1}^{\infty} p_k(z-q)^k = p_0 + \frac{p_{-1}}{z-q}, \tag{A.31}$$

$$Q(z) = \sum_{k=-1}^{\infty} q_k(z-q)^k = q_0 + \frac{q_{-1}}{z-q}. \tag{A.32}$$

Thus, substituting Eqs. (A.29)–(A.32) into Eq. (A.26), we obtain

$$\begin{aligned} g_k k(k-1)(z-q)^{k-2} + p_{-1} g_k k(z-q)^{k-2} \\ + p_0 g_k k(z-q)^{k-1} + q_{-1} g_k (z-q)^{k-1} \\ + q_0 g_k (z-q)^k = 0, \end{aligned} \tag{A.33}$$

from which we arrive at the following recurrence relation

$$k(k+2)g_{k+2} + [p_0(k+1) + q_{-1}]g_{k+1} + q_0g_k = 0, \tag{A.34}$$

where $k \geq 0$. From this recurrence relation, we obtain

$$\begin{aligned} p_{-1}g_1 + q_{-1}g_0 = 0, \\ 2(1 + p_{-1})g_2 + (p_0 + q_{-1})g_1 + q_0 = 0. \end{aligned} \tag{A.35}$$

On the other hand, the values for the coefficients p_k and q_k are directly obtained from Eq. (A.26). They are given by

$$p_{-1} = -1, \tag{A.36}$$

$$p_0 = \frac{\gamma}{q} + \frac{\delta}{q-1} + \frac{\epsilon}{q-t}, \tag{A.37}$$

$$q_{-1} = p, \tag{A.38}$$

$$q_0 = \frac{\alpha\beta}{(q-1)(q-t)} + \frac{ht(t-1)}{q(q-1)(q-t)} - p\left(\frac{1}{q} - \frac{1}{q-1}\right). \tag{A.39}$$

In these terms, Eq. (A.35) take the following form

$$0 = -g_1 + p, \\ 0 = \left(\frac{\gamma}{q} + \frac{\delta}{q-1} + \frac{\epsilon}{q-t} + p\right)p + \frac{\alpha\beta}{(q-1)(q-t)} + \frac{ht(t-1)}{q(q-1)(q-t)} - p\left(\frac{1}{q} - \frac{1}{q-1}\right). \tag{A.40}$$

Therefore, the necessary condition for the point $z = q$ to be an apparent singularity is

$$h(q, p, t) = \frac{-1}{t(t-1)}\{q(q-1)(q-t)p^2 + [(\gamma-1)(q-1)(q-t) + (\delta-1)q(q-t) + \epsilon q(q-1)]p + \alpha\beta q\}. \tag{A.41}$$

It is easy to see that this function $h(q, p, t)$ coincides with the classical Hamiltonian given by Eq. (A.25), concerning to the general Heun equation. However, it differs by an unitary shifting in the parameters γ and δ . It can be written as

$$h(q, p, t; \alpha, \beta, \gamma, \delta, \epsilon) = \mathcal{H}(q, p, t; \alpha, \beta, \gamma - 1, \delta - 1, \epsilon). \tag{A.42}$$

The Slavyanov and Lay’s explanation for this shifting is that the parameters $\alpha, \beta, \gamma, \delta,$ and ϵ satisfy different Fuchs conditions related to the different cases (general and deformed) of the Heun equations. In the case of a deformed Heun equation, the condition is $\gamma + \delta + \epsilon = \alpha + \beta + 3$.

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