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Aspects of the polynomial affine model of gravity in three dimensions

With focus in the cosmological solutions

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Abstract The polynomial affine gravity is a model that is built up without the explicit use of a metric tensor field. In this article we reformulate the three-dimensional model and, given the decomposition of the affine connection, we analyse the consistently truncated sectors. Using the cosmological ansatz for the connection, we scan the cosmological solutions on the truncated sectors. We discuss the emergence of different kinds of metrics.

1 Introduction

Our current understanding of fundamental physics accounts for four different interactions, which split into two pillars. On the one hand, the gravitational interaction is described by General Relativity, which models the spacetime as a Riemannian manifold $(\mathcal{M}, g)$, i.e. its geometrical properties are tied up to the metric tensor field. On the other hand, the remaining three interactions are described by gauge theories, whose fundamental field are connections under transformations of the gauge group, but (covariant) vectors under the group of diffeomorphisms.

The formulation of the gauge theories requires the choice of a background manifold, on which the theories stand, and the standard treatment does not consider back-reaction of the gauge theories to the manifold. Nonetheless, in General Relativity the metric tensor field defines the geometric properties of the spacetime, and also mediates the gravitational interaction. Such dual role of the metric tensor field originates significant differences between these pillars of fundamental physics, being the most highlighted the quantisation program. While the attempts to quantise General Relativity have eluded a consistent culmination, the remaining three interactions are quantisable and renormalisable, raising what is known as the standard model of particles.

In a quest to prevent the *inconsistent* quantisation of the gravitational interaction, a large number of generalisations of General Relativity have been proposed. Inspired by the seminal work by Palatini [1], a large amount of these generalisations modelled the spacetime by manifolds whose connections is not the one of Levi–Civita, dubbed *metric-affine* models of gravity (see for example Ref. [2]). These models rely in the metric as the field mediating the interaction. However, some have attempted to model the gravitational interaction through affine theories, where the mediating field is an affine (linear) connection on the spacetime. The first proposals of affine gravity were considered by Einstein, Eddington and Schrödinger [3–6], but those models were left aside because their manipulation was significantly more complex and offered insufficient novelty from the phenomenological point of view. More contemporaneous affine models of gravity were proposed by Kijowski and collaborators [7–10], Poplawski [11–14], Krasnov and collaborators [15–20], and some of us [21,22].

The analysis of gravitational models in lower dimensions, where the number of parameters decreases in general, serves as a playground to test methods that might be applied to the four-dimensional models, but also for their applications in other branches of physics since vector and tensor gauge theories can be interpreted as the high-temperature limit of four-dimensional models [23]. These *simplified* models stimulate the generation of new ideas and insights into their four-dimensional counterparts.

The three-dimensional version of General Relativity was firstly considered by Staruszkiewicz [24], where a Schwarzschild-like solution was studied, and a relation between the presence of massive point particles and conical

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singularities (which modify the asymptotic behaviour of the spacetime) was found. The interest in three-dimensional gravity diminished due to the lack of propagating degrees of freedom, until in a series of papers Deser, Jackiw and collaborators showed that adding a nontrivial, gauge invariant, topological term to the three-dimensional Einstein–Hilbert action resulted in a massive model for gravity [25–28]. The topological term added in this model, dubbed Deser–Jackiw–Templeton, was the Chern–Simons term associated to the four-dimensional θ-term build from the (metric) Riemannian curvature—also known as the Pontryagin density. A remarkable feature of the Deser–Jackiw–Templeton model is that it was able to induce a mass for the graviton without a Higgs mechanism and preserving the (infinitesimal) gauge invariance.

Witten showed in Ref. [29], that the three-dimensional gravity modified by the Pontryagin Chern–Simons Lagrangian is equivalent to a Yang–Mills theory, called Chern–Simons gravity, and also that its perturbative expansion was renormalisable.[1] Some years later, the first black hole solution was found by Bañados, Teitelboim and Zanelli [31], disproving the *triviality* of classical three-dimensional gravity and revitalising the interest in the search of exact solutions [32] and further development into their quantum aspects [33]. Contemporaneously, Mielke and Baekler generalised the topological massive model of gravity to include torsion [34], and extended further to metric-affine gravity by Tresguerres [35].

An interconnection between three-dimensional gravity with other branches of physics was encountered by Dereli and Verçin in the context of the continuum theory of lattice defects, through the identification of the dislocation and disclination line density tensors with torsion and curvature tensors, and the free-energy density with the Lagrangian of the Deser–Jackiw–Templeton model of gravity [36].[2] More recently, methods of quantum field theory in curved spaces have been applied to the analysis of properties of graphene, viewed as a *membrane* endowed with an metric induced by its embedding into three-dimensional space [38–43].

From a mathematical point of view, the Deser–Jackiw–Templeton model generalises the three-dimensional Einstein–Hilbert model by taking into account global properties of the Riemannian spacetime. The Mielke–Baekler model relaxes the Riemannian condition, by allowing the spacetime to be modelled by a Riemann–Cartan manifold, while in the Tresguerres model the spacetime possesses also non-metricity. In all of these models, the metric plays a fundamental role in their formulation. However, the existence of contexts in which the notion of metric is not helpful, e.g. in a phase space or a moduli space, inspires the search of *gravitational* models which are defined in spaces where the notion of displacement and flow is defined, but not necessarily a notion of length. These spaces are called *affine manifolds*, and they are the underlying structure supporting the polynomial affine model of gravity.

The classification of the affinely connected spaces is shown schematically in the following commutative diagram. The most general affine manifold might not posses a metric, and thus it is characterised by a connection $(\mathcal{M}, \hat{\nabla})$. When an affine manifold is equipped with a metric, dubbed metric-affine manifold, the connection can be decomposed into three contributions—the Levi-Civita connection, the contorsion tensor and the deflection tensor—, and the manifold is classified according to its curvature $(\mathcal{R})$, torsion $(\mathcal{T})$ and non-metricity $(\mathcal{Q})$. In the diagram, Riemannian geometries belong to the sector denoted by $(\mathcal{R})$, Riemann–Cartan geometries to $(\mathcal{R}, \mathcal{T})$, Weizenböck geometries to $(\mathcal{T})$, et cetera.

$$
\begin{tikzcd}
 & (\mathcal{M}, \hat{\nabla}) \arrow[d, "g"] & & \\
 & (\mathcal{Q}, \mathcal{R}, \mathcal{T}) \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathcal{Q}, \mathcal{T}) \arrow[dl] \arrow[dd] \\
(\mathcal{Q}, \mathcal{R}) \arrow[rr] \arrow[dd] & & (\mathcal{Q}) \arrow[dd] & \\
 & (\mathcal{R}, \mathcal{T}) \arrow[rr] \arrow[dl] & & (\mathcal{T}) \arrow[dl] \\
(\mathcal{R}) \arrow[rr] & & (\mathrm{Flat}) &
\end{tikzcd}
$$

[1] An update on the original ideas in this paper can be found in Ref. [30].

[2] For a more recent review, see Ref. [37].

The use of an affine model allows to explore features that should be attributed to the local invariance under coordinate transformations, regardless of the metric structure on the manifold. In this context, the polynomial affine model of gravity emerges naturally. Even though the four-dimensional model is being analysed, the three-dimensional version is expected to be easier to characterise, and eventually solve some issues the affine models have encountered, such as the coupling of matter.

In this paper we re-state the three-dimensional model of polynomial affine gravity, firstly proposed in Ref. [21], analyse their field equations and find explicit cosmological solutions. The paper is organised as follows. Section 2 gives a brief overview of the polynomial affine model of gravity in three dimensions. Then, in Sect. 3 the field equations are derived, and some issues regarding their truncation – i.e. restriction to sectors where only a subset of the fields (irreducible components of the connection) are *turned on* – are discussed in Sect. 4. In Sect. 5 we scan the space of solutions which are compatible with the cosmological principle. Some conclusions are drawn in the Sect. 6. For completeness we include some appendices, including a discussion of our notation in Appendix A.

2 Building the model

The polynomial affine model of gravity was born as an attempt to build up a theory with the affine connection as sole fundamental field [21]. Formally, the idea behind the polynomial affine model of gravity is that spacetime is not a (pseudo-)Riemannian manifold, but an affinely connected manifold $(\mathcal{M}, \hat{\nabla})$. The strategy is then to consider all possible terms, allowed by the invariance under diffeomorphisms, as part of the Lagrangian density. Note that a generic affine connection is a reducible object under the group of diffeomorphisms, and therefore we can decompose it.

In the absence of a metric tensor field, the affine connection decomposes into irreducible components as follows

$$\hat{\Gamma}_{\mu}{}^{\lambda}{}_{\nu} = \hat{\Gamma}_{(\mu}{}^{\lambda}{}_{\nu)} + \hat{\Gamma}_{[\mu}{}^{\lambda}{}_{\nu]} = \Gamma_{\mu}{}^{\lambda}{}_{\nu} + \mathcal{B}_{\mu}{}^{\lambda}{}_{\nu} + \mathcal{A}_{[\mu}\delta^{\lambda}_{\nu]}, \qquad (1)$$

where $\mathcal{B}$ field is the traceless part of the torsion, $\mathcal{A}_{\mu}$ is the trace of the torsion, and $\Gamma_{\mu}{}^{\lambda}{}_{\nu} \equiv \hat{\Gamma}_{(\mu}{}^{\lambda}{}_{\nu)}$ is a renaming of the symmetric part of the affine connection. All these elements transform as tensors under diffeomorphisms, with the exception of the symmetric part of the affine connection, which must be included in the action almost exclusively through the covariant derivative.

In order to build an action, we consider the chart-induced basis of the tangent and cotangent spaces, i.e. $\{\partial_{\mu}\}$ and $\{\mathrm{d}x^{\mu}\}$, and the volume form defined as

$$\mathrm{d}V^{\alpha\beta\gamma} = \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\beta} \wedge \mathrm{d}x^{\gamma}\ . \qquad (2)$$

An interesting type of connection are those *compatible* with the volume, i.e. $\nabla(\mathrm{d}V) = 0$, which are said to be equiaffine. Such compatibility ensures that the Ricci tensor field is symmetric, and the trace of the curvature tensor vanishes [44–46]. Although equiaffinity is not demanded in the following formulation, in Sect. 5 we shall encounter that symmetric connections compatible with the cosmological principle are necessarily equiaffine.

Now, with the aid of the irreducible components of the connection and the volume form, we can write down the most general Lagrangian, i.e. a scalar density in three dimensions. A *dimensional analysis* similar to the one presented in Refs. [21,22,47–49] shows that the most general action (up to boundary terms) is given by[3]

$$\begin{aligned} S = \int \mathrm{d}V^{\alpha\beta\gamma} \Big(& B_1\, \mathcal{A}_{\alpha}\mathcal{A}_{\mu}\mathcal{B}_{\beta}{}^{\mu}{}_{\gamma} \\ & + B_2\, \mathcal{A}_{\alpha}\mathcal{F}_{\beta\gamma} + B_3\, \mathcal{A}_{\alpha}\nabla_{\mu}\mathcal{B}_{\beta}{}^{\mu}{}_{\gamma} \\ & + B_4\, \mathcal{B}_{\alpha}{}^{\mu}{}_{\nu}\mathcal{B}_{\beta}{}^{\nu}{}_{\lambda}\mathcal{B}_{\gamma}{}^{\lambda}{}_{\mu} + B_5 \mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\mu}\mathcal{A}_{\gamma} \\ & + B_6\, \mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu}\mathcal{B}_{\beta}{}^{\nu}{}_{\gamma} + B_7\, \Gamma_{\alpha}{}^{\mu}{}_{\mu}\partial_{\beta}\Gamma_{\gamma}{}^{\nu}{}_{\nu} \\ & + B_8 \left(\Gamma_{\alpha}{}^{\mu}{}_{\nu}\partial_{\beta}\Gamma_{\gamma}{}^{\nu}{}_{\mu} + \frac{2}{3}\Gamma_{\alpha}{}^{\mu}{}_{\nu}\Gamma_{\beta}{}^{\nu}{}_{\lambda}\Gamma_{\gamma}{}^{\lambda}{}_{\mu} \right) \Big). \end{aligned} \qquad (3)$$

In the action, the covariant derivative and the curvature are defined with respect to the symmetric connection, i.e. $\nabla = \nabla^{\Gamma}$ and $\mathcal{R} = \mathcal{R}^{\Gamma}$; and the symbol $\mathcal{F}$ denotes the field strength of the $\mathcal{A}$-fields, i.e. $\mathcal{F}_{\beta\gamma} = \partial_{\beta}\mathcal{A}_{\gamma} - \partial_{\gamma}\mathcal{A}_{\beta}$.

Among the features of the polynomial affine model of gravity we count: (i) The fundamental field is a connection, like the other fundamental interactions; (ii) The coupling constants are dimensionless, which is desirable from the view point of Quantum Field Theory, since the superficial degree of divergence vanishes; (iii) The model seems to exhibit scale invariance; (iv) The number of possible terms in the action is finite (we usually refer to this property as the *rigidity* of the model), giving the impression that in the hypothetical scenario of quantisation all the counter-terms have the form of terms already present in the original action.

3 Field equations

We now focus in obtaining the field equations for the fields Γ, $\mathcal{B}$ and $\mathcal{A}$, by varying the action in Eq. (3). It is important to highlight that although the absence of second-class constraints in polynomial affine gravity has not been proven yet, the structure of the action suggests that the variational problem is well-posed, and therefore the field equations below do

[3] Note that this action is equivalent to the one introduced in Ref. [21].

not consider the existence of affine analogues of the Gibbons–Hawking–York term.[4]

Since the action contains up to first derivatives of the fields, the field equations are obtained through the Euler–Lagrange equations,

$$\begin{aligned}
&\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Gamma_\nu{}^\lambda{}_\rho\right)}\right)-\frac{\partial\mathscr{L}}{\partial\Gamma_\nu{}^\lambda{}_\rho}=0,\\
&\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\mathcal{B}_\nu{}^\lambda{}_\rho\right)}\right)-\frac{\partial\mathscr{L}}{\partial\mathcal{B}_\nu{}^\lambda{}_\rho}=0,\\
&\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\mathcal{A}_\nu\right)}\right)-\frac{\partial\mathscr{L}}{\partial\mathcal{A}_\nu}=0.
\end{aligned}\tag{4}$$

We proceed utilising the formalism introduced by Kijowski in Ref. [7], and following the steps from Ref. [49].

It can be shown with ease that the Euler–Lagrange equations (4) can be rewritten as

$$\begin{aligned}
\nabla_\mu\Pi_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho&=\frac{\partial^*\mathscr{L}}{\partial\Gamma_\nu{}^\lambda{}_\rho},\\
\nabla_\mu\Pi_\mathcal{B}{}^{\mu\nu}{}_\lambda{}^\rho&=\frac{\partial\mathscr{L}}{\partial\mathcal{B}_\nu{}^\lambda{}_\rho},\\
\nabla_\mu\Pi_\mathcal{A}{}^{\mu\nu}&=\frac{\partial\mathscr{L}}{\partial\mathcal{A}_\nu},
\end{aligned}\tag{5}$$

where the quantities Π_X are the canonical momentum associated to the field X – which are tensor densities –, defined as

$$\begin{aligned}
\Pi_\Gamma{}^{\mu\nu}{}_\lambda{}^\rho&=\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\Gamma_\nu{}^\lambda{}_\rho\right)},\\
\Pi_\mathcal{B}{}^{\mu\nu}{}_\lambda{}^\rho&=\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\mathcal{B}_\nu{}^\lambda{}_\rho\right)},\\
\Pi_\mathcal{A}{}^{\mu\nu}&=\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\mathcal{A}_\nu\right)},
\end{aligned}\tag{6}$$

and the asterisk on the right-hand side of the field equation for the symmetric part of the connection in Eq. (5) denotes the partial derivative with respect to the connection that is not contained in the curvature tensor.

The field equations for the fields $\mathcal{A}$, $\mathcal{B}$ and Γ derived from the action in Eq. (3) are respectively:

$$\begin{aligned}
&2B_1\mathcal{A}_\alpha\mathcal{B}_\nu{}^\alpha{}_\rho+2B_2\mathcal{F}_{\nu\rho}+B_3\nabla_\mu\mathcal{B}_\nu{}^\mu{}_\rho\\
&\quad+B_5\mathcal{R}_{\nu\rho}{}^\mu{}_\mu=0,
\end{aligned}\tag{7}$$

$$\begin{aligned}
&2B_1\mathcal{A}_\nu\mathcal{A}_\rho-2B_3\nabla_{(\nu}\mathcal{A}_{\rho)}+3B_4\mathcal{B}_\nu{}^\mu{}_\sigma\mathcal{B}_\rho{}^\sigma{}_\mu\\
&\quad+2B_6\mathcal{R}_{\mu(\nu}{}^\mu{}_{\rho)}=0,
\end{aligned}\tag{8}$$

$$\begin{aligned}
&B_3\mathcal{A}_\mu\mathcal{B}_\rho{}^\nu{}_\sigma+B_5\left(\delta^\nu_\mu\mathcal{F}_{\rho\sigma}+\delta^\nu_{[\rho}\mathcal{F}_{\sigma]\mu}\right)\\
&\quad+B_6\left(2\delta^\nu_\mu\nabla_\tau\mathcal{B}_\rho{}^\tau{}_\sigma+\delta^\nu_\rho\nabla_\tau\mathcal{B}_\sigma{}^\tau{}_\mu+\delta^\nu_\sigma\nabla_\tau\mathcal{B}_\mu{}^\tau{}_\rho\right)\\
&\quad+B_7\left(\delta^\nu_\mu\mathcal{R}_{\rho\sigma}{}^\lambda{}_\lambda+\delta^\nu_{[\rho}\mathcal{R}_{\sigma]\mu}{}^\lambda{}_\lambda\right)\\
&\quad+B_8\left(\mathcal{R}_{\rho\sigma}{}^\nu{}_\mu+\delta^\nu_{[\rho}\mathcal{R}_{\sigma]\lambda}{}^\lambda{}_\mu\right)=0.
\end{aligned}\tag{9}$$

Note that the field equation for the $\mathcal{B}$-field, i.e. Eq. (8), is similar to the Ricci form of the Einstein field equations. Particularly, it has been shown that the tensor $\mathcal{B}_\nu{}^\mu{}_\sigma\mathcal{B}_\rho{}^\sigma{}_\mu$ (when non-degenerated) might be interpreted as a torsion-descendent metric tensor field [14].

4 Truncations

Unlike the analysis of the truncations in the four-dimensional polynomial affine model of gravity [22,48,49], the presence of the Chern–Simons terms in Eq. (3) allows to switch-off certain fields at the level of the action, and not just at the level of the field equations. Evidently, when we set the fields to zero at the action level, we are properly truncating the model. On the contrary, when we set the fields equal to zero at the level of the field equations, we are not truncating but only restricting ourselves to special solutions of the complete model. However, in both cases we called them "truncations" in the absence of a better general terminology. In order to distinguish these two possible *truncations* of the model, we refer to them as off-shell and on-shell limits. Below we list six possible truncations of the model.

4.1 Torsion-free limit ($\mathcal{A}=\mathcal{B}=0$)

On the one hand, taking the off-shell torsion-free limit yields an effective action which is the sum of two Chern-Simons terms, those terms in Eq. (3) accompanying the coefficients B_7 and B_8, whose field equations are

$$\begin{aligned}
&B_7\left(\delta^\nu_\mu\mathcal{R}_{\rho\sigma}{}^\lambda{}_\lambda+\delta^\nu_{[\rho}\mathcal{R}_{\sigma]\mu}{}^\lambda{}_\lambda\right)\\
&\quad+B_8\left(\mathcal{R}_{\rho\sigma}{}^\nu{}_\mu+\delta^\nu_{[\rho}\mathcal{R}_{\sigma]\lambda}{}^\lambda{}_\mu\right)=0.
\end{aligned}$$

Note that the contribution to the field equations of the term with coefficient B_7 vanishes identically for equiaffine connections,[5] and thus the field equations for the symmetric connection are

$$B_8\left(\mathcal{R}_{\rho\sigma}{}^\nu{}_\mu+\delta^\nu_{[\rho}\mathcal{R}_{\sigma]\lambda}{}^\lambda{}_\mu\right)=0.\tag{10}$$

[4] Analysis of affine analogues to the Gibbons–Hawking–York term can be found in Refs. [50–55].

[5] For equiaffine connections the skew-symmetric part of the Ricci tensor field vanishes. From the first (or algebraic) Bianchi identity, it follows that the trace of the curvature vanishes. Therefore, $\mathcal{R}_{\mu\nu}{}^\alpha{}_\alpha=\partial_{[\mu}\Gamma_{\nu]}{}^\alpha{}_\alpha=0$. In the remaining of the article we shall ignore the contribution of this term to the field equations, however, we keep the term in the action because it might contribute to topological quantities.

These field equations require that the connection is projectively Weyl-flat.[6]

On the other hand, when one takes the on-shell torsion-free sector, there are subsidiary equations,

$$\mathcal{R}_{\nu\rho}{}^{\mu}{}_{\mu} = 0 \text{ and } \mathcal{R}_{\mu(\nu}{}^{\mu}{}_{\rho)} = 0,$$

the first is satisfied when the connection is equiaffine, while the second (Ricci-flatness) restricts the solutions of the system to be flat connections.

4.2 Vectorial torsion ($\Gamma = \mathcal{B} = 0$)

The restriction to sole vectorial torsion in the off-shell limit yields a Chern–Simons-like action, whose field equations are

$$\mathcal{F}_{\nu\rho} = 0, \tag{11}$$

while the on-shell restriction raises the auxiliary condition

$$2B_1\mathcal{A}_\nu\mathcal{A}_\rho - 2B_3\partial_{(\nu}\mathcal{A}_{\rho)} = 0. \tag{12}$$

The system of Eqs. (11) and (12) can be solved in general. First, equation (11) implies that locally the field $\mathcal{A}$ is an exact 1-form, $\mathcal{A} = \mathrm{d}\phi$. Secondly, Eq. (12) can be written in the form

$$\partial_\nu\partial_\rho f = 0, \text{ with } f = e^{-\frac{B_1}{B_3}\phi}.$$

The above equation implies that f is a linear function of the coordinates, and allows to solve for ϕ and therefore $\mathcal{A}$,

$$\phi(x) = -\frac{B_3}{B_1}\ln\left(D + C_\mu x^\mu\right) \text{ and } \mathcal{A}_\nu(x) = -\frac{B_3}{B_1}\frac{C_\nu}{D + C_\mu x^\mu}.$$

Note that the field Eq. (11), implies that the $\mathcal{A}$-field inherits a gauge redundancy, i.e. any two configurations of the $\mathcal{A}$-field related by an exact 1-form ($\mathcal{A}' = \mathcal{A} + \mathrm{d}\lambda$) solve the field equations. This statement holds when the $\mathcal{B}$-field is eliminated at the action level. Once the $\mathcal{B}$-field is included in the action, such gauge redundancy disappears with the existence of the $\mathcal{B}$-field, even if $\mathcal{B} = 0$, due to the introduction of additional field equations., i.e. $\mathcal{B}$ breaks the gauge redundancy of $\mathcal{A}$.

4.3 Trace-less torsion ($\Gamma = \mathcal{A} = 0$)

Interestingly, the off-shell truncation to trace-less torsion leaves an effective action whose sole term is that with coefficient B_4, which provides no dynamics to the $\mathcal{B}$-field. The field equations are

$$\mathcal{B}_\nu{}^\mu{}_\sigma\mathcal{B}_\rho{}^\sigma{}_\mu = 0. \tag{13}$$

The on-shell restriction yields the subsidiary equations,

$$\partial_\mu\mathcal{B}_\rho{}^\mu{}_\sigma = 0 \text{ and } \partial_{[\mu}\mathcal{B}_\rho{}^\nu{}_{\sigma]} = 0. \tag{14}$$

Note that these equations are equivalent,[7] and their equivalency can be extended to the covariant version of the equations.

Equation (13) can be rewritten in terms of the quasi-Hodge dual [58],

$$T^{\mu\alpha} = \frac{1}{2}\mathcal{B}_\beta{}^\mu{}_\gamma\epsilon^{\alpha\beta\gamma}, \tag{15}$$

giving a cofactor equation for the tensor T. Hence, $T^{\mu\alpha} = \rho(x)V^\mu V^\alpha$ is the general solution where ρ is a scalar density, and therefore

$$\mathcal{B}_\nu{}^\mu{}_\lambda = \rho(x)V^\mu V^\sigma\epsilon_{\sigma\nu\lambda}.$$

The structure of our tensor $T^{\mu\alpha}$ is equivalent to that of the energy–momentum tensor for cold matter (i.e. *dust*), where ρ is the energy density and V^μ represents the velocity of the matter distribution. Similarly, the Eq. (14) is a generalisation of the continuity equation and energy–momentum conservation,

$$\begin{aligned} 0 &= \partial_\mu\mathcal{B}_\nu{}^\mu{}_\lambda \\ &= \left((\partial_\mu\rho)V^\mu V^\sigma + \rho(\partial_\mu V^\mu)V^\sigma + \rho V^\mu(\partial_\mu V^\sigma)\right)\epsilon_{\sigma\nu\lambda} \\ &= [\rho\nabla_V V + (\nabla\cdot(\rho V))V]^\sigma\,\epsilon_{\sigma\nu\lambda}. \end{aligned}$$

The last line is the equation for a self-parallel vector, written in with a non-canonical affine parameter. The expression in brackets corresponds to the divergence of the tensor T, i.e. $\partial_\mu T^{\mu\sigma}$, which represents – in our analogue with the cold matter energy–momentum tensor – the momentum conservation. However, in General Relativity the each term in the bracket vanishes independently, corresponding to the geodesic and continuity equations respectively.

4.4 Symmetric connection with vectorial torsion ($\mathcal{B} = 0$)

The *off-shell* limit, yielding an effective action that is the sum of the three Chern–Simons terms (whose coupling constants are B_2, B_7 and B_8) plus an interaction coming from the term whose coupling constant is B_5,

$$0 = 2B_2\mathcal{F}_{\nu\rho} + B_5\mathcal{R}_{\nu\rho}{}^\mu{}_\mu, \tag{16a}$$
$$0 = B_5\left(\delta^\nu_\mu\mathcal{F}_{\rho\sigma} + \delta^\nu_{[\rho}\mathcal{F}_{\sigma]\mu}\right)$$

[6] Weyl introduced two different notions of curvature, both of therm are referred as Weyl's tensors, and sometimes they are called *projective* and *conformal* Weyl tensor [56]. As physicists we are accustomed to the conformal Weyl tensor, which might is the curvature without traces (with respect to the metric). The projective Weyl tensor might be defined without requiring a metric, and it is invariant under the projective transformations of the connection, $\tilde{\Gamma}_\mu{}^\lambda{}_\nu = \Gamma_\mu{}^\lambda{}_\nu + \delta^\lambda_\mu V_\nu + \delta^\lambda_\nu V_\mu$. See Refs. [44,46,57].

[7] Contracting the latter with $\epsilon^{\mu\rho\sigma}$ yields the former.

$$+ B_7 \left(\delta^\nu_\mu \mathcal{R}_{\rho\sigma}{}^\lambda{}_\lambda + \delta^\nu_{[\rho} \mathcal{R}_{\sigma]\mu}{}^\lambda{}_\lambda\right) + B_8 \left(\mathcal{R}_{\rho\sigma}{}^\nu{}_\mu + \delta^\nu_{[\rho} \mathcal{R}_{\sigma]\lambda}{}^\lambda{}_\mu\right). \tag{16b}$$

Since Eq. (16a) relates the field strength $\mathcal{F}$ with the trace of the curvature tensor, Eq. (16b) become an equation for just curvature objects. In particular, if the coupling constants satisfy

$$\left(\frac{2B_2B_7 - B_5^2}{B_2B_8}\right) = \frac{1}{2}, \tag{17}$$

Eq. (16b) coincides with Weyl's projective curvature tensor field, which in three dimensions is

$$\mathcal{W}_{\mu\nu}{}^\lambda{}_\rho = \mathcal{R}_{\mu\nu}{}^\lambda{}_\rho - \frac{1}{4}\delta^\lambda_\rho \mathcal{R}_{\mu\nu}{}^\sigma{}_\sigma - \frac{1}{2}\left(\mathcal{R}_{\sigma\nu}{}^\sigma{}_\rho \delta^\lambda_\mu - \mathcal{R}_{\sigma\mu}{}^\sigma{}_\rho \delta^\lambda_\nu\right) - \frac{1}{8}\left(\delta^\lambda_\mu \mathcal{R}_{\nu\rho}{}^\sigma{}_\sigma - \delta^\lambda_\nu \mathcal{R}_{\mu\rho}{}^\sigma{}_\sigma\right).$$

Therefore, in this sector field equations describe a symmetric projectively-flat connection. A projectively-flat connection is *locally* written as

$$\Gamma_\nu{}^\lambda{}_\rho = \delta^\lambda_\nu \psi_\rho + \delta^\lambda_\rho \psi_\nu,$$

where ψ_μ is a generic (differentiable) vector field. From Eq. (16a), the $\mathcal{A}$-field is proportional to ψ_μ,

$$\mathcal{A}_\mu = -\frac{2B_5}{B_2}\psi_\mu.$$

Note that if the coefficients do not satisfy Eq. (17), the two equations in (16) are incompatible unless the trace of the curvature and the field strength vanish independently, i.e.

$$\psi_\mu = \partial_\mu \alpha \quad \text{and} \quad \mathcal{A}_\mu = \partial_\mu \beta.$$

In the on-shell limit the system of field equations are enriched by the additional equation,

$$B_1 \mathcal{A}_\nu \mathcal{A}_\rho - B_3 \nabla_{(\nu} \mathcal{A}_{\rho)} + B_6 \mathcal{R}_{\mu(\nu}{}^\mu{}_{\rho)} = 0, \tag{18}$$

Note that the nontrivial part of the equation comes from the symmetric part, which is an Einstein-like equation.

4.5 Symmetric connection with trace-less torsion ($\mathcal{A} = 0$)

The off-shell restriction to this sector involves the terms in the action, in Eq. (3), with coefficients from B_4, B_6, B_7 and B_8. The field equations on this off-shell limit are,

$$3B_4 \mathcal{B}_\nu{}^\mu{}_\sigma \mathcal{B}_\rho{}^\sigma{}_\mu + 2B_6 \mathcal{R}_{\mu(\nu}{}^\mu{}_{\rho)} = 0, \tag{19}$$

$$B_6\left(2\delta^\nu_\mu \nabla_\tau \mathcal{B}_\rho{}^\tau{}_\sigma + \delta^\nu_\rho \nabla_\tau \mathcal{B}_\sigma{}^\tau{}_\mu + \delta^\nu_\sigma \nabla_\tau \mathcal{B}_\mu{}^\tau{}_\rho\right) + B_7\left(\delta^\nu_\mu \mathcal{R}_{\rho\sigma}{}^\lambda{}_\lambda + \delta^\nu_{[\rho} \mathcal{R}_{\sigma]\mu}{}^\lambda{}_\lambda\right) + B_8\left(\mathcal{R}_{\rho\sigma}{}^\nu{}_\mu + \delta^\nu_{[\rho} \mathcal{R}_{\sigma]\lambda}{}^\lambda{}_\mu\right) = 0. \tag{20}$$

In the on-shell limit, there is an extra subsidiary condition,

$$B_3 \nabla_\mu \mathcal{B}_\nu{}^\mu{}_\rho + B_5 \mathcal{R}_{\nu\rho}{}^\mu{}_\mu = 0. \tag{21}$$

Note that for volume-preserving connections, Eq. (21) is nothing but the continuity equation of the T-tensor [see Eq. (15)] which might be interpreted as a conserved "energy-momentum tensor", or in case of being non-degenerated it admits the interpretation of a compatible *inverse metric* tensor field. Similarly, the term accompanying the coupling constant B_7 in Eq. (20) vanishes.

Equation (19) is an Einstein-like equation, where the term $\mathcal{B}_\nu{}^\mu{}_\sigma \mathcal{B}_\rho{}^\sigma{}_\mu$ behaves like a torsion-descendent metric.[8]

4.6 The torsion sector ($\Gamma = 0$)

The off-shell limit to the torsion sector of our model is given by restricting the action (3) to the terms with coefficients from B_1 to B_4. The field equations in this limit are

$$2B_1 \mathcal{A}_\alpha \mathcal{B}_\nu{}^\alpha{}_\rho + 2B_2 \mathcal{F}_{\nu\rho} + B_3 \partial_\mu \mathcal{B}_\nu{}^\mu{}_\rho = 0, \tag{22a}$$

$$2B_1 \mathcal{A}_\nu \mathcal{A}_\rho - 2B_3 \partial_{(\nu} \mathcal{A}_{\rho)} + 3B_4 \mathcal{B}_\nu{}^\mu{}_\sigma \mathcal{B}_\rho{}^\sigma{}_\mu = 0. \tag{22b}$$

However, the on-shell restriction to the torsion sector yields an extra condition,

$$B_3 \mathcal{A}_\mu \mathcal{B}_\rho{}^\nu{}_\sigma + B_5\left(\delta^\nu_\mu \mathcal{F}_{\rho\sigma} + \delta^\nu_{[\rho} \mathcal{F}_{\sigma]\mu}\right) + B_6\left(2\delta^\nu_\mu \partial_\tau \mathcal{B}_\rho{}^\tau{}_\sigma + \delta^\nu_\rho \partial_\tau \mathcal{B}_\sigma{}^\tau{}_\mu + \delta^\nu_\sigma \partial_\tau \mathcal{B}_\mu{}^\tau{}_\rho\right) = 0. \tag{23}$$

5 A scan of cosmological solutions

5.1 Ansatz for the connection

In order to develop further aspects of the three-dimensional polynomial affine model of gravity, we have to provide an ansatz for the connection. The ansatz are found by solving the equations derived from the vanishing Lie derivative of the connection, i.e.

$$\pounds_V \hat\Gamma_\mu{}^\lambda{}_\nu = V^\sigma \partial_\sigma \hat\Gamma_\mu{}^\lambda{}_\nu - \hat\Gamma_\mu{}^\sigma{}_\nu \partial_\sigma V^\lambda + \hat\Gamma_\sigma{}^\lambda{}_\nu \partial_\mu V^\sigma + \hat\Gamma_\mu{}^\lambda{}_\sigma \partial_\nu V^\sigma + \frac{\partial^2 V^\lambda}{\partial x^\mu \partial x^\nu} = \hat\nabla_\mu \hat\nabla_\nu V^\lambda + \hat{\mathcal{R}}_{\rho\mu}{}^\lambda{}_\nu V^\rho - \hat\nabla_\mu\left(\mathcal{T}_\nu{}^\lambda{}_\rho V^\rho\right) = 0, \tag{24}$$

where V represents a vector associated to the generators of the symmetry group, i.e. each V defines a symmetry flow, and $\mathcal{T}_\nu{}^\lambda{}_\rho$ is the torsion of the affine connection $\hat\Gamma$.

[8] To our knowledge, the analysis of such geometrical object was firstly considered by N. Popławski in Ref. [14].

Two physically interesting cases are the *isotropic* connection, which is required to analyse spherical configurations, e.g. black hole solutions, and the *isotropic and homogeneous* connection, that is compatible with the cosmological principle, and therefore required to build cosmological models.

An (affine) Friedman–Robertson–Walker spacetime can be defined as a geometry containing a constant curvature co-dimension one subspace. The embedding is such that when the self-parallel curves on the spacetime are restricted to the co-dimension one subspace, the path is also a self-parallel curve of the subspace.[9] The three possible constant curvature are customarily differentiated by the parameter κ which take values $+1$, 0 or -1.

In three dimensions, the vector fields generating the isometry groups of the different constant curvature two-dimensional subspace are (in spherical coordinates)

$$\begin{aligned} J &= \begin{pmatrix}0 & 0 & 1\end{pmatrix}, \\ X &= \sqrt{1-\kappa r^2}\begin{pmatrix}0 & \cos\varphi & -\frac{1}{r}\sin\varphi\end{pmatrix}, \\ Y &= \sqrt{1-\kappa r^2}\begin{pmatrix}0 & \sin\varphi & \frac{1}{r}\cos\varphi\end{pmatrix}, \end{aligned} \tag{25}$$

where J is associated with the sole angular momentum defining the isotropy, while X and Y are associated with the translations defining the homogeneity.

5.1.1 Isotropic connection

Since the components of the vector field J are constants, the Lie derivative of the connection along its flow is equal to that of a $\binom{1}{2}$-tensor. The vanishing Lie derivative along the vector field J yields

$$\pounds_J \hat{\Gamma}_\mu{}^\lambda{}_\nu = \partial_\varphi \hat{\Gamma}_\mu{}^\lambda{}_\nu = 0, \tag{26}$$

i.e. this symmetry does not relate different components of the connection, but its dependence on the coordinates.

At this stage, without mentioning the field equations for the connection, if one would try to find an analogue of the Schwarzschild solution, there are *twenty seven* functions to be determined.

5.1.2 Isotropic and homogeneous connection

Now, we require that the restricted components of the connection in Eq. (26), are symmetric with respect to the vector fields X and Y.[10] The procedure to solve the equations $\pounds_X \Gamma_\mu{}^\lambda{}_\nu = 0$ is tedious but straightforward (see for example Refs. [47,48]), hence we only show the results below. This time, we split the affine connection into its irreducible components,

$$\begin{aligned} &\Gamma_t{}^t{}_t = j(t), \qquad \Gamma_i{}^t{}_j = g(t) S_{ij}, \\ &\Gamma_i{}^k{}_j = \gamma_i{}^k{}_j, \quad \Gamma_t{}^i{}_j = \Gamma_j{}^i{}_t = h(t)\delta^i_j + f(t) S^{ik}\epsilon_{kj}\frac{r}{\sqrt{1-\kappa r^2}}, \end{aligned} \tag{27}$$

where f, g, h and j are functions of time, while S_{ij} and $\gamma_i{}^j{}_k$ are the two-dimensional rank two symmetric tensor and connection compatible with isotropy and homogeneity,[11] defined by

$$S_{ij} = \begin{pmatrix} \frac{1}{1-\kappa r^2} & 0 \\ 0 & r^2 \end{pmatrix},$$

and

$$\gamma_r{}^r{}_r = \frac{\kappa r}{1-\kappa r^2}, \quad \gamma_\varphi{}^r{}_\varphi = -r(1-\kappa r^2), \quad \gamma_r{}^\varphi{}_\varphi = \frac{1}{r}, \quad \gamma_\varphi{}^\varphi{}_r = \frac{1}{r}.$$

It is important to highlight the existence of an unexpected function in the components of the affine connection, to know, the f-function, which might be introduced solely in the three-dimensional case. Furthermore, as shown in Appendix C, the function j can be set to zero by a reparametrisation of the time coordinate.

Before moving to the ansatz of other fields, we would like to mention that the connection components in Eq. (27) reduce to the standard Friedman–Robertson–Walker Levi-Civita connection for $f = j = 0$, $g = a\dot{a}$ and $h = \dot{a}/a$, where a is the scale factor from the Friedman–Robertson–Walker metric. Moreover, as shown in the seminar work by Katzin, Levin and Davis [60] the symmetries of the connection would be *improper* collineations of the curvature.

The nonvanishing components of the $\mathcal{B}$-field are

$$\begin{aligned} \mathcal{B}_\varphi{}^t{}_r &= -\mathcal{B}_r{}^t{}_\varphi = \xi(t)\frac{r}{\sqrt{1-\kappa r^2}}, \\ \mathcal{B}_t{}^r{}_\varphi &= -\mathcal{B}_\varphi{}^r{}_t = \psi(t) r\sqrt{1-\kappa r^2}, \\ \mathcal{B}_r{}^\varphi{}_t &= -\mathcal{B}_t{}^\varphi{}_r = \frac{\psi(t)}{r\sqrt{1-\kappa r^2}}, \end{aligned} \tag{28}$$

while the nonvanishing component of the $\mathcal{A}$-field is $\mathcal{A}_t = \eta(t)$.

[9] A similar definition in Riemannian geometry can be found in Ref. [59].

[10] Note that due to the isotropy, calculating both Lie derivatives is redundant, therefore we need to calculate and solve only one of them.

[11] It is worth mentioning that the general two-dimensional isotropic and homogeneous (covariant and contravariant) tensors posses a non-vanishing skew-symmetric component. This off-diagonal component allows us to write the three-dimensional torsion full connection in the same form of Eq. (27) but where the matrices S and S^{-1} are no longer symmetric.

5.2 Curvature of the symmetric connection

From Eq. (27), the components of curvature are calculated, yielding

$$
\begin{aligned}
\mathcal{R}_{ti}{}^{t}{}_{j} &= -\mathcal{R}_{it}{}^{t}{}_{j} = (\dot{g} - gh)\, S_{ij} - fg\frac{r}{\sqrt{1-\kappa r^2}}\epsilon_{ij},\\
\mathcal{R}_{ij}{}^{t}{}_{t} &= 2fg\frac{r}{\sqrt{1-\kappa r^2}}\epsilon_{ij},\\
\mathcal{R}_{ti}{}^{j}{}_{t} &= -\mathcal{R}_{it}{}^{j}{}_{t} = \left(\dot{h} + h^2 - f^2\right)\delta^j_i\\
&\quad + \left(\dot{f} + 2fh\right)\frac{r}{\sqrt{1-\kappa r^2}} S^{jk}\epsilon_{ki},\\
\mathcal{R}_{ij}{}^{k}{}_{l} &= 2(gh+\kappa)S_{l[j}\delta^k_{i]} - fg\frac{r}{\sqrt{1-\kappa r^2}}\epsilon_{ij}\delta^k_l.
\end{aligned}
\tag{29}
$$

It follows that the trace of the curvature vanishes, $\mathcal{R}_{\mu\nu}{}^{\sigma}{}_{\sigma} = 0$, and therefore our connection is *equiaffine*. Hence, the Ricci tensor field is symmetric.

The nonvanishing components of the Ricci tensor field are,

$$
\mathcal{R}_{tt} = -2\left(\dot{h} + h^2 - f^2\right), \quad \mathcal{R}_{ij} = (\dot{g} + \kappa)\, S_{ij}. \tag{30}
$$

The three-dimensional Weyl projective curvature for an equiaffine connection is then,

$$
\mathcal{W}_{\mu\nu}{}^{\lambda}{}_{\rho} = \mathcal{R}_{\mu\nu}{}^{\lambda}{}_{\rho} - \frac{1}{2}\left(\mathcal{R}_{\nu\rho}\delta^\lambda_\mu - \mathcal{R}_{\mu\rho}\delta^\lambda_\nu\right), \tag{31}
$$

Note that since in general three-dimensional connections are not projectively flat, the curvature cannot be resolved in terms of the Ricci tensor, unlike what it is expected from General Relativity.

Before we turn over solving the differential equations, there are some interesting remarks:

- The f^2-function is a *parameter* for the system of ordinary differential equations for isotropic and homogeneous Ricci-flat spaces.
- Isotropic and homogeneous flat spaces require that either f or g vanishes.
- The dynamical equations for f and g are *always* integrable in terms of h and κ.
- The dynamics of h is determined by a Riccati ordinary differential equation, which is the same for both flat and Ricci-flat spaces.

Since the Riccati equation is not solvable in general, we may characterise the solutions of the system of ordinary differential equations by the partial solutions of the Riccati equation.

5.3 Space of solutions

The substitution of the cosmological ansatz for the connection into the field equations in Eqs. (7), (8) and (9), yields the system

$$
2B_8 gf - B_3\xi\eta = 0, \tag{32a}
$$
$$
B_6(2g\psi + \dot{\xi}) - B_8 fg = 0, \tag{32b}
$$
$$
B_8(2gh + \kappa - \dot{g}) = 0, \tag{32c}
$$
$$
B_3\eta\psi + B_8(2hf + \dot{f}) = 0, \tag{32d}
$$

$$
B_1\eta^2 - B_3\dot{\eta} + 3B_4\psi^2 - 2B_6(\dot{h} + h^2 - f^2) = 0, \tag{33a}
$$
$$
B_3 g\eta - 3B_4\psi\xi + B_6(\kappa + \dot{g}) = 0, \tag{33b}
$$
$$
2B_1\eta\xi + B_3(2g\psi + \dot{\xi}) = 0. \tag{33c}
$$

In the above set of equations the absence of the coefficients B_2, B_5 and B_7 is due to the form of the cosmological ansatz, e.g. since $\mathcal{A}(t) = \eta(t)\partial_t$ then its *field strength* is identically zero.

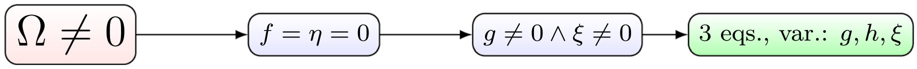
Note that the Eq. (32a) is algebraic, from Eqs. (32b) and (33c) an algebraic relation is obtained after eliminating the expression $2g\psi + \dot{\xi}$, and similarly from Eqs. (32c) and (33b) after eliminating $\dot{g}$. These expressions are

$$
2B_8 gf - B_3\xi\eta = 0, \tag{34a}
$$
$$
\frac{B_8}{B_6} gf + 2\frac{B_1}{B_3}\xi\eta = 0, \tag{34b}
$$
$$
3\frac{B_4}{B_6}\psi\xi - 2gh - \frac{B_3}{B_6} g\eta = 2\kappa. \tag{34c}
$$

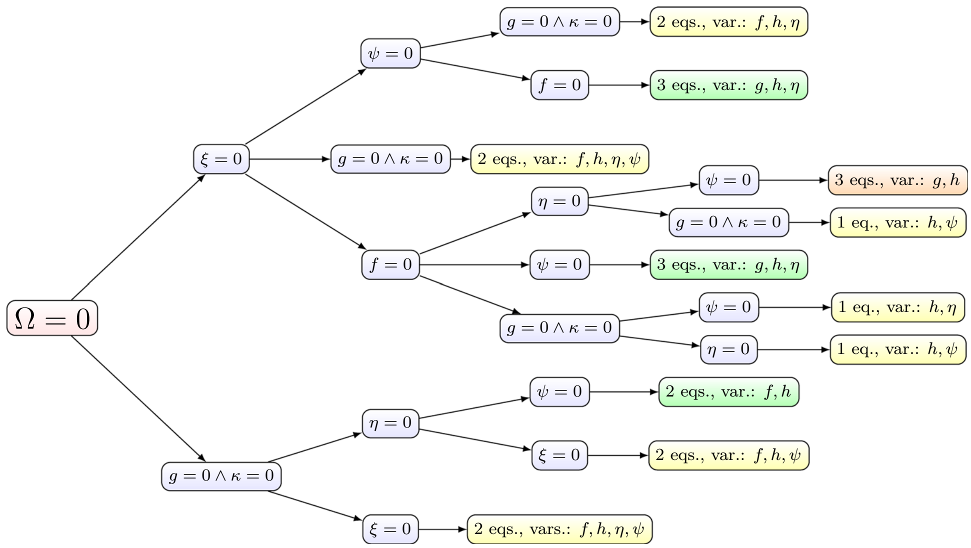
Equations (34a) and (34b) can be seen as a system of equations for the variables f and η as functions of g and ξ, but there independence is dictated by the determinant of the system coefficients,

$$
\Omega = B_8\left[4\frac{B_1}{B_3} + \frac{B_3}{B_6}\right] g\xi. \tag{35}
$$

We explore the space of cosmological solutions considering all possible cases, which can be represented by *decision trees*. On the one hand, when the determinant in Eq. (35) is nonvanishing, it follows that $f = \eta = 0$, and also that both g and ξ are nonvanishing. The field equations allow to parametrise $\psi = \psi(g, h, \xi)$, and thus the final system reduces to a set of three equations and three unknowns.



On the other hand, when the determinant Ω vanishes, following the same approach, the decision tree has more branches.[12]



The solutions obtained after exhausting the branches of the decision trees are presented below, but they are grouped according to the fields that are nontrivial.

5.4 Torsion-free limit ($\mathcal{A} = \mathcal{B} = 0$)

Consider first the field equation (10). Although at first sight the solution seems that spacetime is flat, when working the expression carefully, one notice that the requirements are that,

$$fg = 0, \quad \dot{g} - 2gh - \kappa = 0, \quad \dot{f} + 2fh = 0, \tag{36}$$

since these are the restriction of the field equations in Eq. (32) to elements obtained from the B_8-term.

However, the extra equation obtained when we take the *on-shell* torsion-free limit, i.e. Ricci-flatness, is satisfied simultaneously if and only if the connection is flat.

5.4.1 Projectively-flat solutions

A projectively-flat connection requires to solve the Eq. (36).

There are two *branches* of solutions, with either $g = 0$ or $f = 0$. None of these *branches* requires the connection to be flat. Additionally, since h is a non-dynamical function, it is possible to solve the field equations in terms of an h-parameter function.

The branch $g = 0$, requires vanishing κ. In this scenario, the f-function is solved by

$$f(t) = C_f e^{-2H(t)},$$

where $H(t) = \int_{t_0}^{t} \mathrm{d}\tau\, h(\tau)$.

In the branch $f = 0$, again the h-function plays the role of a parameter function, and the *dynamical* function is given by

$$g(t) = e^{2H(t)} \left(C_g + \kappa \int_{t_0}^{t} \mathrm{d}\tau\, e^{-2H(\tau)} \right),$$

with C_g the integration constant.

Note that the case $f = g = 0$ leaves the function h undetermined, but it is compatible with both of the previous branches.

5.4.2 Flat solutions

As mentioned previously, flat connections solve the field equations obtained from the on-shell torsion-less limit of the polynomial affine model of gravity.

From Eq. (29), the system of ordinary differential equations defining isotropic and homogeneous affinely connected flat spaces are

$$fg = 0, \qquad \dot{g} - gh = 0, \quad gh + \kappa = 0,$$
$$\dot{h} + h^2 - f^2 = 0, \quad \dot{f} + 2fh = 0.$$

[12] Note that in the decision tree we do not include branches for the cases where the parameters of the model are constrained, i.e. the case $B_8 = 0$ and $B_3^2 + 4B_1B_6 = 0$. These cases will be analysed in Sect. 5.10.

As in the previous case, there are two branches of solutions. The branch with $f = 0$, the field equations are solved by

$$h(t) = \frac{1}{t + C_h}, \quad g(t) = -\kappa(t + C_h), \tag{37}$$

with C_h the integration constant of the equation for the h-function.

The second branch requires $g = 0 \wedge \kappa = 0$. Since the case $f = 0$ is included the other branch, we restrict to $f \neq 0$. Therefore, the f-function can be integrated in terms of h,

$$f(t) = e^{-2H(t)}.$$

Then, the field equation for h turns into a second order differential equation. The solution of the system is given by

$$h(t) = \frac{t + c_1}{(t + c_1)^2 + 1}, \quad f(t) = \frac{1}{(t + c_1)^2 + 1}. \tag{38}$$

5.4.3 Ricci-flat solutions

Ricci-flat connections solve the field equations from the on-shell torsion-free limit of polynomial affine gravity solely if the coupling constant B_8 is zero.

From Eq. (30), the system of ordinary differential equations defining isotropic and homogeneous affinely connected Ricci-flat spaces are

$$\dot{h} + h^2 - f^2 = 0, \qquad \dot{g} + \kappa = 0. \tag{39}$$

The solution for the differential equation for g is

$$g(t) = -\kappa t + C_g.$$

The first equation in Eq. (39) is a Riccati ordinary differential equation, in which the f-function plays the role of *parameter* function. A well-known strategy to solve the Riccati equation is to transform it into a second order linear differential equation. The transformation $u(t) = e^{H(t)}$, takes it onto

$$\ddot{u} - f^2 u = 0. \tag{40}$$

Equation (40) can be immediately compared with the one-dimensional time-independent Schrödinger equation, where u would be the wave function, the f^2-function plays the role of the quantum mechanical potential minus the energy eigenvalue.

5.5 Purely vectorial sector ($\Gamma = \mathcal{B} = 0$)

The off-shell limit toward the purely vectorial sector is dominated by the term whose coefficient is B_2. The field equations are identically satisfied, given that for $\mathcal{A}_\mu = \eta(t)\delta^0_\mu$ its field strength vanishes, i.e. the field equations impose no restriction to the function η.

On the other hand, the subsidiary condition coming from the on-shell limit is

$$B_3\,\dot{\eta} - B_1\,\eta^2 = 0,$$

whose solution is

$$\eta(t) = -\frac{B_3}{B_1}\frac{1}{t + C_\eta}. \tag{41}$$

5.6 Purely traceless torsion sector ($\Gamma = \mathcal{A} = 0$)

In this sector, the field equations from the off-shell limit become $\psi^2 = \psi\xi = 0$, whose solution is driven by $\psi(t) = 0$, while $\xi(t)$ remains as an unknown function.

When one considers the on-shell limit, the system of field equations is

$$\dot{\xi} = 0, \quad \kappa = 0, \quad \psi^2 = 0, \quad -3B_4\,\psi\xi + B_6\,\kappa = 0, \tag{42}$$

which is solved by

$$\xi(t) = C_\xi, \qquad \psi(t) = 0, \qquad \kappa = 0. \tag{43}$$

5.7 Connection with vectorial torsion ($\mathcal{B} = 0$)

The field equations, in the *off-shell* limit, are those of the non-Abelian Chern–Simons for the symmetric connection and the Chern–Simons for the $\mathcal{A}$-field. They, however do not interact, and additionally in the cosmological ansatz the field equation for $\mathcal{A}$ is automatically satisfied. Therefore, the cosmological models with vectorial torsion in the off-shell limit, do not differ from those of the torsion-free (off-shell) limit mixed – but non-interacting – with an unconstrained vector field $A_\mu = \eta(t)\delta^0_\mu$.

The field equations obtained in the *on-shell* limit come with an additional condition, which can be written as

$$B_6\mathcal{R}_{\mu\nu} - B_3\nabla_\mu\mathcal{A}_\nu + B_1\mathcal{A}_\mu\mathcal{A}_\nu = 0.$$

The above expression yield two independent field equations,

$$-2B_6\left(\dot{h} + h^2 - f^2\right) - B_3\dot{\eta} + B_1\eta^2 = 0, \tag{44}$$

$$B_6\,(\dot{g} + \kappa) + B_3\,g\eta = 0. \tag{45}$$

Therefore the equations to solve are Eqs. (36), (44) and (45). From Eqs. (36), the branches structure is inherited, i.e. $g = 0 \vee f = 0$.

The branch of solutions with vanishing g requires that $\kappa = 0$. The nontrivial field equations are then Eq. (44) and

$$\dot{f} + 2fh = 0.$$

Since $f(t) = \exp(-2H(t))$, Eq. (44) can be recasted as

$$-2B_6\left(\dot{h} + h^2 - e^{-4H(t)}\right) - B_3\dot{\eta} + B_1\eta^2 = 0.$$

From the last equation it is evident that a solution is given by the flat connection (38), together with

$$\eta(t) = -\frac{B_3}{B_1}\frac{1}{t+C_\eta}.$$

Note in addition that for a generic function h, unrelated to η, we can define

$$\phi(t) = \frac{2B_6}{B_1}\left(\dot{h} + h^2 - e^{-4H(t)}\right),$$

resulting in a Riccati equation for the η-function, which might be transformed into the one-dimensional Schrödinger equation

$$\ddot{u} - \frac{B_3}{B_1}\phi(t)u(t) = 0,$$

with the change of variable, $u(t) = \exp\left(-\frac{B_3}{B_1}\mathcal{H}(t)\right)$ where $\mathcal{H}(t) = \int_{t_0}^{t} \mathrm{d}\tau\, \eta(\tau)$. Summarising, in this branch, given a *parameter* function h, it is (in principle) possible to integrate the field equations to determine the connection.

On the other hand, in the branch $f = 0$, the nontrivial differential equations to solve are

$$\begin{aligned} B_1\eta^2 - B_3\dot{\eta} - 2B_6(\dot{h} + h^2) = 0,\\ B_6(\dot{g} + \kappa) + B_3\eta g = 0,\\ \dot{g} - 2gh - \kappa = 0. \end{aligned} \tag{46}$$

Solving for η and h from the last two expressions and substituting into the first, one obtains a differential equation for the g-function,

$$\frac{B_6}{B_3}\left(B_3^2 - 2B_1B_6\right)\frac{\dot{g}+\kappa}{g} = 0. \tag{47}$$

There are three branches of solutions:

1. For $\dot{g} + \kappa = 0$, it follows that $g = -\kappa t + C_g$, $\eta = 0$ and $h = \frac{\kappa}{2(\kappa t - C_g)}$. Note that for $\kappa = 0$ this solution is valid if $C_g \neq 0$, but the solution with $g = 0$ requires that $h = \frac{1}{t+C_h}$.
2. For $B_6 = 0$ there are two kinds of solutions, both with arbitrary h function: (i) $g = 0 \wedge \kappa = 0$ and $\eta = -\frac{B_3}{B_1}\frac{1}{t+C_\eta}$, or (ii) $\eta = 0$ and $g(t) = e^{2H(t)}\left(C_g + \kappa\int_{t_0}^{t}\mathrm{d}\tau\, e^{-2H(\tau)}\right)$.
3. For $B_3^2 = 2B_1B_6$, the solutions are parametrised by the function g, which is required to be nonvanishing and C^1. Hence, $h = \frac{\dot{g}-\kappa}{2g}$ and $\eta = -\frac{B_3}{B_1}\frac{\dot{g}+\kappa}{g}$.

5.8 Connection with traceless torsion ($\mathcal{A} = 0$)

The field equations in the off-shell limit are obtained from (19) and (20), and yield

$$\begin{aligned} &fg = 0,\quad B_6(2g\psi + \dot{\xi}) - B_8 fg = 0,\quad \dot{g} - 2gh - \kappa = 0,\\ &\dot{f} + 2fh = 0,\quad 3B_4\psi^2 - 2B_6(\dot{h} + h^2 - f^2) = 0,\\ &-3B_4\psi\xi + B_6(\kappa + \dot{g}) = 0. \end{aligned}$$

The *extra* condition obtained from the on-shell limit is

$$\dot{\xi} + 2g\psi = 0,$$

was already satisfied with the off-shell equations, i.e. there is no difference between the off-shell and on-shell $\mathcal{A} \to 0$ limit, for the cosmological ansatz.

The solutions to the system of field equations are categorised in to classes, those with $g = 0$ or $f = 0$.

The branch of solutions with $g = 0$ requires that $\kappa = 0$ and either $\psi = 0$ or $\xi = 0$. On the one hand, for $\psi = 0$, the field equations require that $\xi = C_\xi$, while the nontrivial equations are solved by (38), with the difference that the torsion tensor field is nonvanishing. on the other hand, for $\xi = 0$ the field equations to be solved are

$$\dot{f} + 2fh = 0 \quad \text{and} \quad \dot{h} + h^2 - f^2 - \frac{3B_4}{2B_6}\psi^2 = 0.$$

Since there are three unknowns and only two equations, the solutions are parametrised by one of the unknowns, e.g. ψ. A simple solution is obtained for $\psi \propto f$, with results similar to those in Eq. (38). Note that another solution is given by $f = 0$, in whose case the sole nontrivial field equation is

$$\dot{h} + h^2 - \frac{3B_4}{2B_6}\psi^2 = 0. \tag{48}$$

This is a Riccati equation for h, which is equivalent to a one-dimensional Schrödinger equation, in which the function ψ is the analogous to the quantum mechanical potential.

The system of equations for the branch with $f = 0$ is

$$\begin{aligned} &\psi(t) + \frac{\dot{\xi}}{2g} = 0,\ h(t) - \frac{\dot{g}-\kappa}{2g} = 0,\\ &-3B_4\dot{\xi}^2 + 2B_6(\kappa^2 - \dot{g}^2 + 2g\ddot{g}) = 0,\ 2B_6(\kappa + \dot{g})\\ &+\frac{3B_4\xi\dot{\xi}}{g} = 0 \end{aligned} \tag{49}$$

The system of Eq. (49) is solved by

$$\psi(t) = -\frac{\dot{\xi}}{2g}, \tag{50a}$$

$$h(t) = \frac{\dot{g}-\kappa}{2g}, \tag{50b}$$

$$\xi(t) = \sqrt{C_\xi^2 - \frac{2B_6}{3B_4}\left(g^2 + 2\kappa\int g\,\mathrm{d}t\right)}, \tag{50c}$$

$$\begin{aligned} 0 = g\ddot{g}\left(g^2 + 2\kappa\int g\,\mathrm{d}t\right) + \kappa g^2(\dot{g}+1)\\ +\kappa\int g\,\mathrm{d}t\left(1-\dot{g}^2\right) + \frac{3B_4}{4B_6}C_\xi\left(\dot{g}^2 - 2g\ddot{g} - \kappa^2\right). \end{aligned} \tag{50d}$$

Particularly, for $\kappa = C_\xi = 0$ the explicit solution is given by[13]

$$f(t)=0,\quad g(t)=C_m t+C_g, h(t)=\frac{C_m}{2(C_m t+C_g)},$$
$$\xi(t)=\sqrt{-\frac{2B_6}{3B_4}}\left(C_m t+C_g\right),$$
$$\psi(t)=-\sqrt{-\frac{B_6}{6B_4}\frac{C_m}{C_m t+C_g}}, \eta(t)=0, \tag{51}$$

where $\operatorname{sgn}(B_4)=-\operatorname{sgn}(B_6)$.

For $\kappa \neq 0$ a simplification of the field equations is to propose g as a linear function of t, yielding

$$\begin{aligned} f(t)&=0, \quad g(t)=-\kappa t+C_g, \quad h(t)=\frac{1}{t-C_g\kappa},\\ \xi(t)&=C_\xi \quad \psi(t)=0, \qquad \eta(t)=0. \end{aligned} \tag{52}$$

5.9 Restriction to torsional sector ($\Gamma = 0$)

The solutions to the off-shell restriction to the torsional sector require that either $\psi = 0$ or $\xi = 0$.[14] For $\xi = 0$ the nontrivial field equation is the Riccati-like equation,

$$B_1\eta^2 - B_3\dot{\eta} + 3B_4\psi^2 = 0, \tag{53}$$

which (as mentioned before) is equivalent to a one-dimensional Schrödinger equation whose potential is related to the function ψ^2. For $\psi = 0$, the explicit solution is

$$\psi(t)=0,\quad \eta(t)=-\frac{B_3}{B_1}\frac{1}{t+C_\eta},\quad \xi(t)=C_\xi\left(t+C_\eta\right)^2. \tag{54}$$

In the on-shell limit, the subsidiary conditions are $\xi\eta = 0$, $\dot{\xi} = 0$ and $\psi\eta = 0$. Hence, there is no solution because the field equations require that $\mathcal{B} = 0$.

5.10 Exceptional solutions to the whole model

The cosmological solutions to polynomial affine model of gravity in three dimensions with all the fields turned on are *exceptional*, since we are solving the seven dimensional system of field equations (32), with only six unknowns. Generically, the consistency conditions of the system require that either some of the functions vanish (worsening the well-being of the system) or a relation between the parameters of the model.

A first example of these is given by the case $g = \xi = \kappa = 0$, with nontrivial field equations

$$\dot{f}+2fh=-\frac{B_3}{B_8}\eta\psi, \tag{55a}$$
$$\dot{h}+h^2-f^2=\frac{1}{2B_6}\left(B_1\eta^2-B_3\dot{\eta}+3B_4\psi^2\right). \tag{55b}$$

These field equations are solvable when all functions are inversely proportional to t.

Another branch of solutions is found when the coupling constants are not all independent. As an example, for $B_3^2 + 4B_1B_6 = 0$ the field equations allow to decouple the functions g and ξ – which are unconstrained –, from the equations for f and ψ, which should satisfy the differential equations (55). We were able to find two types of solutions of this system of equations (which do not fall into the previously presented categories).

– Solution with $B_3 = 0$: With this condition, the solution is characterised by the functions

$$\begin{aligned} &f(t)=0,\quad g(t)=-\kappa t+C_g,\quad h(t)=\frac{1}{t-\kappa C_g},\\ &\xi(t)=\sqrt{-\frac{2B_6}{3B_4}}\;C_g,\ \psi(t)=0,\quad \eta(t)=\text{arbitrary}, \end{aligned} \tag{56}$$

for $\kappa \neq 0$. While for $\kappa = 0$, the functions defining the connection are

$$\begin{aligned} &f(t)=0,\quad g(t)=C_m t+C_g,\quad h(t)=\frac{C_m}{2(C_m t+C_g)},\\ &\xi(t)=\sqrt{-\frac{2B_6}{3B_4}}(C_m t+C_g),\quad \psi(t)=\sqrt{-\frac{B_6}{6B_4}\frac{C_m}{C_m t+C_g}},\\ &\qquad \eta(t)=\text{arbitrary}. \end{aligned} \tag{57}$$

– Solutions for $\kappa = 0$: The system of field equations can be solved for the ansatz $g(t) = t^n$ with $n \in \mathbb{R} - \left\{-2, \left[\frac{1-\sqrt{33}}{4}, \frac{1+\sqrt{33}}{4}\right]\right\}$, with the addition condition

$$B_4=-\frac{8B_6^3(n+2)}{3B_8^2\left(2n^3-3n^2-3n+4\right)}.$$

The functions defining the connection are

$$f(t)=\frac{\sqrt{2n^2-n-4}\,\operatorname{sgn}(B_8)}{2t},\quad g(t)=t^n,$$
$$h(t)=\frac{n}{2t},\quad \eta(t)=\frac{2B_6}{B_3 t},$$

[13] Formally, the field equations can be solved for $C_\xi \neq 0$, but the solution for the g-function is expressed in terms of the inverse of an hypergeometric function. Hence, we have omitted the details of such solution.

[14] Note that ψ and ξ cannot vanish at the same time, since it would imply that $\mathcal{B} = 0$.

$$\xi(t) = \frac{\sqrt{2n^2 - n - 4}\,\mathrm{sgn}(B_8) t^n}{2B_6},$$
$$\psi(t) = \frac{(n-1)\sqrt{2n^2 - n - 4}\,\mathrm{sgn}(B_8)}{4tB_6}. \tag{58}$$

6 Discussion and conclusions

In an attempt to bring gravity to the same footing with gauge theories, we have proposed a model of gravity whose sole fundamental field is the affine connection. Such model has been named *polynomial affine gravity*. Our model might be understood as a Schwarz topological theory, in the sense that the metric plays no role in the model building, similar to the case of Chern–Simons theories.

The polynomial affine model of gravity has been built in three and four dimensions (i.e. there are no *ab initio* restrictions on the dimension of the space, unlike for Chern–Simons theories), and possesses attractive features. Firstly, the model is appealing for a quantum theory of gravity, since all the terms in the action are power-counting renormalisable, and in addition the lack of additional invariant forms would forbid the existence of counter-terms. Secondly, the absence of an energy scale, reflected by the fact that all the coupling constants are dimensionless, appears as a hint (of a sort) of conformal invariance (at least at tree level).

Customarily, the conformal transformation is understood (in metric gravitational models) as a point-wise scaling of the metric tensor field, and the invariant curvature under these transformations is the conformal Weyl tensor field, i.e. the g-traceless part of the Riemann–Christoffel curvature. This notion, can be generalised without evoking a metric. The idea is that *self-parallel curves* can be preserved under "generalised" transformations. These are the *projective* transformations, and the invariant curvature under these transformations is the projective Weyl tensor field.

In this article we focus in the three-dimensional version of the polynomial affine model of gravity, where the action and therefore the field equations are simpler than their four-dimensional analogous, expecting the physical interpretation to be clearer. Note that in comparison the three-dimensional action [see Eq. (3)] is determined by eight terms (including the Chern–Simons terms) while the four-dimensional one is composed by twenty terms (disregarding topological terms), unrelated through boundary terms.

Interestingly, the term of the action with coefficient B_3 can be re-written (up to boundary term) as, $\nabla_\mu \mathcal{A}_\alpha \mathcal{B}_\beta{}^\mu{}_\gamma \,\mathrm{d}V^{\alpha\beta\gamma}$. In this case, the $\mathcal{B}$-field can be solved algebraically.

It is worth noticing that unlike the three-dimensional version of General Relativity, in an affine model the (projective) Weyl tensor does not vanish necessarily, and thus there is room to novel phenomenological effects. In particular, the field equation (9) contains terms that can be related to the projective Weyl curvature, and therefore its space of solutions might differ from the one expected in General Relativity, even in the cases where the field equations are alike, e.g. for flat or Ricci-flat manifolds.

The field equations derived from the action in Eq. (3), include a generalisation of the Einstein field equations [see Eq. (8)],

$$\mathcal{R}_{\mu(\nu}{}^\mu{}_{\rho)} = -\frac{B_1}{B_6}\mathcal{A}_\nu \mathcal{A}_\rho + \frac{B_3}{B_6}\nabla_{(\nu}\mathcal{A}_{\rho)} - \frac{3B_4}{2B_6}\mathcal{B}_\nu{}^\mu{}_\sigma \mathcal{B}_\rho{}^\sigma{}_\mu = \tilde{T}_{\nu\rho}, \tag{59}$$

obtained by varying with respect to the $\mathcal{B}$-field is the *analogous* to the Einstein equations written in the Ricci form. Noticeable, the fact that – even for torsion-free truncation – the field equations for the symmetric connection appear from the variation with respect to other field, has been interpreted as a sign of the non-uniqueness of the Lagrangian description of the system [61].

The Eq. (59) represents a non-Riemannian generalisation of the Einstein equations in the Ricci form, where the right-hand side geometrically encodes what in General Relativity is attributed to the presence of matter. However, $\tilde{T}$ does not admit a separation between material and geometrical contributions, unlike its analogous form in General Relativity, which is expressed in terms of the energy–momentum tensor $T_{\mu\nu}$ and its trace T, as

$$\tilde{T}^{GR}_{\nu\rho} \propto T_{\nu\rho} - T g_{\nu\rho}.$$

Furthermore, when $\tilde{T}$ is non-degenerated, it equips the affine manifold with a torsion-descendent notion of metric, and hence Eq. (59) provides a notion of *affine* Einstein manifold. A nice feature of this torsion-descendent metric is that, unlike the *emergent metric* from Ref. [49], it might be well-defined even when the space is Ricci-flat.

However, since the affine connection is a less intuitive geometrical object (in comparison with the metric), we analysed the possible truncations of the model, i.e. sectors where only a subset of the irreducible components of the connections are nontrivial.

Turning to the solutions of the field equations, we found the ansätze of the three-dimensional affine connection compatible with the cosmological principle. Firstly, we found that the symmetric connection is determined by three functions,[15] f, g and h. The f function has no analogous in a cosmological Levi-Civita connection, and therefore it is a non-Riemannian parameter. In addition, the g and h functions describe a non-Riemannian cosmological geometry

[15] When solving the differential equations obtained from the Lie derivative of the connection, there is a fourth parametric function (j) characterising the affine connection, but we show in Appendix C that this parameter can be eliminated by a reparametrisation on the *time* coordinate.

unless they could be parametrised in terms of a scale factor, $a = a(t)$, as

$$g = a\dot{a} \quad \text{and} \quad h = \frac{\dot{a}}{a}. \tag{60}$$

Hence, in the torsion-free sector of the polynomial affine model of gravity the non-Riemannian structure percolates the Einstein-like equations if $f \neq 0$ and/or g and h are not parametrised as in Eq. (60).

Before discussing the cosmological structure of the torsional fields, we would like to briefly mention the geometrical meaning of the f function in the cosmological ansatz of the symmetric connection. Consider the symmetric part of the affine connection defining the polynomial affine model of gravity, $\Gamma_\mu{}^\lambda{}_\nu$, and a generic metric, $g_{\mu\nu}$. Let $\mathring{\Gamma}_\mu{}^\lambda{}_\nu$ be the Levi–Civita connection associated to the metric g. The difference between the two connections, $\Gamma_\mu{}^\lambda{}_\nu - \mathring{\Gamma}_\mu{}^\lambda{}_\nu$, is a tensor defining the Weyl's *congruent transferences*.[16] Such tensor does not have an analogous in Riemannian geometry, since it is related to the non-metricity. The cosmological ansatz for the connection in three dimensions admits certain components of the *transference* tensor, determined by the function f. Interestingly, in the cosmological ansatz in four dimensions requires vanishing *transference* tensor.

The (cosmological) torsion field is characterised by three functions, η, ξ and ψ, the first one defines the $\mathcal{A}$-field and the remaining two define the $\mathcal{B}$-field. In comparison with the four-dimensional case, which is characterised by just two functions, the torsion tensor field is less restricted. Noticeable, the same is true about the characterisation of the (cosmological) symmetric connection, since in the four-dimensional scenario it is determined (after the time reparametrisation) by solely two functions.[17]

Using the cosmological ansatz, the Eq. (59) is written as

$$\begin{aligned} &\dot{h} + h^2 = \Xi(\text{non-Riemannian terms in } \hat{\Gamma}), \\ &\left(\text{in GR: } \dot{H} + H^2 \propto (\rho + 3p) \text{ with } H = \frac{\dot{a}}{a}\right) \end{aligned} \tag{61}$$

where h is one of the functions defining the symmetric connection and its analogous in General Relativity, i.e. H, is the Hubble parameter. Equation (61) is a generalisation of the Friedmann equation, where other geometric fields uphold what in General Relativity would be interpreted as matter effects. Generically, the Ξ function depends on all the other (i.e. non h) functions characterising the affine connection, but it is peculiar that the case where only f and h are nonvanishing, the Eq. (61) becomes a Riccati ordinary differential equation

$$\dot{h} + h^2 - f^2 = 0,$$

which can be expressed through the transformation $u(t) = e^{H(t)}$, where $H(x) = \int_{x_0}^{x} \mathrm{d}y\, h(y)$, onto a one-dimensional "time-independent" Schrödinger equation, $\ddot{u} - f^2 u = 0$, where u would be the wave function, the f^2-function plays the role of the quantum mechanical potential (with the energy eigenvalue subtracted). Note that particularly in the flat and Ricci-flat cases (coming from the torsion-free truncation) the Riccati equation with constant f is analogous to the (acceleration) Friedmann equation describing a Universe filled with a perfect fluid in a dark energy dominated era, which in General Relativity is

$$\dot{h} + h^2 - \frac{8\pi G}{3}\rho_{DE} = 0.$$

In Sect. 5 have found cosmological solutions to the field equation of the polynomial affine model of gravity, characterised by the system in Eq. (32). We proceeded systematically, scanning all the possible kinds of solutions. We noted that the solutions split into two categories, depending on whether the function Ω from Eq. (35) vanishes or not. However, we re-classified the solutions according to the type of truncation they belong to. It is worth mentioning that even though in the classification of solutions we do not consider explicitly the case with vanishing B_8, this branch of solutions lying in the sector $\Omega = 0$, yields no additional solutions to those in Sect. 5, e.g. the solutions with $f = 0$ in Sect. 5.4.2 and the solutions in Sect. 5.8. Furthermore, in the sector with $4B_1B_6 + B_3^2 = 0$ we found solutions to the whole system of field equations, where none of the functions determining the components of the connection vanish.

Now, even though we have found *affine* cosmological solutions to the polynomial affine model of gravity in three dimensions, *real life* applications require the existence of a metric. In purely affine models, although the fundamental field is the connection, it is possible to define various types of (derived) metrics. Let us mention four of these derived metrics:

1. The symmetric part of the Ricci tensor field, when it is non-degenerated, serves as a metric. This was noticed very early in the development of differential geometry (see for example section 5 of Ref. [44]), and used recently in Ref. [49]. A notable disadvantage is that interesting cases, such as Minkowski and Schwarzschild, cannot be described using this notion of metric.
2. The quasi-Hodge dual of the $\mathcal{B}$-field, i.e. $T^{\mu\nu} = \frac{1}{2}\mathcal{B}_\alpha{}^\mu{}_\beta \epsilon^{\nu\alpha\beta}$, is a symmetric $\binom{2}{0}$-tensor density. When T is non-degenerated, it can be used as an *inverse metric density*, similar to that used by Eddington, Einstein, Schrödinger and others to build affine models of General

[16] These transformations might be also called *congruent transplantation*, which is the translation of the original German vocable (kongruente Verpflanzung).

[17] We would like to stress that in Refs. [47–49] we were not aware of the *time* reparametrisation, and therefore the additional function turns the manipulation of the field equations into a more cumbersome process.

Relativity (see Ref. [62]). In Ref. [21] this analogue is used to intuitively relate the three-dimensional action of polynomial affine gravity with General Relativity non-minimally coupled to the $\mathcal{A}$-field (see Eq. (9) of the referred article).
3. The construction $\mathcal{T}_{\mu}{}^{\lambda}{}_{\rho}\mathcal{T}_{\nu}{}^{\rho}{}_{\lambda}$, defined from the torsion, is a symmetric $\binom{0}{2}$-tensor field that serves (when non-degenerated) as a metric. This tensor was introduced by Poplawski in Ref. [14], and it is related to the symmetric part of Eq. (A.7) (since the $\mathcal{S}$-tensor is proportional to the torsion).
4. The symmetric part of $\mathcal{S}_{\sigma\mu}{}^{\sigma}{}_{\nu}$ in Eq. (A.7) can also be interpreted as a metric, when it is non-degenerated.

Each of the examples above serves to endow the affine manifold with a metric structure with respect to which we could measure geodesic distances and compare with the parallel transport obtained using the symmetric connection.

Only a few of the solutions explicitly presented in the paper admit non-degenerated metrics. From the solution to the coupled system Γ–$\mathcal{A}$, the case with vanishing f and B_6 [see the second type of solutions after Eq. (47)] possesses a *Ricci* metric as long as the arbitrary function h is not the reciprocal of t, while the case with vanishing f and $B_3^2 = 2B_1B_6$ [see the third type of solutions after Eq. (47)] possesses a metric of the fourth kind, as long as $g \neq -\kappa t + C_g$. The solutions of the coupled system Γ–$\mathcal{B}$ with $\kappa = 0$ [see Eq. (51)] and the exceptional cases with vanishing κ [see Eqs. (57) and (58)], possess the first three types of metric described above, as long as $C_m \neq 0$ and $n \neq 0$. In addition, one can endow the exceptional solutions with the fourth kind of metric, with the constrain that η is not a constant function in Eqs. (56) and (57).

The above opens the discussion of the background independence of the gravitational model, which is explicitly broken in General Relativity due to the presence of the metric in the Einstein–Hilbert action.

We would like to conclude this article highlighting some interesting questions that are not completely answered or understood, from the point of view of our model.

Firstly, we need to mention a *relation* between the polynomial affine model of gravity and General Relativity. Note that if the field $T^{\mu\nu}$ from Eq. (15), which is a symmetric $\binom{2}{0}$-tensor density, is identified with the inverse metric density from General Relativity, i.e. $T^{\mu\nu} \equiv \sqrt{g}g^{\mu\nu}$,[18] the action in Eq. (3) is analogous to the three-dimensional General Relativity with cosmological constant, nonminimally coupled with a vector field $\mathcal{A}$ and the Chern–Simons terms [21].

Secondly, so far our analysis is based in the *pure gravity* sector, i.e. without the addition of matter. The absence of a fundamental metric tensor in the formulation of the model forbids the standard inclusion of matter through the minimal coupling procedure. Hence, the interaction with matter needs to be reformulated. Even though we have been able of coupling a scalar field with the polynomial affine model of gravity (in four dimensions), we are working in how to generalise such coupling to other types of fields (in particular, fermionic matter).

Thirdly, although our preliminary analysis of the dynamics of the model is not conclusive, we venture to conjecture that a gravitational model based mediated by the most general affine connection contains propagating degrees of freedom, unlike the three-dimensional version of General Relativity.

Our study provides the framework for analysing the four-dimensional version of the polynomial affine model of gravity, since the development of this paper allowed us to focus in deepen our understanding of the geometrical structure of affinely connected manifolds, without distracting ourselves by the complexity introduced on the field equations through additional components available in higher dimensional spaces. Further studies, which take these tools into account, will allow us to complete the classification of the solutions in four dimensions started in Refs. [48,49].

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[18] This identification has been used by Schrödinger [6] and Kijowski [7].

Appendix A: Notions of non-Riemannian geometry

The aim of this appendix is to provide a short summary of results in non-Riemannian geometry, and also intends to fix the notation used through the development of the paper. Readers interested in the subjects are recommended to review the classical texts by Eisenhart [44] and Schouten [45], the final chapter of the book by Synge and Schild [63], the book by Gilkey and collaborators [64], and articles like Refs. [65–67].

A d-dimensional affine manifold $(\mathcal{M}, \hat{\nabla})$ is a d-dimensional differential manifold $\mathcal{M}$ equipped with a linear connection $\hat{\nabla}$. The linear connection is determined by their d^3-independent components $\hat{\Gamma}_\mu{}^\lambda{}_\rho$, such that

$$\hat{\nabla}_\mu = \partial_\mu + \hat{\Gamma}_\mu{}^\bullet{}_\bullet.$$

Since an affine structure does not require the existence of a metric tensor field, the affine connection admits a decomposition in their lower indices, into their symmetric and skew-symmetric parts,

$$\hat{\Gamma}_\mu{}^\lambda{}_\rho = \hat{\Gamma}_{(\mu}{}^\lambda{}_{\rho)} + \hat{\Gamma}_{[\mu}{}^\lambda{}_{\rho]} = \Gamma_\mu{}^\lambda{}_\rho + \mathcal{S}_\mu{}^\lambda{}_\rho.$$

The symmetric component of the connection remains as a connection, which we denote by simply $\Gamma_\mu{}^\lambda{}_\rho$, while the skew-symmetric component is a tensor field proportional to the torsion tensor field (explicitly, it is twice the tensor field).

The curvature of a connection, defined by

$$\hat{\mathcal{R}}(X, Y)Z = \left(\hat{\nabla}_X\hat{\nabla}_Y - \hat{\nabla}_Y\hat{\nabla}_X - \hat{\nabla}_{[X,Y]}\right) Z, \tag{A.1}$$

with X, Y and Z vector fields, can be written in components as

$$\hat{\mathcal{R}}_{\mu\nu}{}^\lambda{}_\rho = \partial_\mu\hat{\Gamma}_\nu{}^\lambda{}_\rho - \partial_\nu\hat{\Gamma}_\mu{}^\lambda{}_\rho + \hat{\Gamma}_\mu{}^\lambda{}_\sigma\hat{\Gamma}_\nu{}^\sigma{}_\rho - \hat{\Gamma}_\nu{}^\lambda{}_\sigma\hat{\Gamma}_\mu{}^\sigma{}_\rho. \tag{A.2}$$

The curvature tensor field is skew-symmetric in the first couple of indices, and therefore the contra-variant index can be contracted in two independent ways, $\hat{\mathcal{R}}_{\lambda\nu}{}^\lambda{}_\rho$ and $\hat{\mathcal{R}}_{\mu\nu}{}^\lambda{}_\lambda$, referred to as the first and second Ricci curvatures. Customarily, the first Ricci curvature is simply called Ricci tensor field, while the second Ricci curvature is also referred to as homothetic curvature or *trace of the curvature*. Note that when restricting oneself to Riemannian connections the Ricci tensor field is symmetric and the trace of the curvature vanishes, but for generic affine connections these properties do not hold.

The curvature in Eq. (A.2) can be decomposed in terms of the symmetric connection $\Gamma_\mu{}^\lambda{}_\rho$ and the tensor $\mathcal{S}_\mu{}^\lambda{}_\rho$, yielding

$$\begin{aligned}\hat{\mathcal{R}}_{\mu\nu}{}^\lambda{}_\rho &= \mathcal{R}_{\mu\nu}{}^\lambda{}_\rho + \hat{\nabla}_\mu\mathcal{S}_\nu{}^\lambda{}_\rho - \hat{\nabla}_\nu\mathcal{S}_\mu{}^\lambda{}_\rho \\ &\quad -\mathcal{S}_\mu{}^\lambda{}_\sigma\mathcal{S}_\nu{}^\sigma{}_\rho + \mathcal{S}_\nu{}^\lambda{}_\sigma\mathcal{S}_\mu{}^\sigma{}_\rho - 2\mathcal{S}_\mu{}^\sigma{}_\nu \\ \mathcal{S}_\sigma{}^\lambda{}_\rho &= \mathcal{R}_{\mu\nu}{}^\lambda{}_\rho + \nabla_\mu\mathcal{S}_\nu{}^\lambda{}_\rho \\ &\quad -\nabla_\nu\mathcal{S}_\mu{}^\lambda{}_\rho + \mathcal{S}_\mu{}^\lambda{}_\sigma\mathcal{S}_\nu{}^\sigma{}_\rho \\ &\quad -\mathcal{S}_\nu{}^\lambda{}_\sigma\mathcal{S}_\mu{}^\sigma{}_\rho = \mathcal{R}_{\mu\nu}{}^\lambda{}_\rho + \mathcal{S}_{\mu\nu}{}^\lambda{}_\rho.\end{aligned} \tag{A.3}$$

Given the split of the curvature of the affine connection in Eq. (A.3), we can analyse the contractions of their components. Let us take first the curvature of the symmetric connection,

$$\mathcal{R}_{\mu\nu}{}^\lambda{}_\rho = \partial_\mu\Gamma_\nu{}^\lambda{}_\rho - \partial_\nu\Gamma_\mu{}^\lambda{}_\rho + \Gamma_\mu{}^\lambda{}_\sigma\Gamma_\nu{}^\sigma{}_\rho - \Gamma_\nu{}^\lambda{}_\sigma\Gamma_\mu{}^\sigma{}_\rho.$$

The Ricci tensor is given by

$$\begin{aligned}\mathcal{R}_{\nu\rho} &= \mathcal{R}_{\lambda\nu}{}^\lambda{}_\rho \\ &= \partial_\lambda\Gamma_\nu{}^\lambda{}_\rho - \partial_\nu\Gamma_\rho{}^\lambda{}_\lambda \\ &\quad +\Gamma_\sigma{}^\lambda{}_\lambda\Gamma_\nu{}^\sigma{}_\rho - \Gamma_\nu{}^\lambda{}_\sigma\Gamma_\rho{}^\sigma{}_\lambda,\end{aligned} \tag{A.4}$$

which splits into symmetric and skew-symmetric parts,

$$\begin{aligned}\mathcal{R}_{(\nu\rho)} &= \partial_\lambda\Gamma_\nu{}^\lambda{}_\rho - \frac{1}{2}\left(\partial_\nu\Gamma_\rho{}^\lambda{}_\lambda + \partial_\rho\Gamma_\nu{}^\lambda{}_\lambda\right) \\ &\quad + \Gamma_\sigma{}^\lambda{}_\lambda\Gamma_\nu{}^\sigma{}_\rho - \Gamma_\nu{}^\lambda{}_\sigma\Gamma_\rho{}^\sigma{}_\lambda, \\ \mathcal{R}_{[\nu\rho]} &= -\frac{1}{2}\left(\partial_\nu\Gamma_\rho{}^\lambda{}_\lambda - \partial_\rho\Gamma_\nu{}^\lambda{}_\lambda\right).\end{aligned}$$

From the transformation of the connection under diffeomorphisms, one finds that the trace of the symmetric connection, $\Gamma_\mu = \Gamma_\mu{}^\lambda{}_\lambda$, transforms like

$$\Gamma'_\mu = \Gamma_\alpha\frac{\partial x^\alpha}{\partial x'^\mu} + \partial_\mu \ln \mathfrak{d},$$

with $\mathfrak{d} = \det\frac{\partial x^\alpha}{\partial x'^\nu}$ is the scalar density defined by the determinant of the transformation. Therefore, the skew-symmetric part of the Ricci tensor is given by the curl of a vector

$$\mathcal{R}_{[\mu\nu]} = \partial_\mu a_\nu - \partial_\nu a_\mu, \quad \text{with } a_\mu = a_\mu{}^\lambda{}_\lambda. \tag{A.5}$$

The tensor $a_\mu{}^\lambda{}_\rho$ is defined by the difference between the connection $\Gamma_\mu{}^\lambda{}_\rho$ and a symmetric connection of reference. In order to illustrate this point, we show two cases: (i) if one chooses the Levi–Civita connection as reference, $a_\mu{}^\lambda{}_\rho$ is the *contorsion* tensor; and (ii) if the reference is given by the parameters of the parallel displacement in an Euclidean manifold, then $a_\mu{}^\lambda{}_\rho = \frac{\partial x^\lambda}{\partial y'^\alpha}\frac{\partial y'^\alpha}{\partial x^{(\mu}\partial x^{\rho)}}$.

The second contraction of the curvature, i.e. the trace of curvature, is given by

$$\mathcal{R}_{\mu\nu}{}^\lambda{}_\lambda = \partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu = -2\mathcal{R}_{[\mu\nu]}.$$

The last step might be obtained from the algebraic Bianchi identity for a torsion free connection.

The second term coming from the splitting of the curvature tensor,

$$\begin{aligned}\mathcal{S}_{\mu\nu}{}^\lambda{}_\rho &= \nabla_\mu\mathcal{S}_\nu{}^\lambda{}_\rho - \nabla_\nu\mathcal{S}_\mu{}^\lambda{}_\rho \\ &\quad +\mathcal{S}_\mu{}^\lambda{}_\sigma\mathcal{S}_\nu{}^\sigma{}_\rho - \mathcal{S}_\nu{}^\lambda{}_\sigma\mathcal{S}_\mu{}^\sigma{}_\rho,\end{aligned} \tag{A.6}$$

admits two contractions

$$\mathcal{S}_{\mu\nu}{}^{\mu}{}_{\rho} = \nabla_\mu \mathcal{S}_\nu{}^{\mu}{}_{\rho} + \nabla_\nu \mathcal{S}_\rho - \mathcal{S}_\sigma \mathcal{S}_\nu{}^{\sigma}{}_{\rho} + \mathcal{S}_\nu{}^{\mu}{}_{\sigma} \mathcal{S}_\rho{}^{\sigma}{}_{\mu}, \tag{A.7}$$

and

$$\mathcal{S}_{\mu\nu}{}^{\lambda}{}_{\lambda} = \nabla_\mu \mathcal{S}_\nu - \nabla_\nu \mathcal{S}_\mu = \partial_\mu \mathcal{S}_\nu - \partial_\nu \mathcal{S}_\mu . \tag{A.8}$$

In the above equations we have introduced the vector $\mathcal{S}_\mu = \mathcal{S}_\mu{}^{\lambda}{}_{\lambda} = -\mathcal{S}_\lambda{}^{\lambda}{}_{\mu}$.

Note that both terms of the homothetic curvature of the linear connection, $\hat{\Gamma}$, is the curl of $\mathcal{S}_\mu - a_\mu$ and thus invariant under the addition of a gradient, i.e. $\mathcal{S}_\mu - a_\mu \mapsto \mathcal{S}_\mu - a_\mu + \partial_\mu \phi$. This reflects a *gauge* redundancy in the trace of the curvature, which is inherited from the freedom of choosing a connection of reference to define the a tensor.

Let us turn to the Bianchi identities. The torsion of the affine connection is defined as

$$\hat{\mathcal{T}}(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y], \tag{A.9}$$

and its derivative is

$$\hat{\nabla}_Z(\hat{\mathcal{T}}(X, Y)) = \hat{\nabla}_Z \hat{\mathcal{T}}(X, Y) + \hat{\mathcal{T}}(\hat{\nabla}_Z X, Y) + \hat{\mathcal{T}}(X, \hat{\nabla}_Z Y). \tag{A.10}$$

The derivative of the vectors in last two terms of Eq. (A.10), are expressible in terms of the torsion, since

$$\hat{\mathcal{T}}(\hat{\mathcal{T}}(X, Y), Z) = \hat{\mathcal{T}}(\hat{\nabla}_X Y, Z) + \hat{\mathcal{T}}(Z, \hat{\nabla}_Y X) - \hat{\mathcal{T}}([X, Y], Z). \tag{A.11}$$

The algebraic Bianchi identity is obtained by adding the cyclic permutation of the vectors X, Y and Z, which shall be denoted by the operator $\mathfrak{S}_{X,Y,Z}$. Therefore, from Eq. (A.11) one gets

$$\mathfrak{S}_{X,Y,Z}\left(\hat{\mathcal{R}}(X, Y, Z) - \hat{\mathcal{T}}(\hat{\mathcal{T}}(X, Y), Z) - \hat{\nabla}_X \hat{\mathcal{T}}(Y, Z)\right) = 0. \tag{A.12}$$

The differential Bianchi identity is obtained from the derivative of the curvature in Eq. (A.1),

$$\hat{\nabla}_Z(\hat{\mathcal{R}}(X, Y)W) = \hat{\nabla}_Z \hat{\mathcal{R}}(X, Y)W + \hat{\mathcal{R}}(\hat{\nabla}_Z X, Y)W + \hat{\mathcal{R}}(X, \hat{\nabla}_Z Y)W + \hat{\mathcal{R}}(X, Y)\hat{\nabla}_Z W,$$

after expressing the derivatives of the vectors (others than W) in terms of the torsion tensor, and the application of the cyclic permutation operator,

$$\mathfrak{S}_{X,Y,Z}\left(\hat{\nabla}_Z \hat{\mathcal{R}}(X, Y)W + \hat{\mathcal{R}}(\hat{\mathcal{T}}(X, Y), Z)W\right) = 0. \tag{A.13}$$

In order to get the last expression one uses that

$$\left[\hat{\nabla}_Z, \hat{\mathcal{R}}(X, Y)\right] W - \hat{\mathcal{R}}([X, Y], Z) = [\hat{\nabla}_Z, [\hat{\nabla}_X, \hat{\nabla}_Y]]W + \hat{\nabla}_{[[X,Y],Z]} W,$$

and the action of the cyclic permutation operator on it vanishes due to the Jacobi identity of the Lie bracket and the commutator.

Appendix B: Dimensional analysis

In Refs. [22,47–49], we used a sort of *dimensional analysis* to find the most general expression satisfying certain conditions. The idea of such analysis was to consider a couple of operators, $\mathscr{W}$ and $\mathscr{N}$ that count the (density) weight and number of free indices of an expression of the form

$$\mathcal{O} = \mathcal{A}^m \mathcal{B}^n \nabla^p \, \mathrm{d}V^q, \tag{B.1}$$

where m, n, p and q are certain power of the terms. Since we have considered terms the form of the generic operator in Eq. (B.1), our model inherits a *polynomial* character.

In order to find the most general action, we should require that the weight of the possible terms is one, $\mathscr{W}(\mathcal{O}) = 1$, which sets $q = 1$, and that there are no free indices,[19] i.e. $\mathscr{N}(\mathcal{O}) = 0$, which restrict that $m + n + p = 3$.

After setting basic structure of possible terms in the action, we have to consider all possible indices contractions, and use the symmetries of the fields to eliminate redundant contributions. Additionally, for presenting the action in Eq. (3), we have dropped terms that are related through the addition of boundary terms.

Appendix C: Reparametrisation of time coordinate

Under a change of coordinates, $x'^a = x'^a(x^i)$, the component of the connection transform as

$$\frac{\partial^2 x^i}{\partial x'^a \partial x'^b} + \Gamma_j{}^i{}_k \frac{\partial x^j}{\partial x'^a} \frac{\partial x^k}{\partial x'^b} = \Gamma'_a{}^c{}_b \frac{\partial x^i}{\partial x'^c}. \tag{C.1}$$

Given the generic form of the isotropic and homogeneous connection, Eq. (27), one notices that the function j has no dynamics in the curvature tensors. A natural question is whether there is a reparametrisation of the time coordinate that allows to set $j = 0$. Since this function comes from the component $\Gamma_0{}^0{}_0$, consider a transformation of the form,

$$t' = t'(t), \qquad r' = r \qquad \varphi' = \varphi.$$

Equation (C.1) for $a = b = c = 0$, and $\Gamma'^{\,0}_{0\,0} = 0$, yields

$$\frac{\partial^2 t}{\partial t'^2} + j \cdot \left(\frac{\partial t}{\partial t'}\right)^2 = 0,$$

19 We have used a convention where positive value of $\mathscr{N}$ denotes the net number of upper indices, while negative values denote the net number of lower indices.

which can be written as a total derivative,

$$\frac{1}{X}\partial_{t'}\left(X\partial_{t'}t\right)=0,$$

for $j=\frac{1}{X}\partial_t X$, or equivalently $X=e^{\int \mathrm{d}t\, j(t)}=e^{J(t)}$. From here, one have that

$$t'=\int \mathrm{d}t\, e^{J(t)}.$$

With the above transformation, it can be checked with ease that the effect of the time reparametrisation is a scaling of the other three functions of time entering in the connection, and thus $f(t)\mapsto f(t')$, $g(t)\mapsto g(t')$ and $h(t)\mapsto h(t')$.

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